EXPONENTIALLY DISTORTED SUBGROUPS IN WREATH PRODUCTS

T. R. RILEY

IN MEMORIAM PETER M. NEUMANN 1940–2020

Abstract. We exhibit exponentially distorted subgroups in $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ and $\mathbb{Z} \wr F_2$.
2020 Mathematics Subject Classification: 20F65, 20F10, 20F16
Key words and phrases: wreath product, subgroup distortion

1. Introduction

The main result of this paper is –

Theorem 1.1. *The subgroup*

$$H = \langle\, x,\ y,\ [x, a]a,\ [y, a]a \,\rangle$$

is exponentially distorted in $\mathbb{Z} \wr F_2$ *where* $\mathbb{Z} = \langle a \rangle$ *and* $F_2 = F(x, y)$. *The same is true in* $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z}) = \langle a \rangle \wr (\langle s \rangle \wr \langle t \rangle)$ *with* $x = ts$ *and* $y = t$.

Subgroup distortion is foundational and widely studied. It compares a subgroup's word metric with the restriction of the word metric of the ambient group. In some basic cases it is well-behaved. Subgroups of finitely generated free groups and of fundamental groups of closed hyperbolic surfaces are undistorted [Pit93, Sho91]. Subgroups of finitely generated nilpotent groups are all at most polynomially distorted [Osi01]. But subgroup distortion can be wild even in some seemingly benign groups. There are subgroups of $F_2 \times F_2$ and of rank-3 free solvable groups whose distortion functions cannot be bounded from above by a recursive function [Mih66, Umi95].

Theorem 1.1 shows that substantial subgroup distortion can arise subtly in wreath products. It is a next step in a direction of inquiry pursued by Davis and Olshanskii [Dav11, DO11]. They proved every subgroup of $\mathbb{Z} \wr \mathbb{Z}$ is

Date: March 11, 2023.

distorted like n^d for some positive integer d and, for each such d, they exhibited a subgroup realizing that distortion. Davis [Dav11] suggested next exploring subgroup distortion in $\mathbb{Z} \wr F_n$ and quoted speculation that an answer would be of interest for the study of von Neumann algebras.

The most novel feature of the work here is the idea behind the exponential lower bound (proved in Section 3). It relies on the observation that F_2 and $\mathbb{Z} \wr \mathbb{Z}$ admit height functions (homomorphisms onto $\mathbb{Z}$) such that for all integers $n > 0$, there are pairs of height-0 elements a distance $2n$ apart with the property that any path from one to the other travels up to height n en route—see Proposition 3.1.

The proof of the exponential upper bound (in Section 4) includes an intrinsic description of H.

The second theorem of this article makes the point that our distorted subgroups of Theorem 1.1 are necessarily delicate. Closely related results can be found in [BLP15], which we recommend for a more detailed treatment than the proof we outline in Section 5.

Theorem 1.2. *(Cf. Burillo–López-Platón* [BLP15]*) Suppose K is a finitely generated group and $G = \mathbb{Z} \wr K$. So, $G = W \rtimes K$ where $W = \bigoplus_K \mathbb{Z}$. Then –*

(1) *All finitely generated subgroups H of W are undistorted in G.*
(2) *If H is a finitely generated subgroup of K, then its distortion in G is the same as its distortion in K (more precisely, $\mathrm{Dist}_H^G \simeq \mathrm{Dist}_H^K$). In particular, K is undistorted in G (meaning $\mathrm{Dist}_K^G(n) \simeq n$).*
(3) *Cases* (1) *and* (2) *give all possible distortion functions of $\mathbb{Z}$-subgroups of G. In more detail, if $\hat{H} \cong \mathbb{Z}$ is a subgroup of G such that $\mathrm{Dist}_{\hat{H}}^G(n) \not\simeq n$, then there exists a subgroup $H \cong \mathbb{Z}$ of G per* (1) *or* (2) *such that $\mathrm{Dist}_H^G \simeq \mathrm{Dist}_{\hat{H}}^G$.*

In the case of $G = \mathbb{Z} \wr F_2$ all the subgroups in this list are undistorted in G, as all finitely generated subgroups of F_2 are undistorted. In the case of $G = \mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ the list includes polynomially distorted subgroups on account of [Dav11, DO11].

