SOME SPECIAL CASES OF THE EISENBUD-GREEN-HARRIS CONJECTURE

RI-XIANG CHEN

Abstract. In this paper we prove some special cases of the Eisenbud-Green-Harris Conjecture, which characterizes the Hilbert functions of homogeneous ideals containing a regular sequence in the polynomial ring.

1. Introduction

Throughout this paper $S = k[x_1, x_2, \ldots, x_n]$ denotes the polynomial ring in n variables over a field k. Given any homogeneous ideal I in S, Macaulay [Ma] proved that there exists a lex ideal L with the same Hilbert function. As a generalization of Macaulay's Theorem, Clements and Lindström [CL] proved that if $I \subset S$ is a homogeneous ideal containing $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ for some integers $2 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $1 \leq r \leq n$, then there exists a lex ideal $L \subset S$ such that $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ has the same Hilbert function as I. Here, $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ is called *a lex-plus-powers ideal* in S. Since $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ is a regular sequence, it is natural to ask what happens if $I \subset S$ is a homogeneous ideal containing a regular sequence of forms $f_1, f_2, \ldots, f_r$ of degrees $a_1, a_2, \ldots, a_r$.

Conjecture 1.1. *(Eisenbud-Green-Harris)*[EGH]*If $I \subset S$ is a homogeneous ideal containing a regular sequence of forms $f_1, f_2, \ldots, f_r$ of degrees $a_1, a_2, \ldots$, a_r where $2 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $1 \leq r \leq n$, then there exists a homogeneous ideal in S containing $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ with the same Hilbert function.*

The above conjecture is called the EGH Conjecture. By the Clements-Lindström Theorem, the EGH Conjecture can be stated in the following equivalent form: If $I \subset S$ is a homogeneous ideal containing a regular sequence of forms $f_1, f_2, \ldots, f_r$ of degrees $a_1, a_2, \ldots, a_r$, then there exists a lex-plus-powers ideal $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ in S with the same Hilbert function.

The following are some known cases of the EGH Conjecture.

Theorem 1.2. *(Mermin)*[Me] *If $I \subset S$ is a homogeneous ideal containing a regular sequence of monomials $m_1, m_2, \ldots, m_r$ of degrees $a_1, a_2, \ldots, a_r$, then*

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there exists a lex-plus-powers ideal $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ *in* S *with the same Hilbert function.*

Note that the above theorem is trivial if $r = n$.

Theorem 1.3. *(Cooper)*[Co1] *Let* k *be an algebraically closed field of characteristic zero. The EGH Conjecture holds if* $I \subset S = k[x_1, x_2, x_3]$ *has minimal generators which are all in the same degree and two of the minimal generators form a regular sequence in* $k[x_1, x_2]$*.*

Cooper [Co2] also studied the conjecture for some cases with $r = n = 3$ in a geometric setting.

In [CM, Propositions 9 and 10], Caviglia and Maclagan proved that if the EGH conjecture holds for all regular sequences of length n, then it holds for all regular sequences of length $r \leq n$. So the rest of the paper will always assume $r = n$.

Definition 1.4. (Caviglia-Maclagan)[CM] Fix integers $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and let d be a non-negative integer. We say that *EGH(d) holds* if for any homogeneous ideal $I \subset S$ containing a regular sequence of forms of degrees $a_1, a_2, \ldots, a_n$, there exists an homogeneous ideal $J \subset S$ containing $x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}$ such that $\dim_k I_d = \dim_k J_d$ and $\dim_k I_{d+1} = \dim_k J_{d+1}$.

Note that given any non-negative integer d, there is a lex-plus-powers ideal $J = L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$ such that $\dim_k I_d = \dim_k J_d$. And the Clements-Lindström Theorem implies that EGH(d) holds if and only if $\dim_k I_{d+1} \geq \dim_k \{S_1 J_d \bigoplus (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})_{d+1}\}$. It follows that the EGH Conjecture holds if and only if EGH(d) holds for all non-negative integers d. In addition, we only need to check if EGH(d) holds for $d < \sum_{i=1}^{n}(a_i - 1)$ because $I_d = S_d$ for $d > \sum_{i=1}^{n}(a_i - 1)$.

Lemma 1.5. *(Caviglia-Maclagan)*[CM] *Fix integers* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ *and set* $N = \sum_{i=1}^{n}(a_i - 1)$*. Then for any* $0 \leq d \leq N - 1$*, EGH(d) holds if and only if EGH(*$N - 1 - d$*) holds.*

Theorem 1.6. *(Caviglia-Maclagan)*[CM] *Fix integers* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$*. If* $a_i > \sum_{j=1}^{i-1}(a_j - 1)$ *for all* $2 \leq i \leq n$ *then the EGH Conjecture holds.*

An immediate consequence of the above theorem is that the EGH Conjecture holds for $n = 2$. Indeed, if $2 \leq a_1 \leq a_2$ then $a_2 > a_1 - 1$. The $n = 2$ case was also obtained by Richert [Ri].

Francisco [Fr] proved the following almost complete intersection case.

