

IMPOSSIBILITY, EXISTENCE, AND GAMES

Some questions ask you to show that something is impossible or that something does not exist. Your first approach to such problems should be pigeonhole principle, then parity or other modular arithmetic. Another approach is to show something is optimal (i.e. you can't do better). Here we introduce another idea and some more interesting mathematics.

Examples:

- A standard 8×8 chessboard can be covered (tiled) by 32 nonoverlapping 2×1 dominoes. If we remove diagonally opposite corner squares from the chessboard, can the remaining board be covered by 31 dominoes?
- The game of Chomp is played by two players as follows. An $n \times m$ rectangular chocolate bar has a poison square in the bottom left corner. Each player takes turns eating one square of the bar along with every remaining square above and to the right of that square. If the chocolate bar is $2 \times n$, is there a sure winning strategy for either player (first player or second player)?
- A line segment is partitioned into a finite number of line segments. Each vertex in the partition is colored either red or blue, with one of the endpoints of the original line segment being colored red and the other colored blue. Show that the number of small "red-blue" line segments in the configuration is odd.
- Show that $\sqrt{2}$ is irrational (not a fraction).

PROBLEMS

- (1) Cucumbers are 99% water by weight. If 500 pounds of cucumbers are left on the sidewalk overnight, and the partially evaporated mass that remains in the morning is 98% water by weight, how much does this mass of cucumber weigh?
- (2) If we remove one corner square from an 8×8 chessboard, can the remaining 63 squares be tiled by 21 3×1 dominoes?
- (3) What squares on an 8×8 chessboard can be removed so that the remaining 63 squares can be tiled by 21 3×1 dominoes?
- (4) What about using three-square L -shaped dominoes?
- (5) If the game of Chomp is played on a square $n \times n$ chocolate bar, is there a sure winning strategy for either player?

- (6) If Chomp is played on a $2 \times \infty$ bar, i.e. two rows of squares stretching forever to the right, is there a sure winning strategy for either player?
- (7) If Chomp is played on a finite $n \times m$ bar, is there a sure winning strategy for either player?
- (8) The Divisor Game is played as follows: two players pick an integer N . The first player names a divisor of N . Then the second player names a divisor of N which is not also a divisor of the number named by the first player. They then take turns, each time naming a divisor of N which does not divide any previously named number. The player who is forced to name N on his move loses. Is there a sure winning strategy for either player?
- (9) A large rectangle is partitioned into smaller rectangles, each of which has at least one side whose length is an integer. Show that the original large rectangle has a side which is an integer.
- (10) Is it possible to partition a three-dimensional cube into a finite number of subcubes of all different sizes?
- (11) (Sperner's Lemma) A triangle is partitioned into a finite number of subtriangles so that adjacent subtriangles share an entire edge. Each vertex of the triangulation is colored red, blue, or yellow with the following conditions: The three vertices of the original large triangle are labeled red, blue, and yellow. The vertices on any edge of the large triangle are colored one of the two colors given to the main vertices of that edge. Show that there is at least one subtriangle whose vertices are colored three different colors.
- (12) Show that the n th root of 2 is irrational, for $n \geq 2$ an integer.
- (13) Are there irrational number a and b such that a^b is rational?
- (14) Suppose a 4×4 matrix A has entries that are all either 1 or -1 . Show that the determinant of A is divisible by 8.
- (15) Show that there aren't any numbers a, b, c such that $a^2 + b^2 - 8c = 6$.
- (16) Show that there aren't any numbers x, y, z, w such that $x^2 + y^2 + z^2 = 8w + 7$.
- (17) (IMO 1986 1) Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.