The results in this paper do not speak to the question of Guba & Sapir [GS99] as to what functions may be distortion functions of finitely generated subgroups of Thompson's group F. While $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ is a subgroup of Thompson's group, $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ and $\mathbb{Z} \wr F_2$ are not [Ble08, Theorem 1.2].

ACKNOWLEDGEMENTS

I am most grateful to Aria Beaupré, Jim Belk, Conan Gillis, and Chaitanya Tappu for helpful and stimulating conversations.

This paper is dedicated to the memory of Peter Neumann in tribute to his foundational work both as a scholar (e.g., pertinently, [Neu64]) and as a teacher.

2. PRELIMINARIES

For a group G with finite generating set S, write $|g|_S$ to denote the length of a shortest word on $S^{\pm 1}$ representing g. The word metric d_S on G is $d_S(g,h) = \left|g^{-1}h\right|_S$.

Suppose a subgroup $H \leq G$ is generated by a finite set $T \subseteq G$. The distortion function $\mathrm{Dist}_H^G : \mathbb{N} \to \mathbb{N}$ for H in G compares the word metric d_T on H to the restriction of d_S to H:

$$\mathrm{Dist}_H^G(n) \;\coloneqq\; \max\left\{\, |g|_T \;\mid\; g \in H \text{ and } |g|_S \leq n \,\right\}.$$

For functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \preceq g$ when there exists $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n. We write $f \simeq g$ when $f \preceq g$ and $g \preceq f$.

Two finite generating sets for a group yield biLipschitz word metrics, with the constants reflecting the minimal length words required to express the elements of one generating set as words on the other. So, up to $\simeq$, the growth rate of a distortion function does not depend on the finite generating sets.

Let $W = \bigoplus_K L$, the direct sum of a K-indexed family of copies of L. The *(restricted) wreath product* $G = L \wr K$ is the semi-direct product $W \rtimes K$ with K acting to shift the indexing. More precisely, given a function $f : K \to L$ that is finitely supported (meaning $f(k) = e$ for all but finitely many $k \in K$) and given $k \in K$, define $f^k : K \to L$ by $f^k(v) = f(vk^{-1})$. Then $L \wr K$ is the set of such pairs (f, k) with multiplication

$$(f,k)(\hat{f},\hat{k}) \;=\; (f + \hat{f}^k, k\hat{k}).$$

A *lamplighter description* helps us navigate G. Suppose $\{a_1, \ldots, a_m\}$ generates L and $\{b_1, \ldots, b_l\}$ generates K. Viewing L and K as subgroups of G, with L being the e-summand of W, the set $S = \{a_1, \ldots, a_m, b_1, \ldots, b_l\}$ generates G. Then W is the normal closure of $a_1, \ldots, a_m$ in G, or equivalently the kernel of the map $\Phi : G \twoheadrightarrow K$ that kills $a_1, \ldots, a_m$. Imagine K as a city. At each street corner (that is, each element of K) there is a lamp whose setting is expressed as an element of L. An $(f, z) \in L \wr K$ records settings $f(k) \in L$ of the lamps $k \in K$ and a location $z \in K$ for the lamplighter. A word w on $S^{\pm 1}$ representing (f, z) describes how at dusk a lamplighter walks the city streets adjusting the lamps to achieve (f, z). He starts at $e \in K$ with all lights off (that is, set to $e \in L$) and, reading w from left to right, moves in K according to the $b_1^{\pm 1}, \ldots, b_l^{\pm 1}$ until finally arrives at z. En route, he adjusts the setting of each lamp where he stands according to the $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$.

Our conventions are that $[x, a] = x^{-1}a^{-1}xa$ and $x^a = a^{-1}xa$.

3. The exponential lower bound on distortion

Proposition 3.1. *Suppose $K = \langle x, y \mid R \rangle$ is a 2-generator group such that mapping x and y to 1 defines an epimorphism $\theta : K \to \mathbb{Z}$ (a 'height function').*

Suppose that for $n \geq 1$, there is a set P_n of elements of K such that:

(i) $x^{n-1} \in P_n$ *but* $x^n y^{-n} \notin P_n$.
(ii) If $p \in P_n$, *then* $px^{-1}, py^{-1} \in P_n$.
(iii) If $k \in K \smallsetminus P_n$ *and either* kx^{-1} *or* ky^{-1} *is in* P_n, *then* $\theta(k) = n$.