Theorem 1.7. *(Francisco)*[Fr] *Fix integers* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ *and let* d *be an integer such that* $d \geq a_1$*. Let* $I \subset S$ *be a homogeneous ideal minimally generated by forms* $f_1, \ldots, f_n, g$ *where* $f_1, \ldots, f_n$ *is a regular sequence,* $\deg f_i = a_i$ *and* $\deg g = d$*. Let* $J = (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}, m)$*, where* m *is the greatest monomial in lex order in degree* d *not in* $(x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$*. Then* $\dim_k I_{d+1} \geq \dim_k J_{d+1}$*.*

In this paper we will focus on the case $a_1 = a_2 = \cdots = a_n = 2$. The EGH Conjecture was originally stated in this case [EGH]. Richert [Ri] says that he verified the EGH Conjecture for $a_1 = a_2 = \cdots = a_n = 2$ and $n \leq 5$, but this result was not published. Herzog and Popescu [HP] proved that if k is a field of characteristic zero and I is minimally generated by generic quadratic forms, then the EGH Conjecture holds.

In section 2 of this paper, we first prove the EGH Conjecture for $a_1 = a_2 = \cdots = a_n = 2$ and $2 \leq n \leq 4$ (Theorem 2.2) by proving EGH(1) and using Lemma 1.5 of Caviglia and Maclagan. Then we show that the EGH Conjecture holds in two other simple cases.

In section 3 we will prove the almost complete intersection case (Theorem 1.7 of Francisco) for $a_1 = a_2 = \cdots = a_n = 2$ by using two different methods.

2. SOME CASES OF THE EGH CONJECTURE

The following proposition implies that EGH(1) holds for the case $a_1 = \cdots = a_n = 2$.

Proposition 2.1. *Let* $I = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ *be an ideal in* S, *where* $f_1, \ldots, f_n$ *is a regular sequence of* 2*-forms and* $g_1, \ldots, g_m$ *are linearly independent* 1*-forms over* k *with* $1 \leq m \leq n$. *Set* $J = (x_1^2, x_2^2, \ldots, x_n^2, x_1, \ldots, x_m) \subset S$. *Then*

$$\dim_k I_2 \geq \dim_k J_2.$$

Proof. Since $J_2 = (x_1, \ldots, x_m)_2 \bigoplus \operatorname{span}\{x_{m+1}^2, \ldots, x_n^2\}$, it follows that

$$\dim_k J_2 = \dim_k (x_1, \ldots, x_m)_2 + (n - m).$$

Without the loss of generality we can assume that $g_1 = x_1, \ldots, g_m = x_m$ and then $I = (x_1, \ldots, x_m, f_1, \ldots, f_n)$. Hence,

$$\dim_k I_2 = \dim_k (x_1, \ldots, x_m)_2 + \dim_k (I/(x_1, \ldots, x_m))_2.$$

Set $t = \dim_k (I/(x_1, \ldots, x_m))_2$. Then there exists $1 \leq i_1 < \cdots < i_t \leq n$ such that $\bar{f}_{i_1}, \ldots, \bar{f}_{i_t}$ form a basis of the k-vector space $(I/(x_1, \ldots, x_m))_2$. Thus we have $I = (x_1, \ldots, x_m, f_{i_1}, \ldots, f_{i_t})$ which implies that $\operatorname{ht}(I) \leq m + t$. Since $f_1, \ldots, f_n$ is a regular sequence it follows that $\operatorname{ht}(f_1, \ldots, f_n) = n$. But $(f_1, \ldots, f_n) \subset I \subset (x_1, \ldots, x_n)$ and $\operatorname{ht}(x_1, \ldots, x_n) = n$, thus $\operatorname{ht}(I) = n$ which implies $n \leq m + t$ and then $t \geq n - m$. So $\dim_k I_2 \geq \dim_k J_2$ and the theorem is proved. □

Theorem 2.2. *If* $a_1 = a_2 = \cdots = a_n = 2$ *and* $2 \leq n \leq 4$ *then the EGH Conjecture holds.*

Proof. Let $N = \sum_{i=1}^{n} (a_i - 1)$. Note that EGH(0) always holds trivially and EGH(1) holds by Proposition 2.1, so we only need to show that EGH(2),..., EGH($N - 1$) hold.

If n=2 then N-1=1 and there is nothing to prove, so that the EGH Conjecture is true.

If n=3 then N-1=2. By Lemma 1.5, EGH(2) holds if and only if EGH(0) holds. So EGH(2) holds and the EGH Conjecture is true.

If n=4 then N-1=3. By Lemma 1.5, EGH(3) holds if and only if EGH(0) holds; EGH(2) holds if and only if EGH(1)holds. Therefore, EGH(2) and EGH(3) hold, and the EGH Conjecture is true. $\square$

Note that if we want to show the cases $n = 5$ and $n = 6$ then EGH(2) needs to be proved directly which is not as simple as Proposition 2.1. Richert [Ri] claimed that he had a proof for $n \leq 5$ but not for $n = 6$ because his proof is different from mine.

The EGH Conjecture also holds in the following two simple cases where regular sequences have nice structures.

