DISTINGUISHING FILLING INVARIANTS ASSOCIATED TO CONJUGACY IN GROUPS

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Abstract. Brick and Corson introduced annular Dehn functions in 1998 to quantify the conjugacy problem for finitely generated groups and gave the fundamental relationships between it, the Dehn function, and the conjugator length function. We furnish the theory with diverse examples groups. In particular, we show that these three invariants are independent—no two of the three functions determine the other.

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1. Introduction

When a word u represents 1 in a finitely presented group G, it admits a disk-diagram (a *van Kampen diagram*) witnessing to how this follows from the defining relations. When words u and v represent conjugate elements, the corresponding witness is an *annular diagram*.

The *annular Dehn function* $\mathrm{Ann} : \mathbb{N} \to \mathbb{N}$ of a finitely presented group G was introduced by Brick and Corson in 1998 [BC98]. They defined $\mathrm{Ann}(n)$ to be the minimal N such that whenever u and v are words that represent conjugate elements of G and whose lengths sum to at most n, there is an annular diagram with at most N faces such the inner boundary is labelled by u and the outer boundary by v.

As Brick and Corson observed, the annular Dehn function is closely related to two other filling functions. One is the *Dehn function* $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$ of G, which is defined so that $\mathrm{Area}(n)$ is the minimal N such that whenever a word u of length at most n represents 1 in G, it admits a van Kampen diagram with at most N faces. The other is the *conjugator length function* $\mathrm{CL} : \mathbb{N} \to \mathbb{N}$ of G, which is defined so that $\mathrm{CL}(n)$ is the minimal N such that whenever u and v are words that represent conjugate elements of G and whose lengths sum to at most n, there is a word w of length at most N such that $uw = wv$ in G. Equivalently, there is an annular diagram for the pair u and v for which there is a path in the 1-skeleton of length at most N from the start of u on one of the two boundary components to the start of v on the other. Section 2 contains more detailed definitions.

Dehn functions have been studied extensively—[Bri02, BRS07, Ger93, Sap11] are surveys. Conjugator length functions have also received considerable attention—[BRS] is a recent account. Much less is known about annular Dehn functions.

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Brick and Corson [BC98] observed the fundamental relationships:

Theorem 1.1 (Brick and Corson's inequalities)**.** *For any finitely presented group G,*

$$\mathrm{Area}(n) \ \leq\ \mathrm{Ann}(n) \ \leq\ \mathrm{Area}(2\mathrm{CL}(n)+n) \tag{1}$$

and

$$\mathrm{CL}(n) \ \leq\ \frac{n}{2} + M \cdot \mathrm{Ann}(n), \tag{2}$$

where $M > 0$ is a constant depending on the presentation for G.

In brief, the second bound in (1) comes from the observation that for a pair of words u and v that represent conjugate elements of G, and for a minimal length conjugator w, we can form an annular diagram by identifying the two w-labelled sides in a minimal-area van Kampen diagram for $w^{-1}uwv^{-1}$. The bound in (2) comes from taking M to be the length of the longest defining relation, since then the right-hand side is an upper bound on the number of edges in an annular diagram.

The purpose of this article is to furnish the theory of annular Dehn functions with some examples which show that it harbors some richness and subtlety: specifically, we show that $\mathrm{Area}(n)$, $\mathrm{CL}(n)$, and $\mathrm{Ann}(n)$ are independent invariants and (1) and (2) need not be sharp. These conclusions will be corollaries of the estimates that are summarized in the following theorem. The conventions are $a^b = aba^{-1}$ and $[a,b] = aba^{-1}b^{-1}$, and for functions $f, g : \mathbb{N} \to \mathbb{N}$, $f \lesssim g$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(Cn+n) + Cn + C$ for all n. We write $f \simeq g$ when $f \lesssim g$ and $g \lesssim f$. Up to $\simeq$, each of $\mathrm{Ann}(n)$, $\mathrm{CL}(n)$, and $\mathrm{Area}(n)$ do not depend on the choice of finite presentation for G.

Theorem 1.2. *Up to $\simeq$, the following finitely presented groups have the following Dehn functions, conjugator length functions, and annular Dehn functions (for all $d, m \geq 1$):*

	$\mathrm{Area}(n)$	$\mathrm{Ann}(n)$	$\mathrm{CL}(n)$
$G_1 = \mathcal{H}_3(\mathbb{Z})$	n^3	n^4	n^2
$G_2 = \mathrm{BS}(1,2)$	2^n	2^n	n
$G_3 = G_1 \times G_2$	2^n	2^n	n^2
G_4	2^n	2^{n^2}	n^2
$G_{5,d} = \mathbb{Z}^d \rtimes \mathbb{Z}$ *filiform*	n^{d+1}		n^d
$G_{6,m}$	n^3		n^{m+1}
$G_7 = G_{5,3} * G_{6,20}$	n^4	$\big\}\simeq$	n^{21}
$G_8 = G_{5,4} * G_{6,20}$	n^5		n^{21}

where

$$\begin{aligned} G_1 &= \langle a,b,c \mid [a,c],[b,c],[a,b]c^{-1}\rangle \\ G_2 &= \langle a,s \mid s^a s^{-2}\rangle \\ G_4 &= \langle a,b,c,d,s \mid [a,b]c^{-1},[a,c],[b,c],[b,d],s^a s^{-2}, s^d s^{-2}\rangle \\ G_{5,d} &= \langle a_1,\ldots,a_d,t \mid a_i^t = a_i a_{i-1} \ \ \forall i>1,\ [a_1,t]=1,\ [a_i,a_j]=1 \ \ \forall i \neq j\rangle \end{aligned}$$

and $G_{6,m}$ is per Definition 1.3 below.

To be clear, the assertions made in the table above include that $\mathrm{Ann}_{G_7}(n) \simeq \mathrm{Ann}_{G_8}(n)$, but we do not determine their growth rates. We also do not determine $\mathrm{Ann}_{G_{5,d}}(n)$ or $\mathrm{Ann}_{G_{6,m}}(n)$.

The Dehn function estimates for G_1 and G_2 in this theorem are well-known. Those for $G_{5,d}$ and $G_{6,m}$ are from [BP94, GHR03]. The conjugator length estimates for G_1, G_2, $G_{5,d}$, and $G_{6,m}$ are from [BRS, Sal16, BRa, BRb]. (We discuss these references in more detail in Section 3.) The remaining estimates are established in this paper. The construction of G_4 is also novel.

Definition 1.3. The groups $G_{6,m}$ of [BRb] are, for all $m \geq 1$, the central extensions of $\mathbb{Z}^{m+2} = \langle a_1, \ldots, a_m, b_1, b_2 \rangle$ by $\mathbb{Z}^m = \langle c_1, \ldots, c_m \rangle$ defined by the relations $b_1 a_i = a_i b_1 c_i$ for $i = 1, \ldots, m$, $b_2 a_i = a_i b_2 c_{i+1}^{-1}$ for $i = 1, \ldots, m-1$, and commutation relations for the pair of generators (b_2, a_m), and all pairs of generators (a_i, a_j), (b_i, b_j), (c_i, c_j), (b_i, c_j), and (a_i, c_j).

Corollary 1.4. *The Dehn function, the conjugator length function, and the annular Dehn function are independent invariants for finitely presented groups. That is, for any two of these invariants, there is a pair of finitely presented groups for which those two invariants agree, but the third differs (all up to $\simeq$).*

Proof. By Theorem 1.2, the groups G_7 and G_8 have different Dehn functions, but the other two functions the same. Likewise, compare G_2 and G_3 for conjugator length function, and compare G_3 and G_4 for annular Dehn function. □

Corollary 1.5. *There exist finitely presented groups for which the inequalities*

$$\mathrm{Area}(n) \overset{\mathrm{i}}{\lesssim} \mathrm{Ann}(n) \overset{\mathrm{ii}}{\lesssim} \mathrm{Area}(n + \mathrm{CL}(n))$$

of Brick and Corson are (1) both sharp, (2) both not sharp, (3) i *is sharp but* ii *is not, and (4)* ii *is sharp but* i *is not, where by 'sharp' we mean that $\preceq$ can be replaced $\simeq$.*

Proof. Our examples are: (1) G_2, (2) G_1, (3) G_3, and (4) G_4. □

An outine of this article. Section 2 provides preliminaries concerning van Kampen diagrams, annular diagrams, Dehn functions, annular Dehn functions, and conjugator length functions. It includes an account (Propositions 2.6 and 2.8) of how each of these three functions behaves with respect to taking direct products and free products. Section 3 is our proof of Theorem 1.2 apart from the following. Section 4 contains a proof that $n^4 \simeq \mathrm{Ann}_{\mathcal{H}_3(\mathbb{Z})}(n)$. Section 5 establishes the three estimates concerning G_4. The group G_4 is an amaglamated free product between an ascending HNN-extension of $\mathcal{H}_3(\mathbb{Z})$ and another amalgamated free product of two copies of $\mathrm{BS}(1,2)$ joined along the exponentially distorted subgroup. These copies of $\mathrm{BS}(1,2)$ are the source of the exponential Dehn function (Corollary 5.3 and Proposition 5.8), and the $H_3(\mathbb{Z})$ seeds the quadratic conjugator length function (Corollary 5.3 and Proposition 5.16). Their composition 2^{n^2} gives an upper bound on the annular Dehn function (by Theorem 1.1). This is matched by a lower bound (Proposition 5.17) because there are, for all n, pairs of conjugate words of total length $\sim n$ for which in any annular diagram, the quadratic distortion within $H_3(\mathbb{Z})$ feeds into the exponential Dehn function in $\mathrm{BS}(1,2)$, so as to create at least 2^{n^2} area.

2. Preliminaries

2.1. **Diagrams.** The following brief account is based on [LS01], to which we refer the reader for full details. A *(singular disc) diagram* Δ is a planar 2-complex that is obtained from a finite 2-complex homeomorphic to the 2-sphere by removing the interior of one face f_∞ ("at infinity"). So the attaching map of the removed face traverses the boundary $\partial\Delta$. An *annular diagram* Ω is obtained from such a Δ by removing the interior of a further

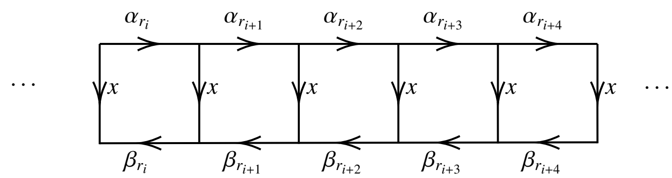

FIGURE 1. A general x-corridor

face f_0. Accordingly, $\partial\Omega$ is the union of the *outer boundary* $\partial\Delta$ and the *inner boundary* traversed by the attaching map of f_0. We also allow the degenerate case where $\Omega = \Delta$, in which case the *inner boundary* is any vertex of Ω.

Let $G = \langle X \mid R \rangle$ be a finitely presented group. We consider singular disc diagrams and annular diagrams Δ such that all edges are directed and are labeled with elements of X. For every directed edge e from vertex u to vertex v, labeled by $x \in X$, we consider e^{-1} to denote the "inverse" edge going from v to u and labeled by x^{-1}. An edge-path p in Δ is a sequence of edges $(e_1^{\delta_1}, \ldots, e_n^{\delta_n})$ with $\delta_i \in \{\pm 1\}$ for all i such that the terminal vertex of $e_i^{\delta_i}$ equals the starting vertex of $e_{i+1}^{\delta_{i+1}}$ for $i = 1, \ldots, n-1$. The *word along p* is the word over X read off the labels of the $e_i^{\delta_i}$.

We say that a singular disc diagram Δ is a *van Kampen diagram* for a word w over $X^{\pm 1}$ read around $\partial\Delta$ (from some starting vertex) if for every face F of Δ, the word read around ∂F (from some starting vertex on F) is in $R^{\pm 1}$.

An annular diagram Ω *for the pair of words u and v over* $X^{\pm 1}$ is when Ω is likewise labelled so that the words read around the two boundary components (from some starting vertices) are u and v, and the words read around the faces are in $R^{\pm 1}$. A van Kampen diagram for a word w is an annular diagram for the pair w and the empty word.

The significance of these diagrams is apparent from the following results, which can be found in [LS01]—see Proposition III.9.2 and Lemmas V.5.1 and V.5.2—and in [BRS].

Lemma 2.1 (van Kampen's Lemma)**.** *Let $G = \langle X \mid R \rangle$ be a finitely presented group. A word w over X represents the identity in G if and only if there exists a van Kampen diagram for w.*

Lemma 2.2. *Let $G = \langle X \mid R \rangle$ be a finitely presented group, and u and v words over X. Then u and v represent conjugate elements in G if and only if there exists an annular diagram Ω for u and v. Moreover, let p_1 and p_2 be vertices on the outer and inner boundaries (respectively) of Ω from which we read u and v (respectively). If γ is an edge-path from p_1 to p_2, and w is the word along γ, then $wuw^{-1} = v$ in G.*

Suppose $G = \langle X \mid R \rangle$ is a finitely presented group and $x \in X$. Suppose that, for all $r \in R$, either r contains no x-letters or contains exactly one x and one x^{-1}. Then, every face in a (van Kampen or annular) diagram Δ will have either two or zero edges labeled x. With the exception of cells adjacent to the boundary of Δ, each cell with two x-edges must be adjacent to other such cells, thereby giving rise to a sequence of these cells which we call an *x-corridor*—see Figure 1; $\cdots \beta_{r_i}^{-1}\beta_{r_{i+1}}^{-1}\beta_{r_{i+2}}^{-1}\beta_{r_{i+3}}^{-1}\beta_{r_{i+4}}^{-1} \cdots$ and $\cdots \alpha_{r_i}\alpha_{r_{i+1}}\alpha_{r_{i+2}}\alpha_{r_{i+3}}\alpha_{r_{i+4}} \cdots$ are the *words along its sides*.

