The Dehn function of Baumslag's metabelian group

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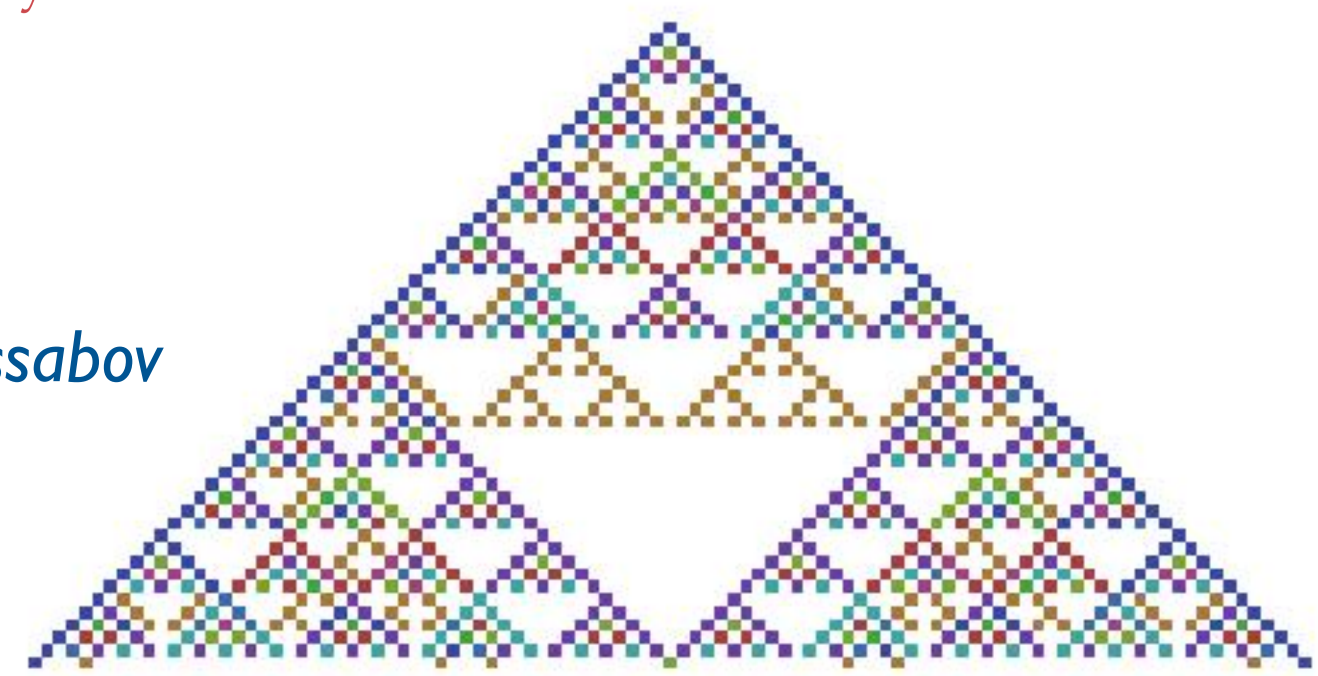
work with Martin Kassabov



Geometry and Topology Seminar – Binghamton

Baumslag, 1972. *A finitely presented metabelian group with an infinite rank free abelian normal subgroup:*

Notation

$$[x,y] = x^{-1}y^{-1}xy$$

$$x^{ny} = y^{-1}x^n y$$

$$\Gamma \;=\; \langle\; a, s, t \;\mid\; [a, a^t] = 1,\;\; [s,t] = 1,\;\; a^s = aa^t \;\rangle$$

$$\langle a, t\rangle \;=\; \mathbb{Z} \wr \mathbb{Z} \;=\; \left(\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\right) \rtimes \mathbb{Z}$$

$$= \;\left\langle\; a, t \;\middle|\; \left[a, a^{t^k}\right] = 1 \;(k \in \mathbb{Z}) \;\right\rangle$$

