

PARITY

The number 2 is very important in mathematical contests. Many problems can be solved simply by finding some quantities for which we can determine the parity; i.e., whether they are even or odd. If this sounds like an exaggeration, consider this problem:

Putnam 1973 A1: Given 9 points in 3-dimensional space with integer coordinates, show that one can select two of these points so that the segment in between them contains another point of integer coordinates.

Here is how one may go about discovering a solution: *“Hmm! A point in the segment between two other points. . . What about the mid-point? At least I know how to find its coordinates; if the coordinates of the two given points are (x_1, y_1, z_1) and (x_2, y_2, z_2) , the mid-point is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$. How could I force these coordinates to be integers? Well, x_1+x_2 must be even, so x_1 must have the same parity as x_2 . For the same reason, I need that y_1 has the same parity as y_2 , and z_1 the same parity as z_2 . Ok, can I guarantee that this will happen for two of the 9 given points? The total of different parity choices (even/odd for x , even/odd for y , and even/odd for z) for 3 coordinates is $2^3 = 8$. Ah! there are 9 points, that is one too many, so two of them must have coordinates of the same parity.”*

Solution in final form:

There are $2^3 = 8$ choices for the parity of the 3 (integer) coordinates of a point in space. Since 9 points are given, two of them, $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, must have the same parity on each coordinate according to the pigeon-hole principle. Then the mid-point of the segment between A and B is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$ and its coordinates are integers.

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The first theorem usually proved in a graph theory course has to do with evenness and oddness of numbers. The proof will illustrate again how to take advantage of parity. The result may also be useful on its own:

Consider a graph G with no loops. Let r be the number of vertices with an odd number of edges. Then r is even.

As an example, the graph on the right has 4 odd vertices. 1 with one edge, 2 with three edges, and 1 with 5 edges.

Proof: We count the number s of “edge tips.” Since each edge has two extremes, s is twice the number of edges, so s is even.

We can also find s by adding the edge tips around each vertex. Of course the vertices with an even number of edges contribute an even amount to the value of s . Since the vertices with an odd number of vertices contribute an odd amount to s , there must be an even number of them so that s is even.

1. Remove the lower-left and upper-right corners of an 8×8 chessboard. Can the resulting board be covered with 31 dominoes so that each domino covers 2 adjacent squares?
2. Suppose $f(x)$ is a polynomial with integer coefficients and such that $f(1) = 3$. Can it happen that $f(3) = 0$?
3. For any pythagorean triple (a, b, c) (i.e. numbers satisfying $a^2 + b^2 = c^2$), prove that abc is even.
4. Let $A = [a_{ij}]$ be the $n \times n$ matrix whose entries are $a_{ii} = 2i + 1$ and $a_{ij} = (i + j)^2 + i - j$ (when $i \neq j$). Prove that A is invertible.
5. (Putnam 2002 A3): Let $n \geq 2$ be an integer and T_n the number of non-empty subsets S of $\{1, 2, \dots, n\}$ with the property that the average of the elements in S is an integer. Prove that $T_n - n$ is even.