Proposition 2.3. *Let $f_1, \dots, f_n$ be a regular sequence of 2-forms in S. Then the EGH Conjecture holds in the following two cases:*

1. *$f_1 = l_1^2, \dots, f_n = l_n^2$, where $l_i = \sum_{j=1}^n a_{ij}x_j$ for $1 \leq i \leq n$, $a_{ij} \in k$ and $\det(a_{ij}) \neq 0$.*
2. *For $1 \leq i \leq n$, $f_i = \sum_{m \in S_2} a_{i,m}m$, where the sum is over all monomials m in S_2, $a_{i,m} \in k$ and $a_{i,m} = 0$ for $m <_{lex} x_i^2$. Here we assume $x_1 > x_2 > \cdots > x_n$ and use the lex order.*

Proof. (1) Note that the k-algebra map $F : S \longrightarrow S$ defined by $F(x_i) = l_i$ for $1 \leq i \leq n$ is an graded isomorphism. So the Hilbert function is preserved under F^{-1}. It follows that the EGH Conjecture holds.

(2) First we claim that $a_{i,x_i^2} \neq 0$ for all $1 \leq i \leq n$. Indeed, if not, then let j be the smallest integer such that $a_{j,x_j^2} = 0$. If $j = 1$ then $f_1 = 0$ which is a contradiction. Hence $j > 1$. Since $a_{i,m} = 0$ for $m <_{lex} x_i^2$, it follows that $(f_1, \dots, f_j) \subseteq (x_1, \dots, x_{j-1})$, so that

$$(f_1, \dots, f_n) \subseteq (x_1, \dots, x_{j-1}, f_{j+1}, \dots, f_n).$$

Since $f_1, \dots, f_n$ is a regular sequence, we have that $\mathrm{ht}(f_1, \dots, f_n) = n$ which implies $\mathrm{ht}(x_1, \dots, x_{j-1}, f_{j+1}, \dots, f_n) = n$, but $(x_1, \dots, x_{j-1}, f_{j+1}, \dots, f_n)$ is generated by $n-1$ elements and its height can not be n. So we get a contradiction and the claim is proved.

Now we consider the initial ideal $\mathrm{in}_{<_{rlex}}(f_1, \dots, f_n)$ with respect to the reverse lex order such that $x_n > \cdots > x_1$. With this monomial order, by the above claim it is easy to see that $\mathrm{in}_{<_{rlex}} f_i = x_i^2$. Thus, $\mathrm{in}_{<_{rlex}}(f_1, \dots, f_n) = (x_1^2, \dots, x_n^2)$. Given any homogeneous ideal I containing $f_1, \dots, f_n$, since $\mathrm{in}_{<_{rlex}}(I)$ contains $\mathrm{in}_{<_{rlex}}(f_1, \dots, f_n) = (x_1^2, \dots, x_n^2)$ and $\mathrm{in}_{<_{rlex}}(I)$ has the same Hilbert function as I, it follows that I has the same Hilbert function as a monomial ideal containing $x_1^2, \dots, x_n^2$. So the EGH Conjecture holds.

$\square$

Remark 2.4. The above proposition is actually an easy consequence of the fact that the Hilbert function is preserved under $GL(n, k)$ actions on the variables or by taking initial ideas. In part (2) of the above proposition, if we replace "lex" by "reverse lex", or replace "$m <_{lex} x_i^2$" by "$m >_{lex} x_i^2$", then the result still holds.

In general, $f_1, \dots, f_n$ do not satisfy the assumptions in the above proposition.

By part (2) of the above proposition, the EGH Conjecture in the case of $a_1 = \cdots = a_n = 2$ can be stated in the following equivalent form: If $I \subset S$ is a homogeneous ideal containing a regular sequence of n 2-forms, then there exists a homogeneous ideal in S containing $f_1, \dots, f_n$ with the same Hilbert function, where $f_1, \dots, f_n$ are some 2-forms satisfying part (2) of the above proposition.

3. Almost complete intersections

This section proves Theorem 1.7 for the case $a_1 = \cdots = a_n = 2$. The key ingredient of any proof of the EGH Conjecture should be about the use of the assumption that $f_1, f_2, \dots, f_n$ is a regular sequence in S. In [Fr], Francisco made use of the fact that if $f_1, f_2, \dots, f_n$ is a regular sequence in S then the minimal free resolution of $S/(f_1, \dots, f_n)$ over S is given by the Koszul complex. In this section we will use the regular sequence assumption in different ways. Before proving Theorem 3.4, we look at some lemmas about regular sequences. The following lemma is a special case of Proposition 7 in [CM], which was originally proved in [DGO].