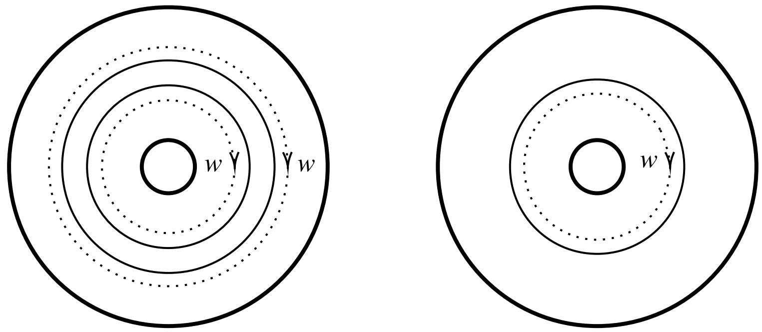

FIGURE 2. Excising an x-corridor

If an x-corridor has no cells adjacent to the boundary, and the words along both sides are the same, then it may be excised from the diagram without increasing the area, as shown in Figure 2 (depicting the case of an x-corridor in an annular diagram). Roughly speaking this excision is done by removing the annulus corridor and then identifying the paths along its two boundaries. (A technical concern here is that this identification may break the planarity of the diagram. How to navigate this issue is explained in [BRS].)

There are some important differences between x-corridors in van Kampen diagrams and in annular diagrams. In a van Kampen diagram Δ, an x-corridor must either form a closed loop, which itself bounds a contractible subcomplex of Δ, or it must run from the boundary of Δ to itself. However, for an annular diagram Ω, there are two boundary components and x-corridor come in four types:

(1) *x-arches* run from one boundary component of Ω to the same component,
(2) *radial x-corridors* run from one boundary component of Ω to the other,
(3) *contractible x-rings* form annuli bounding *contractible* subcomplexes of Ω, and
(4) *non-contractible x-rings* form an annuli not bounding *contractible* subcomplexes.

Definition 2.3. A diagram is *reduced* if there are no cells C_1, C_2 sharing an edge E such that the words along their boundaries, starting at E, are inverses. Two cells C_1, C_2 satisfying this property are called *cancelling cells*. Every word that represents 1 in a finitely presented group admits a reduced van Kampen diagram over that presentations. Every pair of words that represent conjugate elements admits a reduced annular diagram.

2.2. **Dehn functions.** We define Area(Δ) to be the number of faces in a van Kampen diagram Δ. For a word w on X representing the identity of $G = \langle X \mid R\rangle$, we define Area($w$) to be the minimum of Area(Δ) among all van Kampen diagrams for w. A word w on X represents the identity in G if and only if w freely equals a product of the form

$$\prod_{i=1}^{k} w_i r_i^{\delta_i} w_i^{-1},$$

where $r_i \in R$, $\delta_i \in \{\pm 1\}$, and w_i is a word on X for all i. It follows from the standard proof of van Kampen's Lemma that for any word w representing the identity in G, Area(w) is also the smallest k such that w freely equals a word of this form.

Define the *Dehn function* Area : $\mathbb{N} \to \mathbb{N}$ of G relative to a finite presentation $\langle X \mid R \rangle$ for G by

$$\text{Area}(n) = \max \{\text{Area}(w) : |w| \leq n \text{ and } w = 1 \text{ in } G\}.$$

See, for example, [Bri02, BH99, BRS07] for more details.

2.3. **Conjugator length functions and annular Dehn functions.** Suppose G is a group with finite generating set X. Suppose words u and v on X represent conjugate elements in G. Define $\text{CL}(u, v)$ to be min $|\gamma|$ over all words γ such that $\gamma u \gamma^{-1} = v$ in G.

Suppose now that $G = \langle X \mid R \rangle$ is a finitely presented group. By Lemma 3.3 and the argument of Lemma 3.2 in [BRS], $\text{CL}(u, v)$ is equivalently the minimal length L such that there is a annular diagram Ω for u and v, as per Lemma 2.2, for which there is an edge-path of length L from p_1 to p_2.

Define $\text{Ann}(u, v)$ to be the minimum of $\text{Area}(\Omega)$ over all over all annular diagrams Ω for u and v, or similarly equivalently, the minimum of $\text{Area}(\gamma u \gamma^{-1} v^{-1})$ over all words γ such that $\gamma u \gamma^{-1} = v$ in G.

A priori, a diagram witnessing $\text{CL}(u, v)$ may not witness $\text{Ann}(u, v)$.

Define the *conjugator length function* CL : $\mathbb{N} \to \mathbb{N}$ and the *annular Dehn function* Ann : $\mathbb{N} \to \mathbb{N}$ of G by

$$\begin{aligned} \text{CL}(n) &= \max \text{CL}(u, v) \\ \text{Ann}(n) &= \max \text{Ann}(u, v), \end{aligned}$$

where both maxima are taken over all pairs of words u and v such that $|u| + |v| \leq n$ and u and v represent conjugate elements in G.

Up to $\simeq$, $\text{Area}(n)$ and $\text{Ann}(n)$ do not depend on the choice of finite presentation for G, and $\text{CL}(n)$ does not depend on the choice of finite generating set.

We will call on these two technical results.

Lemma 2.4. *If $u = v = w$ in $G = \langle X \mid R \rangle$, then*

$$\text{Area}(uv^{-1}) - \text{Area}(vw^{-1}) \ \leq \ \text{Area}(uw^{-1}) \ \leq \ \text{Area}(uv^{-1}) + \text{Area}(vw^{-1}).$$

Proof. The second inequality follows from the fact that uw^{-1} freely equals $uv^{-1}vw^{-1}$. The first follows from the second after interchanging the roles of v and w. □

Lemma 2.5. *If $u \sim v$ in G and $v = w$ in G, then*

$$\text{Ann}(u, v) - \text{Area}(vw^{-1}) \ \leq \ \text{Ann}(u, w) \ \leq \ \text{Ann}(u, v) + \text{Area}(vw^{-1}).$$

Proof. This follows by a similar argument as the previous lemma, using the fact that $\text{Ann}(u, v) = \text{Area}(\gamma u \gamma^{-1} v^{-1})$ for some γ. □

2.4. **Direct products and free products.** The behaviour of $\text{CL}(n)$, $\text{Ann}(n)$ and $\text{Area}(n)$ for direct and free products of finitely generated groups is summarized in the next two propositions, which in large part are from [BC98]. We will not call on (6), but we include it here as the natural improvement on the upper bound of (4) in circumstances when the optimal conjugator length and annular Dehn function can be simultaneously realized.

Proposition 2.6. *Assuming for (3) that G_1 and G_2 are finitely generated groups, for (4) and (5) they are finitely presented, and further for (5) that* $\mathrm{Area}_{G_1}(n)$ *or* $\mathrm{Area}_{G_2}(n)$ *grows* $\geq n^2$*, we have*

$$\mathrm{CL}_{G_1\times G_2}(n) \;\simeq\; \max\{\mathrm{CL}_{G_1}(n), \mathrm{CL}_{G_2}(n)\}, \tag{3}$$

$$\max\{\mathrm{Ann}_{G_1}(n), \mathrm{Ann}_{G_2}(n)\} \;\leq\; \mathrm{Ann}_{G_1\times G_2}(n) \;\leq\; \max\{n\mathrm{Ann}_{G_1}(n), n\mathrm{Ann}_{G_2}(n)\} + n^2, \tag{4}$$

$$\mathrm{Area}_{G_1\times G_2}(n) \;\simeq\; \max\{\mathrm{Area}_{G_1}(n), \mathrm{Area}_{G_2}(n)\}. \tag{5}$$

Further, suppose there exists a constant D such that for all $u, v \in G_i$ with $|u| + |v| \leq n$, there exists w where $wuw^{-1} = v$, $|w| \leq D\,\mathrm{CL}_{G_i}(n) + D$, and $\mathrm{Area}(wuw^{-1}v^{-1}) \leq D\,\mathrm{Ann}_{G_i}(n) + D$*. Then*

$$\mathrm{Ann}_{G_1\times G_2}(n) \;\leq\; \mathrm{Ann}_{G_1}(n) + \mathrm{Ann}_{G_2}(n) + n\mathrm{CL}_{G_1}(n) + n\mathrm{CL}_{G_2}(n). \tag{6}$$

Proof. The proofs of (3) and (5) are straight-forward.

The first inequality of (4) follows directly from Theorem 2.1 of [BC98]. For the second, let $u, v \in G_1 \times G_2$ be conjugate words such that $|u| + |v| = n$. Then there exist $u_1, v_1 \in G_1$ and $u_2, v_2 \in G$ such that $u = u_1u_2$ and $v = v_1v_2$ in $G_1 \times G_2$, u_1 is conjugate to v_1 in G_1, and u_1 is conjugate to u_2 in G_2. Without loss of generality we may assume $|u_1| + |u_2| + |v_1| + |v_2| = |u| + |v| = n$, and that our presentation for $G_1 \times G_2$ contains a relator $[g_1, g_2]$ for all generators g_1 of G_1 and g_2 of G_2.

For $i = 1, 2$, let w_i be a word that conjugates u_i to v_i in G_i and has $\mathrm{Area}(w_iu_iw_i^{-1}v_i^{-1}) = \mathrm{Ann}(u_i, v_i)$. Then $w = w_1w_2$ conjugates u to v. Then $|w_i| \leq M \cdot \mathrm{Ann}_{G_i}(n) + n$, where M is the length of the longest defining relation in our presentations for G_1 and G_2. We can transform

$$w_1w_2u_1u_2 \;\to\; w_1u_1w_2u_2 \;\to\; w_1u_1v_2w_2 \;\to\; v_1w_1v_2w_2 \;\to\; v_1v_2w_1w_2,$$

at a cost of applying at most $|w_2| \cdot |u_1| \leq nM \cdot \mathrm{Ann}_{G_2}(n) + n^2$, then $M \cdot \mathrm{Ann}_{G_2}(n) + n$, then $M \cdot \mathrm{Ann}_{G_1}(n) + n^2$, and then $|v_2| \cdot |w_1| \leq nM \cdot \mathrm{Ann}_{G_1}(n) + n^2$ defining relations.

The proof of (6) is similar. ◻

The corresponding result for free products requires additional notation.

Definition 2.7. For a finitely presented group G, define $\overline{\mathrm{Ann}_G}(n)$ to be the minimal integer such that $\overline{\mathrm{Ann}_G}(n) \geq \mathrm{Ann}_G(n)$ and $\overline{\mathrm{Ann}_G}(n + \tilde{n}) \geq \overline{\mathrm{Ann}_G}(n) + \overline{\mathrm{Ann}_G}(\tilde{n})$ for all $n, \tilde{n} \in \mathbb{N}$.

Proposition 2.8. *Suppose for (7) that G_1 and G_2 are finitely generated groups, and for (8) that they are finitely presented, and for (9) further suppose that one of* $\mathrm{Area}_{G_1}(n)$ *and* $\mathrm{Area}_{G_2}(n)$ *grows $\geq n$. Let $F_i(n)$ be the smallest function such that $F_i(n) \geq$* $\mathrm{Ann}_{G_i}(n)$ *and $F_i(n + n') \geq F_i(n) + F_i(n')$ for $i = 1, 2$. Then, for $i = 1, 2$,*

$$\mathrm{CL}_{G_1*G_2}(n) \;\simeq\; \max\{\mathrm{CL}_{G_1}(n), \mathrm{CL}_{G_2}(n)\}, \tag{7}$$

$$\mathrm{Ann}_{G_1*G_2}(n) \;\simeq\; \max\{\overline{\mathrm{Ann}_{G_1}}(n), \overline{\mathrm{Ann}_{G_2}}(n)\}, \tag{8}$$

$$\mathrm{Area}_{G_1*G_2}(n) \;\simeq\; \max\{\mathrm{Area}_{G_1}(n), \mathrm{Area}_{G_2}(n)\}. \tag{9}$$

Proof. For (7) and (8), the claim is a restatement [BC98][Corollary 3.3]. The upper bound for (9) is straightforward, and the lower bound follows from the existence of retracts $G_1 * G_2 \to G_i$ for $i = 1, 2$. ◻

3. Proof of Theorem 1.2

That $G_1 = \mathcal{H}_3(\mathbb{Z})$ satisfies $\mathrm{Area}_{G_1}(n) \simeq n^3$ is proved in [ECH+92, Chapter 8.1]. That $\mathrm{CL}_{G_1}(n) \simeq n^2$ is shown in [BRS]. We postpone proof that $\mathrm{Ann}_{G_1}(n) \simeq n^4$ to Section 4.

The Baumslag-Solitar group G_2 is well known to have $\mathrm{Area}_{G_2}(n) \simeq 2^n$ (e.g. [ECH+92, Chapter 7.4], [Ril17, Theorem 8.8]). That $\mathrm{CL}_{G_2}(n) \simeq n$ was proved first by [Sal16]; there is an elementary proof in [BRS]. That $\mathrm{Ann}_{G_2}(n) \simeq 2^n$ then follows from Theorem 1.1.

