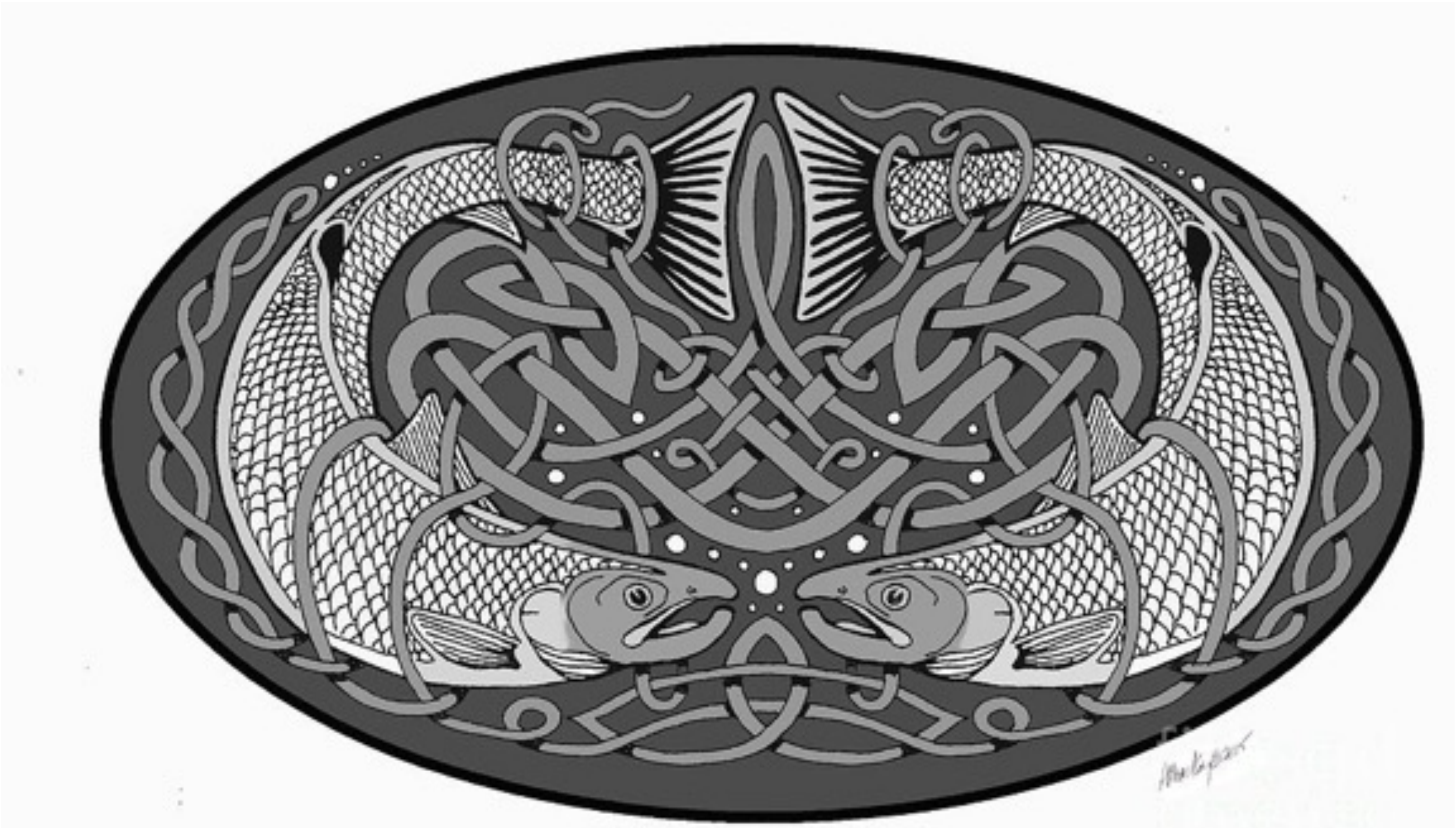
Dehn functions, the word problem, hydra, and magical salmon

Timothy Riley

June 17, 2016



Salmon of Knowledge

GAGTA 2016

Stevens Institute

The word problem

Mathematische Annalen, 71(1), 1911

Über unendliche diskontinuierliche Gruppen.

Von

M. DEHN in Kiel.

1. *Das Identitätsproblem:* Irgend ein Element der Gruppe ist durch seine Zusammensetzung aus den Erzeugenden gegeben. Man soll eine Methode angeben, um mit einer endlichen Anzahl von Schritten zu entscheiden, ob dies Element der Identität gleich ist oder nicht.

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"An element of a group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not."

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Given a word on $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$, try to convert it to the empty word by applying defining relations.

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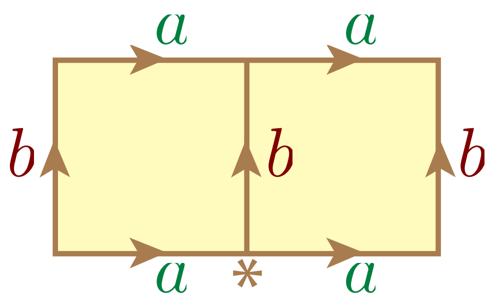

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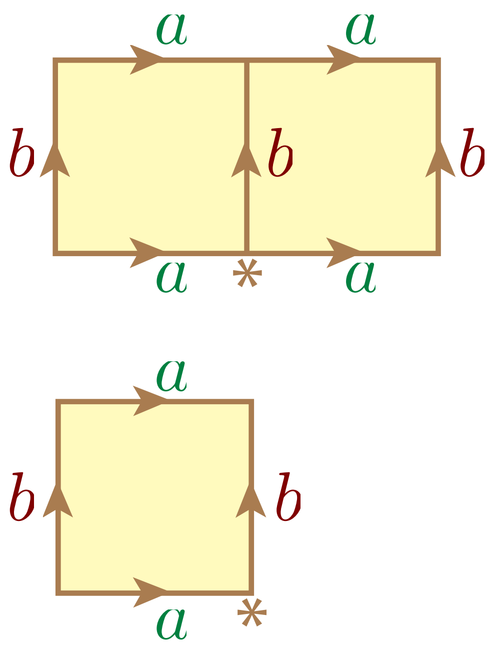

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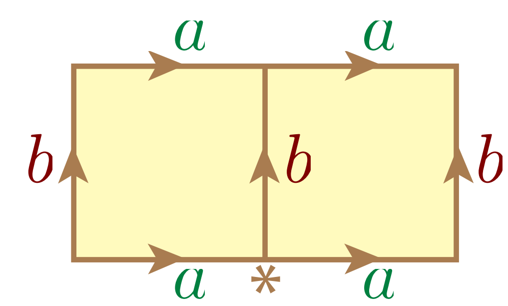

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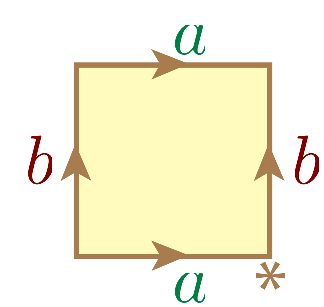

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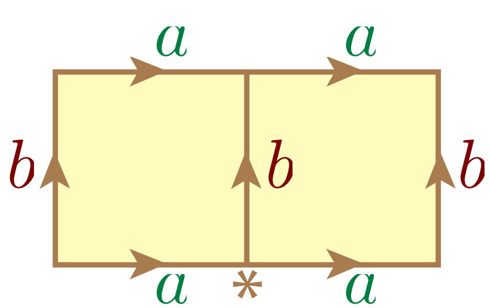

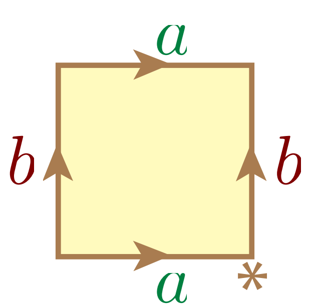

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The minimal such function is the *Dehn function* $\mathrm{Dehn}(n)$.

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- the word problem is solvable,
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- the Dehn function is bounded from above by a recursive function.

But groups with large Dehn function may have efficient solutions to their word problem.

The salmon, now returned to the river, spoke again. "I am the Salmon of Wisdom. I have fed on the Nuts of Knowledge that fall from the hazel tree which leans over the Well at the World's End. Since you have set me free, I will grant you a gift." ...

Now the mathematician had been thinking hard about a certain finite presentation of a group. "What I want", said he, "is a machine that will tell me whether or not a word in the generators equals one in the group I have recently been considering." ...

So the salmon swam deeper into the river, and returned carrying a little machine. It looked very delicate and attractive, and the mathematician was delighted by it. He took it home, and used it with great pleasure ...

"This machine is very nice, but I would like a machine that does more. Instead of one that just shows a green light when the word equals one, I would like one that actually tells me how to write that word as a product of conjugates of the relators and their inverses." ...

"I feel quite sure that, since you have given me one machine, you can also give me the more powerful one I desire."

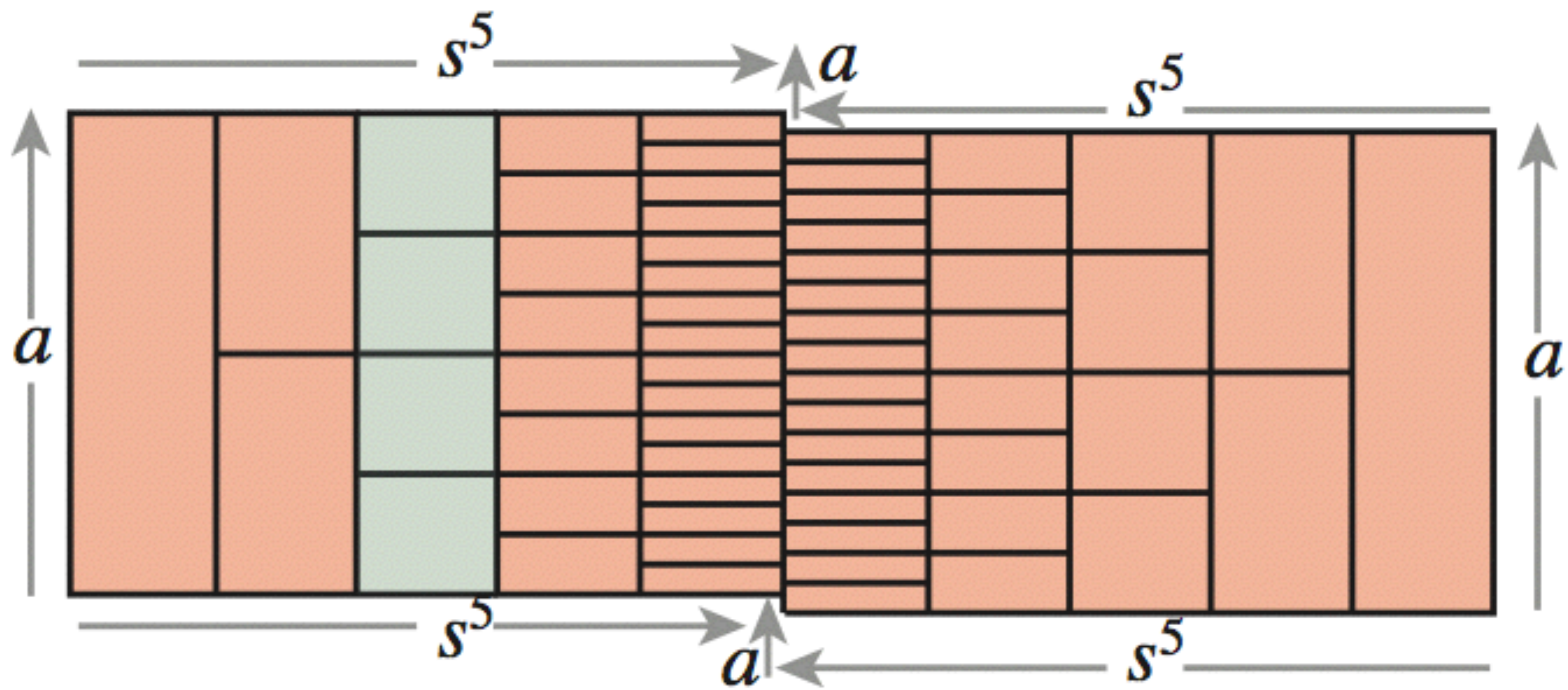
"You are right" said the salmon, and swam deeper into the river. He emerged carrying a heavy and ugly machine. ...

