Partial HW6 Solutions

2.57 If H and K are subgroups of a group G and if $|H|$ and $|K|$ are relatively prime, prove that $H \cap K = \{1\}$.
Solution. By Lagrange's theorem, $|H \cap K|$ is a divisor of $|H|$ and a divisor of $|K|$; that is, $|H \cap K|$ is a common divisor of $|H|$ and $|K|$. But $|H|$ and $|K|$ are relatively prime, so that $|H \cap K| = 1$ and $H \cap K = \{1\}$.

2.68 Prove that a group G is abelian if and only if the function $f: G \to G$, given by $f(a) = a^{-1}$, is a homomorphism.
Solution. If $a, b \in G$, where G is abelian, then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$. Conversely, assume that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Then

$$aba^{-1}b^{-1} = ab(ab)^{-1} = 1,$$

so that $ab = ba$.

2.75 If G is a group and $a, b \in G$, prove that ab and ba have the same order.
Solution. ab and ba are conjugates [$ba = b(ab)b^{-1}$], and hence they have the same order.