STRONG SHORTCUTS, GENERATING SETS, AND ISOMETRIC CIRCLES IN ASYMPTOTIC CONES

NIMA HODA AND TIMOTHY RILEY

Abstract. We show that whether loops can be shortcut in a group's Cayley graph depends on the choice of finite generating set. Our example is the direct product of two rank-2 free groups and a consequence is that this group has asymptotic cones with isometrically embedded circles that are null-homotopic.

1. Introduction

A metric space is *strongly shortcut* [Hod22, Hod24] when any sufficiently long loop can be subdivided into two loops that are shorter by some constant multiplicative factor. In more detail, suppose $\rho : [0,1] \to X$ with $\rho(0) = \rho(1)$ is a loop in a metric space X. A path μ in X from $\rho(p)$ to $\rho(q)$, where $0 \leq p \leq q \leq 1$, subdivides ρ into two loops: the concatenation ρ_1 of the reverse of $\rho|_{[p,q]}$ with μ, and the concatenation ρ_2 of $\rho|_{[q,1]\cup[0,p]}$ with μ. We say X is *strongly shortcut* when there exists $L \geq 0$ and $\lambda < 1$ such that every loop $\rho : [0,1] \to X$ of length $\ell \geq L$ can be *subdivided* into two loops ρ_1 and ρ_2 each of length at most $\lambda\ell$. If X is a graph, it is strongly shortcut if and only if there exists $L \geq 0$ and $\lambda < 1$ such that every combinatorial loop in X can be shortcut into two combinatorial loops in the above manner [Hod24, Theorem B]. The strong shortcut property is the instance of Gromov's loop subdivision property [Gro93, §5.F] where the loop is required to be subdivided into precisely two sub-loops.

A *word* w on an alphabet A is a string $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ where each $a_i \in A$ and each $\varepsilon_i \in \{\pm 1\}$. We write $|w| = n$. A *cyclic conjugate* of w is a word $a_{i+1}^{\varepsilon_{i+1}} \cdots a_n^{\varepsilon_n} a_1^{\varepsilon_1} \cdots a_i^{\varepsilon_i}$ for some $i \in \{1, \ldots, n\}$.

A finitely generated group G is *strongly shortcut* when, for some choice of finite generating set A, its Cayley graph $\mathrm{Cay}(G, A)$ is strongly shortcut; or, equivalently, when G, acts properly and cocompactly on a strongly shortcut graph [Hod22]. For a Cayley graph $\mathrm{Cay}(G, A)$, the strong shortcut property amounts to: there exists $L > 0$ and $\lambda < 1$ such that if w is a word on A that represents the identity and $|w| \geq L$, then w has a cyclic conjugate expressible as a concatenation $w_1 w_2$ of words w_1 and w_2 such that $(1 - \lambda)|w| \leq |w_1| \leq |w_2|$ and for which there exists a word μ that equals w_1 in G and satisfies $|\mu| \leq \lambda |w_1|$.

The purpose of this note is to prove that the choice of finite generating set can matter, answering a question [CHW22, Section 9, Qu. 3] of Cashen, Hoda, and Woodhouse. We prove:

Theorem 1.1. *The Cayley graph* $\mathrm{Cay}(G, A)$ *of the product*

$$G = F_2 \times F_2 = F(a, b) \times F(c, d)$$

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of two rank-2 free groups is strongly shortcut when $A = \{a, b, c, d\}$ *but not when* $A = \{a, b, c, db^{-1}\}$*.*

In [Hod22] Hoda exhibited a group (namely, $\langle a, t \mid t^{-1}at = a^2 \rangle$) such that for one finite generating set (namely, $\{a, t\}$) there is an upper bound on the lengths of isometrically embedded cycles in the Cayley graph (the *shortcut property*), but for another finite generating set (namely, $\{a, t, t^2\}$) there is no such bound. Our examples of Theorem 1.1 strengthen this in that for $A = \{a, b, c, d\}$, there exists $\lambda < 1$ such that every combinatorial loop in $\mathrm{Cay}(G, A)$ of length $\ell > 4$ can be *strongly* shortcut into a pair of loops, each of length at most $\lambda\ell$, but if $A = \{a, b, c, db^{-1}\}$ then there is no upper bound on the lengths of isometrically embedded cycles.