D. E. Cohen, The Mathematician who had little wisdom,
London Mathematical Soc. Lecture Notes, 204, CUP, 1995

Example: $\langle a, s \mid s^{-1}as = a^2 \rangle \ \leq \ \mathrm{GL}_2(\mathbb{Q})$

via $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s \mapsto \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

but $\mathrm{Dehn}(n) \simeq 2^n$. (In fact, WP is in TC^0.)



Cohen–Madlener–Otto 1993

Examples with

$$\mathrm{Dehn}(n) \simeq A_k(n)$$
$$\mathrm{Dehn}(n) \simeq A_n(n)$$

but $WP \leq \exp^{(3)}(n)$ time

$$A_1 : \mathbb{Z} \to \mathbb{Z} \qquad n \mapsto 2n$$

$$A_i(0) = 1 \quad \forall i \geq 2$$

$$A_i : \mathbb{N} \to \mathbb{N} \qquad A_{i+1}(n+1) = A_i A_{i+1}(n)$$

Wilhelm Ackermann

$i \backslash n$	0	1	2	3	4	5	$\cdots$	n
1								
2								
3								
4								
$\vdots$								

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4	1	2	4	65536	$A_3(65536)$	$\cdots$	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				

Kharlampovich–Miasnikov–Sapir 2013

For any given recursive f, an example with

$$\mathrm{Dehn}(n) \geq f(n) \text{ but } WP \text{ in } P$$

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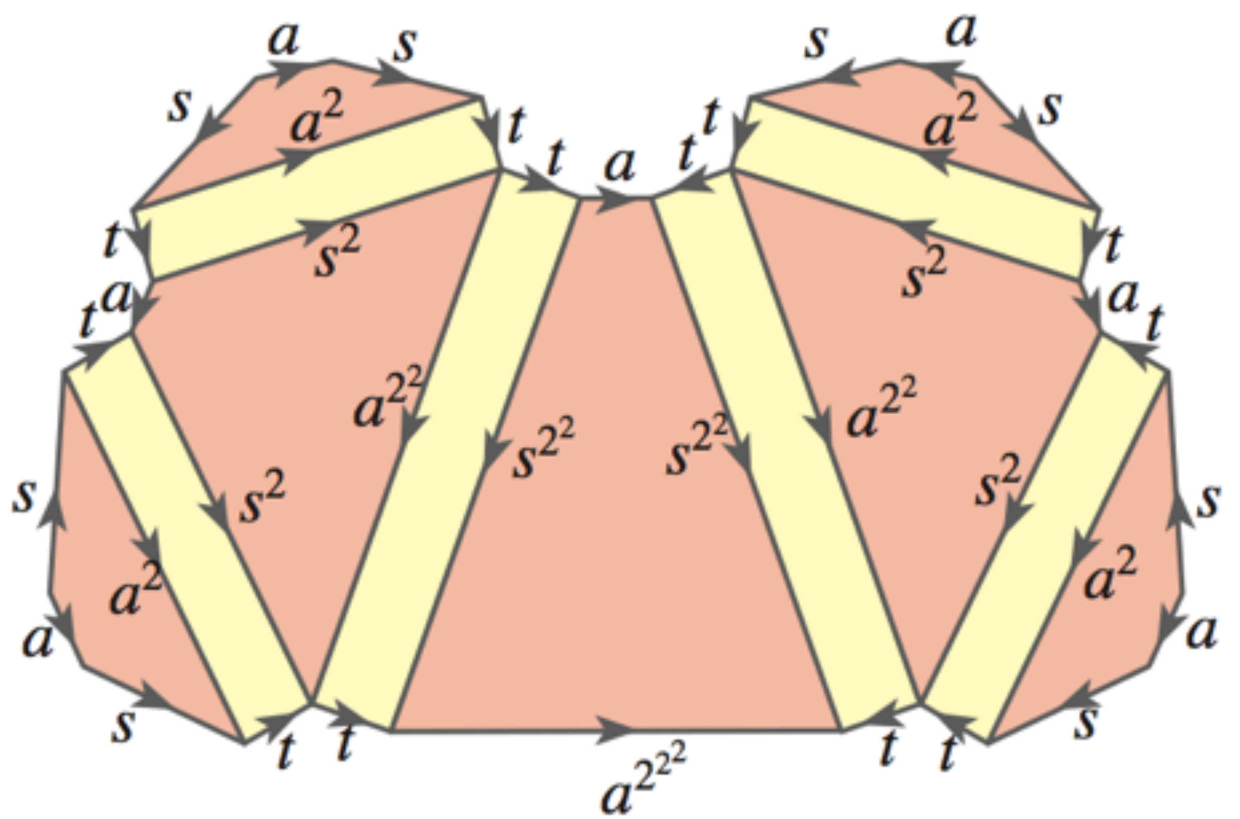

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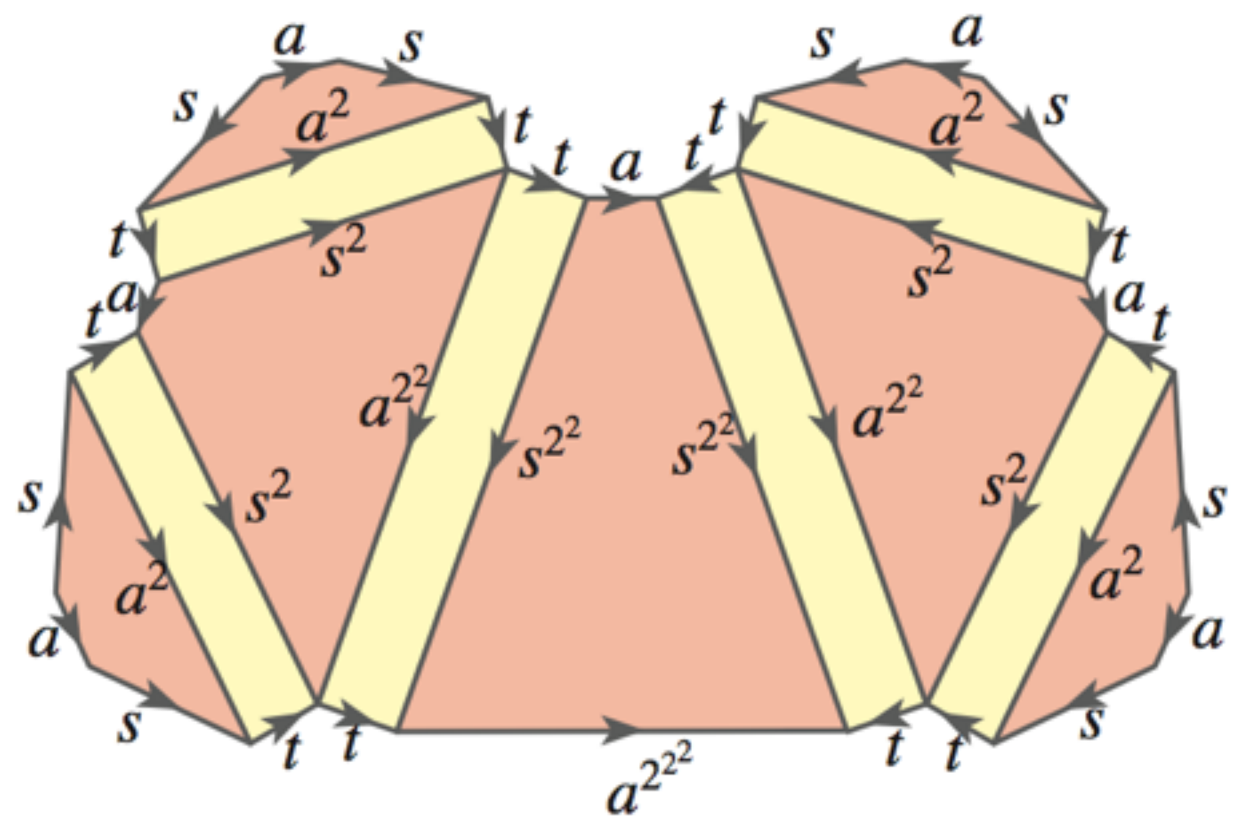

Platonov (cf. Bernasconi & Gersten)

$$\mathrm{Dehn}(n) \simeq \left. 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \right\} \lfloor \log_2 n \rfloor$$

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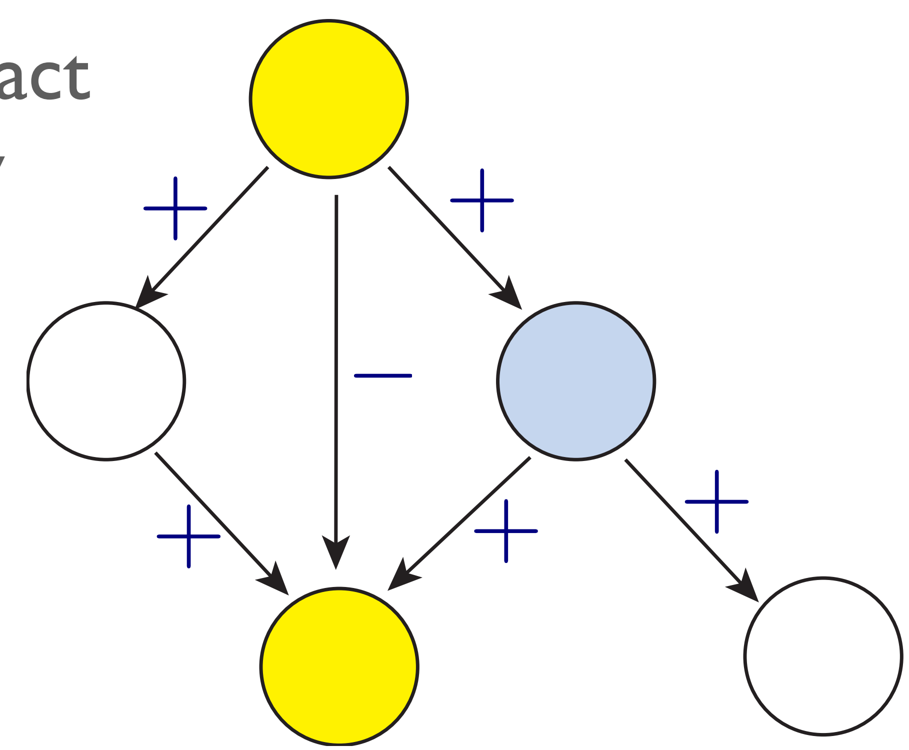