Our estimates for G_3 will follow from those for G_1 and G_2 by Proposition 2.6.

We postpone the estimates concerning G_4 to Section 5.

That $\mathrm{Area}_{G_{5,d}}(n) \simeq n^{d+1}$ is [BP94, Theorem 6.3]. That $\mathrm{Area}_{G_{6,m}}(n) \preceq n^3$ follows from [GHR03] since the groups are class-2 nilpotent. Inspecting the presentation for $G_{6,m}$ given in Definition 1.3, we see that killing all generators other than a_1, b_1, and c_1 retracts $G_{6,m}$ onto a copy of the Heisenberg group G_1, and thereby $\mathrm{Area}_{G_{6,m}}(n) \succeq n^3$ follows from $\mathrm{Area}_{G_1}(n) \simeq n^3$. That $\mathrm{CL}_{G_{5,d}}(n) \simeq n^d$ and $\mathrm{CL}_{G_{6,m}}(n) \simeq n^{m+1}$ are the main results of [BRa] and [BRb], respectively.

The computations of Area_{G_7}, Area_{G_8}, CL_{G_7}, and CL_{G_8} follow via Proposition 2.8. Towards $\mathrm{Ann}_{G_7}(n) \simeq \mathrm{Ann}_{G_8}(n)$, using the notation of Definition 2.7,

$$\overline{\mathrm{Ann}_{G_{6,20}}}(n) \succeq \mathrm{Ann}_{G_{6,20}}(n) \succeq n^{21} \quad \text{and} \tag{10}$$

$$n^{21} \succeq \max\{\mathrm{Ann}_{G_{5,3}}(n), \mathrm{Ann}_{G_{5,4}}(n)\}. \tag{11}$$

where the second inequality of (10) combines Theorem 1.1(2) and $\mathrm{CL}_{G_{6,20}}(n) \simeq n^{21}$ discussed above, and (11) holds because from Theorem 1.1(1) applied to the bounds given above tells us that $\mathrm{Ann}_{G_{5,3}}(n) \preceq (n^3)^4$ and $\mathrm{Ann}_{G_{5,4}}(n) \preceq (n^4)^5$. Because $(n+\tilde{n})^{21} \geq n^{21} + \tilde{n}^{21}$ for all $n, \tilde{n} \in \mathbb{N}$, (11) gives $n^{21} \succeq \max\left\{\overline{\mathrm{Ann}_{G_{5,3}}}(n), \overline{\mathrm{Ann}_{G_{5,4}}}(n)\right\}$. Since $G_7 = G_{5,3} * G_{6,20}$ and $G_8 = G_{5,4} * G_{6,20}$, Proposition 2.8 then gives $\mathrm{Ann}_{G_7} \simeq \overline{\mathrm{Ann}_{G_{6,20}}}(n) = \mathrm{Ann}_{G_8}$.

Aside from the estimates postponed to Sections 4 and 5, this completes our proof of Theorem 1.2

4. The Heisenberg Group

Proposition 4.1. *The annular Dehn function of G_1 satisfies* $\mathrm{Ann}_{G_1}(n) \simeq n^4$.

Proof. We begin by proving that $\mathrm{Ann}_{G_1}(n) \preceq n^4$. Suppose $n > 0$ and that u and v are words on the generators of

$$G_1 = \mathcal{H}_3(\mathbb{Z}) = \langle a, b, c \mid [a,c], [b,c], [a,b]c^{-1} \rangle$$

such that $|u| + |v| \leq n$. By shuffling the a-letters to the front (at the expense of creating a c or c^{-1} whenever an a passes a b) and shuffling all the c-letters to the end, we can transform u to $a^{\alpha_1} b^{\beta_1} c^{\gamma_1}$ and v to $a^{\alpha_2} b^{\beta_2} c^{\gamma_2}$ where $|\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2| \leq n$ and $|\gamma_1| + |\gamma_2| \leq n^2$ using at most $2n^3$ defining relations. This is the well-known normal form for G_1 (see the discussion in [ECH+92]), so $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are uniquely determined by u and v.

Since c is central in G_1, if $\hat{w} = a^x b^y c^z$ satisfies $\hat{w}u = v\hat{w}$ in G_1, then $w = a^x b^y$ satisfies $wu = vw$ in G_1. We follow the strategy of [BRS] for finding x and y to suit our purposes.

Computing the the normal form of wuw^{-1}:

$$wuw^{-1} = a^x b^y a^{\alpha_1} b^{\beta_1} c^{\gamma_1} b^{-y} a^{-x} \tag{12a}$$

$$= a^x b^y a^{\alpha_1} b^{\beta_1} b^{-y} a^{-x} c^{\gamma_1} \tag{12b}$$

$$= a^x a^{\alpha_1} c^{\alpha_1 y} b^y b^{\beta_1} b^{-y} a^{-x} c^{\gamma_1} \tag{12c}$$

$$= a^x a^{\alpha_1} c^{-\alpha_1 y} b^{\beta_1} a^{-x} c^{\gamma_1} \tag{12d}$$

$$= a^x a^{\alpha_1} b^{\beta_1} a^{-x} c^{-\alpha_1 y+\gamma_1} \tag{12e}$$

$$= a^x a^{\alpha_1} a^{-x} b^{\beta_1} c^{-\alpha_1 y-\beta_1 x+\gamma_1} \tag{12f}$$

$$= a^{\alpha_1} b^{\beta_1} c^{-\alpha_1 y-\beta_1 x+\gamma_1}, \tag{12g}$$

we see that $wu = vw$ in G_1 if and only if $\alpha_1 = \alpha_2, \beta_1 = \beta_2$, and

$$\alpha_1 y + \beta_1 x = \gamma_1 - \gamma_2. \tag{13}$$

If $|\alpha_1| = 0 = |\beta_1|$, then $u = v$ and we are done; otherwise we have two cases. For the first case, if $|\alpha_1| = |\beta_1| \neq 0$, then α_1 divides $\gamma_1 - \gamma_2$, so $x = 0, y = (\gamma_1 - \gamma_2)/\alpha_1$, satisfies (13) and $|y| \leq 2n^2$. For the second case, suppose $|\alpha_1| \neq |\beta_1|$. Since there is an automorphism of G_1 interchanging a and b, we may assume without loss of generality that $|\alpha_1| > |\beta_1| \geq 0$. Let $y' = y + \lfloor(\gamma_1 - \gamma_2)/\alpha_1\rfloor$. Then

$$\alpha_1 y' + \beta_1 x = \gamma \tag{14}$$

for some γ satisfying $|\gamma| \leq |\alpha_1|$. Equation (14) has a solution (x, y') with $|x|, |y'| \leq \max\{|\alpha_1|, |\beta_1|\}$ by a quantification of Bezout's identity which can be found in e.g. [BFRT89, BRa, Kor90]. Therefore (13) has a solution (x, y) with

$$|x| \leq \max\{|\alpha_1|, |\beta_1|\} \quad \text{and} \quad |y| \leq \max\{|\alpha_1|, |\beta_1|\} + |(\gamma_1 - \gamma_2)/\alpha_1|.$$

Thus there exist x and y such that $a^x b^y u b^{-y} a^{-x} = v$, $|x| \leq n$, and $|y| \leq 2n^2$.

The equality (12b) holds using $|\gamma_1|(|x| + |y|) \leq Cn^4$ defining relations of the form $[a, c] = 1$ and $[b, c] = 1$, (12c) using $|\alpha_1|^2|y| \leq 2n^4$ defining relations of the form $[a, b]c^{-1} = 1$, and (12d) holds freely. Also, (12e) holds using $|\beta_1||x|^2 \leq n^3$ defining relations of the form $[b, c] = 1$, (12f) holds using $|x||\beta_1|^2 \leq n^3$ defining relations of the form $[a, b]c^{-1} = 1$, and (12g) holds freely. Summing gives $\mathrm{Ann}_{G_1}(n) \preceq n^4$, as desired.

We now turn to proving that $\mathrm{Ann}_{G_1}(n) \succeq n^4$. For $n \in \mathbb{N}$, let $u_n = b$ and $v_n = b[a^n, b^n]$. Then $|u_n| + |v_n| = 4n + 2$ and u_n and v_n are conjugate in G_1:

$$v_n = a^{n^2} u_n a^{-n^2}. \tag{15}$$

We will show that $\mathrm{Ann}(b, bc^{n^2}) \geq n^4$. By applying defining relations for G_1 approximately n^3 times, v_n can be shown to equal bc^{n^2} in G_1, so, via Lemma 2.5, it will follow that $\mathrm{Ann}(u_n, v_n) \succeq n^4$, which will prove that $\mathrm{Ann}_{G_1}(n) \succeq n^4$.

Let Ω be any annular diagram witnessing the conjugacy of u_n and v_n, and let γ be any path from the starting point of u_n to the starting point of v_n such that γ crosses every non-contractible a-corridor precisely once. Let w be the word along γ. By cutting Ω along γ, we obtain a van Kampen diagram Δ for $wbw^{-1}(bc^{n^2})^{-1}$ such that $\mathrm{Area}(\Delta) = \mathrm{Area}(\Omega)$. Say that an a-corridor in Δ is *positively* (resp. *negatively*) *oriented* and vertical if it connects an a (resp. an a^{-1}) in the w-portion of $\partial\Delta$ to an a^{-1} (resp. an a) in the w^{-1}-portion. (There may also be a-corridors that connect an a to an a^{-1} that are both in the w-portion or are both in the w^{-1}-portion, and there may be a-corridors in Δ may form annuli.)

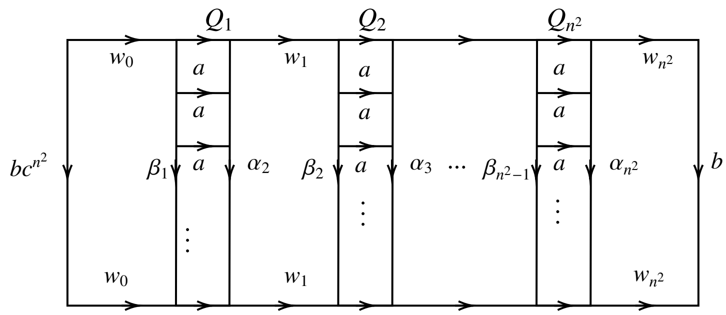

FIGURE 3. A van Kampen diagram Δ for $wbw^{-1}(bc^{n^2})^{-1}$

In [BRS], it is shown that the exponent sum of the a-letters in any word w conjugating b to bc^{n^2} is precisely n^2. It follows that there are precisely n^2 more positively-oriented than negatively-oriented vertical corridors in Δ.

There thus exist n^2 positively-oriented vertical a-corridors $Q_1, Q_2, \ldots, Q_{n^2}$, between any two of which, there are an equal number of positively-oriented and negatively-oriented vertical a-corridors. Let β_i and α_{i+1} be the words along the sides of Q_i, respectively, as shown in Figure 3. Since γ intersects each non-contractible a-corridor only once, the words w_i between Q_{i-1} and Q_i along the top and bottom of Δ are the same for all i.

For all $0 \leq i \leq n^2$, the exponent sum of the $a^{\pm 1}$ in w_i is 0, so $w_i \in \langle b, c \rangle = \mathbb{Z}^2$. This group is abelian, and α_i is a word on $\{b, c\}$, so w_i commutes with α_i. Thus, $\beta_i = \alpha_{i-1}$, and proceeding inductively from $\alpha_1 = b$ we see that $\alpha_i = \beta_{i-1}^a = bc^i$. Every word on $\{b, c\}$ equal to bc^i has at least i many c-letters, therefore Q_i at least i many cells. So $Q_1, \ldots, Q_{n^2}$ have total area at least $1 + 2 + \cdots + (n^2 - 1) + n^2 = n^2(n^2 + 1)/2$, giving the desired quartic lower bound. $\square$

5. COMPUTATIONS OF $\mathrm{Area}_{G_4}(n)$, $\mathrm{CL}_{G_4}(n)$, AND $\mathrm{Ann}_{G_4}(n)$

We begin with a lemma describing the structure of G_4.

