# MATH 413 HONORS INTRODUCTION TO ANALYSIS I
# PRELIM 2.
# (TAKE-HOME)

*This exam is due by the end of the class on Thursday 17 April. Unlike a normal homework, you are not supposed to discuss these problems with your classmates, and you are also not supposed to use any books or resources other than the assigned textbook. You are free to discuss the problems with the lecturer or the TA if you need help, or if you find the wording of the questions ambiguous.*

(1) Let $A \subset \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on $A$ if there exists $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$.

(a) Show that if $f$ satisfies a Lipschitz condition, then $f$ is uniformly continuous.

(b) Show that the function $f(x) = \sqrt{|x|}$ with domain $[-1, 1]$ is uniformly continuous but does not satisfy a Lipschitz condition.

(2) A subset $A$ of $\mathbb{R}$ is called *disconnected* if there exist open subsets $U_1, U_2$ of $\mathbb{R}$ such that $A = (A \cap U_1) \cup (A \cap U_2)$ and $A \cap U_1 \neq \varnothing, A \cap U_2 \neq \varnothing$ but $A \cap U_1 \cap U_2 = \varnothing$. If $A$ is not disconnected, then $A$ is said to be *connected*.

(a) Show that $[0, 1] \cup [2, 3]$ is disconnected.

(b) Show that $\mathbb{Q}$ is disconnected.

(c) Show that $\mathbb{R}$ is connected.

(3) A number $x \in \mathbb{R}$ is said to be *algebraic* if there exist $n \in \mathbb{N}$ and $a_0, \ldots a_n \in \mathbb{Q}$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0.$$

In other words, $x$ is algebraic if and only if $x$ is a root of a polynomial with rational coefficients. For example, $x = 1 + \sqrt{2}$ is algebraic because $(x - 1)^2 - 2 = 0$. We say $x \in \mathbb{R}$ is *transcendental* if $x$ is not algebraic.

(a) Show that the set of algebraic real numbers is countable. You may assume the fact that a polynomial of degree $n$ has at most $n$ roots.

(b) Show that there exists a transcendental real number. (Hint: you do not need to give a specific example; just prove that one exists.)

(4) *In this question, you may use the usual properties of the trigonometric functions which you know from calculus courses.*

(a) Calculate the first-order Taylor approximation to the function

$$f(x) = \frac{1}{\sqrt{\cos(x)}}$$

about the point $x = 0$.

(b) Show that if $0 \leq x \leq \pi/4$ then

$$\left| \frac{1}{\sqrt{\cos(x)}} - 1 \right| \leq \alpha x^2,$$

where

$$\alpha = \frac{1}{2} \sup_{x \in [0,\pi/4]} \left| \frac{2 + 3\sin^2(x)}{4(\cos(x))^{5/2}} \right|.$$

(c) Calculate $\alpha$.