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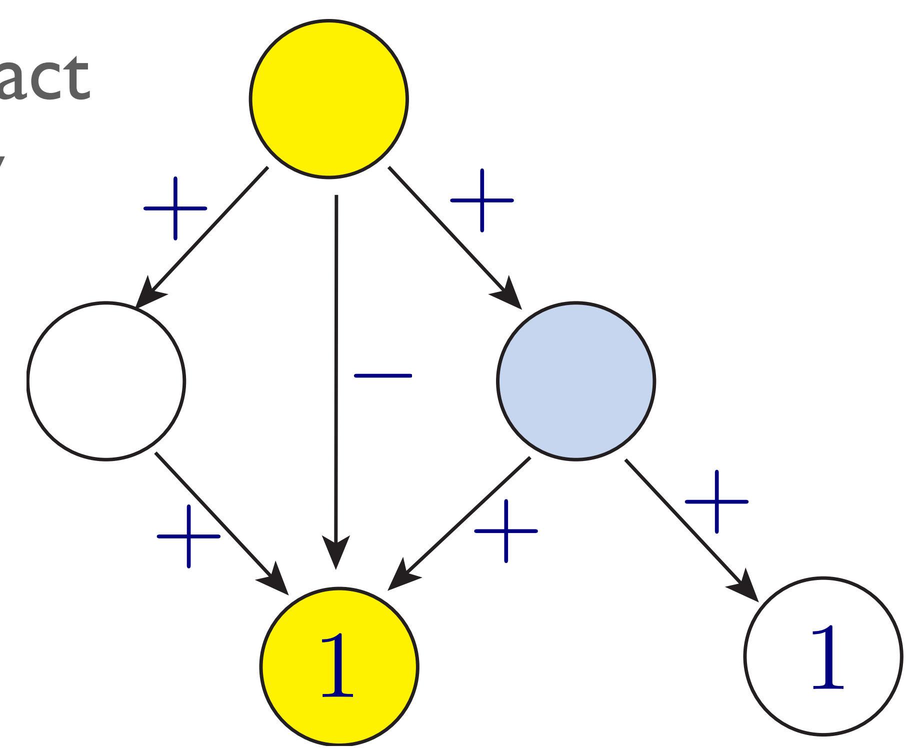

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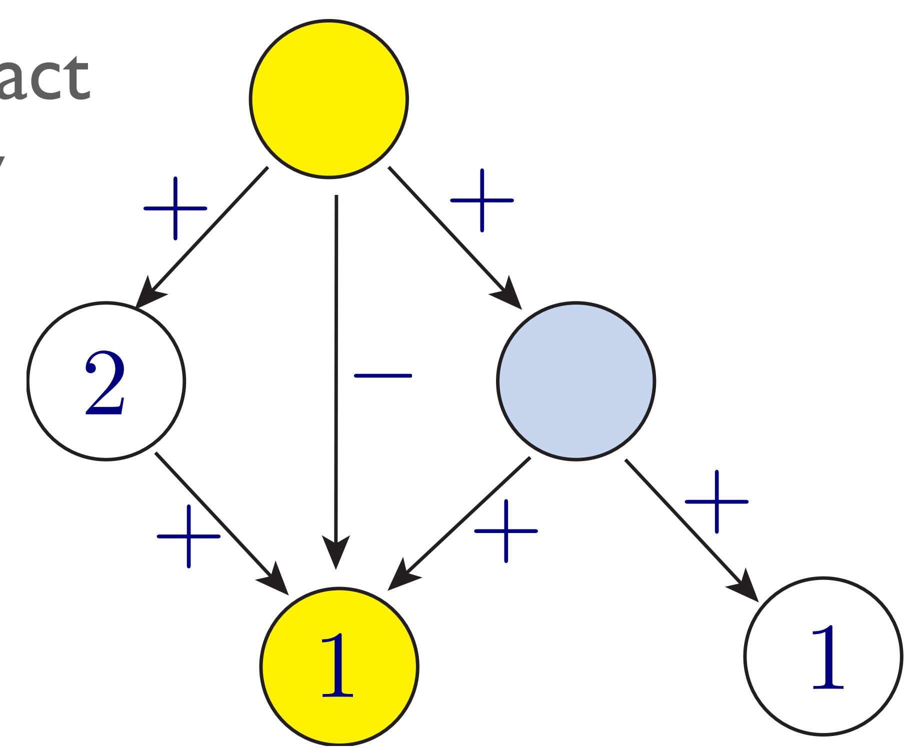

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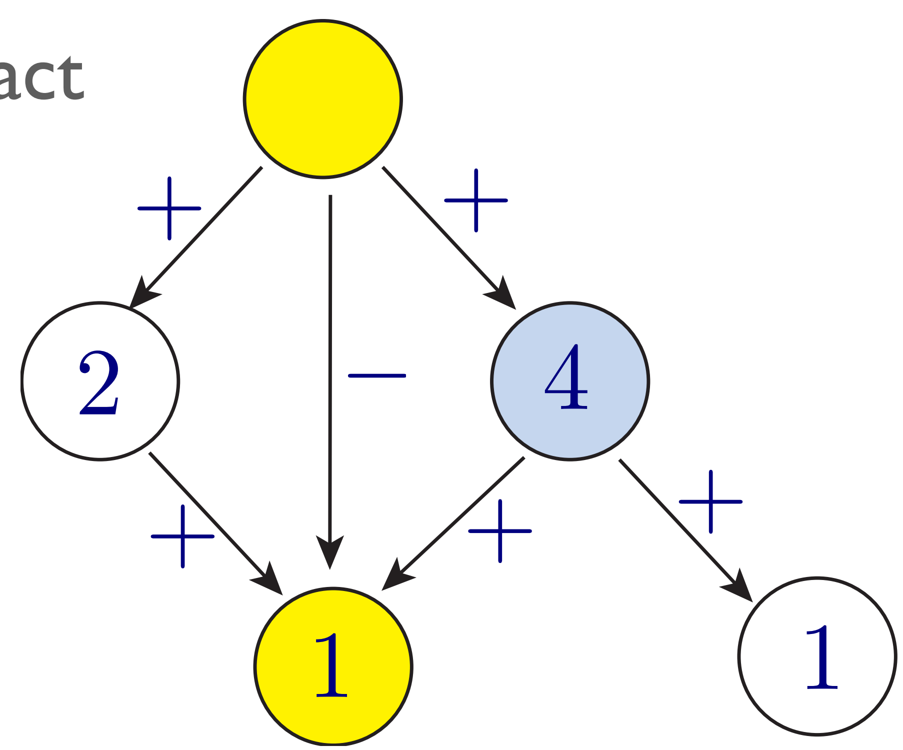

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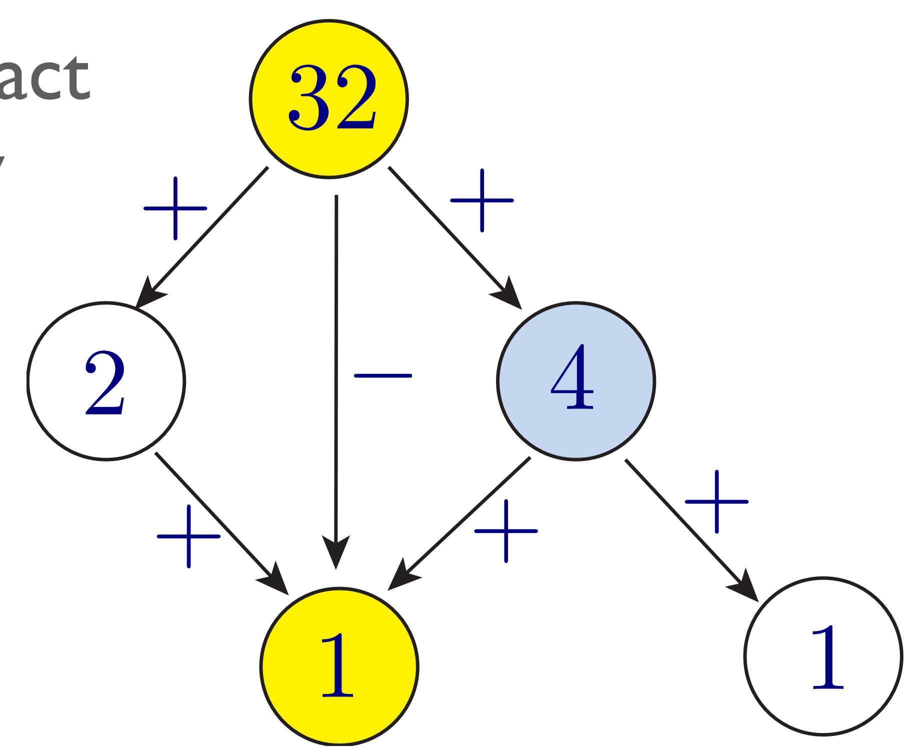

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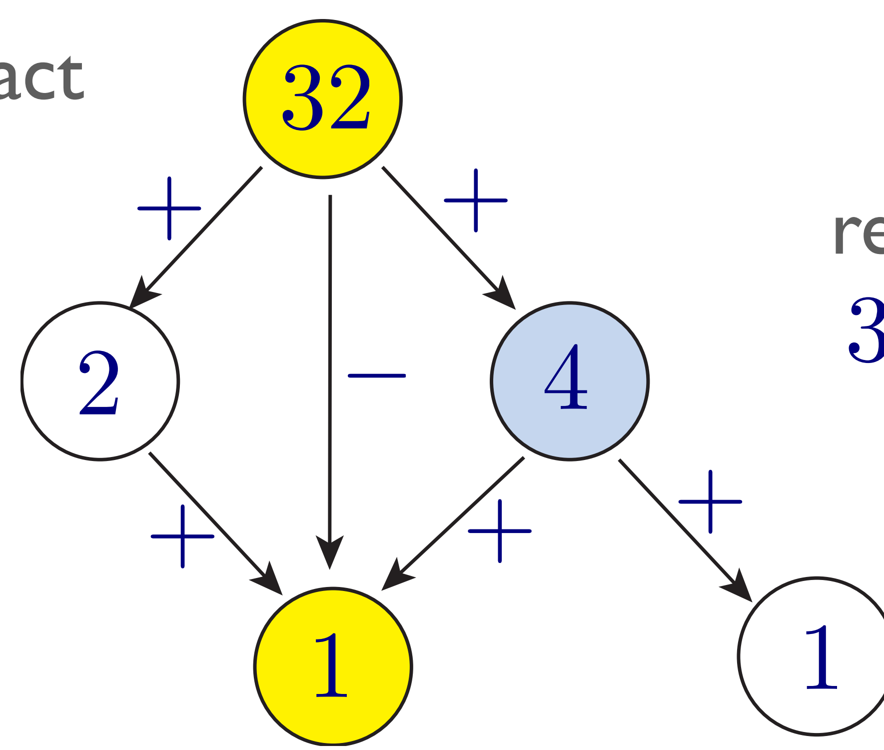

represents $32 + 1 - 4 = 29$

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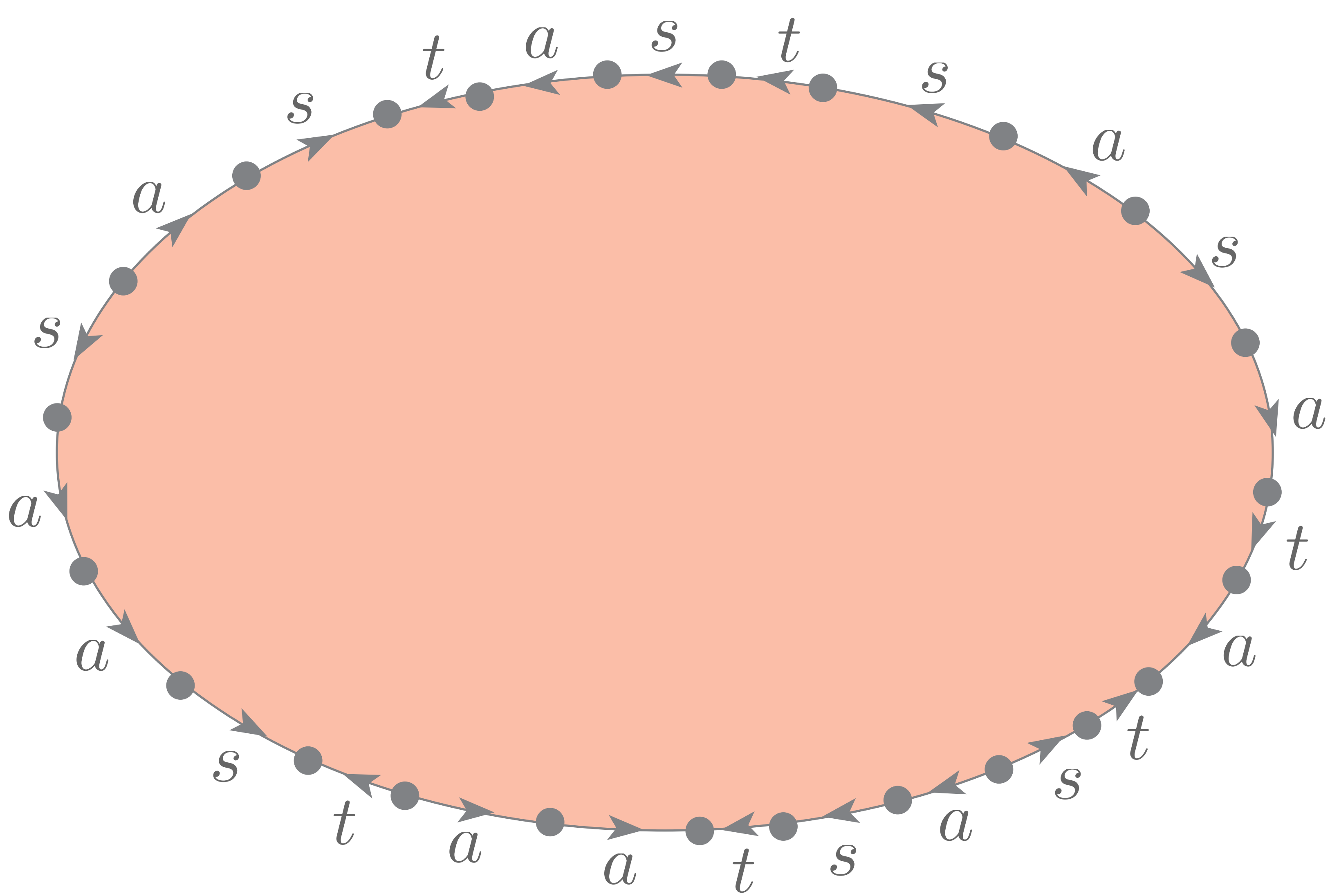

