MATH 413 FINAL EXAM

Math 413 final exam, 13 May 2008. The exam starts at 9:00 am and you have 150 minutes. No textbooks or calculators may be used during the exam. **This exam is printed on both sides of the paper.** *Good luck!*

(1) **(20 marks)** Let $X = (0,1] \subset \mathbb{R}$. State whether each of the following statements about X is true or false, giving a brief reason for each answer.

(a) X is bounded.

(b) X can be written as a countable union of open sets.

(c) X is compact.

(d) There is a point $x_0 \in X$ at which the function $f(x) = \log(x) + x^5 - 8x^4 - 3$ achieves its supremum on X (that is, $f(x_0) = \sup\{f(x) : x \in X\}$).

(2) **(20 marks)** Let $A \subset \mathbb{R}$. Recall that a function $f : A \to \mathbb{R}$ is said to satisfy a Lipschitz condition on A if there is some $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$.

(a) Let $n \in \mathbb{N}$. Show that the function $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = \sqrt{x + \frac{1}{n}}$ satisfies a Lipschitz condition on $[0,1]$.

(Hint: you may wish to use the fact that for all $a, b > 0$, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.)

(b) Show that the sequence of functions $\{f_n\}$ converges uniformly on $[0,1]$ to the function $f(x) = \sqrt{x}$.

(c) Show that $f(x) = \sqrt{x}$ does *not* satisfy a Lipschitz condition on $[0,1]$.

(d) Now suppose $A \subset \mathbb{R}$ and $f_n : A \to \mathbb{R}$ are functions such that there exists $M \in \mathbb{R}$ such that $|f_n(x) - f_n(y)| \leq M|x - y|$ for all $n \in \mathbb{N}$ and all $x, y \in A$. Suppose the sequence of functions $\{f_n\}$ converges uniformly on A to a function $f : A \to \mathbb{R}$. Show that f satisfies a Lipschitz condition on A. Why does this not contradict your answer to part (c)?

[TURN OVER]

(3) **(20 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) State what it means for f to be uniformly continuous on $\mathbb{R}$.

(b) State the Mean Value Theorem.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that the derivative f' is bounded. Show that f is uniformly continuous on $\mathbb{R}$.

(d) Show that $f(x) = e^{-x^2}$ is uniformly continuous on $\mathbb{R}$.

(4) **(20 marks)** Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(a) State what it means for f to be Riemann integrable.

(b) Show that if $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then so is $f + g$.

(c) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is *not* Riemann integrable.

(d) Now let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function. Define $F(x) = \int_0^1 f(x + t)dt$. Show that F is continuous on $\mathbb{R}$.

(5) **(20 marks)** Consider the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} x^{4k+1}.$$

(a) Prove that the series converges absolutely and uniformly on $[-a, a]$ for all $a > 0$. Deduce that this power series defines a C^∞ function $f : \mathbb{R} \to \mathbb{R}$.

(b) Prove that

$$f(x) + f'(x) + f''(x) + f'''(x) = e^x$$

for all $x \in \mathbb{R}$.

(c) Show that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

(d) Show that $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

[END.]