For $A = \{a, b, c, d\}$, every asymptotic cone of $\mathrm{Cay}(G, A)$ is a product of a pair of $\mathbb{R}$-trees, this product having the ℓ_1-metric, and so these cones contain no isometrically embedded circles—loops can be shortened within one factor. The asymptotic cones of a group defined with respect to different finite generating sets are all biLipschitz equivalent, so all asymptotic cones of $G = F_2 \times F_2$ are contractible. Nevertheless, via [Hod24, Theorem A], Theorem 1.1 implies that asymptotic cones of G with $A = \{a, b, c, db^{-1}\}$ have isometrically embedded circles:

Corollary 1.2. *$F_2 \times F_2$ has asymptotic cones with isometrically embedded circles that are homotopically trivial.*

By contrast, it follows from [Cre22, Lemma 1.7] and [Hod23, Proposition 4.4] that there are no isometrically embedded circles in the asymptotic cones of finitely generated abelian groups; there are also none in the asymptotic cones of hyperbolic groups because they are $\mathbb{R}$-trees. Corollary 1.2 shows that relaxing the abelian or hyperbolicity hypothesis in a most elementary manner can give groups with asymptotic cones that have isometrically embedded circles, despite all being simply connected. Cashen, Hoda, and Woodhouse [CHW22] were the first to establish the existence of groups with this property, namely snowflake groups of Brady and Bridson. Corollary 1.2 provides a more elementary example and answers in the affirmative a question they asked [CHW22, Section 9, Qu. 6] as to whether a group with quadratic isoperimetric function can exhibit this behaviour.

2. Proof

We have $G = F(a, b) \times F(c, d)$. That $\mathrm{Cay}(G, \{a, b, c, d\})$ is strongly shortcut is an instance of the part of [Hod22, Theorem C] which states that the 1-skeleton of a finite dimensional CAT(0) cube complex has this property. However, as this is an elementary example, it merits an elementary proof. The following lemma details how loops in $\mathrm{Cay}(F(a, b), \{a, b\})$ admit shortcuts, and afterwards we will explain how to promote this to the product. (The words $u = a^m a^{-m} a^{-m} a^m b^m b^{-m}$ demonstrate the inequality the lemma presents to be sharp.)

Lemma 2.1. *If a word u represents 1 in $F(a, b)$, then it has a cyclic conjugate that can be expressed as a concatenation u_1u_2 of words u_1 and u_2 that both represent 1 in $F(a, b)$ and*

$$\min\{|u_1|, |u_2|\} \geq \lfloor |u|/3 \rfloor.$$

Proof. Given a finite tree T with N vertices, there is a vertex $*$ so that removing $*$ and the incident edges subdivides T into subtrees $T_1, \ldots, T_k$ each with at most $N/2$ vertices. This is known as a *centroid decomposition.* So removing $*$ from T and taking the closures $\widehat{T}_1, \ldots \widehat{T}_k$ of the resulting connected components gives subtrees so that $\widehat{T}_i$ is T_i with one additional edge, $\widehat{T}_i$ has at most $(N/2)+1$ vertices, and there are vertices $*_i$ in $\widehat{T}_i$ so that identifying $*_1, \ldots, *_k$ reconstitutes T.

Given a word u that represents 1 in $F(a, b)$, there is a finite planar tree T with each edge directed and labelled by a or b so that on reads u around T from some base vertex. The number of edges in T is $|u|/2$, and so the number of vertices is $N = (|u|/2)+1$. By centroid decomposition, there exists a cyclic conjugate $w_1 \cdots w_k$ of u so that for all i, w_i represents 1 in $F(a, b)$ and $(|w_i|/2) + 1 \leq (N/2) + 1 = (((|u|/2)+1)/2)+1$. (The point is that w_i is read around $\widehat{T}_i$, which has $(|w_i|/2)+1$ vertices.) That is, $|w_i| \leq (|u|/2) + 1$.

If $k = 2$, then let $u_1 = w_1$ and $u_2 = w_2$, and the result is straight-forward.

Assuming then that $k \geq 3$, we have either $|w_1 w_2| \leq |u|/2$ or $|w_3 \cdots w_k| \leq |u|/2$. In the former case coalesce $w_1 w_2$ and repeat. In the latter case redefine w_3 as $w_3 \cdots w_k$. The result is a cyclic conjugate $w_1 w_2 w_3$ of u such that $|w_i| \leq (|u|/2)+1$ and $w_i = 1$ in $F(a, b)$ for $i = 1, 2, 3$.