Lemma 3.1. *(Davis-Geramita-Orecchia)*[DGO] *Let $f_1, \dots, f_n$ be a regular sequence of 2-forms in S. Let I be a homogeneous ideal containing $f_1, \dots, f_n$. Then for all $0 \leq d \leq n$, we have*

$$\dim_k(S/(f_1, \dots, f_n))_d = \dim_k(S/I)_d + \dim_k(S/((f_1, \dots, f_n) : I))_{n-d},$$

or equivalently,

$$\dim_k(I/(f_1, \dots, f_n))_d = \dim_k(S/((f_1, \dots, f_n) : I))_{n-d}.$$

Lemma 3.2. *Let I be an ideal in S minimally generated by some 2-forms. If the height of I is $r \geq 1$, that is, ht$(I) = r$, then I contains a regular sequence $f_1, \dots, f_r$ of 2-forms.*

Proof. Let s be the maximal integer such that I contains a regular sequence $f_1, \dots, f_s$ of 2-forms. Then it is easy to see that $s \geq 1$ and we have

$$s = \mathrm{ht}(f_1, \dots, f_s) \leq \mathrm{ht}(I) = r.$$

Hence, it suffices to show that $s = r$.

To prove by contradiction, we assume $s < r$. Let $f_1, \dots, f_s$ be a regular sequence of 2-forms contained in I, then $\mathrm{ht}(f_1, \dots, f_s) = s < r$. Let $P_1, \dots, P_l$

be the prime divisors of the ideal $(f_1, \ldots, f_s)$. Since S is Cohen-Macaulay, we have $\operatorname{ht}(P_i) = s$ for $1 \leq i \leq l$. If $I \subseteq P_1 \cup \cdots \cup P_l$, then there exists i such that $I \subseteq P_i$, which implies $\operatorname{ht}(I) \leq \operatorname{ht}(P_i) = s < r$; but $\operatorname{ht}(I) = r$, thus I is not contained in $P_1 \cup \cdots \cup P_l$. Since I is generated by 2-forms, it follows that there exists a 2-form f_{s+1} in I such that $f_{s+1} \notin P_1 \cup \cdots \cup P_l$. Thus, f_{s+1} is a non-zero-divisor of $S/(f_1, \ldots, f_s)$. Therefore, I contains a regular sequence $f_1, \ldots, f_s, f_{s+1}$ of 2-forms, which contradicts the definition of s. So $s = r$ and the lemma is proved. □

Lemma 3.3. *If $f_1, \ldots, f_n$ is a regular sequence of 2-forms in S and $g_1 f_1 + g_2 f_2 + \cdots + g_n f_n = 0$ for some q-forms $g_1, g_2, \ldots, g_n$, then $g_1, g_2, \ldots, g_n \in (f_1, \ldots, f_n)_q$. More precisely, we have $q \geq 2$ and there exists a skew-symmetric $n \times n$ matrix A of $(q-2)$-forms such that*

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix} A.$$

Proof. Let $K(f_1, \ldots, f_n)$ be the Koszul complex with $e_1, \ldots, e_n$ the basis in homological degree 1. Since $f_1, \ldots, f_n$ is a regular sequence, we have $H_1(K(f_1, \ldots, f_n)) = 0$. Thus, if $g_1 f_1 + \cdots + g_n f_n = 0$ then there exists $(q-2)$-forms h_{ij} for $1 \leq i < j \leq n$ such that

$$g_1 e_1 + \cdots + g_n e_n = \sum_{1 \leq i < j \leq n} h_{ij}(f_j e_i - f_i e_j).$$

Comparing the coefficients of $e_1, \ldots, e_n$, we get

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix} A,$$

where A is a skew-symmetric matrix with the $(i,j)^{\text{th}}$ entry given by $-h_{ij}$ for $i < j$. □

Theorem 3.4. *Let $I \subset S$ be a homogeneous ideal minimally generated by a regular sequence of 2-forms $f_1, \ldots, f_n$ and a d-form g with $d \geq 2$. Let $J = (x_1^2, x_2^2, \ldots, x_n^2, m)$, where m is the greatest monomial in lex order in degree d not in $(x_1^2, x_2^2, \ldots, x_n^2)$. Then $\dim_k I_{d+1} \geq \dim_k J_{d+1}$.*

We will prove this theorem by two different methods. The first method uses Lemma 3.1 and Lemma 3.2.

Proof. Note that $(f_1, \ldots, f_n)_{n+1} = (x_1^2, \ldots, x_n^2)_{n+1} = S_{n+1}$, hence $d \leq n$. Since the $d = n$ case is also trivial, we will assume that $2 \leq d \leq n-1$. It is easy to see that $m = x_1 \cdots x_d$ and then $\dim_k J_{d+1} = \dim_k (x_1^2, \ldots, x_n^2)_{d+1} + n - d$. On the other hand,

$$\dim_k I_{d+1} = \dim_k (f_1, \ldots, f_n)_{d+1} + n - \dim_k \left((f_1, \ldots, f_n)_{d+1} \cap S_1 \operatorname{span}\{\mathrm{g}\} \right).$$

Let $r = \dim_k \left((f_1, \ldots, f_n)_{d+1} \cap S_1 \operatorname{span}\{\mathrm{g}\} \right) \leq n$. Since $\dim_k (x_1^2, \ldots, x_n^2)_{d+1} = \dim_k (f_1, \ldots, f_n)_{d+1}$ we need only to show $r \leq d$.