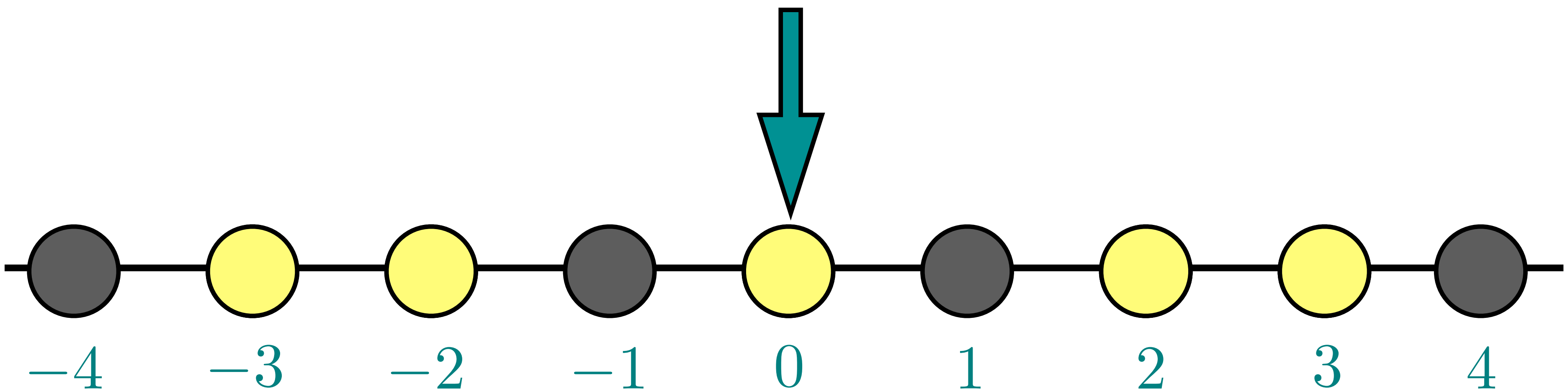
$[\Gamma, \Gamma] = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$, *so* Γ *is metabelian but not polycyclic.*

$$\Gamma_m \;=\; \langle\; a, s, t \;\mid\; a^m = 1, [a, a^t] = 1,\;\; [s,t] = 1,\;\; a^s = aa^t \;\rangle$$

$$\langle a, t\rangle \;=\; C_m \wr \mathbb{Z}$$

$C_2 \wr \mathbb{Z}$ *is the lamplighter group.*

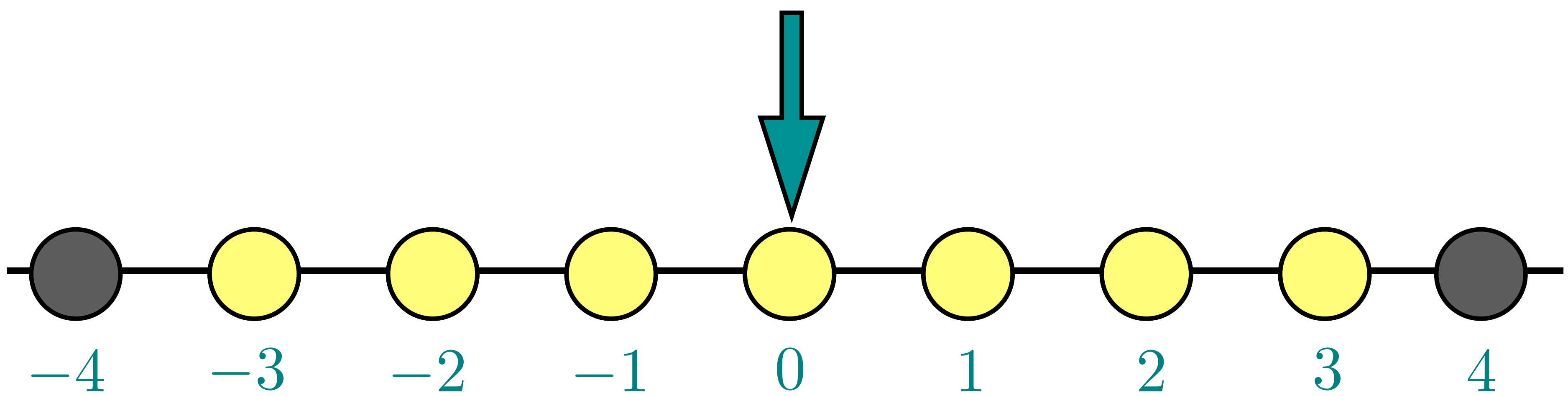
$$C_2 \wr \mathbb{Z} \ = \left\langle \ a, t \ \middle| \ a^2 = 1, \ [a, a^{t^k}] = 1 \ (k \in \mathbb{Z}) \ \right\rangle$$