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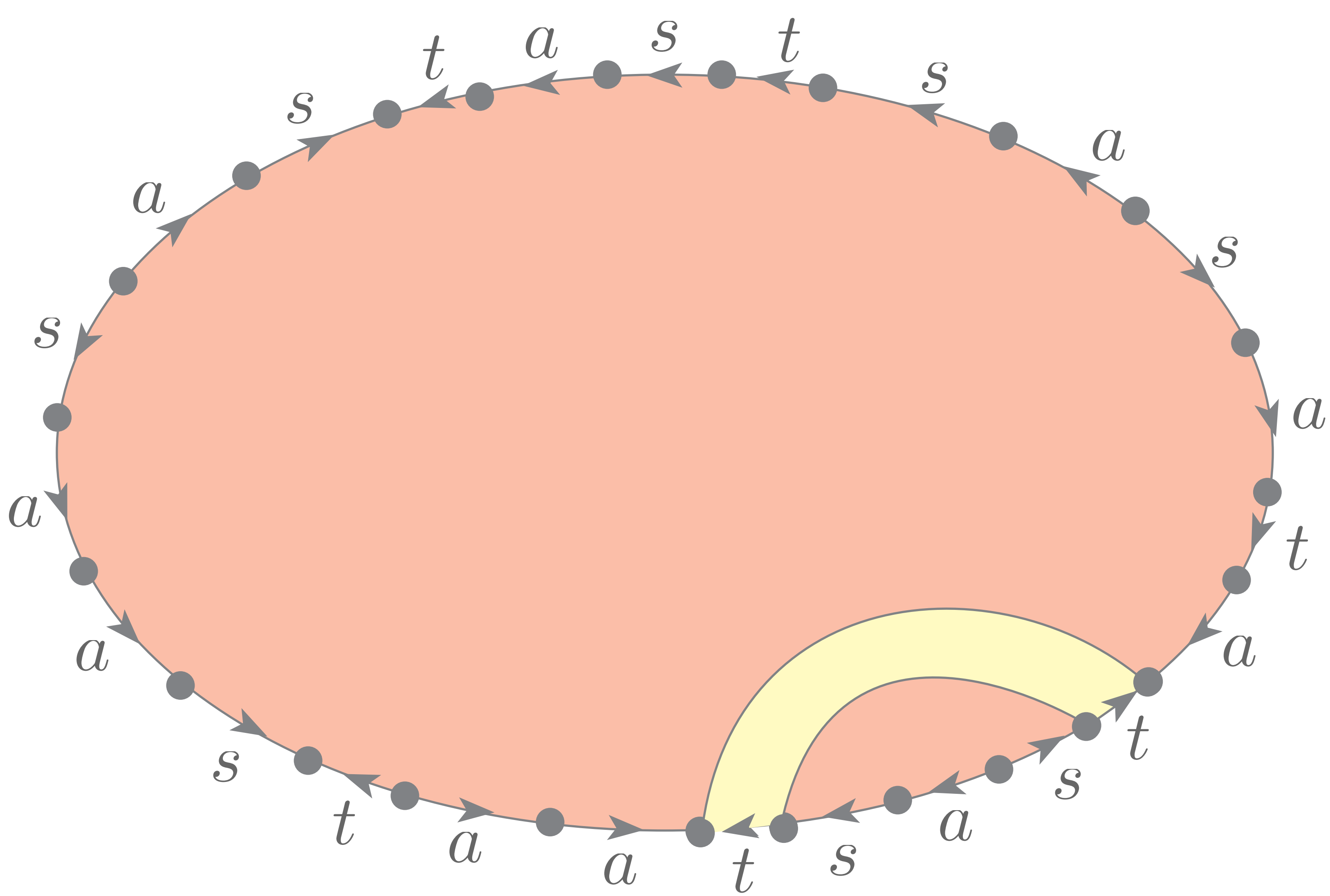

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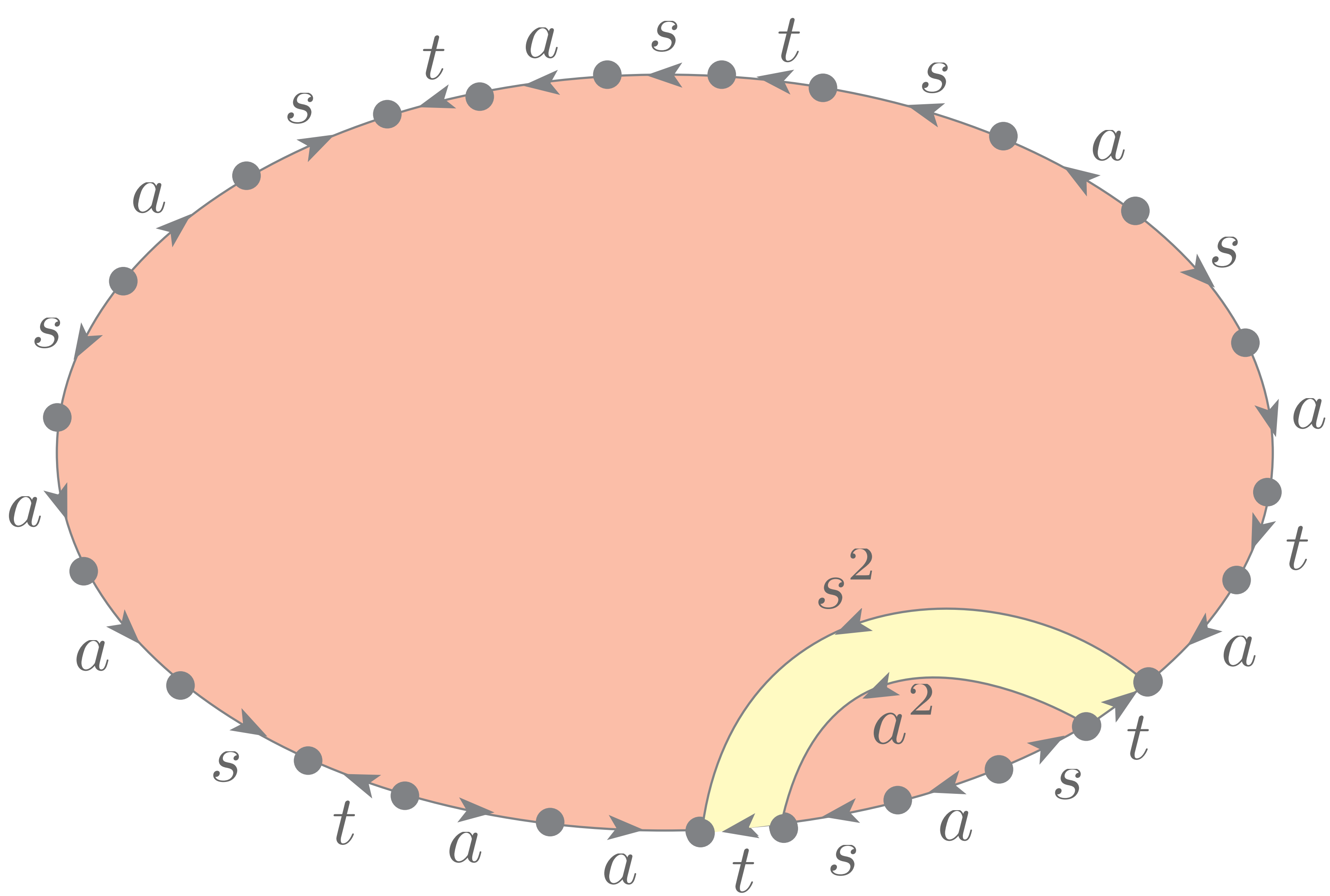