Lemma 5.1. *The group*

$$G_4 \;=\; \langle a, b, c, d, s \mid [a, b]c^{-1}, [a, c], [b, c], [b, d], s^a s^{-2}, s^d s^{-2} \rangle$$

is a free product with amalgamation $A *_C B$ of its subgroups $A = \langle a, b, c, d \rangle$ and $B = \langle a, d, s \rangle$ along the subgroup $C = \langle a, d \rangle$. It is also an HNN-extension of

$$E \;=\; \langle a, c, d, s \mid [a, c], s^a s^{-2}, s^d s^{-2} \rangle$$

with stable letter b and an HNN-extension of

$$\langle a, b, c, s \mid [a, b]c^{-1}, [a, c], [b, c], s^a s^{-2} \rangle$$

with stable letter d. Moreover,

(1) $A = \langle a, b, c, d \mid [a, b]c^{-1}, [a, c], [b, c], [b, d] \rangle$ is the HNN-extension of $K = \langle a, c, d \mid [a, c] \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$ with stable letter b acting via the automorphism of K that maps $a \mapsto c^{-1}a$, $c \mapsto c$, and $d \mapsto d$;

(2) $B = \langle a, d, s \mid s^a s^{-2}, s^d s^{-2} \rangle$*, the amalgamated free product of two copies of* $\mathrm{BS}(1,2)$*, denoted* $D_1 = \langle a, s \mid s^a = s^2 \rangle$ *and* $D_2 = \langle d, s \mid s^d = s^2 \rangle$*, along* $\langle s \rangle \cong \mathbb{Z}$*.*

(3) $C = \langle a, d \mid \ \rangle$ *is free of rank 2.*

(4) Killing d *and* s *retracts* G_4 *onto a subgroup* $\langle a, b, c \mid [a,c], [b,c], [a,b]c^{-1} \rangle$*, which is the Heisenberg group.*

(5) Killing b *and* c *and mapping* $d \mapsto a$ *retracts* G_4 *onto its subgroup* $D_1 = \langle a, s \mid s^a s^{-2} \rangle$*.*

(6) The subgroup $L = \langle b, c, s \rangle$ *is* $\mathbb{Z}^2 * \mathbb{Z} = \langle b, c, s \mid [b,c] \rangle$*.*

Proof. Killing s retracts G_4 onto A, which therefore has the presentation claimed in (1), and the remaining claims of (1) then follow, as does the claim that G_4 is an HNN-extension with stable letter b. Similarly killing b and c retracts G_4 onto B so as to give (2). That $G_4 = A *_C B$ follows from (1) and (2). Killing b and c retracts A onto $C = \langle a, d \mid \ \rangle$, giving (3). That $G_4 \cong A *_C B$ follows. Claims (4) and (5) are straight-forwardly verified. Claim (6) holds on account of a free-product-with-amalgamation normal form for $A *_C B$, given that $[b,c] = 1$ in G_4 and $\langle b, c \rangle \cap C = \langle s \rangle \cap C = \{1\}$. It follows that b and s generate a free subgroup and mapping $b \mapsto b$ and $s \mapsto s^2$ defines an isomorphism between rank-2 free subgroups of G_4, and so G_4 an HNN-extension with stable letter d. □

Corollary 5.2.

(1) A *and* B *are undistorted in* G_4*.*

(2) If $x, y \in A$ *are conjugate in* G_4*, then they are conjugate in* A*.*

(3) If $x, y \in B$ *are conjugate in* G_4*, then they are conjugate in* B*.*

Proof. These properties are consequences of A and B being retracts of G_4. □

Corollary 5.3. $2^n \preceq \mathrm{Area}_{G_4}(n)$ *and* $n^2 \preceq \mathrm{CL}_{G_4}(n)$*.*

Proof. We noted that $2^n \preceq \mathrm{Area}_{\mathrm{BS}(1,2)}(n)$ and $n^2 \preceq \mathrm{CL}_{\mathcal{H}_3(\mathbb{Z})}(n)$ in Section 3. By (5) and (4), respectively, of Lemma 5.1, $D_1 \cong \mathrm{BS}(1,2)$ and $\mathcal{H}_3(\mathbb{Z})$ are retracts of G_4. The claimed bounds follow. □

Next we will prove that $\mathrm{Area}_{G_4}(n) \preceq 2^n$ by means of the following three lemmas. We write $\exp_x(u)$ for the exponent sum of the letters x in a word u, and $|u|_x$ for the number of letters $x^{\pm 1}$.

Lemma 5.4. *If* $g \in \langle a, c, d \rangle \leq G_4$ *and* $|g|_{G_4} = n$*, then there is a word* σ *on* a, c, d *such that* $g = \sigma$ *in* G_4*,* $|\sigma|_a + |\sigma|_d \leq n$*, and* $|\sigma|_c \leq 3n^2$*.*

Proof. Let τ be a minimal length word on a, b, c, d representing g. Then $|\tau| = n$, because $g \in A$ and killing s retracts G_4 onto A.

The subgroup $\langle a, c, d \rangle$ is $\mathbb{Z}^2 * \mathbb{Z}$ with a and c generating the $\mathbb{Z}^2$-factor and d the $\mathbb{Z}$-factor. Since $g \in \langle a, c, d \rangle$, there exists a representative word σ of g of the form $\sigma = \sigma_0 d^{\delta_1} \sigma_1 \cdots d^{\delta_m} \sigma_m$ where $\delta_i \neq 0$ and $\sigma_i = a^{\alpha_i} c^{\gamma_i}$ for $i = 0, \ldots, m$, and $\sigma_j \neq 1$ for $j = 1, \ldots, m-1$. Since $\sigma = \tau$ in A, there exists a reduced van Kampen diagram Δ for $\sigma\tau^{-1}$ over A.

We claim that there is no a- or d-corridor connecting an a to an a^{-1} or a d to a d^{-1} in σ. Suppose, for a contradiction, that there is such a corridor Q. Let σ' be the subword of σ whose first and last letters label the edges at the ends of Q. Then Q bounds a subdiagram

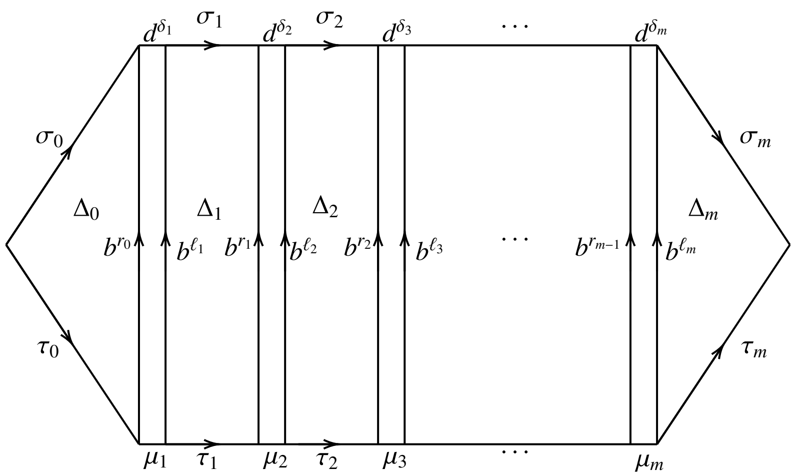

FIGURE 4. The van Kampen diagram Δ for $\sigma\tau^{-1}$.

Δ' of Δ. No a-corridor can cross a d-corridor, so there is an *innermost* such Q—that is, a Q such that no a- or d-corridors connects an a to an a^{-1} or a d to a d^{-1} in σ'. The first and last edges of Q cannot be contained in the same subword σ_i, nor in the same subword d^{δ_i}, since σ_i and d^{δ_i} are both freely reduced. So, if Q is an a-corridor, then σ' contains at least one d-letter, and if Q is a d-corridor then σ' contains at least one a-letter. This a- or d-letter in σ' must be part of an a- or d-corridor Q'. However Q' can neither cross Q nor connect two letters of σ'. This contradiction proves our claim. Thus, every a or d in σ is connected to an a or d in τ by an a- or d-corridor in Δ. So $|\sigma|_a + |\sigma|_d \leq n$.

Per Figure 4, for $i = 1, \ldots, m-1$, let Δ_i be the subdiagram of Δ between the pair of d-corridors starting at the last letter of d^{δ_i} and the first letter of $d^{\delta_{i+1}}$. Let Δ_0 be the subdiagram to the left of the d-corridor starting at the first d-letter of d^{δ_1} and let Δ_m be that to the right of the d-corridor starting at the last d-letter of d^{δ_m}. Thereby, $\sigma = \sigma_0 d^{\delta_1}\sigma_1 \cdots d^{\delta_m}\sigma_m$ and $\tau = \tau_0\mu_1\tau_1\cdots\mu_m\tau_m$ (as words) so that Δ_i is a subdiagram for $b^{\ell_i}\sigma_i b^{-r_i}\tau_i^{-1}$ where the b^{ℓ_i} and b^{r_i} are words along the sides of d-corridors, as shown, and $\ell_0 = r_m = 0$.

Now, for all i, as witnessed by the subdiagram of Δ to the left of the path labelled by b^{ℓ_i}, we have that $b^{\ell_i} = \hat{\tau}_i^{-1}\hat{\sigma}_i$ and $b^{r_i} = \overline{\tau}_i^{-1}\overline{\sigma}_i$ in G_4 for some prefixes $\hat{\sigma}_i$ and $\hat{\tau}_i$ and suffixes $\overline{\sigma}_i$ and $\overline{\tau}_i$ of σ and τ, respectively. So mapping $A \twoheadrightarrow \langle b \rangle = \mathbb{Z}$ by killing a, c, d gives that $|\ell_i| + |r_i| \leq n$.

Let $\overline{\tau_i}$ be τ_i with all d-letters removed. Because $\sigma_i = b^{\ell_i}\tau_i b^{-r_i}$ in G_4, we learn from Lemma 5.1(4) that $\sigma_i = b^{\ell_i}\overline{\tau_i}b^{-r_i}$ in $\mathcal{H}_3(\mathbb{Z})$. Therefore

$$|\sigma_i| \;\leq\; |\tau_i|_c + \left|\exp_a(\overline{\tau_i})\right| \left|\exp_b(b^{\ell_i}\overline{\tau_i}b^{-r_i})\right|.$$

The result follows. □

Lemma 5.5. *Every word w representing the identity G_4 admits a reduced van Kampen diagram with no b-rings. Further, every pair of words representing conjugate G_4 admits a reduced annular diagram with no contractible b- or d-rings.*

Proof. Given that G_4 is, by Lemma 5.1, an HNN-extension with stable letter b and an HNN-extension with stable letter d, this claim then follows results in [BRS] explaining how contractible b- or d-rings can be eliminated. ◻

Lemma 5.6. *If $u = 1$ in $E = \langle a, c, d, s \mid [a, c], s^a s^{-2}, s^d s^{-2}\rangle$, then*

$$\mathrm{Area}_E(u) \leq (|u|_c + |u|_s)2^{(|u|_a+|u|_d)}.$$

Proof. Let $E_0 = \langle c, s \mid \rangle$, let φ_a be the monomorphism $E_0 \hookrightarrow E_0$ given by $c \mapsto c$, $s \mapsto s^2$, and let φ_d be the monomorphism $\langle s\rangle_{E_0}$ given by $s \mapsto s^2$. Then E is the multiple HNN-extension of E_0 along φ_a and φ_d with stable letters a and d, respectively. Our claim now follows from Britton's lemma, observing that φ_a and φ_d increase the length of a word by at most a factor of 2, that, for $e \in \{a, d\}$, $x \in E_0$, a subword of u of the form $e^{\pm 1}\varphi_e^{\mp 1}(x)e^{\pm 1}$ requires $\left|\varphi_e^{\mp 1}(x)\right|$ relations to transform into the word x, and that this transformation strictly decreases the number of a- or d-letters in u. ◻

Lemma 5.7. *If $u = 1$ in A, then $\mathrm{Area}_A(u) \leq \lambda|u|^3$ for some constant $\lambda > 0$.*

Proof. This follows immediately from [BH99, Lemma III.Γ.6.20], the Dehn function for $\mathcal{H}_3(\mathbb{Z})$, and the fact that A is a trivial HNN-extension of $\mathcal{H}_3(\mathbb{Z})$ with stable letter d along the undistorted subgroup $\langle b\rangle_A$. ◻

Proposition 5.8. $\mathrm{Area}_{G_4}(n) \leq 2^n$

Proof. Suppose w is a word of length n representing 1 in G_4. By Lemma 5.5, w admits a reduced van Kampen diagram Δ in which there are no b-rings.

Consider a b-corridor β in Δ. If w_1 is the subword of w along the portion of $\partial\Delta$ between the first and last b-edges of β (inclusive), then w_1 represents an element of $\langle a, c, d\rangle_{G_4}$ and so, by Lemma 5.4, there exists a word w_2 on $\{a, c, d\}$ that equals w_1 in G_4 and has $|w_2|_a + |w_2|_d \leq |w_1|$ and $|w_2|_c \leq 3|w_1|^2$. So $|w_2| \leq 4|w_1|^2$. Then, per Lemma 5.7,

$$\mathrm{Area}(w_2 w_1^{-1}) \ \leq\ \lambda|w_1 w_2^{-1}|^3 \ \leq\ \lambda(|w_1| + |w_2|)^3 \ \leq\ \lambda(|w_1| + 4|w_1|^2)^3 \ \leq\ 5^3\lambda|w_1|^6.$$

There is a family of disjoint subwords in w of the form of w_1 described above that together include all the b-letters in w. After replacing each one with its corresponding w_2, we get a word w' on a, c, d, and s such that $w = w'$ in G_4, $|w'|_a + |w'|_d + |w'|_s \leq n$, $|w'|_c \leq 3n^2$, and $\mathrm{Area}_{G_4}(w'w^{-1}) \leq 5^3\lambda n^6$.

Because w' contains no b-letters, it represents 1 in E (by Lemma 5.1). So Lemma 5.7 applies and tells us that $\mathrm{Area}_E(w') \leq (3n^2 + n)2^n$. But $\mathrm{Area}_{G_4}(w') \leq \mathrm{Area}_E(w')$ and, by Lemma 2.4, $\mathrm{Area}_{G_4}(w) \leq \mathrm{Area}_{G_4}(w'w^{-1}) + \mathrm{Area}_{G_4}(w')$. The result follows. ◻

Lemma 5.9. $\mathrm{CL}_A(n) \leq n^2$ *for $A = \langle a, b, c, d \mid [a, b]c^{-1}, [a, c], [b, c], [b, d]\rangle$.*

Proof. Recall that we denote the Heisenberg group $\langle a, b, c \mid [a, b]c^{-1}, [a, c], [b, c]\rangle$ by G_1. The group A is an HNN-extension of G_1 with stable letter d. Let $\psi : A \to G_1$ be the surjection killing d.

