Modular Arithmetic Solutions

1. If $n \geq 5$, the factorial $n!$ has a factor of 3 and a factor of 5 (at least). Then $5! + \ldots + 100!$ is divisible by 15 and can be canceled modulo 15. The rest of the sum is $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33 \equiv 3 \pmod{15}$.

2. The three consecutive numbers $p-1, p, p+1$ must be congruent to 0,1,2 modulo 3, although not necessarily in that order. Since p is prime, it must be either $p-1$ or $p+1$ that is congruent to 0 modulo 3. Similarily, $p \equiv 1$ or $3 \pmod{4}$, so one of $p-1$ and $p+1$ is divisible by 4 and the other is divisible by 2. Then $p^2 - 1 = (p-1)(p+1)$ has a factor of 3, a factor of 4 and a factor of 2; i.e., $p^2 - 1$ is divisible by 24.

3. Let $n = 24m$. We show first that if d is a divisor of $n - 1 = 24m - 1$, then $d^2 - 1$ is divisible by 24. Clearly d is not a multiple of 3 (because n is), so 3 must divide $d^2 - 1$. Also d must be odd, so $d - 1$ and $d + 1$ are consecutive even numbers, so one must be a multiple of 4, and their product $d^2 - 1$ must be a multiple of 8.

   Now $24m - 1$ cannot be a square (because squares are congruent to 0 or 1 modulo 4), so its divisors come in pairs: d, $(24m-1)/d$. But $d + (24m-1)/d$ is divisible by 24 (because $d^2 - 1$ and $24m$ are, but no factor of 24 can divide d). Hence the sum of all the divisors of $n - 1$ is divisible by 24.

4. The longest tail is 3 and the smallest square with tail 3 is $38^2 = 1444$.

   All squares end in 0, 1, 4, 9, 6, or 5. A square ending in 11, 99, 66, 55 would be congruent to 2 or 3 modulo 4, but squares are congruent to 0 or 1 modulo 4. So for length greater than 1 the square must end in 4.

   A square ending in 4444 would be congruent to 12 modulo 16, but squares are congruent to 0, 1, 4 or 9 modulo 16. So the maximum length is 2 or 3.

   When n^2 ends in 4, then n must end in 2 or 8. This gives two cases:

   - $n^2 = (100a + 10b + 2)^2 = 10000a^2 + 1000(2ab) + 100(4a + b^2) + 10(4b) + 4$. So if this ends in 44, then $b = 1$ or 6. If $b = 1$, then $4a + b^2$ is odd, so the square cannot end in 444. If $b = 6$, then the square is $1000k + (4a + 38)100 + 44$. This will end in 444 if we take $a = 4$ or 9. Thus the smallest numbers ending in 2 whose square ends in 444 are 462 (square 213444) and 962 (square 925444).
   - $n^2 = (100a + 10b + 8)^2 = 1000k + 100(16a^2 + b^2) + 10(16b + 6) + 4$. So if this ends in 44, then $b = 3$ or 8. Now, $(100a + 38)^2 = 1000k + 100(76a) + 1444$ ends in 444 if $a = 0$. This must be the smallest solution.

5. Break the set $\{1, 4, 7, 10, 13, 16, \ldots, 100\}$ in the 18 sets

$$\{1\}, \{52\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \ldots, \{49, 55\}.$$

   By the pigeon-hole principle, one of the sets $\{4, 100\}, \{7, 97\}, \{10, 94\}, \ldots, \{49, 55\}$ has two elements of T. These form the solution.

6. Suppose n has $k+1$ digits, and write it in the form $n = 10M + 6$. Then the transformed number is $6 \cdot 10^k + M$. The problem requires that $6 \cdot 10^k + M = 4(10M + 6)$. Simplifying,

the condition becomes

$$2 \cdot 10^k - 8 = 13M,$$

so $2 \cdot 10^k \equiv 8 \pmod{13}$. Then $10^k \equiv 4 \pmod{13}$ and $10^{k+1} \equiv 40 \equiv 1 \pmod{13}$.

The smallest value of k such that $10^{k+1} \equiv 1 \pmod{13}$ is $k = 6$, so $13M = 2 \cdot 10^6 - 8 = 199992$ and $M = 15384$. Finally, use M to obtain $n = 10 \cdot 15384 + 6 = 153846$.

7. Consider consecutive powers of 2 modulo 7: $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 1$, and the residues repeat after this. Then, $2^n - 1$ can be congruent to 1, 3 or 0 and is 0 precisely when n is a multiple of 3. Similarily, $2^n + 1$ is congruent to 3, 5, or 2 and so it is never a multiple of 7.

8. Suppose n satisfies the conditions, and call the two products a and b. At most one element of $S = \{n, n+1, n+2, n+3, n+4, n+5\}$ can be a multiple of 7, but this would imply that only one of a and b is a multiple of 7. Since the products are equal, it must be the case that the elements of S are congruent in order to 1,2,3,4,5 and 6 modulo 7.

   Consider the product ab. On one hand, $ab \equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \pmod 7$. By Wilson's Theorem, $ab \equiv 6! \equiv -1 \pmod 7$. On the other hand, since $a \equiv b \pmod 7$, we have $ab \equiv a^2 \pmod 7$. But the equation $a^2 \equiv -1 \pmod 7$ has no solution as can be checked by hand.

9. We show by induction on k that $a_{n+k} \equiv 1 \pmod{a_n}$. Obviously true for $k = 1$. Suppose it is true for k. Then for some m, $a_{n+k} = m \cdot a_n + 1$. Hence $a_{n+k+1} = a_{n+k}(ma_n + 1 - 1) + 1 = a_{n+k}ma_n + 1 \equiv 1 \pmod{a_n}$. So the result is true for all k. Hence any pair of distinct a_n are relatively prime.

10. We compute $a_2 = 3^3 = 27$ and $b_2 = 27$. Then $a_3 = 3^{27}$ and we want to compute $b_3 = 3^{27} \mod 100$. Note that

$$\begin{aligned}
3^2 & & &\equiv 9 \pmod{100} \\
3^4 &= 9^2 & &\equiv 81 \pmod{100} \\
3^8 &\equiv 81^2 = 6561 & &\equiv 61 \pmod{100} \\
3^{16} &\equiv 61^2 = 3721 & &\equiv 21 \pmod{100}
\end{aligned}$$

so that $3^{27} = 3^{16+8+2+1} \equiv 3 \cdot 9 \cdot 61 \cdot 21 = 34587 \equiv 87 \pmod{100}$. A similar computation shows that $a_4 = 3^{87} \equiv 87 \pmod{100}$ so $b_4 = 87$. Clearly, the numbers repeat afterwards; in particular $b_{2004} = 87$.