To prove by contradiction, we assume that $r > d$. Then without the loss of generality, we can assume that $x_1 g, \ldots, x_r g \in (f_1, \ldots, f_n)_{d+1}$. Then we have $x_1, \ldots, x_r \in ((f_1, \ldots, f_n) : I)$. Note that

$$\frac{S}{(x_1, \ldots, x_r, f_1, \ldots, f_n)} \cong \frac{k[x_{r+1}, \ldots, x_n]}{(\bar{f}_1, \ldots, \bar{f}_n)},$$

where $\bar{f}_1, \ldots, \bar{f}_n$ are the images of $f_1, \ldots, f_n$ in the quotient ring $S/(x_1, \ldots, x_r) \cong k[x_{r+1}, \ldots, x_n]$. Since $k[x_{r+1}, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_n)$ has dimension zero, we have $\operatorname{ht}(\bar{f}_1, \ldots, \bar{f}_n) = n - r$. Hence, by Lemma 3.2, $(\bar{f}_1, \ldots, \bar{f}_n)$ contains a regular sequence $g_1, \ldots, g_{n-r}$ of 2-forms in the polynomial ring $k[x_{r+1}, \ldots, x_n]$. Thus, for all $i \geq 0$,

$$\dim_k(k[x_{r+1}, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_n))_i \leq \binom{n-r}{i}.$$

Therefore, by Lemma 3.1, we have

$$\begin{aligned} 1 &= \dim_k(I/(f_1, \ldots, f_n))_d \\ &= \dim_k(S/((f_1, \ldots, f_n) : I))_{n-d} \\ &\leq \dim_k(S/(x_1, \ldots, x_r, f_1, \ldots, f_n))_{n-d} \\ &= \dim_k(k[x_{r+1}, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_n))_{n-d} \\ &\leq \binom{n-r}{n-d} \\ &= 0, \text{ since } r > d. \end{aligned}$$

So we get a contradiction and $r \leq d$. ☐

The following proof of Theorem 3.4 uses Lemma 3.3.

Proof. As in the previous proof, we can assume $2 \leq d \leq n - 1$.

First we consider the case $d = 2$ and $n \geq 3$. Now $J = (x_1^2, x_2^2, \ldots, x_n^2, x_1 x_2)$ and $\dim_k J_3 = \dim_k(x_1^2, \ldots, x_n^2)_3 + n - 2$. On the other hand,

$$\dim_k I_3 = \dim_k(f_1, \ldots, f_n)_3 + n - \dim_k\left((f_1, \ldots, f_n)_3 \cap S_1 \mathrm{span}\{\mathrm{g}\}\right).$$

Since $\dim_k(x_1^2, \ldots, x_n^2)_3 = \dim_k(f_1, \ldots, f_n)_3$ we need only to show that

$$\dim_k((f_1, \ldots, f_n)_3 \cap S_1 \mathrm{span}\{\mathrm{g}\}) \leq 2.$$

We prove by contradiction, so assume $\dim_k((f_1, \ldots, f_n)_3 \cap S_1 \mathrm{span}\{\mathrm{g}\}) \geq 3$. Then without the loss of generality we can assume that

$$\begin{aligned} x_1 g &= \vec{f} \cdot \vec{p}_1, \\ x_2 g &= \vec{f} \cdot \vec{p}_2, \\ x_3 g &= \vec{f} \cdot \vec{p}_3, \end{aligned}$$

where $\vec{f}$ is the row vector $\begin{pmatrix} f_1, f_2, \ldots, f_n \end{pmatrix}$ and $\vec{p_1}, \vec{p_2}, \vec{p_3}$ are some column vectors of 1-forms. Hence we have

$$g\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \vec{f} \cdot \begin{pmatrix} \vec{p_1} & \vec{p_2} & \vec{p_3} \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix} = 0,$$

it follows that

$$\begin{aligned} &\vec{f} \cdot \begin{pmatrix} \vec{p_1} & \vec{p_2} & \vec{p_3} \end{pmatrix} \begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix} \\ &= \vec{f} \cdot \begin{pmatrix} x_2\vec{p_1} - x_1\vec{p_2} & x_3\vec{p_1} - x_1\vec{p_3} & x_3\vec{p_2} - x_2\vec{p_3} \end{pmatrix} = 0. \end{aligned}$$

By Lemma 3.3 there are skew-symmetric $n \times n$ matices A_{12}, A_{13}, A_{23} of scalars such that

$$\begin{pmatrix} x_2\vec{p_1} - x_1\vec{p_2} & x_3\vec{p_1} - x_1\vec{p_3} & x_3\vec{p_2} - x_2\vec{p_3} \end{pmatrix} = \begin{pmatrix} A_{12}\vec{f}^{\,T} & A_{13}\vec{f}^{\,T} & A_{23}\vec{f}^{\,T} \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix} \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} = 0,$$

it follows that

$$\begin{pmatrix} A_{12}\vec{f}^{\,T} & A_{13}\vec{f}^{\,T} & A_{23}\vec{f}^{\,T} \end{pmatrix} \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} = 0,$$

so that $(x_3A_{12} - x_2A_{13} + x_1A_{23})\vec{f}^{\,T} = 0$. Since $x_3A_{12} - x_2A_{13} + x_1A_{23}$ is an $n \times n$ matrix of 1-forms, it follows from Lemma 3.3 that $x_3A_{12} - x_2A_{13} + x_1A_{23} = 0$ and then $A_{12} = A_{13} = A_{23} = 0$. Thus, $x_2\vec{p_1} - x_1\vec{p_2} = 0$ which implies that $\vec{p_1}$ can be divided by x_1. So $g = \vec{f} \cdot (\vec{p_1}/x_1)$ and then $g \in (f_1, \ldots, f_n)_2$ which contradicts the assumption that I is minimally generated by $f_1, \ldots, f_n, g$. So we have proved the case $d = 2$.

