Induction

Induction is the tool we use when we want to prove that some property is valid for ALL numbers. It is a mistake to try to prove that 1 has the property, and 2 has the property, and 3 has the ... we would never finish. Instead we prove 2 things:

> a) The number 1 has the property.
> b) IF the number k has the property, then the number $k+1$ has the property.

Let's see how point b) reads when substituting k with 1: "If 1 has the property, then $1+1=2$ has the property." Since we proved that 1 does have the property, point b) shows that 2 also has the property.

Now we can substitute k with 2 in point b) "If 2 has the property, then $2+1=3$ has the property." Since we just proved a moment before that 2 indeed has the property, it follows that 3 will too. This process repeats itself automatically.

As soon as we prove a) and b), we have a machine that shows (one step at a time) that every number has the desired property.

The best way to understand induction is with an example. We will prove a basic fact that is useful all on its own.

For any natural number n, the following equality holds:

$$(*) \qquad 1+2+\ldots+n = \frac{n\cdot(n+1)}{2}.$$

First, let us verify that this is true in some cases.

$$1=\frac{1\cdot 2}{2}\ ,\ 1+2=3=\frac{2\cdot 3}{2}\ ,\ 1+2+3=6=\frac{3\cdot 4}{2}.$$

This helps us to see any patterns that may appear. We also established point a); the number 1 does satisfy property $(*)$. Now we prove point b).

SUPPOSE that $(*)$ is true for k. Then

$$1+2+\ldots+(k+1)=\big(1+2+\ldots+k\big)+(k+1)=\frac{k\cdot(k+1)}{2}+(k+1)=$$
$$(k+1)\left(\frac{k}{2}+1\right)=(k+1)\left(\frac{k+2}{2}\right)=\frac{(k+1)(k+2)}{2}.$$

That is, $(*)$ is also valid for $k+1$.

Some things to remember:

- You must prove both a) and b) for the induction to be complete.
- You may need to start at a number r other than 1. Then the induction proves that the property is valid for all numbers greater than r.
- Sometimes you will need *strong induction*. This means that instead of point b) you prove (see problem 5 for an example)
  b') If all numbers from 1 to k have the property, then $k+1$ has the property.

Problems ⟶

1. Prove that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all n.

2. The Fibonacci numbers are defined by the recurrence $F_1 = 1$, $F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 1$. Show that for every $n \geq 1$ they satisfy
$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

3. Show that all numbers in the sequence 1007, 10017, 100117, 1001117, 10011117,... are divisible by 53.

4. Prove that for $n \geq 6$, a square can be dissected into n smaller squares, not necessarily all of the same size.

5. Show that every positive integer can be written as the sum of distinct Fibonacci numbers.