HW4 Solutions

2.22 Find $\operatorname{sgn}(\alpha)$ and α^{-1}, where

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Solution. In cycle notation, $\alpha = (19)(28)(37)(46)$. Thus, α is even, being the product of four transpositions. Moreover, being a product of disjoint transpositions, $\alpha = \alpha^{-1}$.

2.24 **(i)** If $1 < r \le n$, prove that there are

$$\tfrac{1}{r}[n(n-1)\cdots(n-r+1)]$$

r-cycles in S_n.

Solution. In the notation $(i_1\ i_2\ \ldots\ i_r)$, there are n choices for i_1, $n-1$ choices for $i_2, \ldots, n-(r-1) = n-r+1$ choices for i_r. We conclude that there are $n(n-1)\cdots(n-r+1)$ such notations. However, r such notations describe the same cycle:

$$(i_1\ i_2\ \ldots\ i_r) = (i_2\ i_3\ \ldots\ i_1) = \cdots = (i_r\ i_1\ \ldots\ i_{r-1}).$$

Therefore, there are $\frac{1}{r}[n(n-1)\cdots(n-r+1)]$ r-cycles in S_n.

2.26 Show that an r-cycle is an even permutation if and only if r is odd.
Solution. In the proof of Proposition 2.35, we showed that any r-cycle α is a product of $r-1$ transpositions. The result now follows from Proposition 2.39, for $\operatorname{sgn}(\alpha) = (-1)^{r-1} = -1$.

2.32 If $n \ge 2$, prove that the number of even permutations in S_n is $\frac{1}{2}n!$.
Solution. Let $\tau = (1\ 2)$, and define $f: A_n \to O_n$, where A_n is the set of all even permutations in S_n and O_n is the set of all odd permutations, by

$$f: \alpha \mapsto \tau\alpha.$$

If σ is even, then $\tau\sigma$ is odd, so that the target of f is, indeed, O_n. The function f is a bijection, for its inverse is $g: O_n \to A_n$, which is given by $g: \alpha \mapsto \tau\alpha$.

2.34 If $n \ge 3$, show that if $\alpha \in S_n$ commutes with every $\beta \in S_n$, then $\alpha = (1)$.
Solution. If $\alpha \ne (1)$, then it moves some i; say, $\alpha i = j \ne i$. There is β with $\beta j = j$ and $\beta i = k \ne i$. Then $\beta\alpha i = \beta j = j$, while $\alpha\beta i = \alpha k \ne j$ (for α is injective, and so $k \ne i$ implies $\alpha k \ne \alpha i = j$).

2.35 Can the following 15-puzzle be won?

4	10	9	1
8	2	15	6
12	5	11	3
7	14	13	#

Solution. No, because the associated permutation is odd.