$a \ \ t \ \ t \ \ a \ \ t \ \ a \ \ t^{-1} \ \ t^{-1} \ \ t^{-1} \ \ t^{-1} \ \ t^{-1} \ \ a \ \ t^{-1} \ \ a \ \ t \ \ a \ \ t$

$$C_2 \wr \mathbb{Z} = \left\langle \, a, t \;\middle|\; a^2 = 1, \;\; [a, a^{t^k}] = 1 \;\, (k \in \mathbb{Z}) \, \right\rangle$$

$$\gamma = t^n (a t^{-1})^{2n} a t^n$$



$$d(1, \gamma) = 6n + 1$$

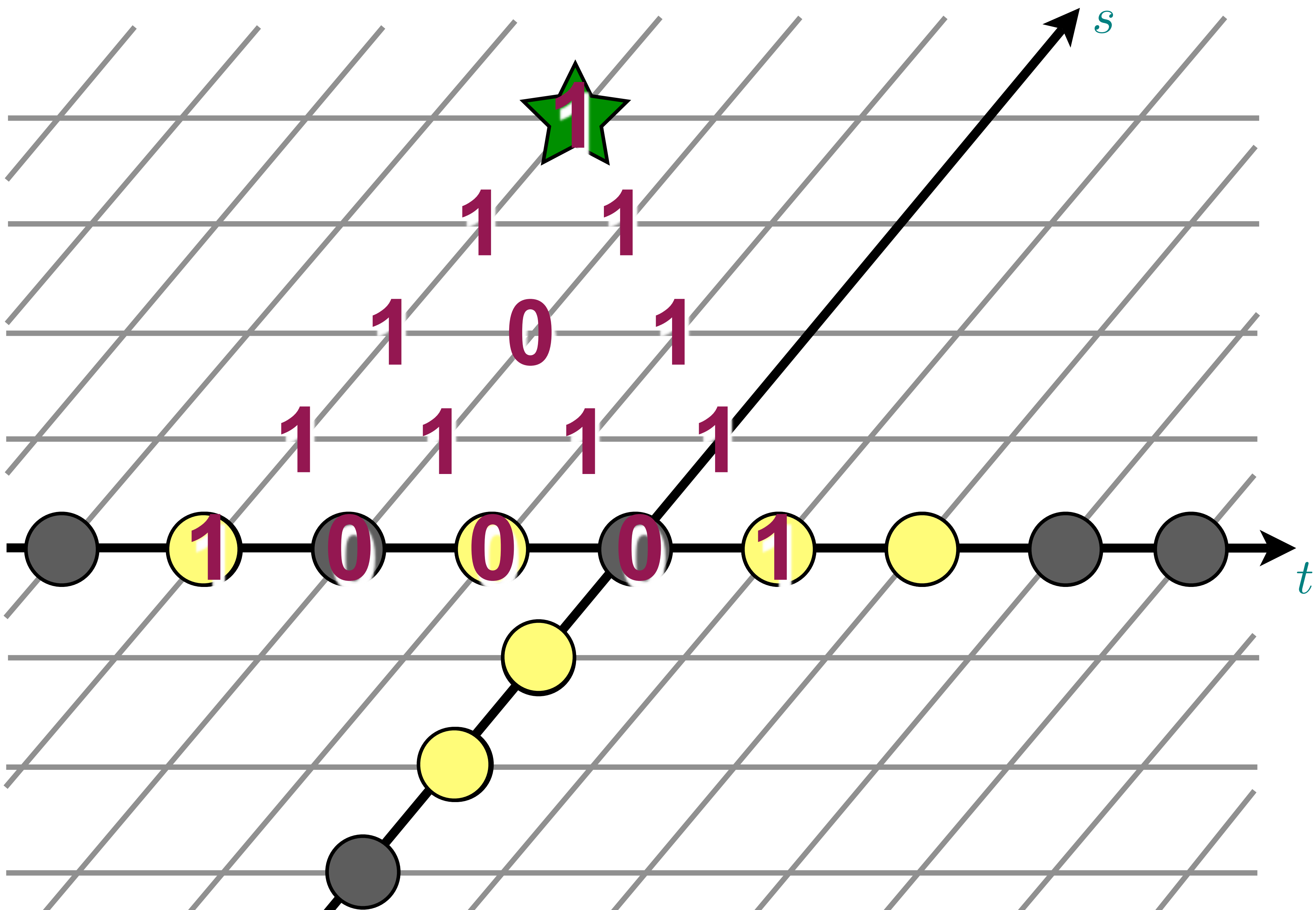
Distance from γ to a group element further than $6n+1$ from 1 is $> n$.

– *"unbounded dead-end depth"* (Cleary–Taback; Erschler)

Cleary–R., 2007. With respect to a suitable generating set, Γ_2 has unbounded dead-end depth.

