1 Introduction

Motivating Question There are uncountably many finitely generated groups, but there are only countably many finitely presented ones. Which countably many finitely generated groups then appear as subgroups of finitely presented ones?

Higman's Embedding Theorem ('60s) A finitely generated group is recursively presentable if and only if it is isomorphic to a subgroup of a finitely presented group.

$$\text{Recursively Presentable:} \qquad G = \langle \underbrace{a_1, \ldots, a_n}_{\text{finite}} | \underbrace{w_1, w_2, w_3, w_4, w_5 \ldots}_{\text{recursively enumerable}} \rangle$$

where a set is **recursively enumerable** if there exists an algorithm (a Turing machine) that can produce a complete list of its elements.

Remark.

1. Higman's embedding theorem provides a complete answer to the motivating question. Its proof however would take us too far afield from the theory of solvable groups.
2. There is, somewhat astonishingly, a related theorem for solvable groups, or more specifically, for metabelian groups.

Definition. A group G is **metabelian** if it has derived length ≤ 2, that is it has an abelian series,

$$1 \lhd H \lhd G$$

The following theorem was proved independently by Baumslag and Remeslennikov,

Baumslag-Remeslennikov's Thorem ('73) Every finitely generated metabelian group embeds into a finitely presented metabelian group.

Remark. In a paper titled 'Finitely Presented Metabelian Groups' by Baumslag he sketches a proof of this theorem in three steps:

(Andrew) Reduce the proof with the Magnus embedding: embed G into W/N where $W = A \wr H$ where A, H abelian

(Margarita) Show that W is embedded into a finitely presented metabelian group in the case

$$W = \mathbb{Z} \wr \mathbb{Z} = \Gamma_1$$

(Amin) Complete the proof...

I am going to use Margarita's step as an example both of the theorem and of how the proof works. Then I will use Andrew's step to reduce the problem, demonstrate Margarita's step in full generality (not just when $A \cong H \cong \mathbb{Z}$, as Baumslag did) and sketch out the remainder of the proof.

2 Baumslag/Margarita's Example: Embedding $\Gamma_1 = \mathbb{Z} \wr \mathbb{Z}$ into Γ_2

In Margarita's first talk we saw the following presentation for the lamplighter group,

$$\Gamma_1 = \langle a, s | [a, a^{s^k}] = 1, k = 1, 2, \ldots \rangle$$

This is clearly an infinitely presented group. I claim it is metabelian:

Claim. Γ_1 is metabelian

Proof. Surjective homomorphism

$$\begin{aligned} \phi : \Gamma_1 &\to \mathbb{Z} \\ a &\mapsto 1 \\ s &\mapsto s \end{aligned}$$

Then $\ker(\phi) \cong \mathbb{Z}^\infty$ is abelian. Thus we have the following abelian series for Γ_1,

$$1 \lhd \langle \ker \phi \rangle \lhd \Gamma_1$$

□

So Γ_1 is an infinitely presented metabelian group. Baumslag-Remeslennikov's theorem says that we should be able to embed this into a finitely presented metabelian group.

Construct G

1. Define an injective endomorphism of Γ_1, $\sigma : \Gamma_1 \to \Gamma_1$ by $\sigma(a) = aa^s, \sigma(s) = s$.
2. Form HNN extensions of Γ_1:
$$G = \langle t, \Gamma_1 | g^t = \sigma(g), g \in \Gamma_1 \rangle$$
Then G has the presentation,
$$G = \langle a, s, t | a^t = aa^s, s^t = s, [a, a^{s^k}] = 1, k = 1, 2, \ldots \rangle$$

Proposition. G is metabelian

Proof. Claim: $G' = \langle a^{s^i} : i \in \mathbb{Z} \rangle$

($\supseteq$) $aa^s = a^t \Rightarrow a^s = a^{-1}t^{-1}at = [a, t] \in G'$. Simple induction to complete (e.g. $s^{-1}a^s s = s^{-1}(a^{-1}t^{-1}at)s = (as)^{-1}t^{-1}(as)t$ since $[s, t] = 1$)

($\subseteq$) Basic fact: G/H abelain $\Rightarrow G' \subseteq H$, since G/G' is the largest abelain quotient of G.

Consider G modulo $\langle a^{s^i} : i \in \mathbb{Z} \rangle$. Get

$$G/\langle a^{s^i} : i \in \mathbb{Z} \rangle = \langle s, t | [s, t] = 1 \rangle = \mathbb{Z}^2$$

Proving the claim.