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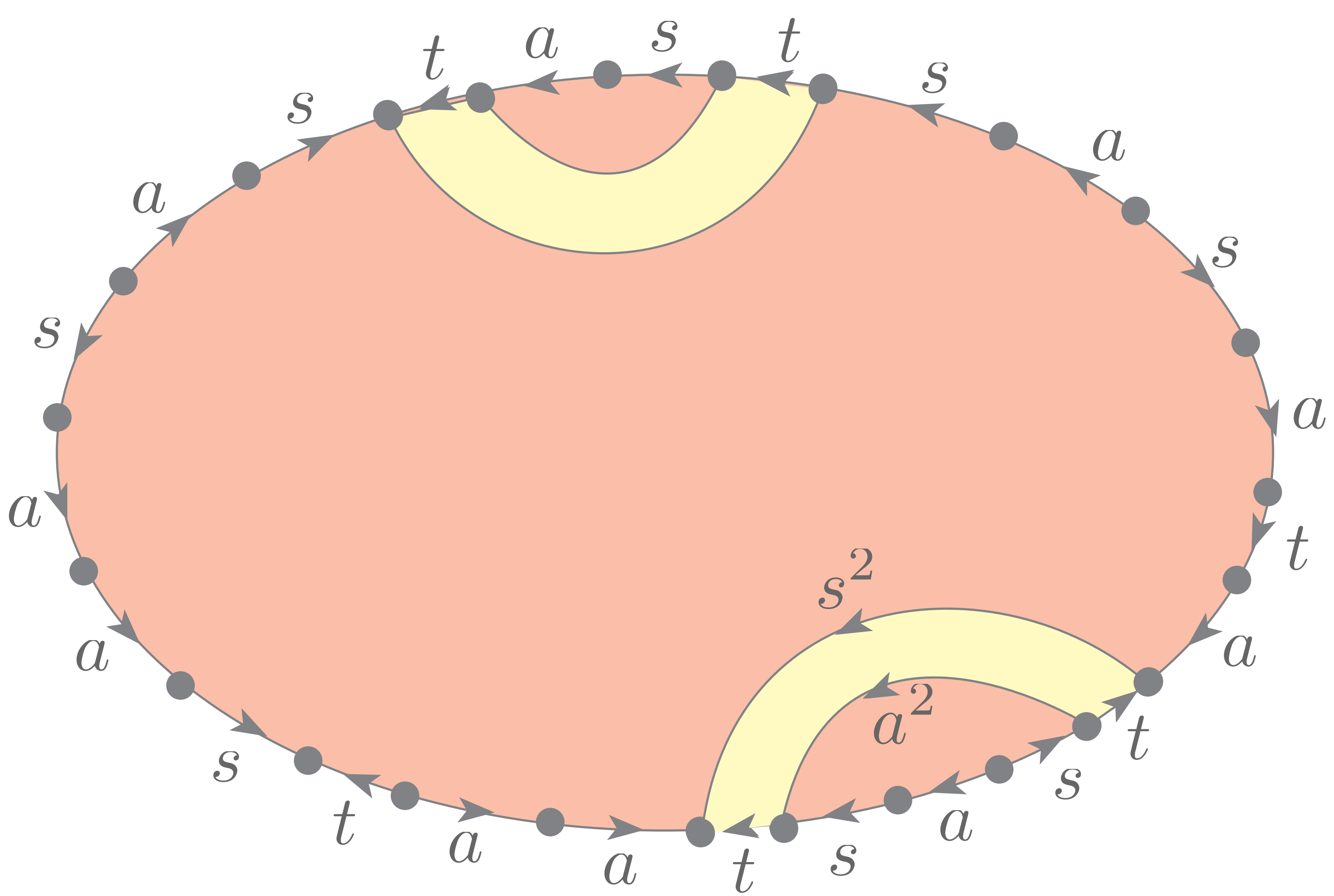
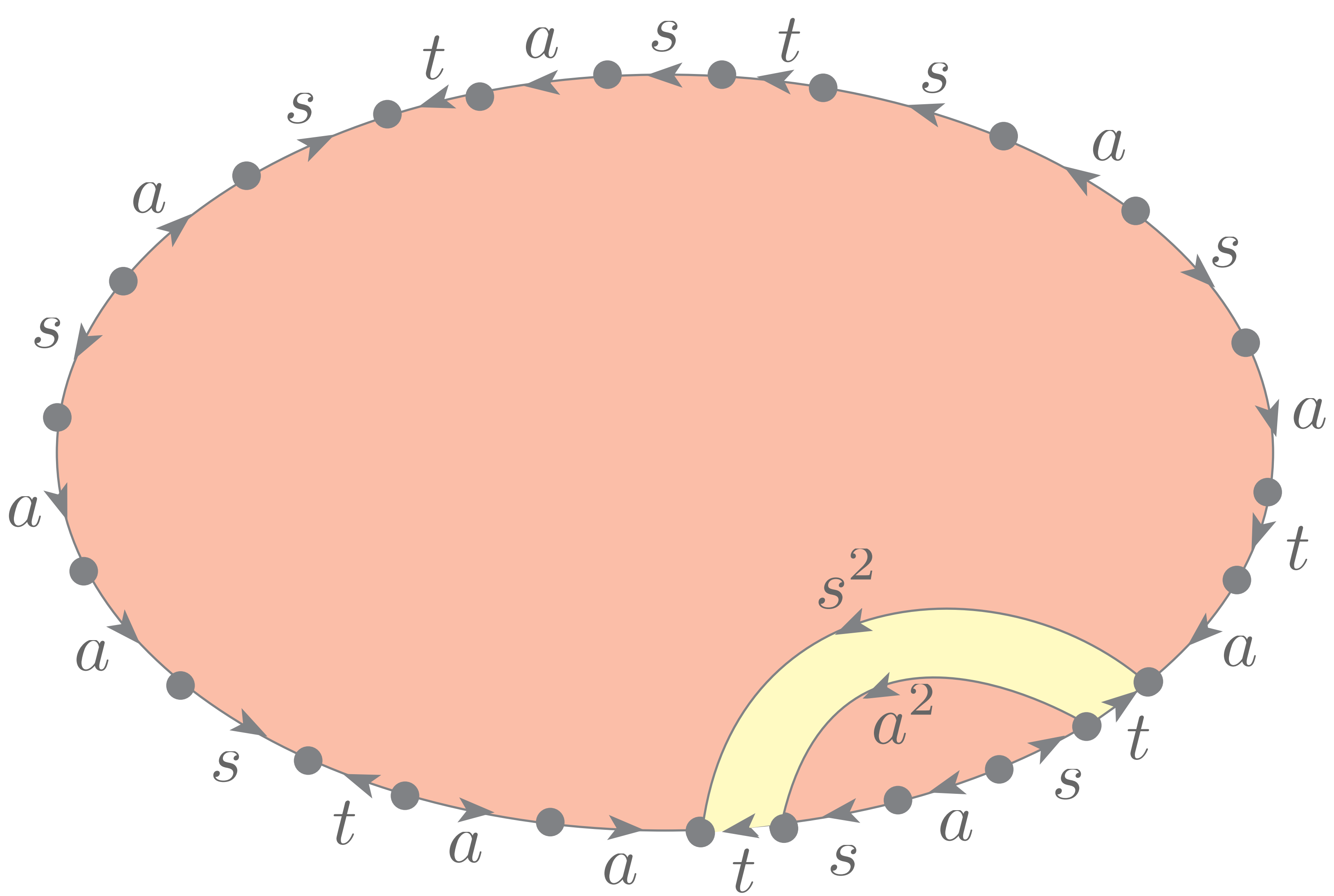


Baumslag's one-relator group

$$\langle a, t \mid (t^{-1}at)^{-1}a(t^{-1}at) \ = \ a^2 \rangle$$

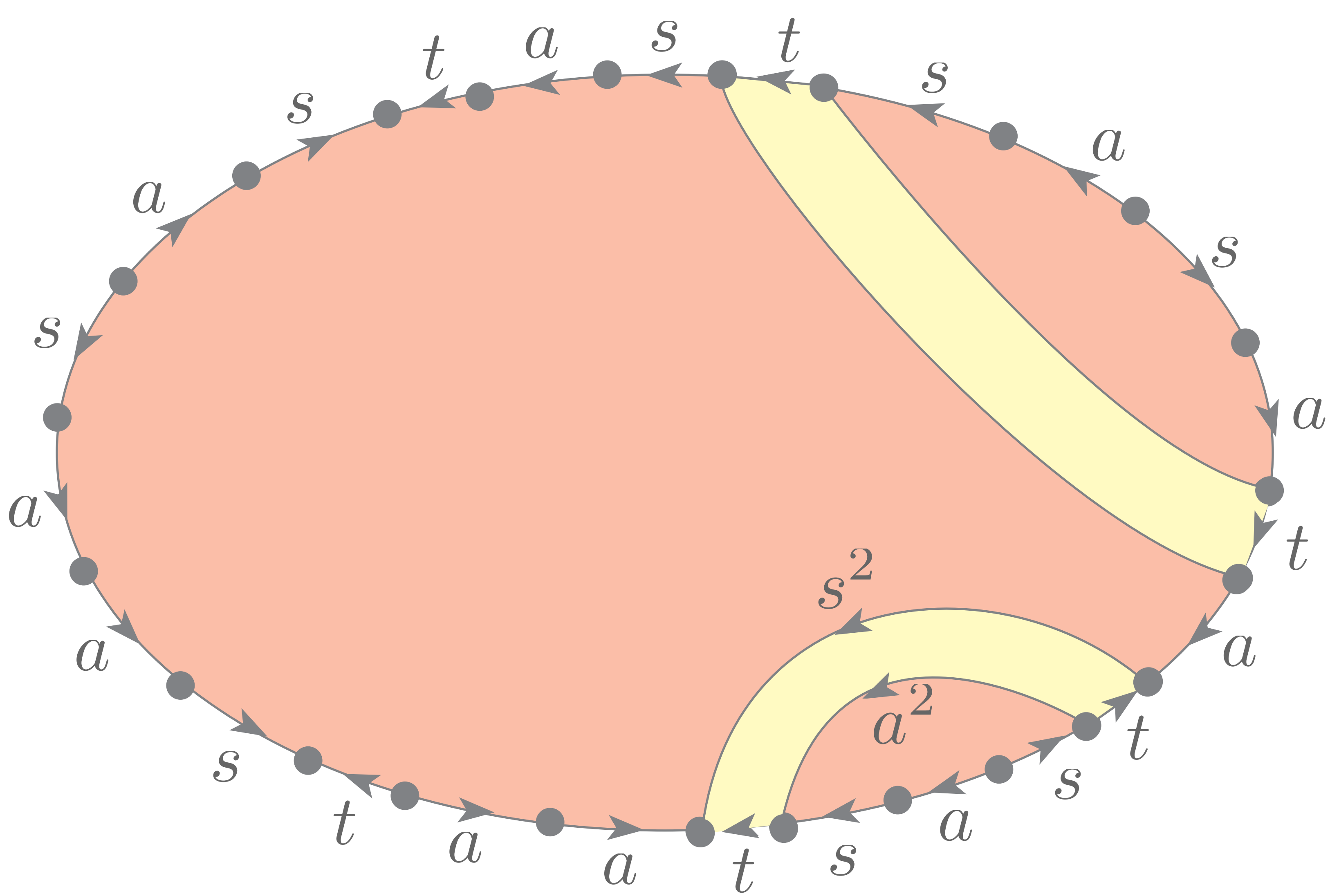
$$= \langle a, s, t \mid s^{-1}as = a^2, \ s = t^{-1}at \rangle$$



Baumslag's one-relator group

$$\langle a, t \mid (t^{-1}at)^{-1}a(t^{-1}at) \;=\; a^2 \rangle$$

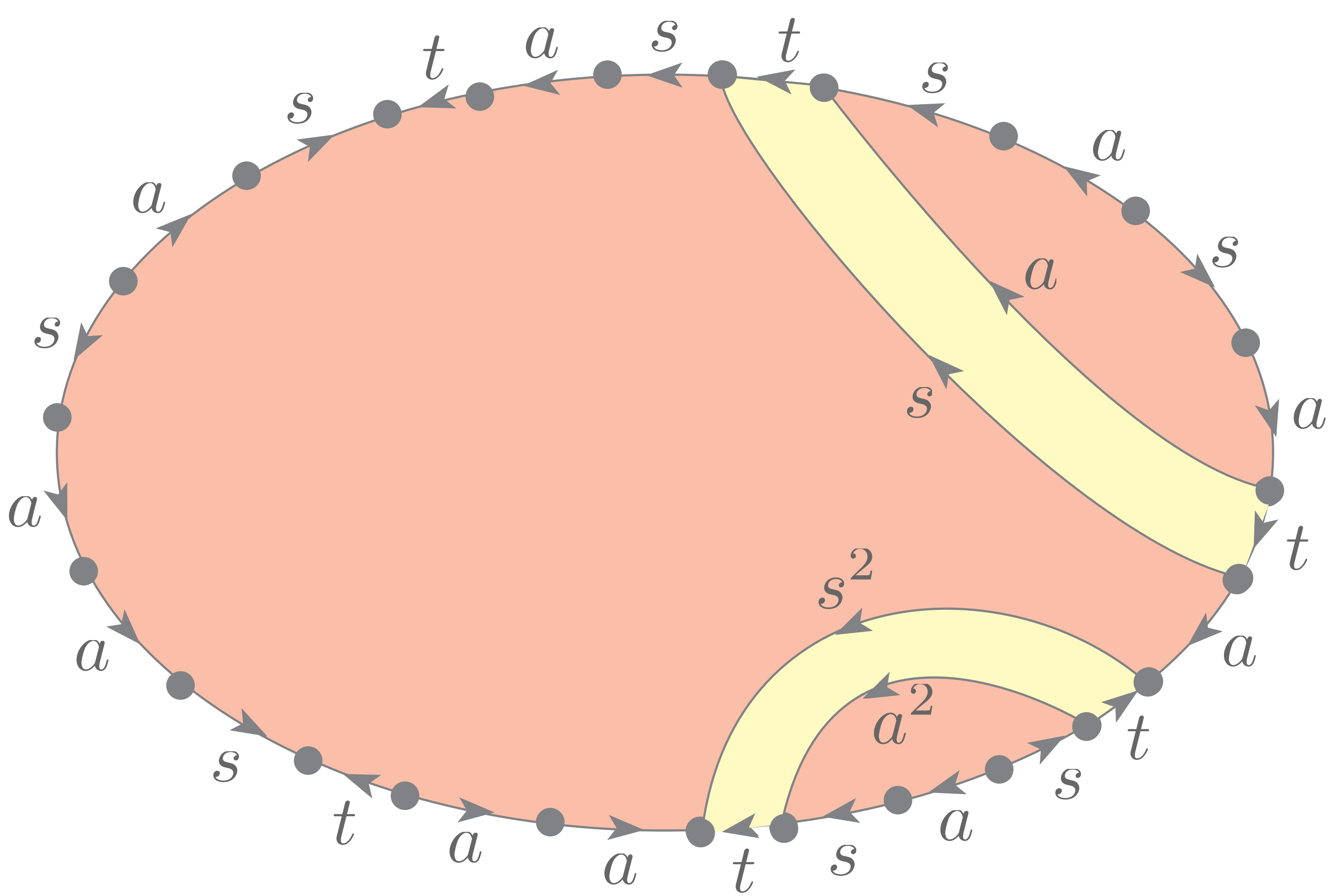
$$= \langle a, s, t \mid s^{-1}as = a^2,\; s = t^{-1}at \rangle$$



