

## GEOMETRY PROBLEMS

*Things to remember:*

- Vector geometry and dot product.
- Complex numbers for planar problems.
- Isoperimetric inequality: If  $C$  is a nonintersecting closed curve of length  $L$  containing area  $A$ , then  $L^2 \geq 4\pi A$  with equality only if  $C$  is a circle.
- High school plane geometry or calculus may be useful.

**Examples:**

- (1) (1998 A1) A right circular cone has a base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
- (2) Given an acute triangle  $ABC$ , let  $P$  be the point that minimizes the sum of distances to the vertices of the triangle. Show that every edge of the triangle appears as a visual angle 120 degrees from  $P$ .

**More Problems:**

- (1) (1998 A2) Let  $s$  be any arc of the unit circle lying entirely in the first quadrant. Let  $A$  be the area of the region lying below  $s$  and above the  $x$ -axis and let  $B$  be the area of the region lying to the right of the  $y$ -axis and to the left of  $s$ . Prove that  $A + B$  depends only on the arc length, and not on the position, of  $s$ .
- (2) (1999 B1) The right triangle  $ABC$  has right angle at  $C$  and  $\angle BAC = \theta$ ; the point  $D$  is chosen on  $AB$  so that  $|AB| = |AD| = 1$ ; the point  $E$  is chosen on  $BC$  so that  $\angle CDE = \theta$ .

The perpendicular to  $BC$  at  $E$  meets  $AB$  at  $F$ . Evaluate

$$\lim_{\theta \rightarrow 0} |EF|.$$

- (3) (1998 B3) Let  $H$  be the unit hemisphere  $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ ,  $C$  the unit circle  $\{(x, y, 0) : x^2 + y^2 = 1\}$ , and  $P$  the regular pentagon inscribed in  $C$ . Determine the surface area of that portion of  $H$  lying over the planar region inside  $P$ , and write your answer in the form  $A \sin \alpha + B \cos \beta$ , where  $A$ ,  $B$ ,  $\alpha$ ,  $\beta$  are real numbers.
- (4) (Napoleon) Given a triangle, erect equilateral triangles on all its edges. Show that the centers of the three equilateral triangles form themselves the vertices of an equilateral triangle.
- (5) Suppose a non-empty compact subset of the plane contains, for any pair of points, a connecting semicircle. Show that the subset is a disc.
- (6) Fix two points  $A$  and  $B$  in the plane. Describe the set of all points that are obtained by reflecting  $A$  about lines through  $B$ .
- (7) (Sylvester) Let  $M$  be a finite set of points in the plane such that for any two of them, there is a third point in  $M$  on the same line. Show that  $M$  is contained in a single straight line.
- (8) Inscribe a rectangle of base  $b$  and height  $h$  in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of  $h$  do the rectangle and triangle have the same area?