Now G/G' is abelian (by proceeding proof), and G' is free abelian of infinite rank. Therefore

$$1 \lhd G' \lhd G$$

is an abelian series for G of derived length 2, that is, G is metabelian.

□

Proposition. G is finitely presented. In fact, $G = \Gamma_2 = \langle a, s, t | a^t = aa^s, [a, a^s] = 1 = [s, t] \rangle$

Proof. The relations $[a, a^{s^i}] = 1$ are redundant for $i > 1$. Indeed, suppose $[a, a^{s^j}] = 1$ for $j = 2, \ldots, i$ follows from $[a, a^s] = 1$. Then

$$1 = [a, a^{s^i}]^t = [a^t, (a^{s^i})^t] = [a^t, (a^t)^{s^i}] = [aa^s, (aa^s)^{s^i}] = [aa^s, a^{s^i} a^{s^{i+1}}] = [a, a^{s^{i+1}}]$$

Where the equalities follow (respectively) from:

1. By hypothesis that $[a, a^{s^i}] = 1$
2. Conjugation is a homomorphism, so $[x, y]^t = [x^t, y^t]$

3. $[s,t]=1$ commute
4. $a^t = aa^s$
5. Again conjugation is a homomorphism
6. By the inductive hypothesis. (Exersize)

$$[aa^s, a^{s^i}a^{s^{i+1}}] = \underline{a}a^s a^{s^i}\underline{a^{s^{i+1}}}(a^s)^{-1}\underline{a^{-1}}(a^{s^i})^{-1}\underline{(a^{s^{i+1}})^{-1}}$$

The underlined terms give $[a, a^{s^{i+1}}]$. Use the inductive hypothesis to show that the other stuff commutes and cancels.

Remark. So we have embedded a finitely generated, infinitely presented metabelian group Γ_1 into a finitely presented metabelian group Γ_2 by constructing an HNN-extension and showing that it was finitely presented. This is the underlying idea behind the proof Baumslag-Remeslennikov theorem. Perhaps knowing now, as we do, that Γ_1 is the horocyclic product of two trees, and Γ_2 is the horocyclic product of three trees, this embedding is not so surprising. It is then perhaps more surprising that this works in general.

3 Proof of Baumslag-Remeslennikov Theorem

Magnus Embedding Theorem F a free group on $\{x_i | \in I\}$, $R \lhd F$. Given an isomorphism $F/R \to H$ by $x_iR \mapsto h_i$. Let A be the free abelian group on $\{a_i | i \in I\}$. Then the assignment $x_iR' \mapsto h_ia_i$ determines an embedding of F/R' into the wreath product $A \wr H$.

Lemma 1. G a finitely generated metabelian group. Then G can be embedded into $G_{ab} \ltimes A$ where A is a finitely generated $\mathbb{Z}G_{ab}$-module.

Proof. $G = \langle g_1, \ldots, g_n\rangle, F = F\langle x_1, \ldots, x_n\rangle, \theta : F \to G, K := \ker\theta$

G metabelian $\Rightarrow F'' \leq K$ (for $f'' \in F''$, $\theta(f'') \in G'' = \{1\}$)
$R := \theta^{-1}(G') \Rightarrow R = F'K$
$R' \leq K$ (indeed, $f \in F', k \in K, \theta[f,f] = \theta[f,k] = \theta[k,k] = 1$)

Let A_0 be the free abelian group on $\{a_1, \ldots, a_n\}$. We can apply the Magnus embedding with $H = G_{ab}$ since $F/R \cong G/G' = G_{ab}$,

$$\psi : F/R' \to A_0 \wr G_{ab} = A_0^{(G_{ab})} \rtimes G_{ab} = W \qquad B := A_0^{(G_{ab})} \text{ base group}$$
$$x_iR' \mapsto \begin{pmatrix} x_iR & * \\ 0 & 1 \end{pmatrix}$$

where $* \in B$, the base group.

Claim 1 W is finitely generated metabelian group
B and G_{ab} both abelian.

