Math 4320 : Introduction to Algebra

First Prelim

(February 27, 2009, 1:25pm-2:15pm)

1. (5 pts each) Define each of the following groups by explicitly describing the set and the operation, and answer the questions **without** proof.
(i) Define the symmetric group S_n. What is the minimum number of elements of S_n we need to generate S_n?
(ii) Define the symmetry group $\Sigma(\pi_n)$ of a regular n-gon π_n. What is the order of $\Sigma(\pi_n)$?

2. (2 pts each) Determine whether each of the following statements is true or false. (Do **not** give a proof or a counter-example.)
(i) The group $\Sigma(\pi_n)$ is isomorphic to the group S_n.
(ii) The permutation (1234) is in A_4.
(iii) The group $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.
(iv) The group S_n is commutative.
(v) Every permutation is a product of transpositions in a unique way.

3. (20 pts) Let H, K be normal subgroups of a finite group G.
(i) Show that $HK = KH$. Show also that HK is a subgroup of G.
(ii) Show that if $H \cap K = \{1\}$, then $HK \cong H \times K$.

4. (20 pts) Find all solutions to the following simultaneous congruences.

$$3x \equiv 2 \mod 5$$
$$2x \equiv 1 \mod 3$$

5. (20 pts) Let $n = p_1^{e_1} p_2^{e_2}$ be the prime decomposition of $n \in \mathbb{N}$. Show that the Euler function φ satisfies

$$\varphi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}).$$

(Hint : show that the group $U(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to a direct product of some groups.)

6. (20 pts) For any $n \geq 3$, show that A_n contains a subgroup isomorphic to S_{n-2}.