MAX/MIN PROBLEMS AND INEQUALITIES

Finding the maximum or minimum of a function: On the Putnam, it's important that your justification of a max or min is rigorous. For instance, it is not enough to solve the equation $f'(x) = 0$ to find a maximum of a function f. Here are some techniques:

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then the maximum value of f is $f(x_0)$.
- Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f achieves both its maximum and minimum on $[a, b]$. These must occur at points x_0 where either $f'(x_0) = 0$, $f'(x_0)$ does not exist, or $x_0 = a$ or b.
- You may be able to show directly for instance that $f(x) \leq M$ for some M, and that $f(x_0) = M$, in which case M is the maximum.

Inequalities: Proving inequalities can be a tough and confusing business. Recall some of the basic tools:

- If $a \geq b$ and $b \geq c$, then $a \geq c$, with equality if and only if $a = b = c$.
- If $a_1 \geq b_1$ and $a_2 \geq b_2$, then $a_1 + a_2 \geq b_1 + b_2$, with equality if and only if $a_1 = b_1$ and $a_2 = b_2$.
- If $a_1 \geq b_1 > 0$ and $a_2 \geq b_2 > 0$, then $a_1 a_2 \geq b_1 b_2$, with equality if and only if $a_1 = b_1$ and $a_2 = b_2$.
- If $a \geq b > 0$, then $\frac{1}{b} \geq \frac{1}{a}$.
- If $a \geq b > 0$ and $\alpha > 0$, then $a^\alpha \geq b^\alpha$.
- $a^2 \geq 0$.

Here are some classical inequalities:

- *Arithmetic-Geometric-Harmonic Mean Inequality:* If $a_1, a_2, \ldots, a_n$ are nonnegative, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n} \geq \left(\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n} \right)^{-1}.$$

- *Jensen's Inequality:* If f is a convex function, i.e. $f'' \geq 0$, then

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right).$$

- *Cauchy–Schwartz Inequality:*

$$(a_1^2 + \cdots a_n^2)(b_1^2 + \cdots b_n^2) \geq (a_1 b_1 + \cdots a_n b_n)^2.$$

  Or, more briefly, if $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then

$$|\vec{a}||\vec{b}| \geq \vec{a} \cdot \vec{b}.$$

- *Triangle Inequality:*

$$\sqrt{a_1^2 + \cdots a_n^2} + \sqrt{b_1^2 + \cdots b_n^2} \geq \sqrt{(a_1 + b_1)^2 + \cdots (a_n + b_n)^2}.$$

  Or, more briefly, if $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then

$$|\vec{a}| + |\vec{b}| \geq |\vec{a} + \vec{b}|.$$

- *Young's Inequality:* If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

- *Hölder's Inequality:* If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and $a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq (a_1^p + \cdots a_n^p)^{1/p} (b_1^q + \cdots b_n^q)^{1/q}.$$

Problems:

(1) Prove the second inequality in the AGH mean inequality, assuming the first.

(2) $x_1^2 + x_2^2 + \cdots + x_n^2 \geq (r_1 x_1 + \cdots r_n x_n)^2$ holds for all real $x_1, \ldots, x_n$ if and only if $r_1^2 + \cdots + r_n^2 \leq 1$.

(3) If in traveling from one town to another, Justin goes half the distance at speed s_1 and half at speed s_2. Kelly travels between the same towns and travels half the time at speed s_1 and half at speed s_2. Who gets to the second town faster?

(4) (1986 A-1) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$

(5) Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where the a_k are real numbers. Suppose that $f(x)$ satisfies $|f(x)| \leq |\sin x|$ for all real x. Show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

(6) Let $y_n = \arctan n$. Prove that for $n = 1, 2, \ldots$, $y_{n+1} - y_n < \frac{1}{n^2+n}$.

(7) (2003 A-2) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$\begin{aligned} &(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \\ &\leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}. \end{aligned}$$

(8) (1973 B-4)

(a) On $[0,1]$, let f have a continuous derivative satisfying $0 < f'(x) \leq 1$. Also suppose that $f(0) = 0$. Prove that

$$\left[\int_0^1 f(x)\,dx\right]^2 \geq \int_0^1 [f(x)]^3\,dx.$$

(b) Show an example in which equality occurs.

(9) (1977 B-5) Suppose that $a_1, a_2, \ldots, a_n$ are real $(n > 1)$ and

$$A + \sum_{i=1}^n a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^n a_i\right)^2.$$

Prove that $A < 2a_i a_j$ for $1 \leq i < j \leq n$.

(10) (1978 A-5) Let $0 < x_i < \pi$ for $i = 1, 2, \ldots, n$ and set $x = \frac{x_1+x_2+\cdots+x_n}{n}$. Prove that

$$\prod_{i=1}^n \frac{\sin x_i}{x_i} \leq \left(\frac{\sin x}{x}\right)^n.$$

(11) (1953 A-1) Show that

$$\frac{2}{3}n^{3/2} < \sum_{i=1}^n \sqrt{i} < \frac{2}{3}n^{3/2} + \frac{1}{2}$$

for all positive integers n.

(12) (1980 B-1) For which real numbers c is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real x?

(13) (1983 A-2) The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

(14) (1978 B-5) Find the largest A for which there exists a polynomial $P(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$ with real coefficients which satisfies $0 \leq P(x) \leq 1$ for $-1 \leq x \leq 1$.

Some Tricks:

- Use logarithm to change products to sums
- Differentiate or integrate and use the fundamental theorem of calculus
- Make choices, e.g. constants, for values in the classical inequalities
- Use Taylor series
- Look for symmetries