PIGEONHOLE PRINCIPLE

The Pigeonhole Principle: If $n + 1$ or more objects are placed in n boxes, then at least one box contains more than one object.

More generally we can say

If $kn + 1$ or more objects are placed in n boxes, than at least one box contains more than k objects.

Examples:
- If a coin is tossed three times, two of the tosses have the same result.
- In a 27 word english sentence, at least two words start with the same letter.
- In a class of 102 students taking an exam to be graded out of 100 points, at least two students get the same grade.
- How many people need to be in a room to guarantee that two of them have the same birthday?

Plan of Attack:
(1) Decide what the pigeons are. These are the things you want more than one of to have some property.
(2) Make the pigeonholes. There are two things to make sure of here. First, if there are two or more pigeons in the same pigeonhole, they have the property you want. Second, there should be fewer pigeonholes than pigeons.
(3) Make a rule for assigning pigeons to pigeonholes. Sometimes it is necessary to make the right rule here in order to get the property you want.

More Examples:
- Given seven distinct numbers between 1 and 11, some pair of them sum to 12.
- If 51 points are placed in a square of side length one, there is a circle of radius $\frac{1}{7}$ that contains at least three of the points.
- Over a 30 day period, Amy walked the dog at least once a day, and a total of 45 times in all. Then there was a period of consecutive days during which she walked the dog exactly 14 times.
- $(a - b)(b - c)(a - c)$ is always an even integer if a, b, and c are integers.
- What is the maximum number of rooks that you can put on an 8×8 chess-board so that no two rooks can hit each other?

PROBLEMS

(1) Suppose $n \geq 2$ baseball teams play in a tournament where no two teams play each other more than once. Prove that two teams have to play the same number of games.

(2) Given a set A of ten distinct integers $\{n_1, \ldots, n_1\}$ between 1 and 100, show that there are two distinct, disjoint subsets of A whose elements sum to the same number.

(3) What is the maximum number of bishops that you can put on an 8×8 chess-board so that no two bishops can hit each other? How about queens?

(4) Show that, given a 7-digit number, you can cross out some digits at the beginning and at the end such that the remaining number is divisible by 7. For example, you can cross out the first 3 and the last 2 digits of 1294961 to get 49.

(5) Prove that every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

(6) Show there are two powers of seven whose difference is divisible by 20042005.

(7) (1993 A4) Let $x_1, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove there exists a (nonempty) sum of some x_i's equal to a sum of some y_i's.

(8) (2000 B6) Let B be a set of more than $2^{n+1}/n$ distinct points of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

(9) Prove that any collection of 31 distinct integers between 1 and 60 has the property that one divides another.

(10) Show that some power of 7 ends in "0000001."

(11) A certain state makes 6-digit license plates so that any two license plates differ in at least two places. How many license plates can they make?