Let $G = \mathbb{Z} \wr K$, generated by a, x, y where $\mathbb{Z} = \langle a \rangle$. Let

$$H = \langle x, y, \sigma, \tau \rangle \leq G$$

where $\sigma = [x, a]a$ and $\tau = [y, a]a$. Then $\mathrm{Dist}_H^G(n) \geq 2^n$.

For motivating examples of K and P_n see Corollary 3.2. We view *(ii)* as saying that when moving in the Cayley graph of K, it is not possible to enter P_n from below, and *(iii)* as saying that P_n can only be entered from above by moving from a height-n element outside P_n to a height-$(n-1)$ element in P_n. Together, *(i)* and *(ii)* imply that P_n contains x^i if and only if $i < n$.

In terms of the lamplighter description, the idea behind this proposition is as follows. Suppose the lights at the elements e and $x^n y^{-n}$ of K are set to 1 and -1, respectively, and all other lights are off (set to 0). How can a lamplighter turn all the lights off using x, y, σ, and τ? The lamplighter has four types of move at his disposal: he can navigate the Cayley graph of K (by using x and y); as $\sigma = [x, a]a = x^{-1}a^{-1}xa^2$, he can decrement by 1 the lamp one step away in the x^{-1}-direction at the expense of incrementing the lamp where he stands by 2; and likewise in the y^{-1}-direction using τ.

The answer is he sets the lamp at e to 0 at the expense of setting that at x to 2. Then he sets that one to 0 at the expense of setting that at x^2 to 4. And so on, until the lamp at x^n is set to 2^n. He then sets that to 0 and, proceeding in the y^{-1} direction, sets the lamp at $x^n y^{-1}$ to 2^{n-1}. Continuing likewise in the y^{-1}-direction he sets the lamp at $x^n y^{-(n-1)}$ to 2. Finally, he adjusts that to zero at the expense of changing the lamp at $x^n y^{-n)}$, but as that was initially set to -1, this results in all lights being off, as required.

The above method takes at least 2^n moves, but could it have been accomplished with fewer? The hypothesis involving P_n, $x^n y^{-n}$, and the epimorphism $K \to \mathbb{Z}$ ensures it cannot. Any path from e to $x^n y^{-n}$ in the Cayley graph must rise to height n to escape P_n and the settings of the lights must be incrementally adjusted on the way up so that the number of σ- and τ-moves grows exponentially with the height.

Here is a proof.

Proof of Proposition 3.1. Fix $n \geq 1$. First we will show that $a^{-1} x^n y^{-n} a \in H$. Define

$$\begin{aligned} \lambda_n &= x\sigma \, x\sigma^2 \, \cdots \, x\sigma^{2^{n-1}} \\ \mu_n &= y\tau \, y\tau^2 \, \cdots \, y\tau^{2^{n-1}}, \end{aligned}$$

which both represent elements of H. In G, the elements a and $x^{-1}ax$ commute, so for all i,

$$a^i \, x\sigma^i \;=\; a^i \, x \, (x^{-1}a^{-1}xa^2)^i \;=\; a^i \, x \, x^{-1}a^{-i}x \, a^{2i} \;=\; x \, a^{2i},$$

and therefore $a\lambda_n = x^n a^{2^n}$. Likewise, $a\mu_n = y^n a^{2^n}$ in G. So $a^{-1}x^n y^{-n} a$ equals $\lambda_n \mu_n^{-1}$ in G and represents an element of H.

The length of $\lambda_n \mu_n^{-1}$ as a word on x, y, σ, τ is $2n + 2^{n+1} - 2$. Next we will argue that the length of *any* word w on x, y, σ, τ that represents $a^{-1}x^n y^{-n} a$ in G is at least $2^n - 1$. The length of $a^{-1}x^n y^{-n} a$ as a word on a, x, y is $2n + 2$. So we will then have that $\mathrm{Dist}_H^G(2n + 2) \geq 2^n - 1$ and the result will follow.

Express $a^{-1}x^n y^{-n} a$ in the form $(f, x^n y^{-n})$ where f is -1 at e, is 1 at $x^n y^{-n}$, and is 0 elsewhere.