Let u and v be words on $\{a, b, c, d\}$ which represent conjugate elements of A, and let $n = |u| + |v|$. Let Ω be a reduced annular diagram witnessing $u \sim v$ in A. We have two cases.

Suppose first that there are no radial d-corridors in Ω. Then $u = \psi(u)$ and $v = \psi(v)$ in A and the result then follows in this case from the fact that $\mathrm{CL}_{G_1}(n) \leq n^2$ and the fact that $|\psi(u)| + |\psi(v)| \leq |u| + |v|$.

Suppose, on the other hand, that Ω has $m \geq 1$ radial d-corridors. Then, after replacing u and v with cyclic conjugates, we can express them as concatenations of subwords

$$\begin{aligned} u &= d^{\epsilon_1} u_1 \cdots d^{\epsilon_m} u_m \\ v &= d^{\epsilon_1} v_1 \cdots d^{\epsilon_m} v_m \end{aligned}$$

where, for all i, $\epsilon_i = \pm 1$ and the edge labelled by the d^{ϵ_i} in u is joined by a radial d-corridor to the edge labelled by the d^{ϵ_i} in v.

Every element of G_1 can be expressed uniquely as $a^\alpha b^\beta c^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$. Let $\overline{u}_i = a^{\alpha_i} b^{\beta_i} c^{\gamma_i}$ and $\overline{v}_i = a^{\alpha'_i} b^{\beta'_i} c^{\gamma'_i}$ be the normal forms of $\psi(u_i)$ and $\psi(v_i)$.

For all i, any d-corridor in Ω emanating from a letter $d^{\pm 1}$ in u_i (or v_i) must be a d-arch ending at some letter $d^{\mp 1}$ in u_i (or v_i). So $\psi(u_i) = u_i$ and $\psi(v_i) = v_i$ in A, and therefore

$$u_i \ = \ \overline{u}_i \ \text{ and } \ v_i \ = \ \overline{v}_i \ \text{ in } \ A \tag{16}$$

for all i. A word on $\{a, b, c\}$ can be converted to its G_1-normal form by shuffling letters, with the proviso that interchanging an a and b introduces a $c^{\pm 1}$. So, because $\overline{u}_i$ and $\overline{v}_i$ can be obtained from u_i and v_i by deleting their d-letters and shuffling letters in this manner,

$$\sum_{i=1}^{m} (|\alpha_i| + |\beta_i| + |\alpha'_i| + |\beta'_i|) \leq n \ \text{ and } \ \sum_{i=1}^{m} (|\gamma| + |\gamma'|) \leq n^2. \tag{17}$$

Let b^{k_i} be the word read along the sides of the i-th radial d-corridor in Ω so that

$$b^{k_i} u_i \ = \ v_i b^{k_{i+1}} \tag{18}$$

in A for all i (subscripts modulo m). Now, (16) and (18) imply that

$$b^{k_i} \ a^{\alpha_i} b^{\beta_i} c^{\gamma_i} \ = \ a^{\alpha'_i} b^{\beta'_i} c^{\gamma'_i} \ b^{k_{i+1}} \tag{19}$$

in A, and therefore in G_1. The normal forms of left and right sides of (19) are $a^{\alpha_i} b^{\beta_i + k_i} c^{\gamma_i - \alpha_i k_i}$ and $a^{\alpha'_i} b^{\beta'_i + k_{i+1}} c^{\gamma'_i}$, respectively. So, for all i (subscripts modulo m),

$$\alpha_i = \alpha'_i, \tag{20}$$

$$\beta_i + k_i = \beta'_i + k_{i+1}, \tag{21}$$

$$\gamma_i - \alpha_i k_i = \ \gamma'_i. \tag{22}$$

Suppose $\alpha_i \neq 0$ for some i. Then (22) gives $k_i = (\gamma'_i - \gamma_i)/\alpha_i$, whence $|k_i| \leq |\gamma'_i| + |\gamma_i| \leq n^2$, which proves the result in this case because b^{k_i} conjugates a cyclic conjugate of u to a cyclic conjugate of v.

Suppose, on the other hand, that $\alpha_i = 0$ for all i. Then $\alpha'_i = 0$ for all i by (20). Define

$$\begin{aligned} \overline{u} &= d^{\epsilon_1} \overline{u}_1 \cdots d^{\epsilon_m} \overline{u}_m \\ \overline{v} &= d^{\epsilon_1} \overline{v}_1 \cdots d^{\epsilon_m} \overline{v}_m. \end{aligned}$$

Then $u = \overline{u}$ and $v = \overline{v}$ in A by (16).

Let $M = \langle b, c, d \mid [b, c], [b, d] \rangle$. Mapping $b \mapsto cb$ and $c \mapsto c$ defines an automorphism of the subgroup $\mathbb{Z}^2 = \langle b, c \rangle$ of M. Thereby, A is an HNN-extension of M with stable letter a. Because there are no a-letters in $\overline{u}$ and $\overline{v}$, they are conjugate in M. The conjugator length function of M is linear per Servatius' solution to the conjugacy problem for RAAGs in [Ser89]. The required bound then follows: for some constant $C > 0$,

$$\mathrm{CL}_A(u,v) \;\leq\; \mathrm{CL}_M(\overline{u},\overline{v}) \;\leq\; C\,(|\overline{u}| + |\overline{v}|) \;\leq\; C(n+n^2),$$

with the final inequality following from (17). □

Lemma 5.10. $\mathrm{CL}_B(n) \leq n$ *for* $B = \langle a,d,s \mid asa^{-1} = s^2, dsd^{-1} = s^2\rangle$.

Proof. Suppose g and h are conjugate elements of B. Let u_0 and v_0 be minimal length words representing g and h, respectively, let $n := |u_0| + |v_0|$, and let Δ be an annular van Kampen diagram for the pair u_0 and v_0.

Case: Δ has no radial a- or d-corridors. In this event, all the a- or d-corridors originating at a boundary component must be arches of Δ. Indeed, we claim that there exist some cyclic conjugates u_1 and v_1 of u_0 and v_0 equal to s^k and s^ℓ, respectively, for some integers k, ℓ such that $|k|+|\ell| \leq 2^n$. To see this, take u_1 and v_1 to be the cyclic conjugates of u_0 and v_0 such that there exists a simple path in the 1-skeleton of Δ from the initial vertex of u_1 to the initial vertex of v_1 which crosses no a- or d-arch. Then u_1 and v_1 can be converted to s^k and s^ℓ, respectively, by eliminating all *pinches*: that is, if u_1 or v_1 contains a subword of the form as^ia^{-1}, ds^id^{-1}, $a^{-1}s^{2i}a$, or $d^{-1}s^{2i}d$, where $i \in \mathbb{Z}$, then replace it with s^{2i}, s^{2i}, s^i, or, s^i, respectively. Arches in Δ correspond to pinches in cyclic conjugates of u_0 and v_0, and our choices of u_1 and v_1 guarantee that they contain all such pinches. Therefore, we may exhaustively freely reduce and make such substitutions until no such subwords remain. In particular, every a- or d-letter will be removed from u_1 and v_1, so the result will be powers s^k, s^ℓ of s; respectively. Moreover, since at most n pinches are present in u_1 and v_1, and every substitution at most doubles the total number of s-letters on both boundary components, so $|k|+|\ell| \leq 2^n$ as desired.

But then there exists a word w such that $ws^kw^{-1} = s^\ell$ in B. So ws^kw^{-1} can be converted to s^ℓ by successively eliminating pinches, and therefore $s^\ell = s^{2^r k} = a^r s^k a^{-r}$ for $r = \exp_a(w) + \exp_d(w)$. So $\ell = 2^r k$ and $r \leq n$, which implies $\mathrm{CL}(u,v) \leq n$.

Case: there is a radial a- or d-corridor in Δ. Take cyclic conjugates u_1 and v_1 of u_0 and v_0, respectively, beginning at the same radial a- or d-corridor Q. Then $u_1 = \alpha_1 e_1 \cdots \alpha_m e_m$ and $v_1 = \beta_1 f_1 \cdots \beta_m f_m$ for some number $m \in \mathbb{N}$, some letters $e_1,\ldots,e_m,f_1,\ldots,f_m \in \left\{a^{\pm 1}, d^{\pm 1}\right\}$, and some words $\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_m$ on a,d,s. The power of s read along one side of Q conjugates u_1 to v_1 in B. Furthermore, for all i, a radial corridor connects the edge labelled by e_i in u_1 to that labelled by f_i in v_1, and each α_i and β_i can be converted to a power of s by eliminating pinches. Accordingly, u_1 and v_1 are equal in B to words $u = s^{\chi_1}e_1 \cdots s^{\chi_m}e_m$ and $v = s^{\xi_1}e_1 \cdots s^{\xi_m}e_m$, respectively, for some $\chi_1,\ldots,\chi_m,\xi_1,\ldots,\xi_m \in \mathbb{Z}$ such that $|u|+|v| = 2^n$. So $u_1 s^k v_1^{-1} = us^kv^{-1} = s^k$ in B for some $k \in \mathbb{Z}$. Let $\mu = \exp_a(u) + \exp_d(u)$.

Subcase: $\mu = 0$. By a propitious choice of which radial a- or d-corridor to read u_1 and v_1 from, we may assume that every suffix $\hat{u}$ of u has the property that $\exp_a(\hat{u}) + \exp_d(\hat{u}) \geq 0$. This ensures that usu^{-1} can be converted to a power of s by replacing the pinch between e_m in u and e_m^{-1} in u^{-1} by a power of s, then likewise replacing the pinch between e_{m-1} and e_{m-1}^{-1}, and so on. Indeed, we get $usu^{-1} = s$. And then, given that $us^kv^{-1} = s^k$, we deduce that $u = v$ in B. This implies $\mathrm{CL}(g,h) \leq n$, as desired.

Subcase: $\mu \neq 0$. Then no non-zero power of s commutes with u and so the $k \in \mathbb{Z}$ such that $us^kv^{-1} = s^k$ is unique. For $i = 1,\ldots,m$ let μ_i be the exponent sum of $e_1 \cdots e_{i-1}$. The existence of the diagram Δ implies that

$$us^k v^{-1} \;=\; s^{\chi_1} e_1 \cdots s^{\chi_m} e_m \; s^k \; e_m^{-1} s^{-\xi_m} \cdots e_1^{-1} s^{-\xi_1}$$

can be converted to s^k by eliminating the pinch bookended by e_m and e_m^{-1}, then eliminating that bookended by e_{m-1} and e_{m-1}^{-1}, and so on. Doing so, we calculate that

$$k \;=\; 2^{\mu} k + \sum_{i=1}^{m} 2^{\mu_i}(\chi_i - \xi_i).$$

Now $|\mu|, |\mu_i| \leq |n|$ for all i and $\sum_{i=1}^m (|\chi_i| + |\xi_i|) \leq 2^n$, and so $k = \frac{1}{1-2^{\mu}} \sum_{i=1}^m 2^{\mu_i}(\chi_i - \xi_i) \leq 2^n$. Therefore $|s^k|_B \leq n$, from which we deduce that $\mathrm{CL}(g, h) \leq n$. $\square$

Lemma 5.11. $\mathrm{CL}_E(n) \leq n$ *for* $E = \langle a, c, d, s \mid [a, c], s^a s^{-2}, s^d s^{-2} \rangle$.

Proof. Suppose u and v are words representing conjugate elements of E and that Δ is a reduced annular diagram for u and v. If there are no radial c-corridors in Δ then, perhaps after replacing u and v by cyclic conjugates, $u, v \in B$. Also, since the only relation involving c is $[a, c] = 1$, we see $|u|_B + |v|_B = |u|_E + |v|_E \leq n$. Thus $\mathrm{CL}(u, v) \leq \mathrm{CL}_B(n)$, which is linear by Lemma 5.10. If there is at least one radial c-corridor, then the word τ along its sides is a reduced power of a, any canceling pair $a^{\pm 1} a^{\mp 1}$ would have to lie on the boundary of two cancelling cells. Up to cyclic conjugation, u is therefore conjugated to v by a power $\tau = a^p$ of a for some $p \in \mathbb{Z}$. If there is a radial d-corridor in Δ, then every letter in τ is connected to a different a-letter in the boundary words u or v by an a-corridor, since they cannot cross the d-corridor nor, because τ is reduced, return to cross the c-corridor. So $|\tau| \leq |u| + |v|$. If there are no radial d-corridors in Δ, then by excising the d-arches of Δ we may replace u, v with u', v' such that u', v' contain no d-letters, $u' = u, v' = v$ in E, $|u'| + |v'| \leq 2^n$, and $|u'|_a + |v'|_a \leq n$. Moving each instance of a in u' and v' to the right and each instance of a^{-1}, we have $u' = a^{-k_1} w_1(c, s) a^{\ell_1}, v' = a^{-k_2} w_2(c, s) a^{\ell_2}$, for some $k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ and words w_1, w_2 on c and s such that $|w_1| + |w_2| \leq 2^{2n}$. Conjugating u' and v' by a^{ℓ_1} and a^{ℓ_2} respectively, we obtain the words $u'' = a^{-k_1+\ell_1} w_1, v'' = a^{-k_2+\ell_2} w_2$, which are also conjugate by a power $a^{p-\ell_1+\ell_2}$ of a. Since $|-\ell_1 + \ell_2| \leq |u'|_a + |v'|_a \leq n$ and $\mathrm{CL}(u, v) \leq p$, to show $\mathrm{CL}(u, v) \leq n$ it suffices to find a linear bound for $p' = p - \ell_1 + \ell_2$. Indeed, since E is an HNN-extension with stable letter a (see Lemma 5.6), we must have $-k_1 + \ell_1 = -k_2 + \ell_2$, so $a^{p'} w_1 a^{-p'} = w_2$. Applying to this equation the retraction $E \twoheadrightarrow B$ given by killing c, we have $a^{p'} s^{q_1} a^{-p'} = s^{q_2}$ in B, where $|q_1| + |q_2| \leq |w_1| + |w_2| \leq 2^{2n}$. This gives $|p'| = \log_2 (|q_1 - q_2|) \leq 2n$, and we are done. $\square$

Per Lemma 5.1(6), the subgroup $L = \langle b, c, s \rangle$ of G_4 is $\mathbb{Z}^2 * \mathbb{Z} = \langle b, c, s \mid [b, c] \rangle$. The following lemma provides quantitative details of the normal form for elements L:

$$u \;=\; b^{\beta_0} c^{\gamma_0} s^{\mu_1} b^{\beta_1} c^{\gamma_1} \cdots s^{\mu_k} b^{\beta_k} c^{\gamma_k}, \tag{23}$$

where $\mu_1, \ldots, \mu_k$ are non-zero and none of $b^{\beta_1} c^{\gamma_1}, \ldots, b^{\beta_{k-1}} c^{\gamma_{k-1}}$ are the identity in $\mathbb{Z}^2$.