– the first finitely presentable example

$$\Gamma_2 = \langle\, a, s, t \mid a^2 = 1,\ [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t \,\rangle$$



$$C_m \wr \mathbb{Z} = \left\langle\, a, t \;\middle|\; a^m = 1,\ [a, a^{t^k}] = 1\ (k \in \mathbb{Z}) \,\right\rangle$$

$$\cong \left\{ \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} \;\middle|\; k \in \mathbb{Z},\ P \in C_m[t, t^{-1}] \right\}$$

$C_m[t, t^{-1}]$ = ring of Laurent polynomials with coefficients in C_m

via $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$

$$\Gamma_m = \left\langle\, a, s, t \;\middle|\; a^m = 1, [a, a^t] = 1,\ [s, t] = 1,\ a^s = a a^t \,\right\rangle$$

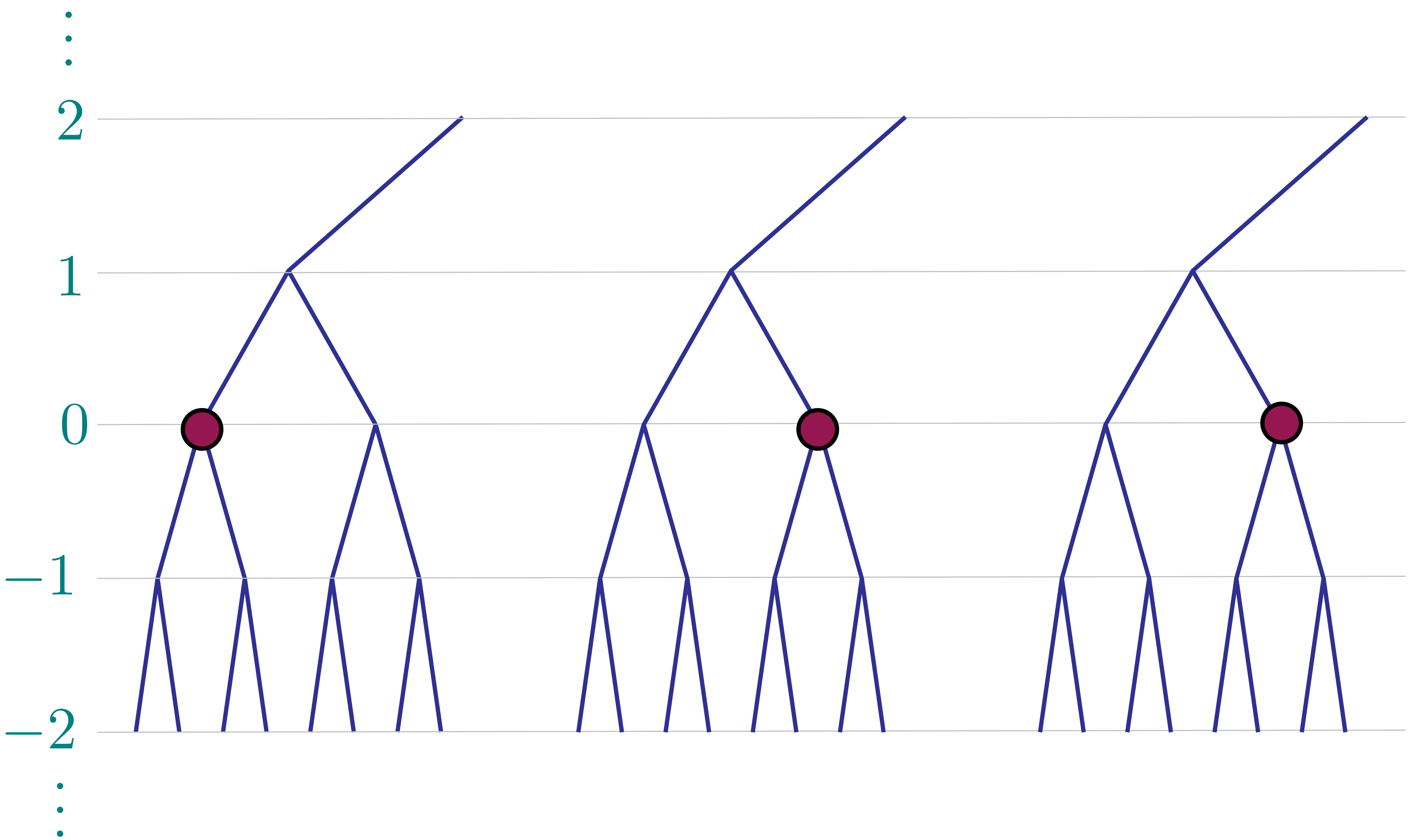
$$\cong \left\{ \begin{pmatrix} t^k (t+1)^l & P \\ 0 & 1 \end{pmatrix} \;\middle|\; k, l \in \mathbb{Z},\ P \in \mathcal{R}_m \right\}$$

$$\mathcal{R}_m := C_m[t, t^{-1}, (1+t)^{-1}]$$

via $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 1+t & 0 \\ 0 & 1 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$