Take u_1 be the longest of w_1, w_2, and w_3 and u_2 to be the concatenation of the other two in the appropriate order so that $u_1 u_2$ is a cyclic conjugate of u. Then $u_1 = u_2 = 1$ in $F(a, b)$. Further, $|u_1| \geq |u|/3$ because it is the maximum of three natural numbers that sum to $|u|$. And $|u_2| = |u| - |u_1| \geq |u| - ((|u|/2) + 1) = (|u|/2) - 1 \geq \lfloor |u|/3 \rfloor$. (The case $|u| = 3$, where the final inequality fails, does not arise because $u = 1$ in $F(a, b)$ implies that $|u|$ is even.) $\square$

Suppose w is a word on a, b, c, d which represents 1 in G. Without loss of generality, we may assume that the total number of $a^{\pm 1}$ and $b^{\pm 1}$ letters in w is at least the number of $c^{\pm 1}$ and $d^{\pm 1}$ letters. So the word u obtained from w by deleting all $c^{\pm 1}$ and $d^{\pm 1}$ letters satisfies $|u| \geq |w|/2$. Now u represents 1 in $F(a, b)$ and, per Lemma 2.1 applied to u, we have $|u_1|, |u_2| \geq \lfloor |w|/6 \rfloor$. So for some $i \in \{1, 2\}$, some cyclic conjugate of w has a subword w_1 of length at most $|w|/2$ that contains all the letters of u_i. Deleting the letters of this u_i from w_1 gives a word that equals w_1 in G and shortcuts w so as to establish the strong shortcut property for $\mathrm{Cay}(G, \{a, b, c, d\})$.

We now turn to showing that for $t = db^{-1}$, the Cayley graph $\Gamma = \mathrm{Cay}(G, \{a, b, c, t\})$ is not strongly shortcut. With this generating set, G has the presentation

$$G = \langle a, b, c, t \mid [a, c],\ [a, tb],\ [b, c],\ [b, t] \rangle. \tag{1}$$

For all $n \in \mathbb{N}$, the length-$(4n + 4)$ word

$$w_n = [t^n c t^{-n}, a] = t^n c t^{-n} a t^n c^{-1} t^{-n} a^{-1}, \tag{2}$$

represents 1 in G, as can be seen per the van Kampen diagram in Figure 1 (illustrating the case $n = 5$), or by observing that $t^n c t^{-n} = (db^{-1})^n c (db^{-1})^{-n} = d^n c d^{-n}$, which commutes with a in G.

We will prove that for all $n \geq 0$, the word w_n defines an isometrically embedded cycle in Γ, and so cannot be shortcut. The "ziggurat" nature of the van Kampen diagram in Figure 1 gives it the property that every edge-path connecting a pair of antipodal vertices on the diagram's perimeter has length (meaning the number of edges comprising the edge-path) greater than or equal to the length of either edge-path around the perimeter connecting them. This is a necessary condition for w_n to define an isometrically embedded cycle in Γ and is what led us to Theorem 1.1.

To prove w_n defines an isometrically embedded cycle in Γ we will show that all words of length $2n+2$ that are subwords of cyclic conjugates of w_n are geodesic. The words in question are for $k \in \{0, 1, 2, \ldots, n\}$,

$$u_k = t^{n-k}ct^{-n}at^k, \qquad u'_k = t^{-(n-k)}at^nc^{-1}t^{-k},$$
$$u''_k = t^{n-k}c^{-1}t^{-n}a^{-1}t^k, \qquad u'''_k = t^{-(n-k)}a^{-1}t^nct^{-k}.$$