Lemma 5.12. *If* u *of* (23) *equals in* G_4 *a word* v *on* $\left\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, s^{\pm 1}\right\}$*, then* $|\beta_0| + \cdots + |\beta_k| \leq |v|$, $|\gamma_0| + \cdots + |\gamma_k| \leq |v|^2$ *and* $|\mu_1| + \cdots + |\mu_k| \leq 2^{|v|}$.

Proof. We proceed by induction on $|v|$. The base case of $|v| = 1$ is trivial. For the inductive step, we have by Britton's Lemma that

$$v = \alpha_1 e_1 \beta_1 e_1^{-1} \alpha_2 \cdots \alpha_m e_m \beta_m e_m^{-1} \alpha_{m+1},$$

where α_i is a word on $\{b, c, s\}$, $e_i \in \{a^{\pm 1}, d^{\pm 1}\}$, and β_i is a word representing:

- an element of $\langle b, c, s\rangle_{G_4}$ if $e_i = a$,
- an element of $\langle b, c, s^2\rangle_{G_4}$ if $e_i = a^{-1}$,
- an element of $\langle b, s\rangle_{G_4}$ if $e_i = d$, or
- an element of $\langle b, s^2\rangle_{G_4}$ if $e_i = d^{-1}$

for all i. In any of these cases, the inductive hypothesis gives that β_i is equal to a word on $\{b, c, s\}$ with $|\beta_i|$ many b-letters, $|\beta_i|^2$ many c-letters, and $2^{|\beta_i|}$ many s-letters. Conjugating by e_i adds at most $|\beta_i|$ many c-letters and $2^{|\beta_i|}$ many s-letters, so $e_1\beta_1 e_1^{-1}$ can be written as a word on $\{b, c, s\}$ with $|\beta_i|$ many b-letters, $|\beta_i|^2 + |\beta_i| \le (|\beta_i| + 1)^2 \le |e_1\beta_1 e_1^{-1}|^2$ many c-letters, and $2^{|\beta_i|+1} \le 2^{|e_1\beta_1 e_1^{-1}|}$ many s-letters. The claim then follows by the total number of each letter and by the structure of L. ◻

Our next lemma concerns manipulations of words in the normal form (23).

Lemma 5.13. *Suppose $z = f_m f_{m-1} \cdots f_1$ is a reduced word where $f_1, \dots, f_m \in \left\{a^{\pm 1}, d^{\pm 1}\right\}$. Suppose that $u_0, \dots, u_m$ are words in the normal form (23) and that*

$$u_0 \;=\; b^{\beta_0} c^{\gamma_0} s^{\mu_1} b^{\beta_1} c^{\gamma_1} \cdots s^{\mu_k} b^{\beta_k} c^{\gamma_k}. \tag{24}$$

Assume that $u_i = f_i u_{i-1} f_i^{-1}$ in G_4 for $i = 1, \dots, m$ and that some (and so every) u_i is not an element of $\langle s\rangle$. Then

- *either $z = a^\lambda$ (as words) for some $\lambda \in \mathbb{Z}$ such that $m = |\lambda|$ and*
$$u_m \;=\; b^{\beta_0} c^{\gamma_0 + \lambda\beta_0} s^{2^\lambda \mu_1} b^{\beta_1} c^{\gamma_1 + \lambda\beta_1} \cdots s^{2^\lambda \mu_k} b^{\beta_k} c^{\gamma_k + \lambda\beta_k},$$
- *or $z = a^{\lambda_2} d^{\xi} a^{\lambda_1}$ (as words) for some $\lambda_1, \xi, \lambda_2 \in \mathbb{Z}$ such that $m = |\lambda_1| + |\xi| + |\lambda_2|$ and*
$$\begin{aligned} u_{|\lambda_1|} &= b^{\beta_0} s^{2^{\lambda_1}\mu_1} b^{\beta_1} \cdots s^{2^{\lambda_1}\mu_k} b^{\beta_k}, \\ u_{|\lambda_1|+|\xi|} &= b^{\beta_0} s^{2^{\lambda_1+\xi}\mu_1} b^{\beta_1} \cdots s^{2^{\lambda_1+\xi}\mu_k} b^{\beta_k}, \\ u_m &= b^{\beta_0} c^{\gamma'_0} s^{2^{\lambda_1+\xi+\lambda_2}\mu_1} b^{\beta_1} c^{\gamma'_1} \cdots s^{2^{\lambda_1+\xi+\lambda_2}\mu_k} b^{\beta_k} c^{\gamma'_k} m \end{aligned}$$

 and $\gamma_i = -\beta_i\lambda_1$ and $\gamma'_i = \beta_i\lambda_2$ for $i = 0, \dots, k$.

Proof. Here are the key points. When $f_i = d^{\pm 1}$, the relation $u_i = f_i u_{i-1} f_i^{-1}$ in G_4 necessitates that there be no c-letters in the normal form of u_{i-1}. And when $f_i = a^{\pm 1}$, we can relate the c-letters in u_{i-1} to those in u_i—for example, because $u_{i-1}, u_i \notin \langle s\rangle$, at least one of u_{i-1} and u_i contains c-letters. ◻

We are now ready to bound the lengths of conjugators.

Lemma 5.14. *Suppose x and y are words on $\left\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, s^{\pm 1}\right\}$. Let $n = |x| + |y|$. If there exists a word w on $\left\{a^{\pm 1}, d^{\pm 1}\right\}$ such that $wx = yw$ in G_4, then there exists such a w for which $|w| \le 2n + 8n^2$.*

Proof. Suppose w is a word on $\left\{a^{\pm 1}, d^{\pm 1}\right\}$ such that $wx = yw$ in G_4. Further assume that w is of minimal length among all such w (so in particular is reduced). Let Δ be a van Kampen diagram for $wxw^{-1}y^{-1}$.

No two a- or d-corridors in Δ can cross. No pair of a-edges in one of the two portions of $\partial\Delta$ labelled by w can be connected by an a-corridor, and ditto no pair of d-edges by a d-corridor. So, per Figure 5, w can be expressed in two ways as concatenations of subwords:

$$w \ = \ w_0 z w_1 \ = \ \tilde{w}_0 z \tilde{w}_1$$

where $m = |z|$ and the $z = f_m f_{m-1} \cdots f_1$ subwords label two portions of $\partial\Delta$ so that for all $1 \leq i \leq m$, the letter f_i in the z in one copy of w is joined by an a- or d-corridor to the f_i in the other, and all a- and d-corridors starting from w_0 or $\tilde{w}_0$ end in y, and those starting from w_1 or $\tilde{w}_1$ end in x. The latter implies that for $x_1 := w_1 x \tilde{w}_1^{-1}$ and $y_1 := w_0^{-1} y \tilde{w}_0$ (as words),

$$|w_0| + |w_1| + |\tilde{w}_0| + |\tilde{w}_1| \ \leq \ n, \text{ and} \tag{25}$$

$$|x_1| + |y_1| \ \leq \ 2n. \tag{26}$$

For $i = 0, 1, \ldots, m$, the word

$$f_i f_{i-1} \cdots f_1 \, x_1 \, f_1^{-1} \cdots f_{i-1}^{-1} f_i^{-1}$$

around part of $\partial\Delta$ equals in G_4 a word along one side of one such corridor, so represents an element of L. Let u_i be its normal form per (23). Then $x_1 = u_0$ and $y_1 = u_m$ in G_4 and $z u_0 = u_m z$ in L.

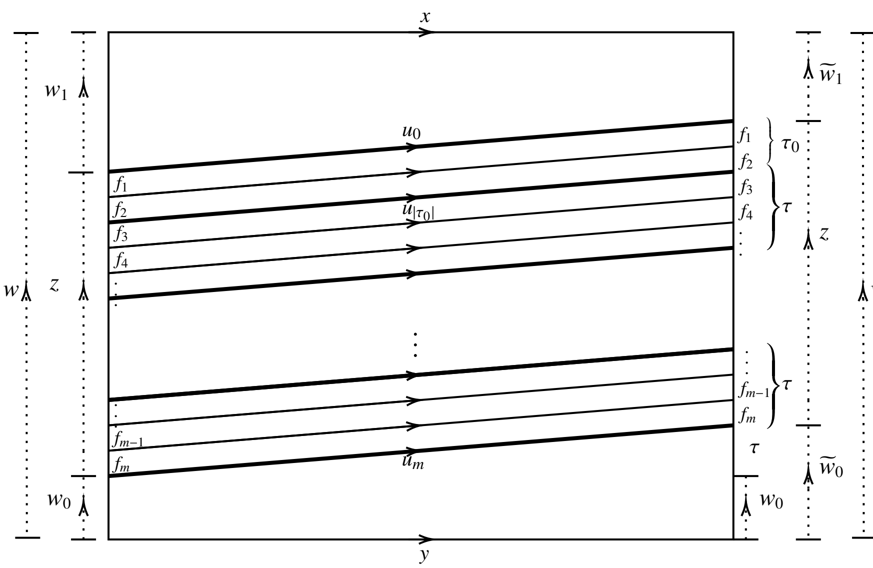

Figure 5. The van Kampen diagram Δ for Lemma 5.14.

Let $\lambda = \exp(z)$. Here is an estimate we will call on multiple times. It is contingent on the exponent μ_1 of the first power of s in the normal form (24) for u_0:

$$\mu_1 \neq 0 \quad \Longrightarrow \quad |\lambda| \ \leq \ \max\{|x_1|, |y_1|\} \ \leq \ 2n, \tag{27}$$

To establish this, let μ_1' be exponent of the first power of s in the normal form of u_m. Then $\mu_1' = 2^{\lambda}\mu_1$ and, by Lemma 5.12, $2^{\lambda}\mu_1 \leq 2^{|y_1|}$. So $\lambda \leq |y_1|$ because $\mu_1 \in \mathbb{Z} \smallsetminus \{0\}$. Also $2^{-\lambda}\mu' = \mu$ and $\mu' \in \mathbb{Z} \smallsetminus \{0\}$, so likewise, $-\lambda \leq |x_1|$. So (27) follows, with (26) giving the final inequality.

We will argue bounds on $|w|$ in cases according to details of the normal form (23) of u_0.