Define $N = \psi(K/R')$

Claim 2 $N \leq B$
$\psi(wR') \mapsto \begin{pmatrix} wR & * \\ 0 & 1 \end{pmatrix}$. But $R = F'K$ contains K. So if $wR' \in N$ then $w \in K \subset R$, thus upper left entry is trivial:

$$\begin{pmatrix} 1R & * \\ 0 & 1 \end{pmatrix} \xrightarrow{\cong} * \leq B$$

Claim 3 $N \lhd Im(\psi)$
Third isomorphism theorem: $K/R' \lhd F/R'$. Hence $N = \psi(K/R') \lhd \psi(F/R') = Im(\psi)$.

Claim 4 $N \lhd W$
$Im(\psi)$ contains all $x_i R$. W is generated by B (abelian) and by $(x_i R)$.

Finally, by the third isomorphism theorem again,

$$G \cong (F/R')/(K/R')$$

so ψ induces an embedding of G into $W/N = G_{ab} \ltimes (B/N)$ where B/N is a finitely generated $\mathbb{Z}G_{ab}$-module.

□

Remark The key point here is that we embedded G into a finitely generated metabelian group of the form $H \ltimes A$ where H, A were abelian. So if we can prove Baumslag-Remeslennikov theorem in that case we are done.

The idea of the proof is to embed G into ascending HNN-extensions and show that eventually we are left with a finitely presented metabelian group. We need a way to construct these HNN-extensions, namely, we need an endomorphism:

Lemma 2. H a finitely generated abelian group, A a finitely generated $\mathbb{Z}H$-module. For each $h \in H$ there exists a polynomial

$$p = 1 + c_1 x + \cdots + c_{r-1}x^{r-1} + x^r \in \mathbb{Z}[x]$$

such that $a \mapsto ap(h)$ is an injective $\mathbb{Z}H$-endomorphism of A.

Proof. A is a $\mathbb{Z}H$-module, so multiplication by $p(h)$ is clearly an endomorphism. Need to show it is injective. Call polynomials of the form above 'special polynomials'. Define

$$A_0 = \{a \in A : ap(h) = 0 \text{ some special } p\}$$

Then A_0 is a $\mathbb{Z}H$-submodule (indeed multiplication of two special polynomials is again special).

Fact: G virtually polycyclic then $R = \mathbb{Z}G$ is a Noetherian R-module.
Since H is abelian, and since A is finitely generated, then A_0 is finitely generated. Say it is generated by

$$b_1, \ldots, b_s$$

Then there exist special polynomials p_i such that $b_i p_i(h) = 0$. Define

$$p = xp_1 \cdots p_s + 1$$

This is clearly a special polynomial. Suppose $ap(h) = 0$ some $a \in A$. Then $a \in A_0$. Therefore

$$a = b_1 f_1 + \cdots + b_s f_s$$

some $f_i \in \mathbb{Z}H$. Notice that

$$b_i p(h) = b_i(hp_1(h) \cdots p_s(h) + 1) = hp_1(h) \cdots \underbrace{b_i p_i(h)}_{=0} \cdots p_s(h) + b_i = b_i$$

Therefore

$$0 = ap(h) = (b_1 f_1 + \cdots + b_s f_s)p(h) = b_1 f_1 + \cdots + b_s f_s = a$$

Thus $a = 0$ and we see that the map is injective.

□

Proof of Baumslag-Remeslennikov Theorem
Assume $G = H \ltimes A$ where A, H abelian, is a finitely generated metabelian group. If H were finite then G would be polycyclic and therefore finitely presented. So assume H infinite,

$$H = \langle h_1 \rangle \times \cdots \times \langle h_r \rangle \times \cdots \times \langle h_n \rangle$$

where $h_1, \ldots, h_r$ have infinite order, and $h_i^{q_i} = 0, i = r+1, \ldots, n$. By lemma 2 there exist special polynomials,

$$p_1, \ldots, p_r$$

such that $a \mapsto ap_i(h_i)$ determines an injective $\mathbb{Z}H$-endomorphism of A, say τ_i.

Constructing the HNN-extensions

1. $G_0 = G = H \ltimes A$
   Extend τ_1 to an injective endomorphism of G_0 acting as the identity on H (H abelian). Define,
   $$G_1 = \langle t, G_0 | g_0^{t_1} = g_0^{\tau_1}, g_0 \in G_0 \rangle$$
2. Extend τ_2 to G_1 by requiring it to act as identity on the abelian subgroup $\langle H, t_1 \rangle$. Define,
   $$G_2 = \langle t_2, G_1 | g_1^{t_2} = g_1^{\tau_2}, g_1 \in G_1 \rangle$$
3. Repeat this r times resulting in G_r.