Then we consider the case $d = 3$ and $n \geq 4$. Now $J = (x_1^2, \ldots, x_n^2, x_1x_2x_3)$ and $\dim_k J_4 = \dim_k(x_1^2, \ldots, x_n^2)_4 + n - 3$. On the other hand,

$$\dim_k I_4 = \dim_k(f_1, \ldots, f_n)_4 + n - \dim_k((f_1, \ldots, f_n)_4 \cap S_1\text{span}\{\text{g}\}).$$

Since $\dim_k(x_1^2, \ldots, x_n^2)_4 = \dim_k(f_1, \ldots, f_n)_4$ we need only to show that

$$\dim_k((f_1, \ldots, f_n)_4 \cap S_1\text{span}\{\text{g}\}) \leq 3.$$

We prove by contradiction, so assume $\dim_k((f_1, \dots, f_n)_4 \cap S_1 \mathrm{span}\{\mathrm{g}\}) \geq 4$. Then without the loss of generality we can assume that

$$\begin{aligned} x_1 g &= \vec{f} \cdot \vec{p}_1, \\ x_2 g &= \vec{f} \cdot \vec{p}_2, \\ x_3 g &= \vec{f} \cdot \vec{p}_3, \\ x_4 g &= \vec{f} \cdot \vec{p}_4, \end{aligned}$$

where $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ are some column vectors of 2-forms. Hence we have

$$g \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \vec{f} \cdot \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \vec{p}_4 \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix} = 0,$$

it follows that

$$\begin{aligned} &\vec{f} \cdot \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \vec{p}_4 \end{pmatrix} \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix} \\ &= \vec{f} \cdot \begin{pmatrix} x_2 \vec{p}_1 - x_1 \vec{p}_2 & \cdots & x_4 \vec{p}_3 - x_3 \vec{p}_4 \end{pmatrix} = 0. \end{aligned}$$

By Lemma 3.3 there are skew-symmetric $n \times n$ matices $A_{12}, A_{13}, \dots, A_{34}$ of 1-forms such that

$$\begin{pmatrix} x_2 \vec{p}_1 - x_1 \vec{p}_2 & \cdots & x_4 \vec{p}_3 - x_3 \vec{p}_4 \end{pmatrix} = \begin{pmatrix} A_{12} \vec{f}^{\,T} & \cdots & A_{34} \vec{f}^{\,T} \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix} \begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix} = 0,$$

it follows that

$$\begin{pmatrix} A_{12} \vec{f}^{\,T} & \cdots & A_{34} \vec{f}^{\,T} \end{pmatrix} \begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix} = 0,$$

that is,

$$\Big((x_3A_{12}-x_2A_{13}+x_1A_{23})\vec{f}^{\,T} \quad \cdots \quad (x_4A_{23}-x_3A_{24}+x_2A_{34})\vec{f}^{\,T}\Big)=0.$$

By Lemma 3.3 there are skew-symmetric $n\times n$ matices $B_{123,1},\dots,B_{123,n},\dots,$ $B_{234,n}$ of scalars such that

$$x_3A_{12}-x_2A_{13}+x_1A_{23}=\begin{pmatrix}\vec{f}B_{123,1}\\ \vdots\\ \vec{f}B_{123,n}\end{pmatrix},$$

$$x_4A_{12}-x_2A_{14}+x_1A_{24}=\begin{pmatrix}\vec{f}B_{124,1}\\ \vdots\\ \vec{f}B_{124,n}\end{pmatrix},$$

$$x_4A_{13}-x_3A_{14}+x_1A_{34}=\begin{pmatrix}\vec{f}B_{134,1}\\ \vdots\\ \vec{f}B_{134,n}\end{pmatrix},$$

$$x_4A_{23}-x_3A_{24}+x_2A_{34}=\begin{pmatrix}\vec{f}B_{234,1}\\ \vdots\\ \vec{f}B_{234,n}\end{pmatrix}.$$