Given a finitely supported function $f : K \to \mathbb{Z}$ and an integer $i < n$, define

$$p_i(f) \;=\; \sum_{\substack{g \in P_n \\ \theta(g) = i}} f(g).$$

So the sequence

$$\mathcal{P}_n(f) \;=\; \big(\ldots, \; p_{-1}(f), \; p_0(f), \; p_1(f), \; \ldots, \; p_{n-2}(f), \; p_{n-1}(f) \big)$$

is all zeroes apart from $p_0(f) = -1$, since e is P_n and $x^n y^{-n}$ is not.

Now consider the effect on $\mathcal{P}_n(f)$ of changing f via the action of σ or τ when the lamplighter is located at some $k \in K$. Let $i = \theta(k)$. If $k \in P_n$, then (by hypothesis) kx^{-1} and ky^{-1} are in P_n and so $p_{i-1}(f)$ is lowered by 1 and $p_i(f)$ is increased by 2. And if $k \notin P_n$, then kx^{-1} and ky^{-1} can only be in P_n if $i = n$ (again, by hypothesis) and if so, $p_{n-1}(f)$ (only) decreases by 1. The actions of σ^{-1} and τ^{-1} are the same, instead of lowering lamp settings by 1 they increase then by 1, and instead of increasing by 2 they decrease by 2.

We can read off w^{-1} a sequence of applications of $\sigma^{\pm 1}$ and $\tau^{\pm 1}$ (and lamplighter movements around K) that convert $a^{-1}x^n y^{-n} a$ to e and so convert $\mathcal{P}_n(f)$ to the sequence of all zeroes. This process must display a doubling effect that gives us the claimed lower bound $2^n - 1$. The net effect of the (at least 1) moves that change $p_0(f)$ must be to convert it from -1 to 0, and so in the process they convert $p_1(f)$ from 0 to -2. There must therefore be (at least 2) moves that increment $p_1(f)$ at the expense of converting $p_2(f)$ from 0 to -4, and so on. □

(In fact, when K is F_2 or $\mathbb{Z} \wr \mathbb{Z}$ as per the following corollary, the rolls of x and y are interchangeable and the above proof shows $\lambda_n \mu_n^{-1}$ is a geodesic word.)

Corollary 3.2. *The subgroups of $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ and $\mathbb{Z} \wr F_2$ of Theorem 1.1 are both at least exponentially distorted.*

Proof. For $K = F_2 = F(x, y)$, because the Cayley graph is a tree, the proposition applies with P_n the set of reduced words whose prefixes π all satisfy $\theta(\pi) < n$.

For $K = \mathbb{Z} \wr \mathbb{Z} = \langle s, t \mid [s, s^{t^i}] = 1 \ \forall i \rangle$ with respect to the generating set $x = ts$ and $y = t$, mapping $x, y \mapsto 1$ defines a homomorphism $\theta : K \to \mathbb{Z}$.

In the lamplighter model for $\mathbb{Z} \wr \mathbb{Z}$, the integer $\theta(k)$ is the position of the lamplighter. Take P_n to be the set of all $k \in \mathbb{Z} \wr \mathbb{Z}$ such that $\theta(k) < n$ and the lamp settings is supported on $\{...., n-2, n-1\}$. Then kx^{-1} and ky^{-1} are in P_n for all $k \in P_n$ since x^{-1} decrements the light at the lamplighter's location and then moves one step in the negative direction, and y^{-1} only moves one step in the negative direction. The elements x^i for $0 \leq i \leq n-1$ are in P_n since they have the lights at positions $1, 2, \ldots, i$ set to 1 and locate the lamplighter at position i, but $x^n y^{-n} = (ts)^n t^{-n}$ has the lights at positions $1, 2, \ldots, n$ set to 1 (and at all others positions set to 0), so is not in P_n. And if $k \in K$ and either kx^{-1} or ky^{-1} is in P_n, then $\theta(k) = n$. □

The same proof works for $\mathbb{Z} \wr (C \wr \mathbb{Z})$ for any finite cyclic group $C \neq \{1\}$.

It may be illuminating to see an example where Proposition 3.1 does not apply. The hypotheses on P_n imply that any path from e to $x^n y^{-n}$ in the Cayley graph of K must climb to height n en route. If $K = \mathbb{Z}^2 = \langle x, y \mid [x,y] \rangle$, then this is not so, because $x^n y^{-n}$ can be expressed as $(xy^{-1})^n$. Indeed, in $\mathbb{Z} \wr \mathbb{Z}^2$ we find that $a^{-1} x^n y^{-n} a = \left(a^{-1} x y^{-1} a\right)^n = ((x\sigma)(y\tau)^{-1})^n$, a word of length $4n$ on the generators of H.

