HW5 Solutions

2.39 (i) How many elements of order 2 are there in S_5 and in S_6?

Solution. In S_5, there are $\frac{1}{2}(5 \times 4) = 10$ transpositions and

$$\tfrac{1}{2}[\tfrac{1}{2}(5 \times 4) \times \tfrac{1}{2}(3 \times 2)] = 15$$

products of two disjoint transpositions (the extra factor $\frac{1}{2}$ so that $(a\ b)(c\ d) = (c\ d)(a\ b)$ not be counted twice).

In S_6, there are $\frac{1}{2}(6 \times 5) = 15$ transpositions,

$$\tfrac{1}{2}[\tfrac{1}{2}(6 \times 5) \times \tfrac{1}{2}(4 \times 3)] = 45$$

products of two disjoint transpositions, and

$$\tfrac{1}{6}[\tfrac{1}{2}(6 \times 5) \times \tfrac{1}{2}(4 \times 3) \times \tfrac{1}{2}(2 \times 1)] = 15$$

products of three disjoint transpositions.

(ii) How many elements of order 2 are there in S_n?

Solution.

$$\tfrac{1}{2}n(n-1) + \tfrac{1}{2!}[\tfrac{1}{2}n(n-1)\tfrac{1}{2}(n-2)(n-3)] \\ + \tfrac{1}{3!}[\tfrac{1}{2}n(n-1)\tfrac{1}{2}(n-2)(n-3)\tfrac{1}{2}(n-4)(n-5)] + \cdots.$$

2.40 Let y be a group element of order m; if $m = dt$ for some $d \geq 1$, prove that y^t has order d.

Solution. Let $x = y^t$. Now $x^d = (y^t)^d = t^{td} = y^m = 1$, and so the order k of x is a divisor of d, by Lemma 2.53. But if x has order 1, then $x = 1$ and so $y^t = 1$. This contradicts $m = pt$ being the smallest positive integer with $y^m = 1$. Therefore, x has order p.

2.42 Let $G = \mathrm{GL}(2, \mathbb{Q})$, and let $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right]$. Show that $A^4 = I = B^6$, but that $(AB)^n \neq I$ for all $n > 0$. Conclude that AB can have infinite order even though both factors A and B have finite order.

Solution.

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } (AB)^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}.$$

2.44 If G is a group in which $x^2 = 1$ for every $x \in G$, prove that G must be abelian.

Solution. If $x \in G$, then $x^2 = 1$ implies $x^{-1} = x$. Let $y \in G$. Since $1 = (xy)^2 = xyxy$, we have $xy = y^{-1}x^{-1} = yx$.

2.47 What is the largest order of an element in S_n, where $n = 1, 2, \ldots, 10$?
Solution. Denote the largest order of an element in S_n by $\mu(n)$. There is

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	1	2	3	4	6	6	12	15	20	~~[illegible]~~ 30

no known formula for $\mu(n)$, though its asymptotic behavior is known, by a theorem of E. Landau.

2.50 Prove that every element in a dihedral group D_{2n} has a unique factorization of the form $a^i b^j$, where $0 \leq i < n$ and $j = 0$ or 1.
Solution. The cyclic subgroup $\langle a \rangle$ has order n, hence index 2, so that D_{2n} is the disjoint union $\langle a \rangle \cup \langle a \rangle b$.

This solution will make more sense after learning about indices.

As an alternate solution, note that $|D_{2n}| = 2n$ and $|\{a^i b^j \mid i = 0, \ldots, n-1 \text{ and } j = 0, 1\}| = 2n$. Since $\{a^i b^j \mid i = 0, \ldots, n-1 \text{ and } j = 0, 1\} \subseteq D_{2n}$ they must be equal.