Since

$$\begin{pmatrix} x_3 & x_4 & 0 & 0\\ -x_2 & 0 & x_4 & 0\\ 0 & -x_2 & -x_3 & 0\\ x_1 & 0 & 0 & x_4\\ 0 & x_1 & 0 & -x_3\\ 0 & 0 & x_1 & x_2 \end{pmatrix}\begin{pmatrix} x_4\\ -x_3\\ x_2\\ -x_1\end{pmatrix}=0,$$

it follows that for any $1\le i\le n$,

$$\vec{f}(x_4B_{123,i}-x_3B_{124,i}+x_2B_{134,i}-x_1B_{234,i})=0.$$

Since $x_4B_{123,i}-x_3B_{124,i}+x_2B_{134,i}-x_1B_{234,i}$ is an $n\times n$ matrix of 1-forms, it follows from Lemma 3.3 that

$$x_4B_{123,i}-x_3B_{124,i}+x_2B_{134,i}-x_1B_{234,i}=0,$$

and then $B_{123,1}=\cdots=B_{234,n}=0$. Thus, $x_3A_{12}-x_2A_{13}+x_1A_{23}=0$ which implies that $x_2A_{13}-x_1A_{23}$ can be divided by x_3. Let A'_{13} and A'_{23} be the skew-symmetric matrices of 1-forms obtained from A_{13} and A_{23} by keeping only the terms containing x_3, then we have

$$\begin{aligned} A_{12} &= \frac{1}{x_3}(x_2A_{13} - x_1A_{23}) \\ &= \frac{1}{x_3}(x_2A'_{13} - x_1A'_{23}) \\ &= x_2\frac{A'_{13}}{x_3} - x_1\frac{A'_{23}}{x_3}. \end{aligned} \tag{1}$$

Thus,

$$x_2\vec{p}_1 - x_1\vec{p}_2 = A_{12}\vec{f}^{\,T} = (x_2\frac{A'_{13}}{x_3} - x_1\frac{A'_{23}}{x_3})\vec{f}^{\,T},$$

and then,

$$x_1(\vec{p}_2 - \frac{A'_{23}}{x_3}\vec{f}^{\,T}) = x_2(\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T}),$$

so that $\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T}$ can be divided by x_1. Note that $\frac{A'_{13}}{x_3}$ is an $n \times n$ skew-symmetric matrix of scalars, which implies that $\vec{f}\frac{A'_{13}}{x_3}\vec{f}^{\,T} = 0$. So we have $x_1g = \vec{f}\cdot(\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T})$ and then $g = \vec{f}\cdot\frac{1}{x_1}(\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T}) \in (f_1, \dots, f_n)_3$ which contradicts the assumption that I is minimally generated by $f_1, \dots, f_n, g$. So we have proved the case $d = 3$.

Proceeding in the same way we can prove the theorem for all $2 \leq d \leq n-1$ and we are done. □

The second proof actually uses the minimal free resolution (Koszul complex) of $S/(x_1, x_2, \dots, x_i)$. This is because we add only *one* polynomial g in degree d. If we add two or more polynomials in degree d, things get very complicated and the second proof does not work any more. The first proof also depends heavily on adding just *one* polynomial g. If we add two or more polynomials in degree d, then $((f_1, \dots, f_n) : I)$ will not always contain many variables as in our first proof.

After proving theorem 3.4, it is natural to consider the following problem, which is a special case of the EGH Conjecture.

Problem 3.5. *Let* $f_1, \dots, f_n$ *be a regular sequence of* 2*-forms in* S *with* $n \geq 3$. *Let* $g, h \in S$ *be* 2*-forms such that* $\dim_k(f_1, \dots, f_n, g, h)_2 = n + 2$. *Is it true that* $\dim_k(f_1, \dots, f_n, g, h)_3 \geq \dim_k(x_1^2, \dots, x_n^2, x_1x_2, x_1x_3)_3 = n^2 + 2n - 5$?

From section 2, we know that it is true if $3 \leq n \leq 4$, or if $f_1, \dots, f_n$ satisfy the assumption of Proposition 2.3. From [HP], we know that it is true if g and h are generic 2-forms and $\mathrm{Char}(k) = 0$.

By theorem 3.4 we see that $\dim_k((f_1, \dots, f_n)_3 \cap S_1\mathrm{span}\{\mathrm{g}\})$ can only be 0, 1 or 2. In the next proposition we study the case $\dim_k((f_1, \dots, f_n)_3 \cap S_1\mathrm{span}\{\mathrm{g}\}) = 2$ by using a combination of techniques used in the two proofs of Theorem 3.4.

Proposition 3.6. *Let* $f_1, \dots, f_n$ *be a regular sequence of* 2*-forms in* S *with* $n \geq 3$. *Let* g, h *be* 2*-forms such that* $\dim_k(f_1, \dots, f_n, g, h)_2 = n + 2$. *If* $\dim_k((f_1, \dots, f_n)_3 \cap S_1 span\{g\}) = 2$, *then*

$$\dim_k(f_1, \dots, f_n, g, h)_3 \geq n^2 + 2n - 5.$$

Proof. Since $\dim_k((f_1, \dots, f_n)_3 \cap S_1\text{span}\{\text{g}\}) = 2$, there exists linearly independent 1-forms l_1 and l_2 such that

$$l_1 g = \vec{f} \cdot \vec{p}_1,$$
$$l_2 g = \vec{f} \cdot \vec{p}_2,$$

where $\vec{f}$ is the row vector $\big(f_1, f_2, \dots, f_n\big)$ and $\vec{p}_1, \vec{p}_2$ are some column vectors of 1-forms.

