Math 4320 : Introduction to Algebra

Prelim II (Chapter 2& 3)

(due April 17 , 2009, 1:25pm)

You are not allowed to discuss your answers with others. Books, lecture notes and calculators are allowed.

Part I. Group theory

1. (2.88) Show that a finite group G generated by two elements of order 2 is isomorphic to a dihedral group D_{2n} for some n.

2. Let G be a group of order n, and let F be any field. Prove that G is isomorphic to a subgroup of $GL_n(F)$.

3. Rule out as many of the followings as possible as Class Equations for a group of order 10:

$$3+2+5,\ 1+2+2+5,\ 1+2+3+4,\ 2+2+2+2+2.$$

(Note that the first term in each expression corresponds to the center $Z(G)$ in $|G| = |Z(G)| + \sum [G : C_G(x)]$.)

4. Determine the class equation for each of the following groups.

(1) D_6, (2) D_{10}, (3) D_{2n}

(4) the group of upper triangular matrices in $GL_2(\mathbb{F}_3)$

(You may choose not to write down part (1) and part (2) if you know part (3).)

5. Show that A_n is a simple group for all $n \geq 5$ by showing Exercise **2.127**.

6. Determine all finite groups which contain at most three conjugacy classes.

Part II. Rings and fields

The following set of problems is to show that the ring

$$R = \mathbb{Z}[\theta] = \{a + b\theta : a, b \in \mathbb{Z}\},$$

where $\theta = \frac{1+\sqrt{-19}}{2}$, is a principal ideal domain (PID) that is not a Euclidean domain (ED) (a result of Motzkin).

7. Let $F = \{a + b\sqrt{-19} : a, b \in \mathbb{Q}\} \subset \mathbb{C}$.

(a) Show that R is a ring, $R \subset F$ and F is a field. Conclude that R is an integral domain. Show that F is the field of fractions of R.

(b) Define $N(a + b\sqrt{-19}) = a^2 + 19b^2$. Prove that $N(\alpha) > 0$ for $\alpha \in F - \{0\}$, and that N is multiplicative, i.e. $N(\alpha\beta) = N(\alpha)N(\beta)$. Also prove that $N(\alpha)$ is a positive integer for every $\alpha \in R$.

(c) Prove that ± 1 are the only units in R.

8. *(Criterion of Dedekind and Hasse)* Let S be an integral domain and let N denote any function from S to $\mathbb{Z}$ which satisfies $N(\alpha) > 0$ for $\alpha \neq 0$. Suppose that for ever $\alpha, \beta \in S$ with $N(\alpha) \geq N(\beta)$, either β divides α in S, or there exist $s, t \in S$ with

$$0 < N(s\alpha - t\beta) < N(\beta). \qquad (*)$$

Show that S is a PID. (Hint : Let I be a nonzero idea in S and let β be a nonzero element of I with $N(\beta)$ minimal. If $\alpha \in I$, then $s\alpha - t\beta$ is also in I for all $s, t \in S$. Use minimality of β.)

9. *(R is a PID)* Show that the ring R, with the function N defined in problem 7 satisfies the criterion of Dedekind and Hasse. (Hint : Since N is multiplicative, the condition $(*)$ is equivalent to

$$0 < N(\frac{\alpha}{\beta}s - t) < 1.$$

Suppose $\beta \nmid \alpha$. Write $\frac{\alpha}{\beta} = \frac{a+b\sqrt{-19}}{c}$ in F with integers a, b, c having no common divisor and with $c > 1$. Divide into four cases, $c \geq 5$ and $c = 2, 3, 4$.)

10. *(R is not a ED)* Let D be an integral domain. Recall that a non-zero non-unit element $u \in D$ is called a *universal side divisor* if for every $x \in D$, there is some unit $z \in D$ such that u divides $x - z$ in D. Prove that the ring R above has no universal side divisors, hence is not a ED. (Hint : Use part (c) of problem **7**. Use first $x = 2$, then $x = \theta$. What are the divisors of $2, 3$ in R?)