4. The exponential upper bound on distortion

Let $G = \mathbb{Z} \wr K$ where K is F_2 or $\mathbb{Z} \wr \mathbb{Z}$ as per Theorem 1.1. Let $\theta : K \to \mathbb{Z}$ be the epimorphism mapping x and y to 1.

Lemma 4.1. *The subgroup H of G of Theorem 1.1 is the set of all $g = (f, k) \in G$ such that*

$$\sum_{i \in \mathbb{Z}} 2^{-i} \sum_{v \in K,\ \theta(v) = i} f(v) \;=\; 0. \tag{1}$$

Proof. The four generators x, y, $\sigma = [x, a]a$ and $\tau = [y, a]a$ of H satisfy (1). And any $g = (f, k) \in G$ satisfying (1) can be expressed as a word u on $x^{\pm 1}$, $y^{\pm 1}$, $\sigma^{\pm 1}$, $\tau^{\pm 1}$ since it can can be transformed to the identity element as follows.

Let $n = d_G(e, g)$, the length of the shortest word on $a^{\pm 1}, x^{\pm 1}, y^{\pm 1}$ representing g. The cardinality of $\operatorname{supp} f$ is at most n. Every $h \in \operatorname{supp} f$ can be joined to e in the Cayley graph of K (with respect to x and y) by a path of length at most n. The lamp setting $f(h)$ at h has absolute value at most n. By moving along this path (using $x^{\pm 1}$ and $y^{\pm 1}$) and successively adjusting lamps along it (using $\sigma^{\pm 1}$ and $\tau^{\pm 1}$), the lamplighter can reset the lamp at h to 0 at the expense of changing the lamp at e by at most 2^n while, in the process, the lamp settings always satisfy (1). Once all the other lights have been extinguished the light at e is also at 0 on account of (1). □

The above argument is quantified in such a way that a couple of further observations suffice to complete the exponential upper bound proof for Theorem 1.1. The absolute values of the settings of the lamps along the at most n paths will grow to at most $n + n2^n$ in the course of the transformation of the lamp settings. The number of $x^{\pm 1}$ and $y^{\pm 1}$ (the movement) is at most n^2. So u has length at most a constant times 2^n, establishing the exponential upper bound on Dist_H^G.

5. Elementary subgroups of $\mathbb{Z} \wr K$

Here we prove Theorem 1.2. We have $G = \mathbb{Z} \wr K$, where K is a finitely generated group. So $G = W \rtimes K$, where $W = \bigoplus_K \mathbb{Z}$.

We begin by proving (1). Suppose H is a finitely generated subgroup of W. Then H is a subgroup of the product of finitely many of the summands in $W = \bigoplus_K \mathbb{Z}$ and there exists $C \geq 1$ such that for all $g = (f, e) \in H$, both $d_G(e, g)$ and $d_H(e, g)$ (word metrics with respect to the generating sets for G or for H, respectively) are between $\frac{1}{C} \max_{i \in K} |f(i)|$ and $C \max_{i \in K} |f(i)|$. So H is undistorted in G.

Case (2) is straight-forward on account of the map $G \twoheadrightarrow K$ killing W.

For (3), suppose $\hat{H} = \langle t \rangle$ is an infinite cyclic subgroup of G. Then $t = (f, k)$ for some $f \in W$ and some $k \in K$.

If k has finite order r, then $t^r = (f', e)$ for some $f' \in W$, and $H = \langle t^r \rangle$ is a subgroup of W per Case (1) such that $\mathrm{Dist}_H^G \simeq \mathrm{Dist}_{\hat{H}}^G$.

Suppose, on the other hand, k has infinite order. Roughly speaking, we will show that for all j, either t^j illuminates lights close to most of $e, k, \ldots, k^j$ and $\hat{H}$ is therefore undistorted in G, or it only illuminates lights close to e and k^j, and $\hat{H}$ is therefore distorted in G similarly to $\langle (\mathbf{0}, k) \rangle$.