Baumslag's one-relator group

$$\langle a, t \mid (t^{-1}at)^{-1}a(t^{-1}at) \;=\; a^2\rangle$$

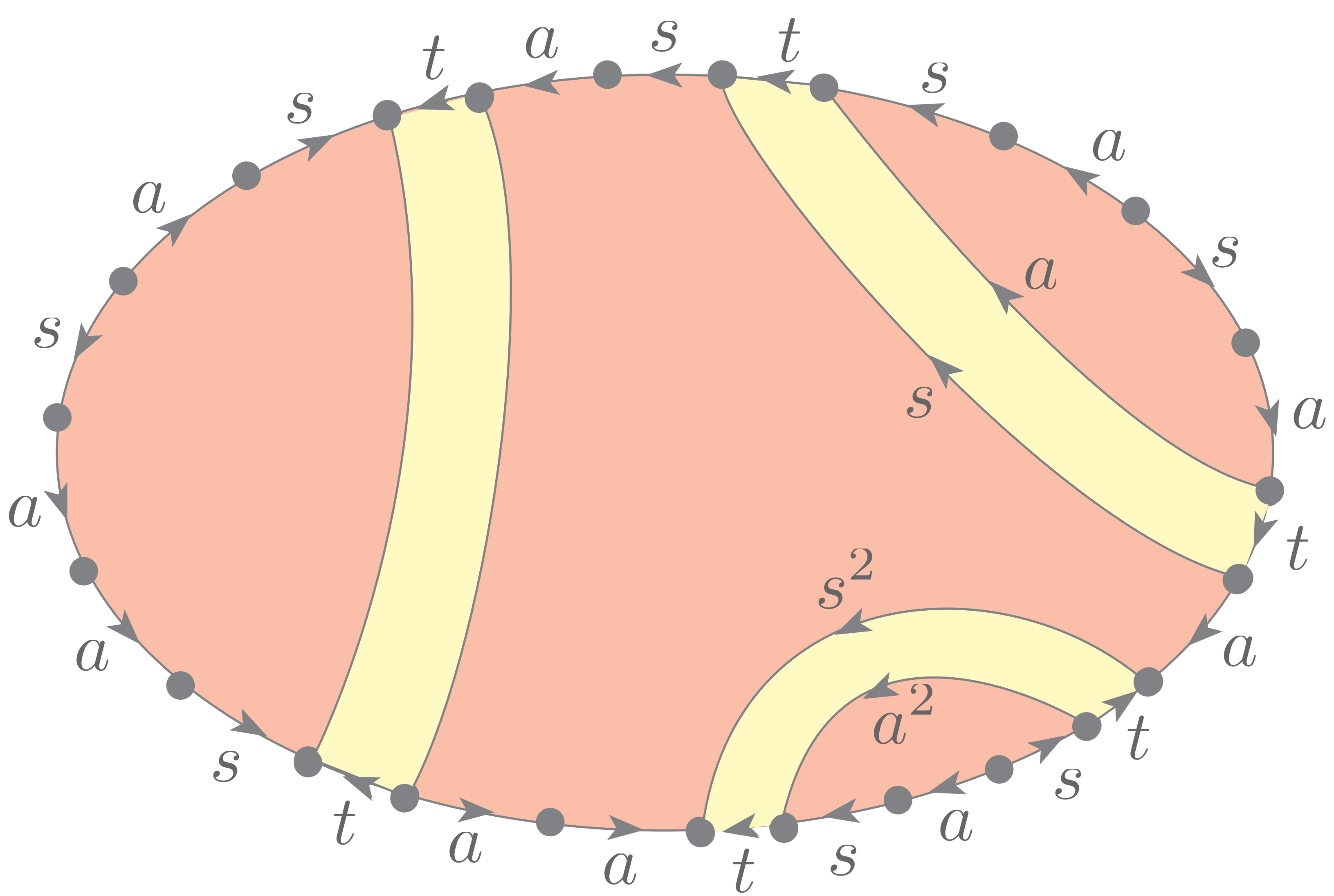
$$= \langle a, s, t \mid s^{-1}as = a^2,\; s = t^{-1}at\rangle$$



Baumslag's one-relator group

$$\langle a, t \mid (t^{-1}at)^{-1}a(t^{-1}at) \ = \ a^2 \rangle$$

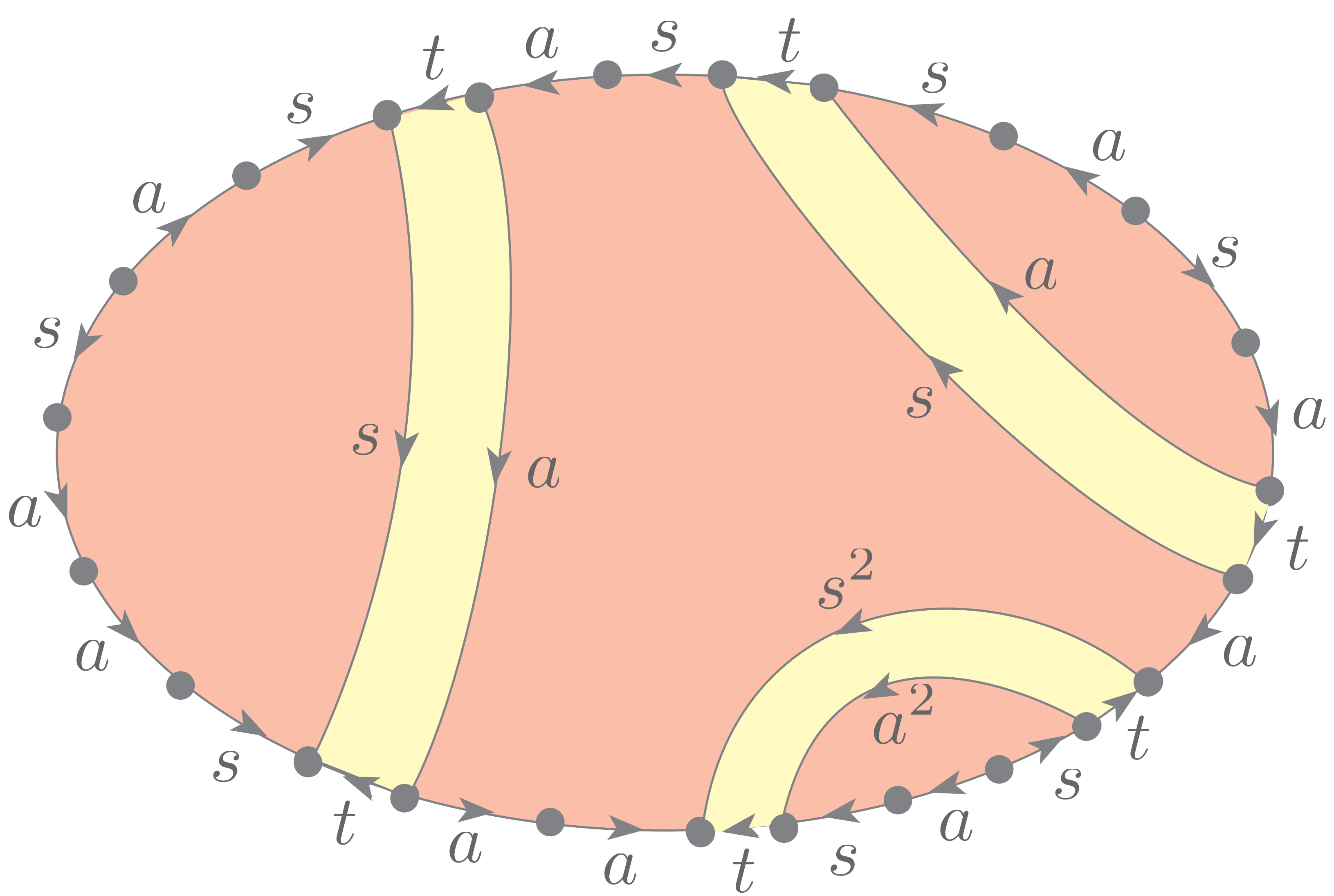
$$= \langle a, s, t \mid s^{-1}as = a^2, \ s = t^{-1}at \rangle$$



Baumslag's one-relator group

$$\langle a, t \mid (t^{-1}at)^{-1}a(t^{-1}at) \;=\; a^2\rangle$$

$$= \langle a, s, t \mid s^{-1}as = a^2,\; s = t^{-1}at\rangle$$



$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 strikes

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$ $\qquad\qquad$ $a\,a\,a$

$a\,b$

$b\,c$

c

4 strikes

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

$a\,a\,a$

$a\,b\,a\,b$

4 strikes

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 strikes

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$c\,a\,b\,b\,c\,b\,c\,c$

4 strikes

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 strikes

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$c\,a\,b\,b\,c\,b\,c\,c$

$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$	$a\,a\,a$
$a\,b$	$a\,b\,a\,b$
$b\,c$	$b\,c\,a\,b\,b\,c$
c	$c\,a\,b\,b\,c\,b\,c\,c$
	$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$
	$b\,c\,b\,c\,c\,b\,c\,c\,c\,b\,c\,c\,c\,c$

4 strikes

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 strikes

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$c\,a\,b\,b\,c\,b\,c\,c$

$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$

$b\,c\,b\,c\,c\,b\,c\,c\,c\,b\,c\,c\,c\,c$

$c\,b\,c\,c\,c\,b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 strikes

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$c\,a\,b\,b\,c\,b\,c\,c$

$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$

$b\,c\,b\,c\,c\,b\,c\,c\,c\,b\,c\,c\,c\,c$

$c\,b\,c\,c\,c\,b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c$

$b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 **strikes**

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$c\,a\,b\,b\,c\,b\,c\,c$

$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$

$b\,c\,b\,c\,c\,b\,c\,c\,c\,b\,c\,c\,c\,c$

$c\,b\,c\,c\,c\,b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c$

$b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$

$a\,b$

$b\,c$

c

4 **strikes**

$a\,a\,a$

$a\,b\,a\,b$

$b\,c\,a\,b\,b\,c$

$c\,a\,b\,b\,c\,b\,c\,c$

$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$

$b\,c\,b\,c\,c\,b\,c\,c\,c\,b\,c\,c\,c\,c$

$c\,b\,c\,c\,c\,b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c$

$b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

a a
a b
b c
c

4 strikes

a a a
a b a b
b c a b b c
c a b b c b c c
a b b c b c c b c c c
b c b c c b c c c b c c c c
c b c c c b c c c c b c c c c c
b c c c c b c c c c c b c c c c c c
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c c c c c c c c c c c
c c c c c c c c c c