To prove by contradiction, we assume that $\dim_k(f_1, \dots, f_n, g, h)_3 < n^2 + 2n - 5$. Since

$$\begin{aligned}
&\dim_k(f_1, \dots, f_n, g, h)_3 \\
&= \dim_k(f_1, \dots, f_n, g)_3 + n - \dim_k((f_1, \dots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}) \\
&= (\dim_k(f_1, \dots, f_n)_3 + n - 2) + n - \dim_k((f_1, \dots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}) \\
&= n^2 + 2n - 2 - \dim_k((f_1, \dots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}),
\end{aligned}$$

it follows that $\dim_k((f_1, \dots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}) \geq 4$. Without the loss of generality, we can assume that

$$x_1 h = l_3 g + \vec{f} \cdot \vec{p}_3,$$
$$x_2 h = l_4 g + \vec{f} \cdot \vec{p}_4,$$
$$x_3 h = l_5 g + \vec{f} \cdot \vec{p}_5,$$
$$x_4 h = l_6 g + \vec{f} \cdot \vec{p}_6,$$

where l_3, l_4, l_5, l_6 are some 1-forms and $\vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6$ are some column vectors of 1-forms. Multiplying the above 4 equations by l_1, because $l_1 g = \vec{f} \cdot \vec{p}_1$, we get that

$$x_1(l_1 h), x_2(l_1 h), x_3(l_1 h), x_4(l_1 h) \in (f_1, \dots, f_n)_4.$$

By the second proof of Theorem 3.4, we conclude that $l_1 h \in (f_1, \dots, f_n)_3$. Similarly, we have $l_2 h \in (f_1, \dots, f_n)_3$. Thus,

$$l_1, l_2 \in ((f_1, \dots, f_n) : (f_1, \dots, f_n, g, h)).$$

Without the loss of generality we can assume that $l_1 = x_1$ and $l_2 = x_2$. Therefore, similar to the first proof of Theorem 3.4, we have

$$\begin{aligned} 2 &= \dim_k((f_1, \dots, f_n, g, h)/(f_1, \dots, f_n))_2 \\ &= \dim_k(S/((f_1, \dots, f_n) : (f_1, \dots, f_n, g, h)))_{n-2} \\ &\le \dim_k(S/(x_1, x_2, f_1, \dots, f_n))_{n-2} \\ &= \dim_k(k[x_3, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_n))_{n-2} \\ &\le \binom{n-2}{n-2} \\ &= 1, \end{aligned}$$

which is a contradiction. So $\dim_k(f_1, \dots, f_n, g, h)_3 \ge n^2 + 2n - 5$ and we are done. $\square$

Remark 3.7. The key point of the above proof is that there exist two 1-forms l_1 and l_2 such that $l_1, l_2 \in ((f_1, \dots, f_n) : (f_1, \dots, f_n, g, h))$, which is not the case if

$$\dim_k((f_1, \dots, f_n)_3 \cap S_1 \mathrm{span}\{\mathrm{g}\}) \ne 2 \text{ and } \dim_k((f_1, \dots, f_n)_3 \cap S_1 \mathrm{span}\{\mathrm{h}\}) \ne 2.$$

It would be interesting to study the other two cases of Problem 3.5.

We end this section by looking at two criteria and one example about regular sequences. Here we do not assume that $f_1, f_2, \dots, f_n$ are of degrees 2. One simple criterion for $f_1, f_2, \dots, f_n$ being a regular sequence in S is the following:

$$f_1, f_2, \dots, f_n \text{ is a regular sequence } \iff \mathrm{Rad}(f_1, \dots, f_n) = (x_1, \dots, x_n).$$

The other criterion follows easily from [Mt, Corollary on Page 161], which says: $f_1, \dots, f_n$ is a regular sequence in S if and only if the following condition holds:

$$\text{if } g_1 f_1 + \cdots + g_n f_n = 0 \text{ for some } g_1, \dots, g_n \in S, \text{ then } g_1, \dots, g_n \in (f_1, \dots, f_n).$$

In general, given homogeneous polynomials $f_1, \dots, f_n$ of degrees 2 in S, it is hard to check by hand whether $f_1, \dots, f_n$ form a regular sequence, although generically $f_1, \dots, f_n$ form a regular sequence. The following example gives a characterization of a special class of regular sequences.

Example 3.8. Let $f_1 = x_1 l_1, \dots, f_n = x_n l_n$ be a sequence of homogeneous polynomials in S, where $l_i = \sum_{j=1}^n a_{ij} x_j$ with $a_{ij} \in k$ and $i = 1, \dots, n$. Let A be the $n \times n$ matrix (a_{ij}). For any $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$, let $A[i_1, \dots, i_r]$ be the submatrix of A formed by rows $i_1, \dots, i_r$ and columns $i_1, \dots, i_r$. By looking at the primary decomposition of the ideal $(f_1, \dots, f_n)$, we see that $f_1, \dots, f_n$ is a regular sequence if and only if $\det(A[i_1, \dots, i_r]) \ne 0$ for all $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$. It would be interesting to know if the EGH Conjecture holds in this special case.

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Department of Mathematics, Cornell University, Ithaca, NY 14853
E-mail address: `rxc@math.cornell.edu`