Partial HW8 Solutions

2.120 Prove that $S_4/\mathbf{V} \cong S_3$.

Solution. $S_4/\mathbf{V}$ is a group of order 24/4=6, hence Proposition 2.135 shows that it is isomorphic to either S_3 or $\mathbb{I}_6$. But $S_4/\mathbf{V}$ is not abelian: for example, $(1\ 2)\mathbf{V}(1\ 3)\mathbf{V} \neq (1\ 3)\mathbf{V}(1\ 2)\mathbf{V}$ because

$$(1\ 2)(1\ 3)[(1\ 3)(1\ 2)]^{-1} = (1\ 2)(1\ 3)(1\ 2)(1\ 3) = (2\ 3) \notin \mathbf{V}.$$

2.121 **(i)** Prove that $A_4 \not\cong D_{12}$.

Solution. A_4 has no element of order 6, while D_{12} does have such an element.

(ii) Prove that $D_{12} \cong S_3 \times \mathbb{I}_2$.

Solution. We may suppose that $D_{12} = \langle a, b \rangle$, where $a^6 = 1 = b^2$ and $bab = a^{-1}$. We know that the subgroup $\langle a \rangle$ has order 6, hence index 2, and so there are two cosets:

$$D_{12} = \langle a \rangle \cup b\langle a \rangle.$$

Thus, every element $x \in D_{12}$ has a unique factorization $x = b^i a^j$, where $i = 0, 1$ and $0 \leq j < 6$. Define $H = \langle a^2, b \rangle$; now $H \cong S_3$, for it is a nonabelian group of order 6; note that $H \lhd D_{12}$ because it has index 2. If we define $K = \langle a^3 \rangle$, then $|K| = 2$ and $K \lhd D_{12}$: it suffices to prove that $aa^3a^{-1} \in K$ (which is, of course, obvious) and $ba^3b \in K$; but $ba^3b = a^{-3} = a^3 \in K$. It is plain that $H \cap K = \{1\}$ and $HK = D_{12}$, and so $D_{12} \cong H \times K \cong S_3 \times \mathbb{I}_2$, by Proposition 2.127.

2.125 **(i)** Show that there are two conjugacy classes of 5-cycles in A_5, each of which has 12 elements.

Solution. The hint shows that $|C_{S_5}(\alpha)| = 5$. Since $|\langle\alpha\rangle| = 5$ and $\langle\alpha\rangle \le C_{S_5}(\alpha)$, we have $\langle\alpha\rangle = C_{S_5}(\alpha)$. By (i),

$$C_{A_5}(\alpha) = A_5 \cap C_{S_5}(\alpha) = A_5 \cap \langle\alpha\rangle = \langle\alpha\rangle,$$

so that $|C_{A_5}(\alpha)| = 5$. Therefore, the number of conjugates of α in A_5 is $60/|C_{A_5}(\alpha)| = 60/5 = 12$.

(ii) Prove that the conjugacy classes in A_5 have sizes 1, 12, 12, 15, and 20.

Solution. There are exactly 4 cycle structures in A_5: (1); (1 2 3); (1 2 3 4 5); (1 2)(3 4). Using Example 2.30, these determine conjugacy classes in S_5 of sizes 1, 20, 24, and 15, respectively. In part (ii), we saw that the class of 5-cycles splits, in A_5, into two conjugacy classes of size 12. The centralizer $C_{S_5}(1\ 2\ 3)$ consists of

$$(1), (1\ 2\ 3), (1\ 3\ 2), (4\ 5), (4\ 5)(1\ 2\ 3), (4\ 5)(1\ 3\ 2);$$

Only the first 3 of these are even, and so $|C_{A_5}((1\ 2\ 3))| = 3$. It follows from Corollary 2.145 that the conjugacy class of (1 2 3) in A_5 has the same size as in S_5, namely, 20. Finally, (1 2)(3 4) has 15 conjugates in S_5. By part (ii), there must be 15 conjugates in A_5, for the other alternative, $\frac{15}{2}$, is obviously impossible.

(iii) Prove that every normal subgroup H of a group G is a union of conjugacy classes of G, one of which is $\{1\}$.

Solution. It follows from Proposition 2.142 that a group G is a disjoint union of its conjugacy classes. Since a normal subgroup of G contains all the conjugates of its elements, it follows that H is a union of conjugacy classes of G.

(iv) Use parts (ii) and (iii) to give a second proof of the simplicity of A_5.

Solution. Since H contains 1, the order of H is a sum of 1 together with some of the numbers 12, 12, 15, and 20. The only such sum that divides 60 is 60 itself.