Let $F : K \to \mathbb{Z}$ be the map $\sum_{i \in \mathbb{Z}} f^{k^i}$. In terms of the lamplighter model, F tells us the settings of the lights after the lamplighter acts per f at k^i for every $i \in \mathbb{Z}$. As k has infinite order, F is well-defined—for any $h \in K$, $f^{k^i}(h) = 0$ for all but finitely many i—but it may be that F is not finitely supported and so does not represent and element of W. Indeed, as F is invariant under the action of k, either $F = \mathbf{0}$ (the zero-map) or F has infinite support.

For $j \geq 1$, let $f_j = \sum_{i=0}^{j-1} f^{k^i}$, so that $t^j = (f_j, k^j)$.

Let $L > 0$ be sufficiently large that $\mathrm{supp} f \subset N_L(e)$—that is, the radius-$L$ neighbourhood of e in the Cayley graph of K contains the support of f. Then $\mathrm{supp} F \subseteq N_L(\hat{H})$ and $\mathrm{supp} f_j \subseteq N_L(\{k^0, k^1, \ldots, k^j\})$ for all $j \geq 1$.

As k has infinite order, for all $R > 0$, there exists i such that $k^i, k^{i+1}, \ldots$ are a distance greater than R from e in the Cayley graph of K. It follows that there exists $C > 0$ such that for all $j > 0$, the functions F and f_j agree on $\mathcal{N} := N_L(\{k^C, k^{C+1}, \ldots, k^{j-C}\})$. So, as F is k-invariant, f_j follows the same repeating pattern as F along $\mathcal{N}$—more precisely, the restrictions of f_j to $N_L(k^C)$, to $N_L(k^{C+1})$, $\ldots$, and to $N_L(k^{j-C})$ all agree after translations by successive powers of k. And therefore, if $\mathrm{supp} F \neq \emptyset$, there exists $\lambda, \mu > 0$

such that for all $j > 0$ we have $d_G(e, t^j) \geq \lambda j - \mu$, because to achieve the element $t^j \in G$, the lamplighter must visit every one of these neighbourhoods. So $\hat{H}$ is undistorted in G and satisfies (3) of the theorem. And if, on the other hand, $\mathrm{supp} F = \emptyset$, then there exists $\nu > 0$ such that for all j and all $g \in K \smallsetminus N_\nu(\{e, k^j\})$, we have $f_j(g) = e$. So $d_G(e, t^j) \leq d_G(e, u^j) + C$ where $u = (\mathbf{0}, k)$. So $\mathrm{Dist}_H^G \simeq \mathrm{Dist}_{\hat{H}}^G$ where $H := \langle u \rangle$, which is a subgroup per Case (2).

References

[Ble08] C. Bleak. A geometric classification of some solvable groups of homeomorphisms. *J. Lond. Math. Soc. (2)*, 78(2):352–372, 2008.

[BLP15] J. Burillo and E. López-Platón. Metric properties and distortion in wreath products, 2015. arxiv.org/abs/1506.06935.

[Dav11] T. C. Davis. *Subgroup distortion in metabelian and free nilpotent groups*. PhD thesis, Vanderbilt University, 2011.

[DO11] T. C. Davis and A. Yu. Olshanskii. Subgroup distortion in wreath products of cyclic groups. *J. Pure Appl. Algebra*, 215(12):2987–3004, 2011.

[GS99] V. S. Guba and M. V. Sapir. On subgroups of the R. Thompson group F and other diagram groups. *Mat. Sb.*, 190(8):3–60, 1999.

[Mih66] K.A. Mihailova. The occurence problem for direct products of groups. *Mat. Sbornik*, 70:241–251, 1966.

[Neu64] P. M. Neumann. On the structure of standard wreath products of groups. *Math. Z.*, 84:343–373, 1964.

[Osi01] D. V. Osin. Subgroup distortions in nilpotent groups. *Comm. Algebra*, 29(12):5439–5463, 2001.

[Pit93] Ch. Pittet. Surface groups and quasi-convexity. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 169–175. Cambridge Univ. Press, Cambridge, 1993.

[Sho91] H. Short. Quasiconvexity and a theorem of Howson's. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 168–176. World Sci. Publ., River Edge, NJ, 1991.

[Umi95] U. U. Umirbaev. The occurrence problem for free solvable groups. *Algebra i Logika*, 34(2):211–232, 243, 1995.

Timothy R. Riley

Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, NY 14853, USA

tim.riley@math.cornell.edu