Spring 2010

MATH 4320: Final Exam

Instructor: Yuri Berest

The exam is due **6 pm, Thursday, May 20**. Please turn in your exam in 439 Mallot Hall.

Problem 1. (*35 points*)

Let A be a commutative ring with 1.

a. An element $a \in A$ is called *nilpotent* if $a^n = 0$ for some $n \in \mathbf{N}$. Prove that the set $\mathfrak{R} = \mathfrak{R}(A)$ of all nilpotent elements is an ideal in A and the quotient ring $A/\mathfrak{R}$ has no nonzero nilpotent elements. Give an example of A with $\mathfrak{R}(A) \neq \{0\}$. (The ideal $\mathfrak{R}(A)$ is called the *nilradical* of A.)

b. Let $\mathfrak{I}(A)$ be the intersection of all maximal ideals in A. Show that $\mathfrak{I}(A)$ is an ideal in A. Prove that $a \in \mathfrak{I}(A)$ if and only if $1 - ab$ is invertible for all $b \in A$. Conclude that $\mathfrak{R}(A) \subseteq \mathfrak{I}(A)$, where $\mathfrak{R}(A)$ is the nilradical of A.

Problem 2. (*40 points*)

Let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in a commutative ring A. Let $f = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in A[x]$. Prove that

a. f is a unit $\Leftrightarrow$ a_0 is a unit in A and $a_1, a_2, \ldots, a_n$ are nilpotent;

b. f is a zero divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.

Problem 3. (*35 points*)

a. Prove that $x^2 + 1$ is irreducible over $\mathbf{Z}_3$. Conclude that $\mathbf{F} = \mathbf{Z}_3[x]/(x^2+1)$ is a field. List all the elements of $\mathbf{F}$. Show that the class of the polynomial $x+1$ in $\mathbf{F}$ is a generator in the (multiplicative) group of units of $\mathbf{F}$, but the class of x is not.

b. Let $\mathbf{Z}[\frac{1}{2}]$ be the smallest subring of $\mathbf{Q}$ containing $\mathbf{Z}$ and $\frac{1}{2}$. Let $(2x-1)$ be the ideal of $\mathbf{Z}[x]$ generated by the polynomial $2x-1$. Show that $\mathbf{Z}[x]/(2x-1) \cong \mathbf{Z}[\frac{1}{2}]$. Is the ideal $(2x-1)$ maximal in $\mathbf{Z}[x]$? If not, find an ideal $I \subset \mathbf{Z}[x]$ such that $(2x-1) \subsetneq I \subsetneq \mathbf{Z}[x]$.

Bonus Problem.

a. Let $f(x) \in k[x]$ be an irreducible polynomial with coefficients in a field k. Suppose that $f(x)$ has a root of multiplicity greater than one in some extension of k. Prove that $\mathrm{char}(k) = p$ for some prime p and $f(x) = g(x^p)$ for some $g(x) \in k[x]$.

b. Find all the values of $a \in \mathbf{Z}$ for which the polynomial $f(x) = x^5 - ax - 1$ is *not* irreducible in $\mathbf{Z}[x]$. For each such a find the corresponding factorization of $f(x)$ into a product of irreducible polynomials in $\mathbf{Z}[x]$.