The Putnam Competition

What: The Putnam exam tests originality, technical competence and familiarity with undergraduate mathematics. Most questions require only the standard concepts of Calculus and linear algebra.

The competition usually takes place in early December and is open only to regularly enrolled undergraduates, in colleges and universities of the United States and Canada, who have not yet received a college degree.

Prizes: Competition is strictly individual. The 5 highest ranked students are designated "Putnam Fellows" and receive $2500. There are other cash prizes for the next 20 students. Additionally, each school can select 3 students to form that school's team. There are prizes awarded to the 5 top-ranked teams.

The Elizabeth Lowell Putnam Prize will be awarded periodically to a woman whose performance on the Competition has been deemed particularly meritorious. This prize would be in addition to any other prize she might otherwise win.

Exam format and Grading: The exam consists of two sessions in a single day. Six problems are given on both the morning and the afternoon sessions.

Each problem is graded on a basis of 0 to 10 points, for a maximum possible score of 120 points. *All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Some partial credit may be given, but only when a contestant has shown significant and substantial progress toward a solution.*

History: The table below shows Putnam fellows associated to Cornell (past students and current faculty) and top ranked Cornell teams.

STUDENTS	FACULTY	TEAMS
Jeremy L. Bem 1996 1994 Robert D. Kleinberg 1996 David J. Wright 1992 Leonard Evens 1954	Robert S. Strichartz 1962	1st 1954 1951 2nd 1995 1994 3rd 1957 1953 5th 1992 1960 1958

Meetings: Attendance is not mandatory. In fact, we encourage you to prepare at your own pace. However, we will give a lot of useful tips and discuss interesting problems every Wednesday at 4:00PM in room 532, Malott Hall.

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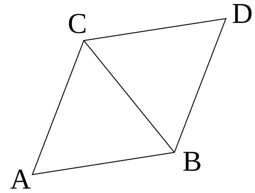
solutions to the poster problems ⟶

(1) Can you paint the plane with red, blue and green inks so that any pair of points 1 inch apart has 2 different colors?

A: NO. Suppose such coloring was possible. Then any equilateral triangle of side 1 in must have a different color on each vertex, so in any copy of



the vertices A and D have the same color. Now fix the vertex A and rotate the above figure around A. The vertex D will trace a circle $\mathcal{O}$ of radius $\sqrt{3}$ in, and all points on $\mathcal{O}$ have the same color as A. Since 1 in is less than the diameter of $\mathcal{O}$, we can easily find two points on the circle at distance 1 in. These two points have the same color.

(2) Let A be the sum of the digits of the number 4444^{4444}. Let B be the sum of the digits of A. What is the sum of the digits of B?

A: The answer is 7. Call C the sum of the digits of B. First we need to find an upper bound for C.

Clearly $4444^{4444} < 10000^{4444} = 10^{17776}$, so 4444^{4444} has 17776 or less digits, each smaller or equal to 9. That means that

$$A \leq 9 \cdot 17776 = 159984.$$

Similarly, A has 6 or less digits, so $B \leq 9 \cdot 6 = 54$. Finally, among all numbers from 1 to 54, the largest digit sum is attained by 39, with digit sum equal to 12. In particular, $C \leq 12$.

To finish the proof, we need to use *modular arithmetic*. This is a fun tool to simplify numerical computations and we will explain it in a problem session.

$$C \equiv B \equiv A \equiv 4444^{4444} \equiv 7^{4444} = 7^{6 \cdot 740+4} \equiv 7^4 = 2401 \equiv 7 \pmod{9}.$$

Since $C \leq 12$ and $C \equiv 7 \pmod{9}$, we deduce that C must be 7.

(3) How much is $\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$?

A: The answer is 2, but in this case we played a little trick to catch your attention, because we did not explain the exact meaning of $\sqrt{2+\dots}$ This will NEVER be the case in a Putnam problem where all questions are clearly stated.

In any case... The sensible interpretation is that we look at the limit of the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \text{etc.}$$

so we must show that this limit really does exist. We will do that in a problem session. Assuming that the limit exists, we show here how to compute its value.

Let $x = \sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$ Then we can square both sides to get

$$x^2 = 2 + \sqrt{2+\sqrt{2+\dots}} = 2 + x.$$

The only positive solution of $x^2 = 2 + x$ is $x = 2$.