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

aa
ab
bc
c

4 strikes

aaa
$abab$
$bcabbc$
$cabbcbcc$
$abbcbccbccc$
$bcbccbcccbcccc$
$cbcccbccccbccccc$
$bccccbcccccbcccccc$
$ccccbccccccbccccccc$
$cccbcccccccbcccccccc$
$ccbccccccccbccccccccc$
$cbcccccccccbcccccccccc$
$bccccccccccbccccccccccc$
$ccccccccccbcccccccccccc$
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$cccccccc$
$ccccccc$
$cccccc$
$ccccc$
$cccc$
ccc
cc

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

a a
a b
b c
c

4 strikes

a a a
a b a b
b c a b b c
c a b b c b c c
a b b c b c c b c c c
b c b c c b c c c b c c c c
c b c c c b c c c c b c c c c c
b c c c c b c c c c c b c c c c c c
c c c c b c c c c c c b c c c c c c c
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c c c c c c
c c c c c
c c c c
c c c
c c
c

46 strikes

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

aa
ab
bc
c

4 **strikes**

aaa
$abab$
$bcabbc$
$cabbcbcc$
$abbcbccbccc$
$bcbccbcccbcccc$
$cbcccbccccbccccc$
$bccccbcccccbcccccc$
$ccccbccccccbccccccc$
$cccbcccccccbcccccccc$
$ccbccccccccbccccccccc$
$cbcccccccccbcccccccccc$
$bccccccccccbccccccccccc$
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$cccccccc$
$ccccccc$
$cccccc$
$ccccc$
$cccc$
ccc
cc
c

46 **strikes**

$aaaa$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

a a
a b
b c
c

4 strikes

a a a
a b a b
b c a b b c
c a b b c b c c
a b b c b c c b c c c
b c b c c b c c c b c c c c
c b c c c b c c c c b c c c c c
b c c c c b c c c c c b c c c c c c
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c c c c c
c c c c
c c c
c c
c

46 strikes

a a a a
a b a b a b

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

aa

ab

bc

c

4 **strikes**

aaa

$abab$

$bcabbc$

$cabbcbcc$

$abbcbccbccc$

$bcbccbcccbcccc$

$cbcccbccccbccccc$

$bccccbcccccbcccccc$

$ccccbccccccbccccccc$

$cccbcccccccbcccccccc$

$ccbccccccccbccccccccc$

$cbcccccccccbcccccccccc$

$bccccccccccbccccccccccc$

$ccccccccccbcccccccccccc$

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$cbccccccccccccccccccccc$

$bcccccccccccccccccccccc$

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$cccccccccccc$

$ccccccccccc$

$cccccccccc$

$ccccccccc$

$cccccccc$

$ccccccc$

$cccccc$

$ccccc$

$cccc$

ccc

cc

c

46 **strikes**

$aaaa$

$ababab$

$bcabbcabbc$

$a \mapsto a\,b \quad b \mapsto b\,c \quad c \mapsto c$

$a\ a$

$a\ b$

$b\ c$

c

4 **strikes**

$a\ a\ a$

$a\ b\ a\ b$

$b\ c\ a\ b\ b\ c$

$c\ a\ b\ b\ c\ b\ c\ c$

$a\ b\ b\ c\ b\ c\ c\ b\ c\ c\ c$

$b\ c\ b\ c\ c\ b\ c\ c\ c\ b\ c\ c\ c\ c$

$c\ b\ c\ c\ c\ b\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c$

$b\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$b\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c\ c$

$c\ c\ c\ c\ c$

$c\ c\ c\ c$

$c\ c\ c$

$c\ c$

c

46 **strikes**

$a\ a\ a\ a$

$a\ b\ a\ b\ a\ b$

$b\ c\ a\ b\ b\ c\ a\ b\ b\ c$

$c\ a\ b\ b\ c\ b\ c\ c\ a\ b\ b\ c\ b\ c\ c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

$a\,a$
$a\,b$
$b\,c$
c

4 **strikes**

$a\,a\,a$
$a\,b\,a\,b$
$b\,c\,a\,b\,b\,c$
$c\,a\,b\,b\,c\,b\,c\,c$
$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$
$b\,c\,b\,c\,c\,b\,c\,c\,c\,b\,c\,c\,c\,c$
$c\,b\,c\,c\,c\,b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c$
$b\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$b\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
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$c\,c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c\,c$
$c\,c\,c\,c\,c$
$c\,c\,c\,c$
$c\,c\,c$
$c\,c$
c

46 **strikes**

$a\,a\,a\,a$
$a\,b\,a\,b\,a\,b$
$b\,c\,a\,b\,b\,c\,a\,b\,b\,c$
$c\,a\,b\,b\,c\,b\,c\,c\,a\,b\,b\,c\,b\,c\,c$
$a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c\,a\,b\,b\,c\,b\,c\,c\,b\,c\,c\,c$

$a \mapsto ab \quad b \mapsto bc \quad c \mapsto c$

---

aa
ab
bc
c

4 **strikes**

aaa
$abab$
$bcabbc$
$cabbcbcc$
$abbcbccbccc$
$bcbccbcccbcccc$
$cbcccbccccbccccc$
$bccccbcccccbcccccc$
$ccccbccccccbccccccc$
$ccc\,b\,ccccccc\,b\,cccccccc$
$cc\,b\,cccccccc\,b\,ccccccccc$
$c\,b\,ccccccccc\,b\,cccccccccc$
$b\,cccccccccc\,b\,ccccccccccc$
$cccccccccc\,b\,cccccccccccc$
$ccccccccc\,b\,ccccccccccccc$
$cccccccc\,b\,cccccccccccccc$
$ccccccc\,b\,ccccccccccccccc$
$cccccc\,b\,cccccccccccccccc$
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$cccc\,b\,cccccccccccccccccc$
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$cccccccccccc$
$ccccccccccc$
$cccccccccc$
$ccccccccc$
$cccccccc$
$ccccccc$
$cccccc$
$ccccc$
$cccc$
ccc
cc
c

46 **strikes**

$aaaa$
$ababab$
$bcabbcabbc$
$cabbcbccabbcbcc$
$abbcbccbcccabbcbccbccc$
$\vdots$

211106232532990 **strikes**

Dison–Riley

Dison–Riley

The group generated by

$$a_1, \ldots, a_k, p, t$$

subject to

$$t^{-1} a_i t = a_i a_{i-1} \text{ for all } i > 1$$

$$t^{-1} a_1 t = a_1$$

$$[p, a_i t] = 1 \text{ for all } i$$

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$[p, a_i t] = 1$ for all i

$$G_k = F(a_1, \ldots, a_k) \rtimes \mathbb{Z}$$

$$H_k = \langle a_1 t, \ldots, a_k t \rangle$$

$$\langle G_k, p \mid [p, H_k] \rangle$$

HNN Extensions of Hydra Groups

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has Dehn function like the k-th of Ackermann's functions

$$n \mapsto A_k(n)\,.$$

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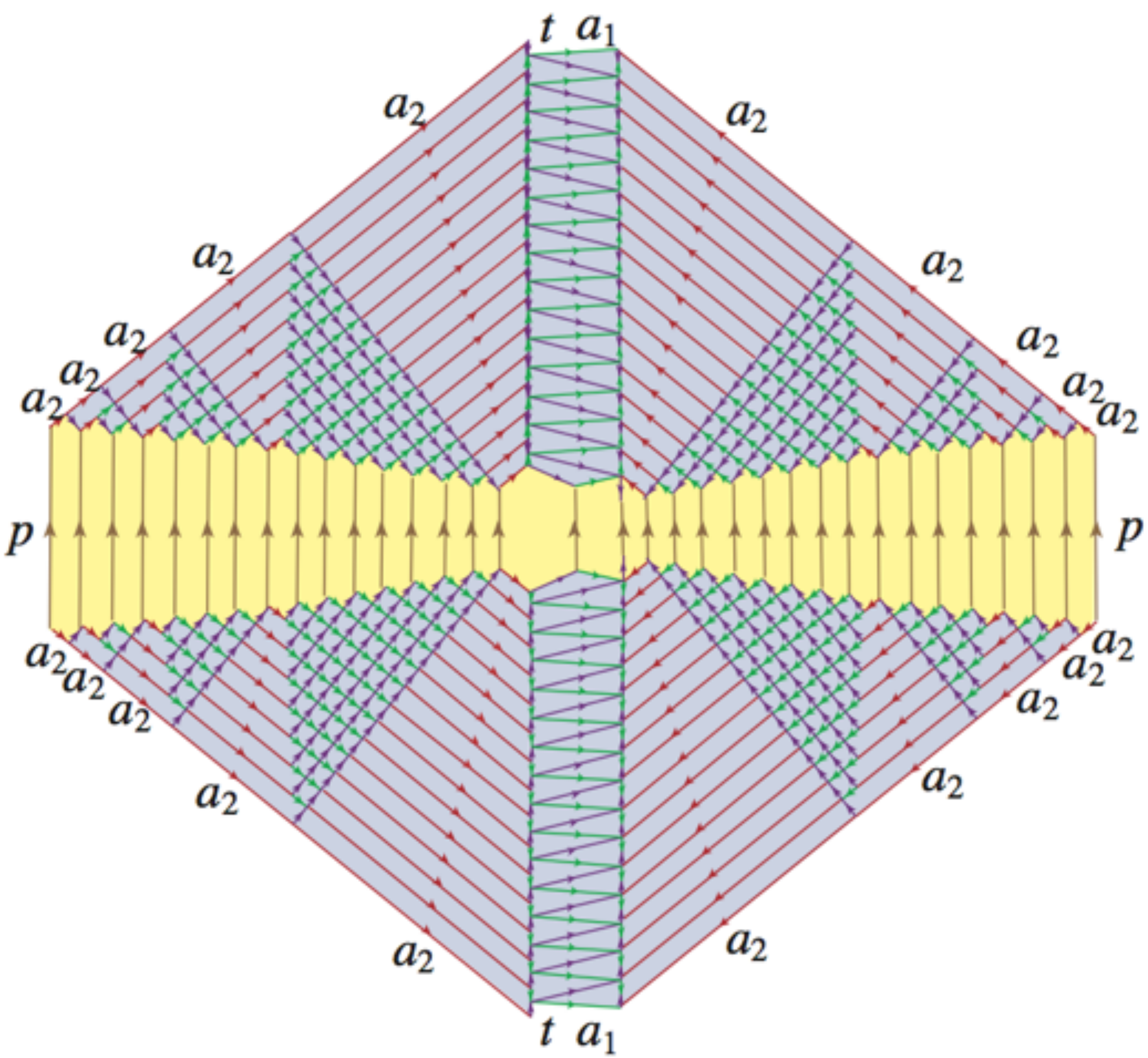