$\{t^i, (1+t)^j \mid i \in \mathbb{Z}, j < 0\}$ is a basis for $\mathcal{R}_m$

Horocyclic products of trees

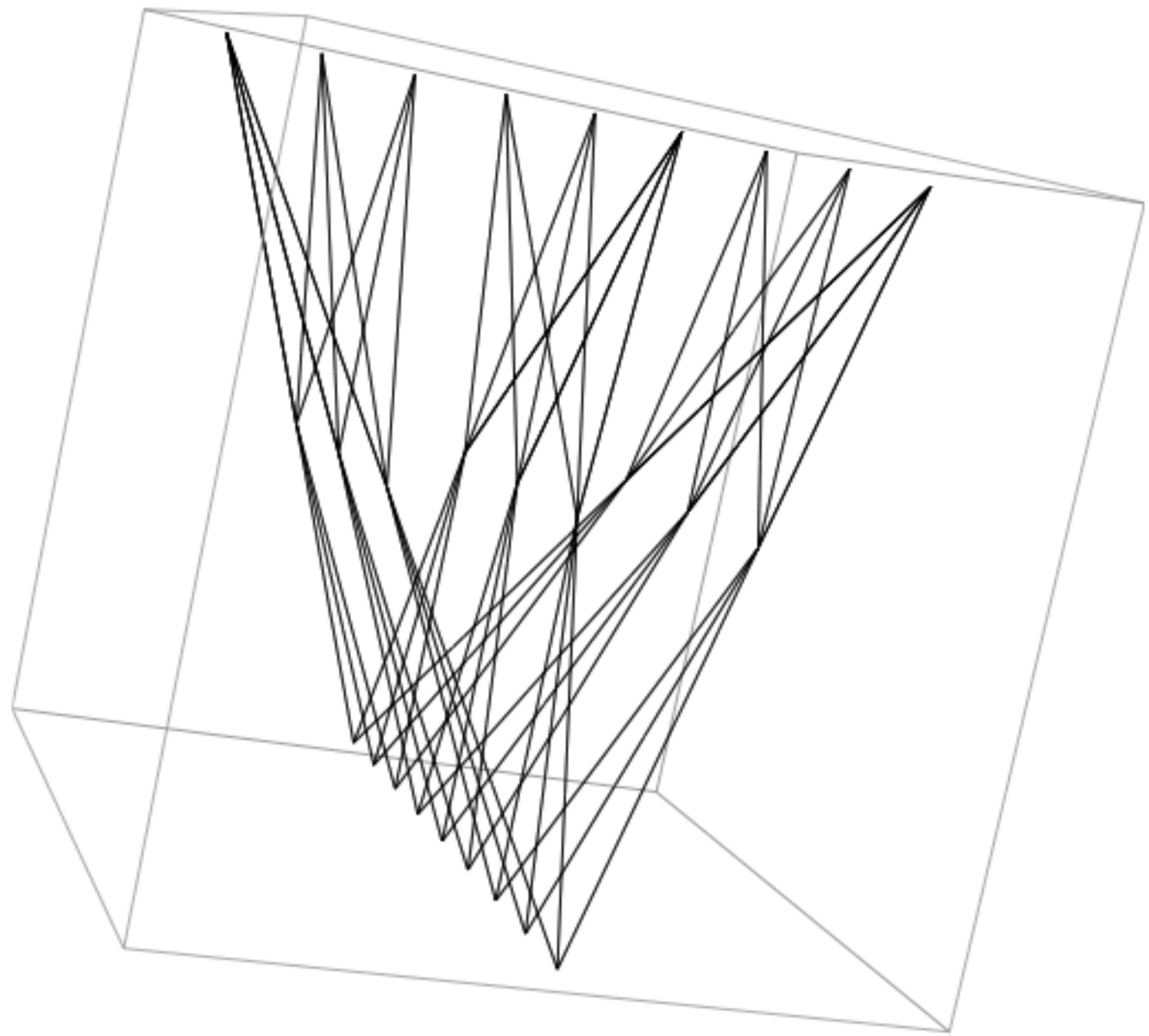


$$C_2 \wr \mathbb{Z} = \langle x, y \mid \left(x^k y^{-k}\right)^2 = 1 \ (k \in \mathbb{Z}) \ \rangle$$

$$x = t$$
$$y = at$$

$$\Gamma_2$$

Bartholdi, Neuhauser, Woess



Picture of a portion of the horocyclic product of two 3-valent trees by Tullia Dymarz

For p prime,

$C_p \wr \mathbb{Z}$ *embeds as a cocompact lattice in* $\mathrm{SOL}_3(\mathbb{F}_p((t))\,)$.

Γ_p *embeds as a cocompact lattice in* $\mathrm{SOL}_5(\mathbb{F}_p((t))\,)$.

Grigorchuk, Linnell, Schick, Zuk. Γ_2 is a counterexample to a strong version of the Atiyah Conjecture on L^2-Betti numbers.

