Math 4320 — Final Exam

2:00pm–4:30pm, Friday 18th May 2012

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection. Hermann Weyl, *Symmetry*, 1980.

This exam contains eight questions. Choose ONLY FOUR to answer — if you attempt more than four questions, you must indicate which four you would like to be graded. Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

Write your name on all exam booklets. Do not hand in any scratch paper. Unless otherwise indicated, all answers should be justified.

1. (a) State and prove Lagrange's Theorem on subgroups of finite groups.

   (b) Suppose a is an element of a group G. Show that $\langle a\rangle = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G.

   (c) Recall that the order n of an element a in a finite group G is the least integer $n \geq 1$ such that $a^n = 1$. Show that $|\langle a\rangle| = n$ and explain why n divides $|G|$.

   13 + 6 +6 = 25 pts

2. (a) When a group G acts on set X, what are meant by the *orbit* $\mathcal{O}(x)$ and the *stabilizer* G_x of $x \in X$? What formula gives $|G|$ in terms of $|\mathcal{O}(x)|$ and $|G_x|$ when G is finite?

   (b) Show that when a group G acts on set X, the orbits partition X.

   (c) Explain how the Class Equation for a finite group G:

   $$|G| \;=\; |Z(G)| + \sum_i [G : C_G(x_i)]$$

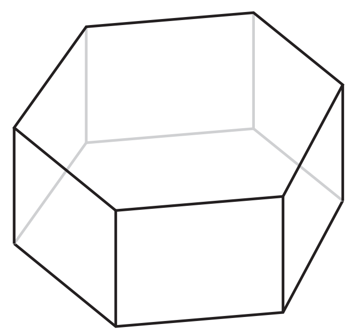
   where in the sum one x_i is selected from each conjugacy class of size at least 2, follows from parts (a) and (b). [*Recall that $Z(G)$ denotes the* center *of G—that is, the elements that commute with all elements of the group—and that $C_G(x_i)$ denotes the centralizer of x_i—that is, all elements of G that commute with x_i.*]

   (3+3+3) + 8 + 8 = 25 pts

3. (a) Burnside's Lemma gives what formula for the number of orbits of a finite group G acting on a finite set X?

   (b) How many ways are there to color the eight faces of a regular hexagonal prism (see below) up to rotational symmetry using the colors red and blue?

5 + 20 = 25 pts



4. (a) State and prove the First Isomorphism Theorem for groups.

   (b) Show that the index of $\mathrm{SL}_2(\mathbb{F}_q)$ in $\mathrm{GL}_2(\mathbb{F}_q)$ is $q-1$, where q is a prime power and $\mathbb{F}_q$ denotes the finite field with q elements. [*You may assume that the determinant map* $\det : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{F}_q \setminus \{0\}$ *is a group homomorphism.*]

18 + 7 = 25 pts

5. (a) What is meant by an *ideal* in a commutative ring R?

   (b) When is an ideal *prinipal*? What does it mean to say that an integral domain R is a *principal ideal* domain (PID)?

   (c) Give, with justification, an example of a non–zero commutative ring that is a PID.

   (d) Give, with justification, an example of a commutative ring that is not a PID.

   (e) Show that if $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ are ideals in a PID R, then there exists n such that $I_m = I_n$ for all $m > n$.

6 + (2+2) + 4 + 4 + 7 = 25 pts

6. (a) Show that the following two characterizations of what it means for a commutative ring R to be a *domain* are equivalent.

   (i) For all $a, b, c \in R$ with $c \neq 0$, if $ca = cb$, then $a = b$.

   (ii) For all $a, b \in R$, if $ab = 0$ then $a = 0$ or $b = 0$.

Recall that a commutative ring R is a *Euclidean ring* if it is a domain and there is a function $\partial : R \setminus 0 \to \mathbb{N}$ such that

- $\partial(f) \leq \partial(fg)$ for all $f, g \in R \setminus 0$, and
- for all $f, g \in R$ with $f \neq 0$, there exists $q, r \in R$ such that $g = qf + r$ and either $r = 0$ or $\partial(r) < \partial(f)$.

(b) Give an example of a *Euclidean ring*. What is ∂ for your example? [*You are not asked to prove that ∂ satisfies the above axioms.*]

(c) Show that if R is a Euclidean ring, then it is a principal ideal domain (PID).

10 + 5 + 10 = 25 pts

7. (a) Recall that a polynomial in $\mathbb{Z}[x]$ is *primitive* when the gcd of its coefficients is 1. Show that the product of two primitive polynomials in $\mathbb{Z}[x]$ is primitive.

(b) State Eisenstein's Criterion for the irreducibility in $\mathbb{Q}[x]$ of a polynomial with integer coefficients.

(c) Show that $f(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$. [*Hint: consider $f(x+1)$.*]

10 + 6 + 9 = 25 pts

8. (a) Suppose k is a field and $I = (p(x))$ where $p(x)$ is a non–constant polynomial in $k[x]$. Show that if $p(x)$ is irreducible in $k[x]$, then $k[x]/I$ is a field. (*You can use facts about primes and irreducibles, provided you quote them correctly.*)

(b) By applying the First Isomorphism Theorem for Rings to the homomorphism $\mathbb{R}[x] \to \mathbb{C}$ given by $f(x) \mapsto f(i)$, show that

$$\mathbb{R}[x]/(x^2+1) \;\cong\; \mathbb{C}.$$

(c) Show that

$$\mathbb{R}[x]/(x^2-2x+2) \;\cong\; \mathbb{C}.$$

13 + 6 + 6 = 25 pts

TRR, 12 May 2012