HNN Extensions of Hydra Groups

Dison–Riley

The group generated by

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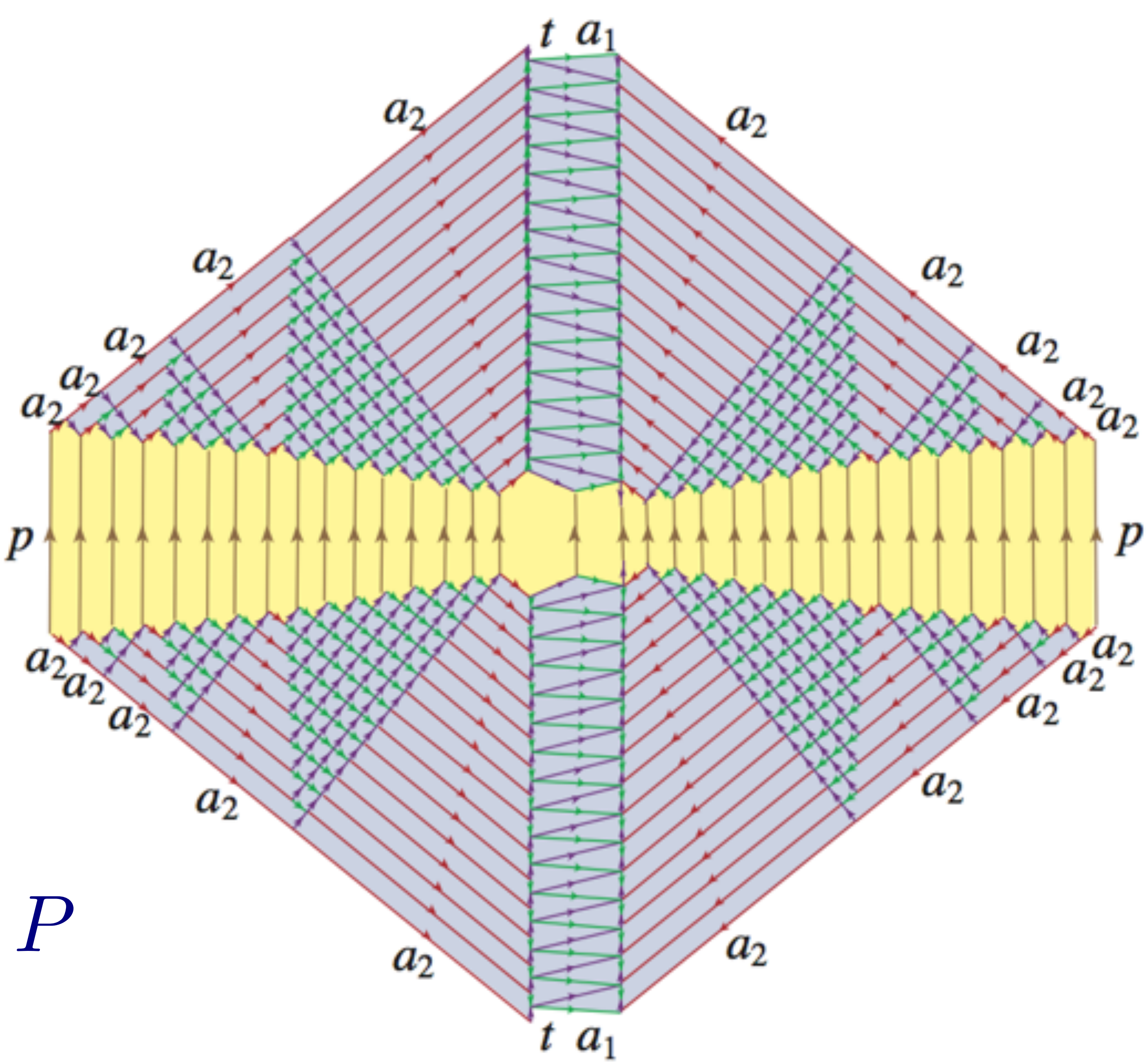
subject to

$$t^{-1} a_i t = a_i a_{i-1} \quad \text{for all } i > 1$$
$$t^{-1} a_1 t = a_1$$
$$[p, a_i t] = 1 \quad \text{for all } i$$

has Dehn function like the k-th of Ackermann's functions

$$n \mapsto A_k(n) .$$

$$G_k = F(a_1, \ldots, a_k) \rtimes \mathbb{Z}$$
$$H_k = \langle a_1 t, \ldots, a_k t \rangle$$
$$\langle G_k, p \mid [p, H_k] \rangle$$



Dison–Einstein–Riley

WP in P

Key idea: compute with compact representations of integers by strings of Ackermann functions.

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$$A_0 : \mathbb{Z} \to \mathbb{Z} \qquad n \mapsto n + 1$$

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$$A_1^{-1} A_2 A_0^3(0)$$
$$= A_1^{-1} A_2(3)$$

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A word w on $A_0^{\pm 1}, \cdots, A_k^{\pm 1}$ may represent an integer $w(0)$.

$$
\begin{aligned}
&A_1^{-1} A_2 A_0^3(0) \\
&\quad = A_1^{-1} A_2(3) \\
&\quad = A_1^{-1}(8)
\end{aligned}
$$

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$$\begin{aligned} &A_1^{-1} A_2 A_0^3(0) \\ &\quad = A_1^{-1} A_2(3) \\ &\quad = A_1^{-1}(8) \\ &\quad = 4 \end{aligned}$$

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$$
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& \quad = A_1^{-1} A_2(3) \\
& \quad = A_1^{-1}(8) \\
& \quad = 4
\end{aligned}
$$

$$A_3^2 A_0^4(0)$$

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$$\begin{aligned} & A_3^2 A_0^4(0) \\ &\quad = A_3^2(4) \end{aligned}$$

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$$\begin{aligned} &A_3^2 A_0^4(0) \\ &\quad = A_3^2(4) \\ &\quad = A_3(65536) \end{aligned}$$

Key idea: compute with compact representations of integers by strings of Ackermann functions.

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$$A_2^{-1} A_1^2 A_0^3(0)$$

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$$\begin{aligned} &A_3^2 A_0^4(0) \\ &= A_3^2(4) \\ &= A_3(65536) \\ &= \text{ENORMOUS} \end{aligned}$$

$$\begin{aligned} &A_2^{-1} A_1^2 A_0^3(0) \\ &= A_2^{-1} A_1^2(3) \\ &= A_2^{-1}(12) \end{aligned}$$

Key idea: compute with compact representations of integers by strings of Ackermann functions.

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Fix $k > 0$.

A word w on $A_0^{\pm 1}, \cdots, A_k^{\pm 1}$ may represent an integer $w(0)$.

$$A_1^{-1} A_2 A_0^3(0) = A_1^{-1} A_2(3) = A_1^{-1}(8) = 4$$

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$$A_2^{-1} A_1^2 A_0^3(0) = A_2^{-1} A_1^2(3) = A_2^{-1}(12) = \text{INVALID!}$$

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$$A_2^{-1} A_1^2 A_0^3(0) = A_2^{-1} A_1^2(3) = A_2^{-1}(12) = \text{INVALID!}$$

Dison–Einstein–Riley.
We can decide in $O(\ell(w)^{k+4})$ time whether w is valid and, if so, whether $w(0) = 0$.

The membership problem for H_k in G_k

E.g.: is $a_3 t a_2 {a_3}^{-1} t a_1 {a_2}^{-1} {a_3}^{-1} a_1$ in H_3?

$$G_3 = \left\langle\, t, a_1, a_2, a_3 \;\middle|\; \begin{array}{l} t^{-1}a_3t = a_3a_2 \\ t^{-1}a_2t = a_2a_1 \\ t^{-1}a_1t = a_1 \end{array} \right\rangle$$

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1. Collect all $t^{\pm 1}$ at the front:

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$$H_3 = \langle a_1 t, a_2 t, a_3 t \rangle$$

1. Collect all $t^{\pm 1}$ at the front:

$$t^2 a_3 a_2 a_2 a_1 a_2 a_1 {a_2}^{-1} {a_3}^{-1} a_1 {a_2}^{-1} {a_3}^{-1} a_1$$

The membership problem for H_k in G_k

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2. Partition into "pieces":

The membership problem for H_k in G_k

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The membership problem for H_k in G_k

$$G_3 = \left\langle t, a_1, a_2, a_3 \;\middle|\; \begin{array}{l} t^{-1}a_3t = a_3a_2 \\ t^{-1}a_2t = a_2a_1 \\ t^{-1}a_1t = a_1 \end{array} \right\rangle$$

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E.g.: is $a_3ta_2{a_3}^{-1}ta_1{a_2}^{-1}{a_3}^{-1}a_1$ in H_3?

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….but it has length $2^{47} \cdot 3 - 1$ there!

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Idea:

– Words on Ackermann-like functions ψ_i record the power of t as it advances.

– The validity of words on the ψ_i determines whether the power of t can advance.

– Then whether the final word on the ψ_i represents 0 tells us whether $w \in H_k$.

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What next?

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Miasnikov–Ushakov–Won (Baumslag's group):

54 +18 pages

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But (19) and the hypothesis that $\epsilon_p = 1$ allow us to determine $e_{p+1}, \dots, e_{q-1}$ from e_p and $m_1, \dots, m_\lambda$, so as to deduce that

$$u = a_1^{m_1}\theta^{e_p-m_1}(a_2^{-1})a_1^{m_2}\theta^{e_p-m_1-m_2+1}(a_2^{-1})\cdots a_1^{m_\lambda}\theta^{e_p-m_1-\cdots-m_\lambda+\lambda-1}(a_2^{-1}) \tag{27}$$

$$= a_1^{-e_p+2m_1}a_2^{-1}a_1^{-1-e_p+m_1+2m_2}a_2^{-1}\cdots a_1^{-\lambda+1-e_p+m_1+\cdots+m_{\lambda-1}+2m_\lambda}a_2^{-1}. \tag{28}$$

Comparing the powers of a_1 here with those in (24), we get:

$$\begin{cases} -2+i_p = 2m_1 \\ -1+i_p = -1 + m_1 + 2m_2 \\ i_p = -2 + m_1 + m_2 + 2m_3 \\ \vdots \\ \lambda-3+i_p = 1-\lambda + m_1 + m_2 + \cdots + m_{\lambda-1} + 2m_\lambda, \end{cases} \tag{29}$$

which simplifies to

$$i_p + 2^{j+1} - 6 = 2^j m_j \qquad \text{for } j = 1, \dots, \lambda. \tag{30}$$

2.2.3.1. *Case* $\lambda = 0$. This is a case we have previously addressed: u is the empty word.

So we can assume that $\lambda \geq 1$, and then the $j = 1$ instance of (30) tells us that i_p is even, and so

$$i_p \geq 4. \tag{31}$$

2.2.3.2. *Case* $\lambda = 1$. By (26),

$$e_p - i_p - 1 = e_q - i_q. \tag{32}$$

Also

$$z = \theta^{e_p}(a_{i_p})\underbrace{\theta^{e_{p+1}}(a_1^{\text{sign}(m_1)})\cdots\theta^{e_{p+|m_1|}}(a_1^{\text{sign}(m_1)})}_{|m_1|}\theta^{e_p-m_1}(a_2^{-1})\theta^{e_q}(a_{i_q}^{-1})$$

by (27), and so (19) applied to $\theta^{e_p-m_1}(a_2^{-1})$ and $\theta^{e_q}(a_{i_q}^{-1})$ tells us that $e_q = e_p - m_1 + 1$. But $i_p - 2 = 2m_1$ by the $j = 1$ case of (30), and so

$$e_q = e_p - \frac{i_p - 2}{2} + 1. \tag{33}$$

By (32) and (33),

$$i_p + 1 = i_q + \frac{i_p - 2}{2} - 1,$$

and so

$$i_p + 6 = 2i_q. \tag{34}$$

So (31) implies $i_q \geq 5$. And we can assume that it is not the case that $e_p - i_p + 3 = e_q - i_q + 3 = 0$, else (32) would be contradicted. So $e_p - i_p + 3 > 0$ or $e_q - i_q + 3 > 0$. If $e_p - i_p + 3 > 0$, there are at least two a_3 in $\theta^{e_p}(a_{i_p})$ (because $i_p \geq 4$) and hence at least two a_3^{-1} in $\theta^{e_q}(a_{i_q}^{-1})$. Likewise, if $e_q - i_q + 3 > 0$, then there are at least two a_3^{-1} in $\theta^{e_q}(a_{i_q}^{-1})$ (because $i_q \geq 4$), and so two a_3 in $\theta^{e_p}(a_{i_p})$. In either case, using Lemma 4.4 to identify the relevant suffix of $\theta^{e_p}(a_{i_p})$ and prefix of $\theta^{e_q}(a_{i_q}^{-1})$, there is a subword

$$\theta^{e_p-i_p+2}(a_3)\theta^{e_p-i_p+3}(a_3)u\theta^{e_q-i_q+3}(a_3^{-1})\theta^{e_q-i_q+2}(a_3^{-1}), \tag{35}$$

of z, which contains exactly two a_3 and two a_3^{-1}. If (35) freely reduces to the empty word, then, once the inner a_3 and a_3^{-1} pair have cancelled, it reduces to $\theta^{e_p-i_p+2}(a_3)\theta^{e_q-i_q+2}(a_3^{-1})$, which must

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