Bartholdi, Neuhauser, Woess

- Poisson boundary; random walks
- Finiteness properties

Cleary. $\mathbb{Z} \wr \mathbb{Z}$ is exponentially distorted in Γ .

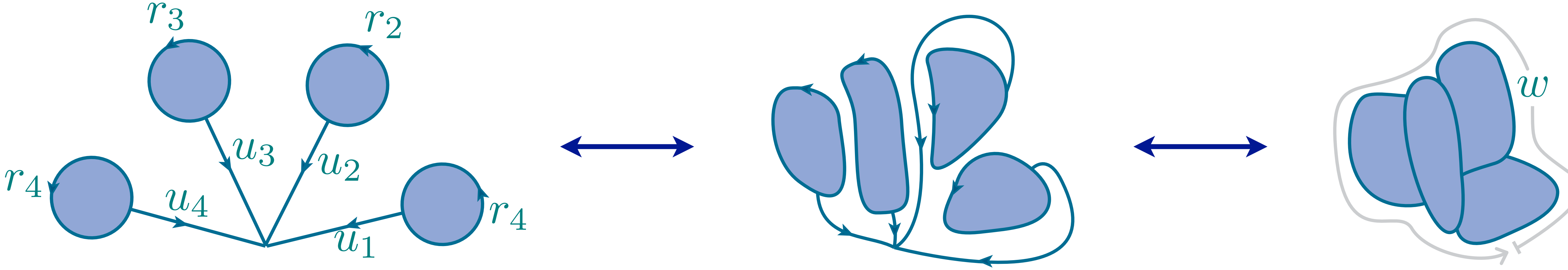
Dehn functions

$\langle A \mid R \rangle$ *a finite presentation for a group* Γ

If w *represents* 1 *in* Γ*, then* $\mathrm{Area}(w)$ *is the minimal* N *such that*

$$w = \prod_{i=1}^{N} {u_i}^{-1} r_i u_i \text{ in } F(A)$$

for some $u_i = u_i(A)$ *and* $r_i \in R^{\pm 1}$.



$$\mathrm{Area}(n) := \max\{\, \mathrm{Area}(w) \mid w = 1,\ \ell(w) \le n \,\}$$

a quasi–isometry invariant

– in particular, invariant under passing to a subgroup of finite index.

Kassabov–R. The Dehn function of Γ satisfies $\mathrm{Area}(n) \simeq 2^n$.

Kassabov–R. The Dehn function of Γ_m satisfies $\mathrm{Area}(n) \preceq n^4$.

de Cornulier, Tessera. The Dehn function of Γ_m satisfies $\mathrm{Area}(n) \preceq n^2$.

Gromov. Cocompact lattices in $\mathrm{SOL}_5(\mathbb{R})$ have Dehn function $\preceq n^2$.

Central extensions, distortion and Dehn functions

$H \leq G$ *a central subgroup*

$$\begin{array}{ccccccccc} 1 & \to & H & \to & G & \to & G/H & \to & 1 \\ & & \| & & \| & & \| & & \\ & & \langle B \rangle & & \langle A \rangle & & \langle A \mid R \rangle & & \end{array}$$

$|A|, |B|, |R| < \infty$
$R \subseteq B$

Suppose $w = w(A)$ *represents* 1 *in* G/H.

Then

$$w = \prod_{i=1}^{N} {u_i}^{-1} r_i u_i \text{ in } F(A)$$

for some $u_i = u_i(A)$, $r_i \in R^{\pm 1}$, $n \in \mathbb{N}$.

So $w = \underbrace{\prod_{i=1}^{N} r_i}_{\text{word on } B^{\pm 1}}$ *in* G .

Example. Crazy proof that the Dehn function of $\mathbb{Z}^2$ is $\succeq n^2$.

$$\mathcal{H}_3 = \left\langle \ a, b, c \ \middle| \ \begin{array}{ll} [a,c]=1, & [b,c]=1 \\ [a,b]=c, & \end{array} \ \right\rangle$$

