Spring 2010

MATH 4320: Solutions to Prelim 2

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Problem 1.

(a) This is straightforward: for example, we have

$$\pi_1[(x_1,\, x_2)\cdot(y_1,\, y_2))] = \pi_1[(x_1y_1,\, x_2y_2)] = x_1y_1 = \pi_1[(x_1,\, x_2)]\,\pi_1[(y_1,\, y_2)]\ ,$$

and similarly for π_2. Both π_1 and π_2 are surjective, because given any $x_1 \in H_1$ and any $x_2 \in H_2$, we have $\pi_1[(x_1,\, x_2)] = x_1$ and $\pi_2[(x_1,\, x_2)] = x_2$. Finally, $\mathrm{Ker}(\pi_1) = (e_1,\, H_2)$ and $\mathrm{Ker}(\pi_2) = (H_1,\, e_2)$.

(b) Given any group G with two homomorphisms $f_1 : G \to H_1$ and $f_2 : G \to H_2$, we define $\varphi : G \to H_1 \times H_2$ by $\varphi(x) := (f_1(x),\, f_2(x))$ for $x \in G$. Then, we see at once that $(\pi_1\circ\varphi)(x) = \pi_1[\varphi(x)] = \pi_1[(f_1(x),\, f_2(x))] = f_1(x)$ and $(\pi_2 \circ \varphi)(x) = \pi_2[\varphi(x)] = \pi_2[(f_1(x),\, f_2(x))] = f_2(x)$ for all $x \in G$. In other words, $f_1 = \pi_1 \circ \varphi$ and $f_2 = \pi_2 \circ \varphi$ as required.

Now, suppose there is another homomorphism, say $\psi : G \to H_1 \times H_2$, satisfying $f_1 = \pi_1 \circ \psi$ and $f_2 = \pi_2 \circ \psi$. By definition of $H_1 \times H_2$, the elements $\psi(x)$ in the image of ψ can be written as $\psi(x) = (g_1(x),\, g_2(x))$, where $g_1(x) \in H_1$ and $g_2(x) \in H_2$. Since $f_1 = \pi_1 \circ \psi$ and $f_2 = \pi_2 \circ \psi$, we have $f_1(x) = (\pi_1 \circ \psi)(x) = \pi_1[\psi(x)] = \pi_1[(g_1(x),\, g_2(x))] = g_1(x)$ and $f_2(x) = (\pi_2 \circ \psi)(x) = \pi_2[\psi(x)] = \pi_2[(g_1(x),\, g_2(x))] = g_2(x)$ for all $x \in G$. Thus, $g_1(x) = f_1(x)$ and $g_2(x) = f_2(x)$, and therefore $\psi(x) = \varphi(x)$ for all $x \in G$. This proves that $\psi = \varphi$, which means the uniqueness of φ.

Problem 2.

(a) For each $h \in H$ and for all x_1 and x_2 in N, we have

$$f(h)(x_1\, x_2) := h\,(x_1\, x_2)\, h^{-1} = h\, x_1\, (h^{-1}h)\, x_2\, h^{-1} = (h\, x_1\, h^{-1})\,(h\, x_2\, h^{-1})\ .$$

So $f(h)(x_1\, x_2) = f(h)(x_1)\, f(h)(x_2)$, which means that $f(h)$ is a group homomorphism $N \to N$. Moreover, $f(h)$ is bijective for each $h \in H$, because it has an inverse (namely $f(h^{-1})$). Thus, $f : H \to \mathrm{Aut}(N)$ is a well-defined map, assigning to elements of H the group automorphisms of N. It remains to show that this map is a homomorphism of groups.

Fix any h_1 and h_2 in H. Then $f(h_1\, h_2) : N \to N$ is given by

$$f(h_1\, h_2)(x) := (h_1\, h_2)\, x\, (h_1\, h_2)^{-1} = (h_1\, h_2)\, x\, (h_2^{-1} h_1^{-1}) = h_1\, (h_2\, x\, h_2^{-1})\, h_1^{-1} \ ,$$

where $x \in N$. Since $h_2\, x\, h_2^{-1} = \mathrm{Ad}_{h_2}(x) = f(h_2)(x)$, we see that

$$f(h_1\, h_2)(x) = h_1\, [f(h_2)(x)]\, h_1^{-1} = f(h_1)[f(h_2)(x)] = [f(h_1) \circ f(h_2)](x)\,.$$

This holds for all $x \in N$, so we conclude $f(h_1\, h_2) = f(h_1) \circ f(h_2)$ as functions $N \to N$. Since the composition "$\circ$" is precisely the group operation on $\mathrm{Aut}(N)$, the last equality implies that f is a group homomorphism.