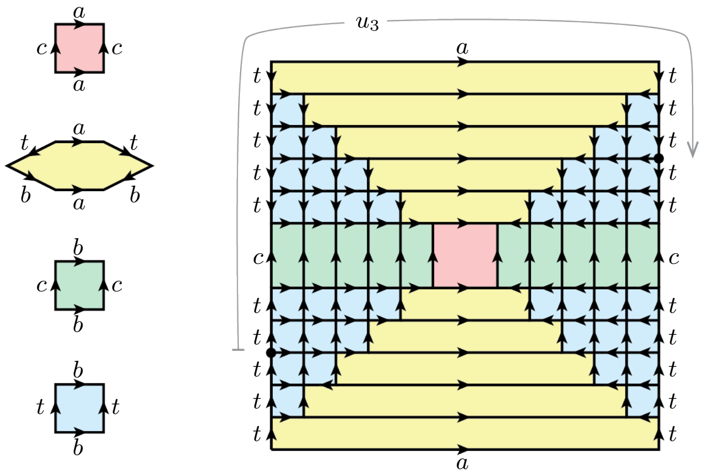

Figure 1. Left: the four defining relators of the presentation (1) for G. Right: a "ziggurat" van Kampen diagram for the word $w_5 = t^5ct^{-5}at^5c^{-1}t^{-5}a^{-1}$.

Claim 2.2. *It suffices to show that u_k is a geodesic word for all $k \in \{0, 1, 2, \ldots, n\}$.*

Proof. Let ϕ be the automorphism of G that sends a to a^{-1} and fixes b, c and t. Let ψ be the automorphism of G that sends c to c^{-1} and fixes a, b and t. Both induce isomorphisms of Γ. Now, $u'_{n-k} = \phi(u_k^{-1})$, $u''_k = \phi(\psi(u_k))$, and $u'''_{n-k} = \psi(u_k^{-1})$, and so if u_k is a geodesic word, then so are u'_{n-k}, u''_k, and u'''_{n-k}. $\square$

Let $D \subset \mathbb{R}^2$ be a van Kampen diagram with respect to the presentation in (1). Take the union of all open edges labeled a and open faces having a-labeled edges on their boundary. An *a-corridor* of D is a connected component of this union. We similarly define *c-corridors*. For example, there is an a-corridor in Figure 1 running vertically through the diagram and a c-corridor running horizontally.

Each defining relator in the presentation in (1) either has no appearance of a or has exactly two appearances: one of each sign. Moreover, each relator in which a appears is a commutator of the form $[a, y]$ where $y = tb$ or $y = c$. So any a-corridor consists of a chain of edges and faces. Moreover, the chain is either linear in which case the corridor is an open disk with its boundary word having the form $[a, y_1 \cdots y_m]$ or the chain is cyclic in which case the corridor is an open annulus with its two boundary words equal and having the form $y_1 \cdots y_m$, where each y_i is either $(tb)^{\pm 1}$ or $c^{\pm 1}$. The same statements hold for c-corridors with each y_i instead being equal to either $a^{\pm 1}$ or $b^{\pm 1}$. For example, the a- and c-corridors in Figure 1 have boundary words $[a, (tb)^5c(tb)^{-5}]$ and $[c, b^5ab^{-5}]$, respectively.

Lemma 2.3. *The subgroup $\langle b, t\rangle \leq G$ is an isometrically embedded copy of $\mathbb{Z}^2 = \langle\, b, t \mid [b, t]\,\rangle$.*

Proof. Killing a and c retracts G onto $\langle\, b, t \mid [b, t]\,\rangle$. □

Lemma 2.4. *Suppose D is a minimal area van Kampen diagram for the presentation in* (1)*.*

(a) *If x is the boundary word of an a- or c-corridor of D, then x is cyclically reduced or $|x| = 2$.*
(b) *There are no annular a- or c-corridors in D.*
(c) *The intersection of an a-corridor and an c-corridor in D is either empty or equal to an open face with boundary word $[a, c]$.*

Proof. (a) A cancellable pair of letters in x with $|x| > 2$ would correspond to a cancellable pair of faces in the corridor, contradicting minimality of D.

(b) The two boundary words of an annular a- or c-corridor of D would be equal. Thus we could obtain a new disk diagram of lesser area by deleting the corridor and glueing together these two boundary components along a label and orientation preserving isomorphism.