Claim 1 $G_r = \overline{G} = Q \ltimes \overline{A}$, where $Q = H \times \langle t_1 \rangle \times \cdots \times \langle t_r \rangle$ and $\overline{A} = A^{\langle t_1, \ldots, t_r \rangle}$ is the normal closure of A in $\langle A, t_1, \ldots, t_r \rangle$.

Pf: Starting with $G_0 = H \ltimes A$. Adding t_i's and forcing them to commute with H and with a defined action of t_i on A. So we expect a semi-direct product of this form. The slight question is, why $\overline{A}$. To see this consider conjugation by a negative power of t_i. For example, if $g = (h, a) \in G_1$

$$t_1 g t_1^{-1} = \tau_1^{-1}(g)$$

τ_i are not necessarily surjective. Hence we take the normal closure. This is easier to visualise with the simpler example $BS(1,2) = \langle a, t | a^t = a^2 \rangle$.

□

Claim 2 G embeds in $\overline{G}$ and $\overline{G}$ is metabelian.

Pf: $\overline{G}$ is generated by the elements

$$h_1, \ldots, h_n \qquad t_1, \ldots, t_r \qquad \underbrace{a_1, \ldots, a_m}_{\text{generators for A}}$$

Similarly to before, $1 \lhd Q \lhd \overline{G}$ is an abelian normal series for $\overline{G}$ because both Q and $\overline{A}$ are abelian.

□

Constructing G^*
What relations do we have?

1. $[h_i, h_j] = [t_i, t_j] = [h_i, t_j] = [a_i, a_j] = 1$
2. $h_i^{q_i} = 1$ for $i = r+1, \ldots, n$
3. $a_i^{t_j} = a_i p_j(h_j)$ for $i = 1, \ldots, m, j = 1, \ldots, r$

Fact: $\mathbb{Z}H$ Noetherian $\Rightarrow A$ (a $\mathbb{Z}H$-module) is finitely presented as finitely generated is equivalent to finitely presented for modules over Noetherian rings. Put these relations in,

4. $a_1^{r_{i1}} a_2^{r_{i2}} \cdots a_m^{r_{im}} = 1$ for $i = 1, \ldots, k$ with $r_{ij} \in \mathbb{Z}H$

Finally need relations to ensure the normal closure of $\langle a_1, \ldots, a_m \rangle$ in $\overline{G}$ is abelian,

5. $[a_i^\nu, a_j^\mu] = 1$ for ν, μ of the form $h_1^{u_1} \cdots h_n^{u_n}$ where $0 \leq u_i \leq d_i$ for d_i the degree of p_i, when $1 \leq i \leq r$, and $0 \leq u_i < q_i$ for $r + 1 \leq i \leq n$

Define

$$G^* = \langle h_1, \ldots, h_n, t_1, \ldots, t_r, a_1, \ldots, a_m | 1, 2, 3, 4, 5 \rangle$$

The Conclusion There is a surjective homomorphism

$$G^* \to \overline{G}$$

Hall ('54) Finitely generated metabelian groups satisfy max-n, the maximum condition on normal subgroups. That is, every normal subgroup is finitely generated.

1. Prove G^* metabelian
2. Then G^* satisfies max-n
3. Hence $\overline{G}$ is finitely presented: Indeed, $\overline{G} = G^*/N$ some normal subgroup $N \lhd G^*$. Now since G^* satisfies max-n N is finitely generated. Therefore $\overline{G}$ is too.

Lemma. G^* is metabelian.

Sketch of Proof:

1. If we show that $A^* = \langle a_1, \ldots, a_m \rangle^{G^*}$ is abelian, then

$$G^* = \langle h_1, \ldots, h_n, t_1, \ldots, t_r, a_1, \ldots, a_m | 1, 2, 3, 4, 5 \rangle$$
$$G^*/A^* = \langle h_1, \ldots, h_n, t_1, \ldots, t_r | [h_i, h_j] = [h_i, t_j] = [t_i, t_j] = 1 \rangle \text{ is abelain}$$

   hence $1 \lhd A^* \lhd G^*$ is an abelian series of derived length 2 and therefore G^* is metabelian.

2. To prove A^* is abelian one uses the special polynomials p_i. It involves bashing out even larger commutators then in the example of embedding Γ_1 into Γ_2. The idea is similar enough.

□