(*b*) The map $\pi : N \times H \to NH$, $(x, h) \mapsto xh$, is obviously surjective. To show that π is injective consider (x_1, h_1) and (x_2, h_2) in $N \times H$, such that $x_1 h_1 = x_2 h_2$. The last equation is equivalent to $x_1^{-1} x_2 = h_1\, h_2^{-1}$ in G. Now, since $x_1^{-1} x_2 \in N$, while $h_1\, h_2^{-1} \in H$, we must have $x_1^{-1} x_2 = e$ and $h_1\, h_2^{-1} = e$, because $N \cap H = \{e\}$. It follows that $x_1 = x_2$ in N and $h_1 = h_2$ in H, and therefore $(x_1, h_1) = (x_2, h_2)$ in $N \times H$.

The map π is an isomorphism of groups iff

$$\pi[(x_1, h_1)\, (x_2, h_2)] = \pi[(x_1 x_2, h_1 h_2)] = \pi[(x_1, h_1)]\, \pi[(x_2, h_2)] \ .$$

for all (x_1, h_1) and (x_2, h_2) in $N \times H$. This last equation says that

$$x_1\, x_2\, h_1\, h_2 = x_1\, h_1\, x_2\, h_2 \tag{1}$$

for all $x_1, x_2 \in N$ and $h_1, h_2 \in H$. Letting $x_1 = h_2 = e$ in (1), we see that $x_2\, h_1 = h_1\, x_2$ for all $x_2 \in N$ and $h_1 \in H$. Conversely, if $x_2\, h_1 = h_1\, x_2$ for all $x_2 \in N$ and for all $h_1 \in H$, then (1) obviously holds. Thus, π being a group homomorphism is equivalent to the condition $x\, h = h\, x$ for all $x \in N$ and for all $h \in H$. The latter can be written as $h\, x\, h^{-1} = x$, or equivalently $f(h)(x) = x$. The last equation simply says that the map f sends every element $h \in H$ to the identity map Id_N on N.

(*c*) It is routine to check that $(G, *)$ satisfies the axioms of a group. For example, let's check the associativity of $*$:

$$\begin{aligned}
[(x_1,\, h_1) * (x_2,\, h_2)] * (x_3,\, h_3) &= (x_1 \cdot \alpha_{h_1}(x_2),\, h_1 \cdot h_2\,) * (x_3,\, h_3) \\
&= ((x_1 \cdot \alpha_{h_1}(x_2)) \cdot \alpha_{h_1 \cdot h_2}(x_3),\, (h_1 \cdot h_2) \cdot h_3\,) \\
&= (x_1 \cdot \alpha_{h_1}(x_2) \cdot (\alpha_{h_1} \circ \alpha_{h_2})(x_3),\, h_1 \cdot h_2 \cdot h_3\,) \ .
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(x_1,\, h_1) * [(x_2,\, h_2) * (x_3,\, h_3)] &= (x_1,\, h_1) * (x_2 \cdot \alpha_{h_2}(x_3),\, h_2 \cdot h_3\,) \\
&= (x_1 \cdot \alpha_{h_1}(x_2 \cdot \alpha_{h_2}(x_3)),\, h_1 \cdot (h_2 \cdot h_3)) \\
&= (x_1 \cdot \alpha_{h_1}(x_2) \cdot \alpha_{h_1}(\alpha_{h_2}(x_3)),\, h_1 \cdot h_2 \cdot h_3) \\
&= (x_1 \cdot \alpha_{h_1}(x_2) \cdot (\alpha_{h_1} \circ \alpha_{h_2})(x_3),\, h_1 \cdot h_2 \cdot h_3)\,.
\end{aligned}
$$

Comparing the expressions in the right-hand sides, we conclude that

$$[(x_1,\, h_1) * (x_2,\, h_2)] * (x_3,\, h_3) = (x_1,\, h_1) * [(x_2,\, h_2) * (x_3,\, h_3)]\ .$$

Note, in the above calculations we used the formulas $\alpha_{h_1 \cdot h_2}(x) = (\alpha_{h_1} \circ \alpha_{h_2})(x)$ and $\alpha_h(x \cdot y) = \alpha_h(x) \cdot \alpha_h(y)$, which follow from the fact that $\alpha:\, H \to \mathrm{Aut}(N)$ is a group homomorphism.

The maps $N \to G$, $x \mapsto (x,\, e_H)$, and $H \to G$, $h \mapsto (e_N, h)$, are obviously injective. Let's check that these are group homomorphisms. For example, under the first map the product $x_1 \cdot x_2 \in N$ goes to $(x_1 \cdot x_2,\, e_H)$, while by definition of the $*$-product, we have

$$(x_1 \cdot x_2,\, e_H) = (x_1 \cdot \alpha_{e_H}(x_2),\, e_H \cdot e_H) = (x_1,\, e_H) * (x_2,\, e_H)\ .$$

Note, here we use the fact that $\alpha_{e_H} = \mathrm{Id}_N$, which is again a consequence of α being a group homomorphism. A similar argument works for $H \to G$.