(c) Since the only defining relator in (1) containing both a and c letters is $[a, c]$, the intersection of an a-corridor with an c-corridor is a disjoint union of open faces each of which has boundary word $[a, c]$. If there is more than one open face in this intersection then there is a subdiagram R of D that is bounded by the union of the two corridors. The boundary word of an innermost such R has no a- or c-letters and so, by (b), has the form $(tb)^{\ell}b^{m}$. But this word represents the identity in G, so by lemma 2.3, we have $\ell = m = 0$. That is, the region R consists of a single vertex of D, contained in a cancellable pair of faces each having boundary word $[a, c]$. This contradicts minimality of D. □

Claim 2.5. *If v is a geodesic word representing the same element as u_k in G, then a and c each appear exactly once in v and a^{-1} and c^{-1} do not appear in v.*

Proof. Let D be a minimal area van Kampen diagram for the word $u_k v^{-1}$, which represents 1 in G. Since a^{-1} does not appear in u_k, the a-corridor of D starting at the a in u_k must terminate at some a in v. Thus a appears at least once in v. Any other occurrence of $a^{\pm 1}$ in v would correspond to a distinct a-corridor. Since u_k has no other occurrence of $a^{\pm 1}$, this corridor must both start and terminate in v. Since the corridor has boundary word of the form $[a, y]$ we can obtain a new disk diagram by deleting the corridor and gluing the y and y^{-1} parts of the boundary together. The new disk diagram has boundary word $u_k(v')^{-1}$ for the word v' obtained from v by deleting the $a^{\pm 1}$ and $a^{\mp 1}$ of the deleted corridor, contrary to geodicity of v.

The proof for c is the same. □

We can now prove that u_k is a geodesic word for all $k \in \{0, 1, 2, \ldots, n\}$. Suppose v is a geodesic word representing u_k in G. We will first consider the case where c occurs before a in v. Let D be a minimal area van Kampen diagram for $u_k v^{-1}$. By Lemma 2.4 and Claim 2.5, there is exactly one corridor of each type (a or c) in D. These two corridors do not cross: were they to cross, they would do so at least twice, but by Lemma 2.4(c) that does not happen. Thus, by Lemma 2.4(a), the a-corridor has boundary word $[a, (tb)^m]$, for some m, and the c-corridor has boundary word $[c, b^{\ell}]$, for some ℓ. Without loss of generality, a path along the boundary of the a-corridor spelling out $(tb)^m$ starts on u_k in ∂D and ends on v and the same holds for a path along the boundary of the c-corridor spelling out b^ℓ. See Figure 2.

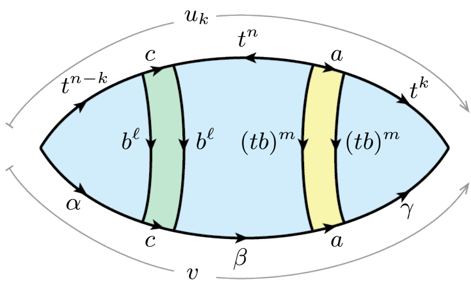

FIGURE 2. The van Kampen diagram D for the word $u_k v^{-1}$.

Let α, β, and γ be the subwords such that the word v is the concatenation $\alpha c \beta a \gamma$. Then $\alpha = t^{n-k} b^\ell$, $\beta = b^{-\ell} t^{-n} (tb)^m$ and $\gamma = (tb)^{-m} t^k$ in G. By Lemma 2.3 these are equal in G to $b^\ell t^{n-k}$, $b^{m-\ell} t^{m-n}$ and $b^{-m} t^{k-m}$, respectively. Since these are geodesic representatives in $\langle b, t \mid [b,t] \rangle$, and therefore (by Lemma 2.3) in G, it follows that

$$\begin{aligned} |u_k| \geq |v| &\geq 2 + (|\ell| + n - k) + (|m - \ell| + |m - n|) + (|m| + |k - m|) \\ &= 2 + n - k + |n - m| + |m - \ell| + |\ell| + |k - m| + |m| \\ &\geq 2 + n - k + (n - m) + (m - \ell) + \ell + (k - m) + m \\ &= 2 + 2n, \end{aligned}$$

so that $|u_k| = |v|$ and u_k is geodesic.

The other case is where a occurs before c in v. The word $u_k u_k''$ is a cyclic conjugate of w_n, and so $(u_k'')^{-1} = u_k$ in G. So $(u_k'')^{-1} = t^{-k} a t^n c t^{-(n-k)}$ is a length-$(2n+2)$ word which equals v in G. Comparing them using the same argument as the previous case establishes that $|u_k| = 2n + 2$. This completes our proof.

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853, USA
Email address: `nima@nimahoda.net`

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853, USA
Email address: `tim.riley@math.cornell.edu`