1. *Case: $u_0 = s^{\mu_1}$ for some $\mu_1 \in \mathbb{Z}$.*
   a. *Subcase: $w_0 = \tilde{w}_0$ (as words).* If $\mu_1 = 0$, then $u_0 = u_m = 1$ and z is the empty word, for otherwise we could remove it and $wx = yw$ in G_4 would remain true. So, $|w| = |w_0 w_1| = |w_0| + |w_1| \leq 2n$ by (25).
   Assume, then, that $\mu_1 \neq 0$. Then for $\lambda := \exp(z)$, we have $|\lambda| \leq 2n$ by (27), and
   $$u_m \;=\; z u_0 z^{-1} \;=\; z s^{\mu_1} z^{-1} \;=\; s^{2^{\lambda}\mu_1} \;=\; a^{\lambda} s^{\mu_1} a^{-\lambda}.$$
   So we may take $z = a^{\lambda}$ because doing so could only shorten w and it will remain true that $wx = yw$ in G_4. So, using (25), we get $|w| = |w_0 a^{\lambda} w_1| = |w_0| + |w_1| + |\lambda| \leq 3n$.
   b. *Subcase: there is some non-empty word τ such that $\tilde{w}_0 = w_0\tau$ (as words).* This is the situation illustrated in Figure 5. We have $z = \tau^{\ell}\tau_0$ (as words) for some $\ell \geq 0$ and some proper (perhaps empty) prefix τ_0 of τ. Let $e = \exp(\tau)$.
      i. *Assume $e = 0$ or $\mu_1 = 0$.* Define $w' := w_0\tau_0 w_1$. We claim that $w'x = yw'$. If $\mu_1 = 0$, then $u_0 = u_1 = \cdots = u_m$. If $e = 0$, then $\tau^{-1}u_m\tau = u_m$ and so $u_{|\tau_0|} = \tau^{-\ell}u_m\tau^{\ell} = u_m$. Either way, removing the region whose boundary is labelled by $z u_{|\tau_0|} z^{-1} u_m^{-1}$ from the van Kampen diagram of Figure 5 and identifying the path labelled by u_m with that labelled by $u_{|\tau_0|}$ demonstrates that $w'x = yw'$. So, because $|w|$ is minimal, $\ell = 0$ and $w = w_0\tau_0 w_1$. But $|w_0\tau_0| \leq |w_0\tau| \leq |y|$ and $|w_1| \leq |x|$, so $|w| \leq n$.
      ii. *Assume, instead, that $e \neq 0$ and $\mu_1 = 0$.* Let $\lambda := \exp(z)$. Then $|\lambda| \leq 2n$ because $\mu_1 \neq 0$ means (27) applies. But $\lambda = e\ell + \exp(\tau_0)$, so, because $e \neq 0$ and $|\exp(\tau_0)| \leq |\tau| \leq n$, we deduce that $|\ell| \leq 3n$. So
$$|w| \;=\; |w_0\tau^{\ell}\tau_0 w_1| \;\leq\; |w_0| + |w_1| + (\ell+1)|\tau| \;\leq\; n + (3n+1)n \;=\; 3n^2 + 2n.$$
   c. *Subcase: there is some non-empty word τ such that $w_0 = \tilde{w}_0\tau$ (as words).* We argue that $|w| \leq n$ or that $|w| \leq 3n^2 + 2n$ like in the prior subcase, mutatis matantis.
2. *Case, $u_0 = c^{\gamma_0}s^{\mu_1}c^{\gamma_1}\cdots s^{\mu_k}c^{\gamma_k}$ with $\mu_1 \neq 0$ and ($\gamma_0 \neq 0$ or $\gamma_1 \neq 0$).* In this case all the normal form words $u_0, \ldots, u_m$ contain at least one c-letter and so there are no $d^{\pm 1}$ letters in z and we are in the $z = a^{\lambda}$ case of Lemma 5.13. So, by (25) and (27), $|w| = |w_0| + |z| + |w_1| \leq n + |\lambda| \leq 3n$.
3. *Case, $u_0 = b^{\beta_0}c^{\gamma_0} \neq 1$.* Lemma 5.13 applies and we divide into subcases accordingly.
   a. *Subcase, $z = a^{\lambda_2}d^{\xi}a^{\lambda_1}$ with $\xi \neq 0$.*
      i. *Assume $w_0 = \tilde{w}_0$ (as words).* Due to the absence of s-letters in u_0, $u_{|\lambda_1|} = u_{|\lambda_1|+|\xi|}$ and $\xi = 0$, because otherwise we could shorten w. So this case does not arise.
      ii. *Assume, instead, that there is some non-empty word τ such that $\tilde{w}_0 = w_0\tau$ (as words).* Then $z = \tau^{\ell}\tau_0$ (as words) for some $\ell \geq 0$ and some proper (perhaps empty) prefix τ_0 of τ. There are now two possibilities. The first is that $a^{\lambda_2}d^{\xi}$ is a prefix of τ, in which case $|z| \leq 2|\tau| \leq 2n$, and so $|w| = |w_0| + |z| + |w_1| \leq 4n$. The second is that $z = d^{\xi}$ (as words), but then $\gamma_0 = 0$ and $u_0 = \cdots = u_m = b^{\beta_0}$, and like in Case 1.b.i., $\ell = 0$, $w = w_0\tau_0 w_1$, and $|w| \leq n$.
      iii. *Assume, instead, that there is some non-empty word τ such that $w_0 = \tilde{w}_0\tau$ (as words).* The same argument applies, mutatis mutantis.
   b. *Subcase, $z = a^{\lambda}$.* In this event, $m = |\lambda|$ and $u_m = b^{\beta_0}c^{\gamma_0+\lambda\beta_0}$.

i. *Assume* $\beta_0 = 0$. Then $m = \lambda = 0$, and $u_0 = \cdots = u_m = c^{\gamma_0}$, and like in Cases 1.b.i. and 3.a.ii., $\ell = 0$, $w = w_0 \tau_0 w_1$, and $|w| \leq n$.

ii. *Assume, instead, that* $\beta \neq 0$. We can use the bounds $|\gamma| \leq |x_1|^2$ and $|\gamma_0 + \lambda\beta_0| \leq |y_1|^2$ from Lemma 5.12, to get that $m = |\lambda| \leq |\lambda||\beta_0| \leq |\gamma_0| + |y_1|^2 \leq |x_1|^2 + |y_1|^2$. Finally, $|w| \leq n + (2n)^2 \leq 5n^2$ by (25) and (26).

4. *Case,* $u_0 = b^{\beta_0} c^{\gamma_0} s^{\mu_1} b^{\beta_1} c^{\gamma_1} \cdots s^{\mu_k} b^{\beta_k} c^{\gamma_k}$ *with* $\mu_1 \neq 0$ *and* $\beta_i \neq 0$ *for some* i.
   Lemma 5.13 applies and tells us that $z = a^{\lambda_2} d^{\xi} a^{\lambda_1}$ for some $\lambda_1, \xi, \lambda_2 \in \mathbb{Z}$.
   We have $\gamma_i = -\beta_i \lambda_1$ and $\gamma_i' = \beta_i \lambda_2$. Lemma 5.12 applies to $x_1 = u_0$ and to $y_1 = u_m$ and gives us that $\gamma_i \leq |x_1|^2$ and $\gamma_i' \leq |y_1|^2$. So, because $\beta_i \neq 0$, we learn that $|\lambda_1| \leq |x_1|^2$ and $|\lambda_2| \leq |y_1|^2$.
   Now, the exponent of the first power of s in the normal form of u_m is $2^{\lambda_1 + \xi + \lambda_2} \mu_1$. By Lemma 5.12, $|2^{\lambda_1 + \xi + \lambda_2} \mu_1| \leq 2^{|y_1|}$. And, because $\mu_1 \neq 0$, we deduce that $|\lambda_1 + \xi + \lambda_2| \leq |y_1|$. So $|\xi| \leq |y_1| + |\lambda_1| + |\lambda_2| \leq |y_1| + |x_1|^2 + |y_1|^2$ and
$$|z| \;=\; |\lambda_1| + |\xi| + |\lambda_2| \;\leq\; |y_1| + 2|x_1|^2 + 2|y_1|^2 \;\leq\; n + 8n^2.$$
   Finally, $|w| \leq n + |z| \leq 2n + 8n^2$ by (25) and (26).

□

Remark 5.15. The defining relations of G_4 that contain c are $[a, b] = c, [a, c] = 1$, and $[b, c] = 1$. The last two of these form corridors in diagrams, except that they may begin or end at a cell labelled $[a, b] = c$ instead of at the boundary of the diagram. We will call these sequences of cells *c-segments*, and the cells corresponding to $[a, b] = c$ the endpoints of a c-segment. A c-segment can close up and form an annulus (which could be contractible or non-contractible within an annular diagram). In general, if a c-segment has no endpoints, then they behave exactly like corridors, so we will refer to them as c-arches, radial c-segments, and so on.

Proposition 5.16. $\mathrm{CL}_{G_4}(n) \;\leq\; n^2$.

Proof. Suppose words u and v represent conjugate elements of G_4. Let $n = |u| + |v|$ and let Δ be an annular diagram for u and v. We can assume Δ contains no contractible b-corridor—any such corridor would have along its outer boundary δ a word on $\{a, c, d\}$ which represents the identity in G_4; by Lemma 5.1(1) we could replace the subdiagram bounded by δ with a van Kampen diagram over $\langle a, c, d \mid [a, c] \rangle$, without changing u or v. Additionally, by Lemma 5.4, we may replace every b-arch in Δ to obtain boundary words u' and v' (equal, up to cyclic conjugacy, to u and v respectively) such that $|u'| + |v'| \leq 3n^2$. More specifically, we also have $|u'|_a + |v'|_a + |u'|_d + |v'|_d \leq n$.

We consider three cases.

1. *Case:* Δ *has no radial or non-contractible b-corridor.* In this event, Δ contains no b-edges whatsoever, so u' and v' represent conjugate elements of $E = \langle a, c, d, s \mid [a, c], s^a s^{-2}, s^d s^{-2} \rangle$. By Lemma 5.11, $\mathrm{CL}(u', v')$ is at most a constant times $|u'|+|v'|$, and so $\mathrm{CL}(u, v) \leq \lambda n^2$ for a suitable constant $\lambda > 0$.

2. *Case:* Δ *has at least one non-contractible b-annulus.*

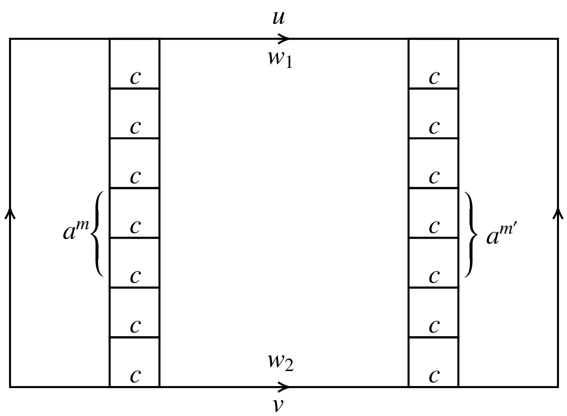

Figure 6. Two radial c-segments

We claim that there exists a constant $\lambda > 0$ (independent of u' and v') and words $\tilde{u}$ and $\tilde{v}$ on $\{a, b, c, d\}$ such that $u' \sim \tilde{u}$ and $v' \sim \tilde{v}$ in G_4, and

$$|\tilde{u}|_A, |\tilde{v}|_A \leq \lambda n + \lambda, \tag{28}$$

$$\mathrm{CL}(u', \tilde{u}), \mathrm{CL}(v', \tilde{v}) \leq 3\lambda n^2. \tag{29}$$

We will argue this for $\tilde{v}$ in three subcases. The argument for $\tilde{u}$ is the same mutatis mutantis.

Let β be the outermost b-corridor.

a. Suppose there is a c-segment running from the outer side of β to the outer boundary, as in Figure 7. Then, up to cyclic permutation, v' is equal in G_4 to the word $\tilde{v}$ along the dotted line, which is an element of A. Every s-letter in v' corresponds lexically to an s-letter in v, so there is a path in v' from the c-edge of the given c-segment to some vertex belonging to our original word v which does not contain any s-edges. Thus we may assume that $\tilde{v}$ is conjugate to v' (and thus to v) in A. Since A is undistorted in G_4 (Corollary 5.2), there therefore exists a constant $\lambda > 0$ such that $|\tilde{v}|_A \leq \lambda|v| + \lambda \leq \lambda n + \lambda$.
b. Suppose there are no such c-segments. Then every c-segment leaving β must return to it, so we have the situation illustrated in Figure 8. Let $\tilde{v} \in C = F(a, d)$ (recall, Lemma 5.1(3)) be the word along the dotted line. Every a- or d-letter of the reduced form of $\tilde{v}$ is part of a corridor running from β to the outer boundary, so $|\tilde{v}|_C \leq |v'|_a + |v'|_d \leq n$.
Now, every $c^{\pm 1}$ in v' labels the initial edge of a c-arch in Δ', and so excising these, we learn that that a cyclic conjugate of v' equals in G_4 a word $v'' \in \langle a, d, s \rangle = B$. So $\mathrm{CL}(v', v'') \leq |v'| + |v''|$ and, since the words along the sides of a c-arch are the same, $|v''| \leq |v'|$. Since v'' and $\tilde{v}$ represent elements of B that are conjugate in G_4, they are conjugate in B by Corollary 5.2(3). By Lemma 5.10, $\mathrm{CL}(v'', \tilde{v})$ is at most a constant times $|v''| + |\tilde{v}|$ (in B and therefore in G_4). Combining these estimates give that $\mathrm{CL}(v', \tilde{v}) \leq \mathrm{CL}(v', v'') + \mathrm{CL}(v'', \tilde{v})$ is at most $\lambda|v'|$, and so at most $3\lambda n^2$, for a suitable constant $\lambda > 0$.

