Spring 2010

MATH 4320: Prelim 1

Instructor: Yuri Berest

Problem 1. (15 points)

a. Find all integer solutions to the congruence $72x \equiv 36 \pmod{376}$.

b. Find the smallest positive integer which leaves remainders 1, 3, 4 after dividing by 9, 7, 5 respectively.

Problem 2. (30 points)

a. Prove that if $(a, b) = 1$, then $(ab, c) = (a, c)(b, c)$ for all $a, b, c \in \mathbf{Z}$.

b. Prove that if $(a, b) = 1$, the equation $(a + bx, c) = 1$ is solvable in integers for any $c \in \mathbf{Z}$.

Problem 3. (20 points)

a. Let $f : X \to Y$ be a map between two finite sets of the same size. Prove that f is injective if and only if f is surjective.

b. Let $X = \{0, 1, 2, \ldots, 9, 10\}$. Define a map $\sigma : X \to X$ by the rule:

$$\sigma(n) = \text{the remainder after dividing } 4n^2 - 3n^7 \text{ by } 11 .$$

Show that σ is a permutation of X. Find its complete factorization into a product of disjoint cycles and factorization into a product of transpositions. Compute $\mathtt{sign}(\sigma)$ and σ^{-1}.

Problem 4. (35 points)

For a permutation $\sigma \in S_n$, define $|\sigma|$ to be the least integer $r > 0$ such that $\sigma^r = (1)$. ($|\sigma|$ is called the *order* of σ in S_n.)

a. If $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ is a product of disjoint cycles, show that

$$|\sigma| = \operatorname{lcm}\{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_k|\} .$$

b*. Suppose that $\sigma \in S_n$ has k_1 cycles of length 1, k_2 cycles of length 2, k_3 cycles of length 3, …, k_r cycles of length r, so that $n = k_1 + 2k_2 + 3k_3 + \ldots + rk_r$. Find $|\sigma|$.