Let's now identify N and H with their images $(N,\, e_H)$ and $(e_N,\, H)$ in G. Then it is obvious that $N \cap H = \{e_G\}$ in G, where $e_G := (e_N,\, e_H)$ is the identity element in G. On the other hand, we have $G = N * H$, because every element $(x,\, h) \in G$ can be written as

$$(x,\, h) = (x \cdot e_N, e_H \cdot h) = (x \cdot \alpha_{e_H}(e_N),\, e_H \cdot h) = (x,\, e_H) * (e_N,\, h)\ ,$$

where $(x,\, e_H) \in N$ and $(e_N,\, h) \in H$. Thus, we conclude that $G \cong N \rtimes H$.

Problem 3.

First, we observe that $N \cap H = \{e\}$ in G. Indeed, $N \cap H$ is a subgroup in both N and H, and hence, by Lagrange's Theorem, its order $|N \cap H|$ must divide both $|N|$ and $|H|$. But $|N|$ and $|H|$ are relatively prime. Hence, $|N \cap H| = 1$, meaning that $N \cap H = \{e\}$.

Now, take any elements $x \in N$ and $h \in H$ and consider their commutator $[x, h] := x\,h\,x^{-1}h^{-1}$ in G. Since H is normal, we have $h \in H \Rightarrow x\,h\,x^{-1} \in H$ and $h^{-1} \in H$. Whence $[x, h] = (x\,h\,x^{-1})h^{-1} \in H$. On the other hand, N is also normal, so $x \in N \Rightarrow x^{-1} \in N \Rightarrow h\,x^{-1}h^{-1} \in N$, whence $[x, h] = x\,(h\,x^{-1}h^{-1}) \in N$. Thus, we see that $[x, h] \in N \cap H$ and therefore $[x, h] = e$.

Problem 4.

It is straightforward to check that the compostion of two functions $f_1(z) = (a_1 z + b_1)/(c_1 z + d_1)$ and $f_2(z) = (a_2 z + b_2)(c_2 z + d_2)$ is given by

$$(f_1 \circ f_2)(z) := f_1(f_2(z)) = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)} .$$

So, letting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) = \frac{az + b}{cz + d} ,$$

and taking into account that the assumption $ad - bc \neq 0$ we impose on $f(z)$ is equivalent to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ being invertible, we get a surjective group homomorphism $\mathrm{GL}(2, \mathbf{C}) \to \mathcal{F}(\mathbf{C})$. Its kernel consists of matrices which correspond to the identity function $f(z) = z$. These are exactly the scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ with $\lambda \neq 0$. By the First Isomorphism Theorem, we conclude that $\mathcal{F}(\mathbf{C}) \cong \mathrm{GL}(2, \mathbf{C})/\mathbf{C}^*$, where $\mathbf{C}^*$ is identified with a subgroup of (nonzero) scalar matrices in $\mathrm{GL}(2, \mathbf{C})$.

Problem 5.

This is an immediate consequence of Theorem 2.133 on p. 193.

Problem 6.

Since $|H| = p$ is prime, H must be a *cyclic* group (see Corollary 2.87 on p. 157). Now, due to HW Problem 2.94 on p. 171 (see also your lecture notes), we know that the order of the automorphism group $\mathrm{Aut}(H)$ of H is $|\mathrm{Aut}(H)| = \phi(p) = p - 1$.

On the other hand, since H is normal in G, we can define a map $f : G \to \mathrm{Aut}(H)$ by $g \mapsto \mathrm{Ad}_g$, where $\mathrm{Ad}_g(h) = g\,h\,g^{-1}$ for $h \in H$. As shown in Problem $2(b)$ above, this map is a group homomorphism. Since G is a p-group, the image of f is also a p-group, so that $|\mathrm{Im}(f)| = p^k$ for some $k \geq 0$. By Lagrange's Theorem, $|\mathrm{Im}(f)|$ must divide $|\mathrm{Aut}(H)| = p - 1$, which is possible only if $k = 0$. Thus, $\mathrm{Im}(f)$ is a trivial subgroup of $\mathrm{Aut}(H)$ (consisting only of the identity map Id_H). In other words, we have $\mathrm{Ad}_g = \mathrm{Id}_H$ for all $g \in G$. This means that $\mathrm{Ad}_g(h) = h$, and hence $g\,h = h\,g$ for all $g \in G$ and all $h \in H$. Whence $H \subseteq Z(G)$.