Spring 2010

MATH 4320: Prelim 2

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The exam is due **Wednesday, April 14**. Please write clearly and concisely.

Problem 1. (15 points)
If H_1 and H_2 are two groups, define their *direct product* $H_1 \times H_2$ to be the set of ordered pairs $\{(x_1, x_2) \ : \ x_1 \in H_1 , x_2 \in H_2\}$ equipped with the operation

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2) \ .$$

(a) Show that the natural maps $\pi_1 : H_1 \times H_2 \to H_1$, $(x_1, x_2) \mapsto x_1$, and $\pi_2 : H_1 \times H_2 \to H_2$, $(x_1, x_2) \mapsto x_2$, are surjective group homomorphisms. What are the kernels of π_1 and π_2?
(b) Prove the following property of $(H_1 \times H_2, \pi_1, \pi_2)$: given any group G together with two homomorphisms $f_1 : G \to H_1$ and $f_2 : G \to H_2$, there is a unique homomorphism $\varphi : G \to H_1 \times H_2$ such that $f_1 = \pi_1 \circ \varphi$ and $f_2 = \pi_2 \circ \varphi$.

Problem 2. (20 points)
Let N and H be subgroups of a group G with N normal. Let $\mathrm{Ad}_g : G \to G$, $x \mapsto gxg^{-1}$, be conjugation by an element $g \in G$.
(a) Show that the assignment $h \mapsto \mathrm{Ad}_h$ defines a group homomorphism $f : H \to \mathrm{Aut}(N)$.
(b) If $N \cap H = \{e\}$, show that the map $N \times H \to NH$ given by $(x, h) \mapsto xh$ is a bijection, and this map is an isomorphism of groups if and only if f is trivial, i. e. $f(h) = \mathrm{Id}_N$ for all $h \in H$.
We say that G is the *semidirect product* of its subgroups N and H (Notation: $G = N \rtimes H$), if $G = NH$ and $N \cap H = \{e\}$.
(c) Conversely, let N and H be any groups, and let $\alpha : H \to \mathrm{Aut}(N)$, $h \mapsto \alpha_h$, be a group homomorphism. Construct a semidirect product as follows. Denote by G the set of ordered pairs $\{(x, h) \ : \ x \in N , h \in H\}$, and define an operation in G by the formula

$$(x_1, h_1) * (x_2, h_2) = (x_1 \cdot \alpha_{h_1}(x_2), h_1 \cdot h_2) \ ,$$

where "$\cdot$" denote the given operations in H and N.
Check that $(G, *)$ is a group. Show that the maps $N \to G\,,\ x \mapsto (x,\, e_H)\,,$ and $H \to G\,,\ h \mapsto (e_N, h)\,,$ are injective group homomorphisms. Identifying N and H with their images in G under these homomorphisms, prove that $G \cong N \rtimes H\,.$

Problem 3. (15 points)
Let N and H be two normal subgroups in a finite group G. Assume that the orders of N and H are relatively prime. Prove that $xh = hx$ for all $x \in N$ and $h \in H\,.$ Conclude that $N \rtimes H \cong N \times H\,.$

Problem 4. (10 points)
Let $\mathcal{F}(\mathbf{C})$ be the set of linear fractional functions

$$f : \mathbf{C} \to \mathbf{C}\,, \quad z \mapsto \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbf{C}$ and $ad - bc \neq 0\,.$ Check that $\mathcal{F}(\mathbf{C})$ is a group with respect to the usual composition of functions. Show that $\mathcal{F}(\mathbf{C}) \cong \mathrm{GL}(2, \mathbf{C})/\mathbf{C}^*\,,$ where $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$ is regarded as a subgroup of $\mathrm{GL}(2, \mathbf{C})$ via the inclusion

$$\mathbf{C}^* \hookrightarrow \mathrm{GL}(2, \mathbf{C})\,, \quad \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\,.$$

Problem 5. (20 points)
If a group G has a subgroup of finite index, then it has also a *normal* subgroup of finite index. Prove this.

Problem 6. (20 points)
Let G be a p-group (p prime). Let H be a normal subgroup of order p. Show that H is contained in the center of G.