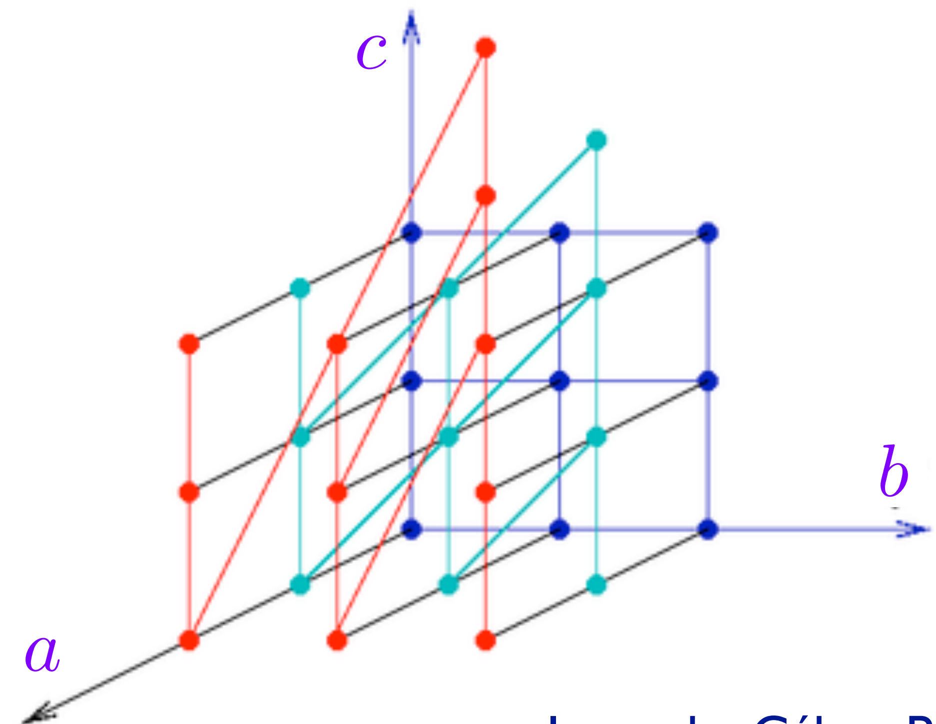


Image by Gábor Pete

$$\begin{array}{c} 1 \to \mathbb{Z} \to \mathcal{H}_3 \to \mathbb{Z}^2 \to 1 \\ \hspace{4em} \| \\ \langle \ a, b, c \ \mid \ c = [a,b] = 1 \ \rangle \end{array}$$

$[a^n, b^n] = 1$ *in* $\mathbb{Z}^2$.

$c^{n^2} = [a^n, b^n]$, *so* $\mathrm{Area}[a^n, b^n] \geq n^2$

$$\Gamma = \langle\, a, s, t \mid [a, a^t] = 1,\ [s,t] = 1,\ a^s = aa^t \,\rangle$$

$$\bar{\Gamma} =$$

$$\left\langle a, p, q, s, t \;\middle|\; \begin{array}{rrrrr} [a, a^t] = p, & aa^t = a^s q, & s^{-1}ps = p^{-1}, & t^{-1}pt = p^{-1}, & [a,p] = 1 \\ [s,t] = 1, & [p,q] = 1, & s^{-1}qs = q^{-1}, & t^{-1}qt = q^{-1}, & [a,q] = 1 \end{array} \right\rangle$$

an extension of Γ by $H = \langle p, q \rangle$. NOT CENTRAL!

But we can pass to index–2 subgroups and get a genuine central extension: the kernels of the parity map for the sum of the exponents of s and t.

Note for later: $\left[a, a^{t^n} \right]$ is in the kernel.

$$\left\langle a, p, q, s, t \;\middle|\; \begin{array}{ccccc} [a, a^t] = p, & aa^t = a^s q, & s^{-1}ps = p^{-1}, & t^{-1}pt = p^{-1}, & [a,p] = 1 \\ [s,t] = 1, & [p,q] = 1, & s^{-1}qs = q^{-1}, & t^{-1}qt = q^{-1}, & [a,q] = 1 \end{array} \right\rangle$$

Headache. Check $H = \langle p, q \rangle$ is $\mathbb{Z}^2$.

$R := \mathbb{Z}[x, x^{-1}, (x+1)^{-1}]$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & -2x-1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & -x-1 \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ & x+1 & 0 \\ & & -x^2-x \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ & x & 0 \\ & & -x^2-x \end{pmatrix}$$

$$\tau = (-1+\sqrt{5})/2 \qquad \begin{array}{rcl} R & \to & \mathbb{Z}[\tau] \\ f(x) & \mapsto & f(\tau) \end{array} \qquad \bar{\Gamma} \to \begin{pmatrix} 1 & R & \mathbb{Z}[\tau] \\ & R^* & R \\ & & R^* \end{pmatrix}$$

The image of H is $\begin{pmatrix} 1 & 0 & \mathbb{Z}[\tau] \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cong \mathbb{Z}^2$.

Lemma. In $\bar{\Gamma}$

$$\left[a, a^{t^n}\right] = p^{(-1)^{n+1} F_n},$$

where F_n is the n-th Fibonacci number.

$$\left[A, A^{T^n}\right] = \begin{pmatrix} 1 & 0 & (-x-1)^n - x^n \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & (-1)^n \sqrt{5} F_n \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

Conclusion. In Γ,

$$\operatorname{Area}\left[a, a^{t^n}\right] \succeq F_n \succeq 2^n$$

So the Dehn function of Γ is $\succeq 2^n$.