With these two cases complete, note that Lemma 5.9 applies to $\tilde{u}$ and $\tilde{v}$, since they represent conjugate elements of A. Moreover, this Lemma combines with

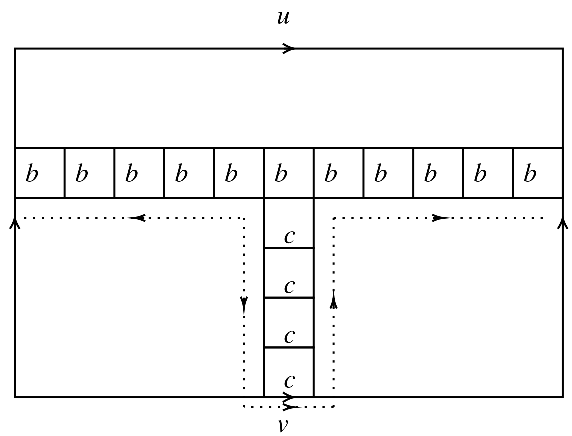

FIGURE 7. c-segment meeting a non-contractable b-corridor

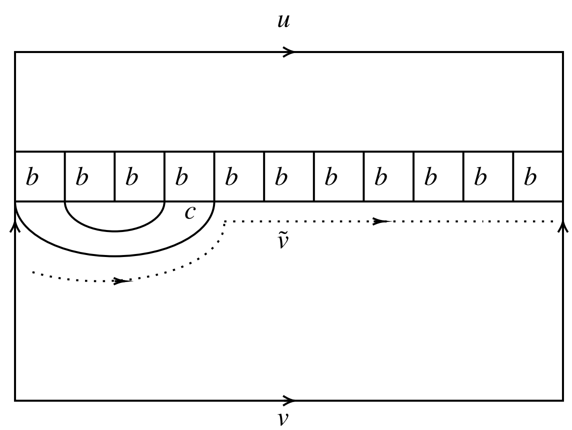

FIGURE 8. no c-segment running between a non-contractable b-corridor and the boundary

(28) and (29) to give

$$\mathrm{CL}(u, v) \ \leq\ \mathrm{CL}(u, \tilde{u}) + \mathrm{CL}(\tilde{u}, \tilde{v}) + \mathrm{CL}(\tilde{v}, v) \ \leq\ \mu n^2,$$

for a suitable constant $\mu > 0$, as required.

3. *Case: Δ contains a radial b-corridor.* If there is a c-segment running between each pair of consecutive radial b-corridors, then we have the situation illustrated in Figure 9. Up to cyclic conjugation, u and v are equal in G_4 to the words along the two dotted lines, which are words on $\{a, b, c, d\}$. So there are cyclic conjugates u_0 and v_0 of u and v, respectively, that represent elements of $\langle a, b, c, d\rangle$ and the minimal length words $\tilde{u}$ and $\tilde{v}$ on $\{a, b, c, d\}$ equaling u_0 and v_0 in G_4, respectively, have $|\tilde{u}| + |\tilde{v}| \leq n$ by Corollary 5.2(1). And because $u \sim v$ in G_4 and A is a retract of G_4, we have $\tilde{u} \sim \tilde{v}$ in A. And then, by Lemma 5.9, there exists a word w such that $\tilde{u}w = w\tilde{v}$ in A and $|w| \leq \lambda n^2$, for a suitable constant $\lambda > 0$. But then, $u_0 w = w v_0$ in G_4, and the claim follows.

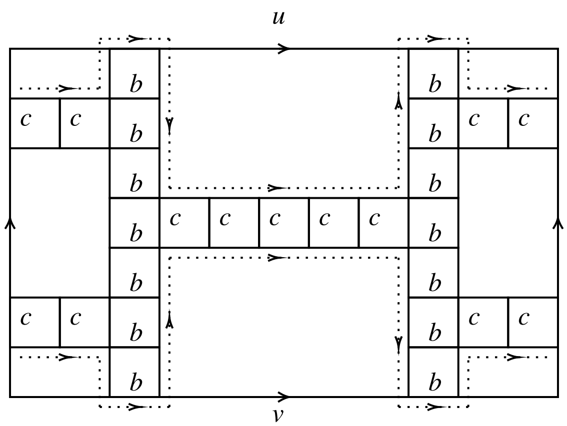

FIGURE 9. c-segments running between consecutive radial b-corridors

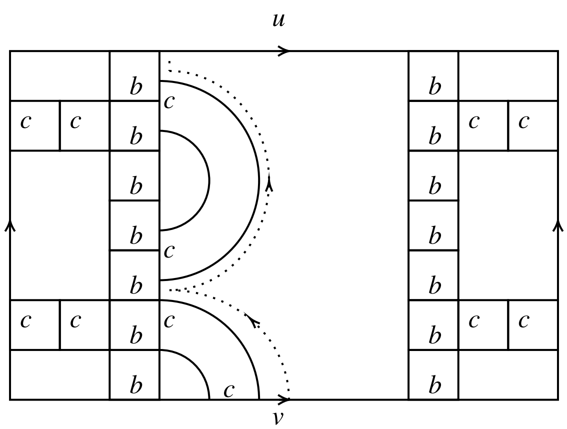

FIGURE 10. radial b-corridors not connected by a c-segment

Alternatively, suppose some pair of radial b-corridors has no c-segment running between them, as shown in Figure 10. Then, up to cyclic conjugation, u and v are *conjugate* via a word w on a and d (the word read along the dotted line in the figure). The claim then follows from Lemma 5.14 (applied to the equality $uw = wv$).

□

The final bound we need for Theorem 1.2 is:

Proposition 5.17. $2^{n^2} \leq \mathrm{Ann}_{G_4}(n)$.

Proof. Suppose $n \in \mathbb{N}$. Let $u = b^{-1}c^{n^2}s$, $v = b^{-1}s$, and $w = a^{-n^2}d^{n^2}$. We claim that in G_4,

$$uw = b^{-1}c^{n^2}s\,a^{-n^2}d^{n^2} = b^{-1}c^{n^2}a^{-n^2}d^{n^2}s = a^{-n^2}b^{-1}d^{n^2}s = a^{-n^2}d^{n^2}b^{-1}s = wv,$$

and so u and v are conjugate in G_4. The second of these equalities holds because $s^a = asa^{-1} = s^2$ and $s^d = dsd^{-1} = s^2$ in G_4 imply that for all $k \in \mathbb{N}$ we have $sa^{-k} = a^{-k}s^{2^k}$ and $d^k s = s^{2^k} d^k$, and therefore s commutes with $a^{-k}d^k$. The third uses $[a,b] = aba^{-1}b^{-1} = c$ and $[a,c] = [b,c] = 1$, from which it follows that $a^{-1}b^{-1} = b^{-1}ca^{-1}$ and then that $a^{-k}b^{-1} = b^{-1}c^k a^{-k}$ for all $k \in \mathbb{Z}$. The fourth uses $[b,d] = 1$.

Also, $|u|_{G_4} \le n$ and

$$\mathrm{Area}(u\,(b^{-1}[a^n,b^n]s)^{-1}) \ \le\ \mathrm{Area}(b^{-1}c^{n^2}\,[a^n,b^n]^{-1}b) \ \le\ n^3,$$

because $c^{n^2} = [a^n,b^n]$ in G_4 as a consequence of apply defining relations $[a,b] = c$ and $[a,c] = 1$ and $[b,c] = 1$ at most n^3 times. So, by Lemma 2.5, it suffices to show that $2^{n^2-1} \le \mathrm{Ann}(u,v)$.

Well, suppose Ω is any annular diagram for $u \sim v$. A b-corridor β connects the edge labelled by the b^{-1} in the u-boundary component to the b^{-1} in the v-boundary component. Let w be the word on a, c, and d read along one side of that b-corridor, so that cutting the diagram along that side gives a van Kampen diagram Δ for $uwv^{-1}w^{-1}$—see Figure 11.

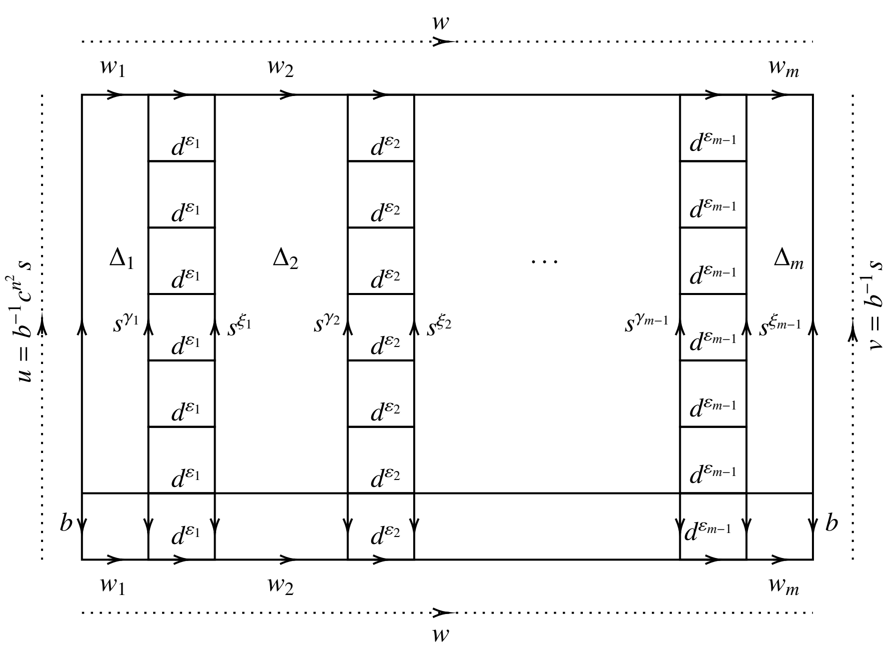

FIGURE 11. A van Kampen diagram Δ for $uwv^{-1}w^{-1}$ per Proposition 5.17.

Killing d and s maps G_4 onto the Heisenberg group $H = \langle a,b,c \mid [a,c],[b,c],[a,b]c^{-1}\rangle$ and sends $w \mapsto a^\alpha c^\gamma$, where $\alpha = \exp_a(w)$ and $\gamma = \exp_c(w)$. Thereby, $uw = wv$ in G_4 implies that $b^{-1}c^{n^2}a^\alpha c^\gamma = a^\alpha c^\gamma b^{-1}$ in H, and so $\alpha = -n^2$, because $\langle c\rangle \cong \mathbb{Z}$ is central and $a^\alpha b^{-1} = b^{-1}a^\alpha c^{-\alpha}$ in H.

Killing b and c maps G_4 on $B = \langle a,d,s \mid s^a s^{-2}, s^d s^{-2}\rangle$ and deleting these letters takes w to the word $\overline{w}$, which represents an element of the free subgroup $F(a,d)$ of B. The equality $uw = wv$ in G_4 implies that $s\overline{w} = \overline{w}s$ in B. Then on mapping B to $\langle a,s \mid s^a s^{-2}\rangle$ by sending

$d \mapsto a$, we learn that $\exp(\overline{w}) = 0$, and so $\exp_d(w) = -\exp_a(w)$, which we previously evaluated to be n^2.

Lemma 5.5 allows us to assume that Ω is reduced and contains no contractible b- or d-rings. And then because there are no d-edges in $\partial\Omega$, any d-corridor δ in Ω must close up as a non-contractible d-ring. We claim that δ can have at most one 2-cell in common with β. Otherwise there would (by an *innermost* argument) be a d-corridor that crosses β twice so as to enclose a disc-subdiagram whose boundary is labelled by a word σs^{ν} (where $\nu \in \mathbb{Z}$) such that $\sigma = \sigma(a, c)$ follows part of one side of β and s^{ν} follows part of one side of a d-corridor. But there can be no b-edges or d-edges in this subdiagram, so $\sigma s^{\nu} = 1$ in $\langle a, c, s \mid [a, c], s^a s^{-2}\rangle$. Mapping to $\langle a, s \mid s^a s^{-2}\rangle$ by killing c, we learn that $\nu = 0$, and there are adjacent 2-cells, both labelled by $[b, d]$, where the b- and d-corridors cross, contrary to Ω being reduced.

So the d-corridors in Ω form a family of nested non-contractible d-rings, one for each $d^{\pm 1}$ in w. Therefore Δ is as shown in Figure 11: $w = w_1 d^{\epsilon_1} w_2 d^{\epsilon_2} \cdots d^{\epsilon_{m-1}} w_m$, where each $\epsilon_i = \pm 1$, $\nu_m = 1$, and, for $i = 1, \ldots, m-1$, $w_i = w_i(a, c)$, the words along the sides of the d-corridors are $b^{-1} s^{\nu_i}$ and $b^{-1} s^{\xi_i}$, and $\nu_i = 2^{\epsilon_i} \xi_i$, and Δ consists of one b-corridor and the d-corridors, and, in between, subdiagrams $\Delta_1, \ldots, \Delta_m$ over $\langle a, c, s \mid [a, c], s^a s^{-2}\rangle$.

For $i = 2, \ldots, m$, the boundary of the sub-diagram Δ_i is labelled by

$$\kappa_i \;=\; s^{\xi_{i-1}}\, w_i(a, c)\, s^{-\nu_i}\, w_i(c^{-1} a, c)^{-1}.$$

Killing s maps $\langle a, c, s \mid [a, c], s^a s^{-2}\rangle \twoheadrightarrow \mathbb{Z}^2 = \langle a, c\rangle$, so $\exp_c(\kappa_i) = 0$, and therefore $\exp_a(w_i) = 0$. Killing c maps $\langle a, c, s \mid [a, c], s^a s^{-2}\rangle \twoheadrightarrow \langle a, s \mid s^a s^{-2}\rangle$ and $w_i \mapsto 1$, and so $s^{\xi_{i-1}} = s^{\nu_i}$ in $\langle a, s \mid s^a s^{-2}\rangle$, which implies that $\xi_{i-1} = \nu_i$.

So $\nu_i = 2^{\epsilon_i} \nu_{i+1}$ for $i = 1, \ldots, m-1$, and because $\exp_d(w) = \epsilon_1 + \cdots + \epsilon_{m-1} = n^2$ and $\nu_m = 1$, we deduce that $\nu_j = 2^{n^2}$ for some j. A count of the 2-cells comprising the d-corridor along one side of which we read s^{ν_j} gives $\mathrm{Area}(\Delta) \geq 2^{n^2-1}$, and therefore $\mathrm{Ann}(u, v) \geq 2^{n^2-1}$, as required. □

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