Our upper bound on the Dehn function of Γ

$$\Gamma = \langle\, a, s, t \mid [a, a^t] = 1,\ [s,t] = 1,\ \boxed{a^s = aa^t}\,\rangle$$

mitosis

$$\begin{array}{ccccccc}
 & & a^s = aa^t & & a^s = a^t a & & \\
a & = & a & = & a & & 1 \\
a^s & = & a\ a^t & = & a^t\ a & & 1\ 1 \\
a^{s^2} & = & a\ a^{2t} a^{t^2} & = & a^{t^2} a^{2t} a & & 1\ 2\ 1 \\
a^{s^3} & = & a\ a^{3t}\ a^{3t^2}\ a^{t^3} & = & a^{t^3} a^{3t^2} a^{3t}\ a & & 1\ 3\ 3\ 1 \\
a^{s^4} & = & a\ a^{4t}\ a^{6t^2} a^{4t^3} a^{t^4} & = & a^{t^4}\ a^{4t^3} a^{6t^2} a^{4t}\ a & & 1\ 4\ 6\ 4\ 1
\end{array}$$

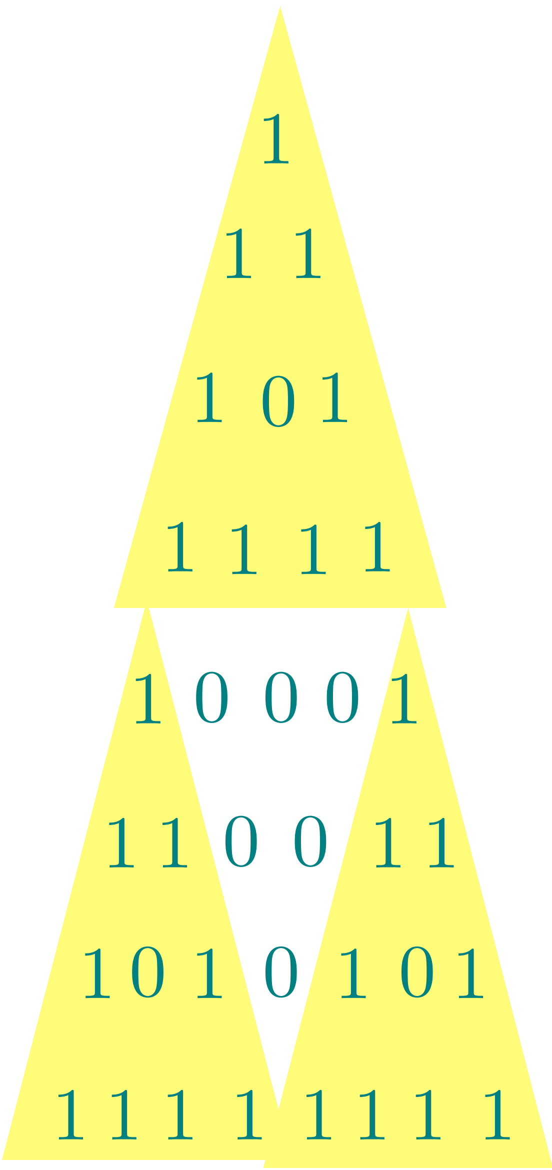
This leads to $\mathrm{Area}\left[a, a^{t^n}\right] \preceq 2^n$.

From there, one can get that the Dehn function of Γ is $\preceq 2^n$.

Our upper bound on the Dehn function of Γ_2

$$\Gamma_2 = \langle\, a, s, t \mid a^2 = 1,\ [a, a^t] = 1,\ [s, t] = 1,\ a^s = a a^t \,\rangle$$

$$\begin{array}{llll}
 & & a^s = a a^t & a^s = a^t a \\
a & = & a & = \quad a \\
a^s & = & a \; a^t & = \quad a^t \; a \\
a^{s^2} & = & a \quad a^{t^2} & = \quad a^{t^2} \quad a \\
a^{s^3} & = & a \; a^t \; a^{t^2} a^{t^3} & = \quad a^{t^3} a^{t^2} \; a^t \; a \\
a^{s^4} & = & a \qquad\qquad a^{t^4} & = \quad a^{t^4} \qquad\qquad a
\end{array}$$



This leads to $\mathrm{Area}\left[a, a^{t^n}\right] \preceq n^2$ when n is a power of 2.

This is the beginning of a way to get a quartic upper bound on the Dehn function of Γ_2 and similarly Γ_m.