MINIMAL FREE RESOLUTIONS OF LINEAR EDGE IDEALS

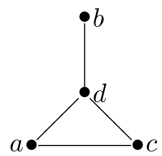
RI-XIANG CHEN
DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853

ABSTRACT. We construct minimal free resolutions for all edge ideals which have a linear free resolution.

1. INTRODUCTION

Let $S = k[x_1, \dots, x_n]$ be the polynomial ring over a field k. In this paper we consider minimal free resolutions of quadratic monomial ideals in S. By polarization, the study of such resolutions is equivalent to the study of the resolutions of square-free quadratic monomial ideals, that is, edge ideals. Such an ideal can be easily encoded in a graph as follows: let G be a simple graph with vertices $x_1, \dots, x_n$, then the edge ideal I_G of the graph G is the monomial ideal in S generated by $\{x_i x_j \mid \{x_i, x_j\}$ is an edge of $G\}$. The general goal is to relate the properties of the minimal free resolution of I_G and the combinatorial properties of the graph G. In 1990, Fröberg [Fr] proved that I_G has a linear free resolution if and only if the complement graph $\overline{G}$ is chordal (see Definition 2.1). Because of this, I_G is called a linear edge ideal if $\overline{G}$ is chordal.

Minimal free resolutions were constructed for the following two classes of linear edge ideals. In [CN], Corso and Nagel used cellular resolutions to get the minimal free resolutions of the linear edge ideals I_G where G is a Ferrers graph. In [Ho], Horwitz constructed the minimal free resolutions of the linear edge ideals I_G provided that G does not contain an ordered subgraph of the form



with $\quad a < b < c < d,$

which is called the pattern Γ in [Ho]. However, from Example 3.18 in [Ho], we see that if $\overline{G}$ is complicated, then it may be impossible to satisfy the Γ avoidance condition. In Construction 3.4 and Theorem 3.7 we provide the minimal free resolutions of *all* linear edge ideals. The construction is different than the one in [Ho] and the following paragraph explains the difference.

In 1990, Eliahou and Kervaire [EK] constructed the minimal free resolutions of Borel ideals. In 1995, Charalambous and Evans [CE] noted that the Eliahou-Kervaire resolution can be obtained by using iterated mapping cones. Then in 2002, Herzog and Takayama [HT] used the iterated mapping cone construction to obtain the minimal free resolutions of monomial ideals which have linear quotients and satisfy some regularity condition. Following this idea, in 2007, Horwitz [Ho] constructed the minimal free resolutions of a class of linear edge ideals. In [HT] and

[Ho], the constructions are based on induction on the number of generators of the monomial ideal and the resolutions are similar to the Eliahou-Kervaire resolution. In this paper we use the mapping cone construction in a new way: (1) we use induction on the number of variables, that is the number of vertices of G; (2) in each induction step, we use the mapping cone construction twice. Consequently, the minimal free resolution in this paper is very different from the Eliahou-Kervaire resolution and is not a modification of the resolution obtained in [Ho] (See Remark 3.12).

Another thing that plays an important role in our construction is the notion of a perfect elimination order (See Definition 2.1) of a chordal graph. From [Di] and [HHZ], we know that every chordal graph has a perfect elimination order on the set of vertices; conversely, it is easy to see that if a simple graph has a perfect elimination order then it is chordal. Therefore, a simple graph is chordal if and only if it has a perfect elimination order. In general, given a chordal graph, there are many perfect elimination orders. In section 2 we give an algorithm (Algorithm 2.2) to produce a special perfect elimination order on the vertices of a chordal graph. This special perfect elimination order has a nice property (Lemma 3.2) and will be used in the construction of the minimal free resolutions of linear edge ideals.

In section 3 we construct the minimal free resolutions of linear edge ideals and Theorem 3.7 is the main result of this paper.

In section 4 we prove $d^2 = 0$ case by case, where d is the differential defined in Construction 3.4. The proof is not difficult but very long.

Section 5 gives a nice formula (Corollary 5.2) for calculating the Betti numbers of linear edge ideals and the formula works for any perfect elimination order of $\overline{G}$. Finally, in Corollary 5.4, we use our method to prove another Betti number formula obtained by Roth and Van Tuyl in [RV] (see also [HV]).

Acknowledgments. The author is thankful to Irena Peeva for providing the idea of using induction on the number of vertices and two mapping cones to find minimal free resolutions of all linear edge ideals. The author also thanks the referee for the valuable suggestions.

2. Perfect Elimination Orders

In this section we use H to denote a chordal graph. In the other sections of this paper, we have $H = \overline{G}$.

Definition 2.1. Let H be a simple graph with vertices $x_1, \ldots, x_n$. We write $x_i x_j \in H$ if $\{x_i, x_j\}$ is an edge of H. We say that $C = (x_{j_1} x_{j_2} \ldots x_{j_r})$ is a *cycle* of H of length r if $x_{j_i} \neq x_{j_l}$ for all $1 \leq i < l \leq r$ and $x_{j_i} x_{j_{i+1}} \in H$ for all $1 \leq i \leq r$(where $x_{j_{r+1}} = x_{j_1}$). A *chord* in the cycle C is an edge between two non-consecutive vertices in the cycle. We say that H is a *chordal graph* if every cycle of length > 3 in H has a chord. The order $x_1, \ldots, x_n$ on the vertices of H is called a *perfect elimination order* if the following condition is satisfied: for any $1 \leq i < j < l \leq n$, if $x_i x_j \in H$ and $x_i x_l \in H$, then $x_j x_l \in H$.

The perfect elimination orders we will use in sections 3 and 4 are given by the following algorithm.

Algorithm 2.2. *Let H be a chordal graph with vertices $x_1, \ldots, x_n$. Let Σ be a set containing a sequence of sets.*
Input: $\Sigma = \{\{x_1, \ldots, x_n\}\}$, $i = n + 1$.

Step 1: Choose and remove a vertex v from the first set in Σ. Set $i := i-1$ and $v_i := v$. If the first set in Σ is now empty, remove it from Σ. Go to setp 2.
Step 2: If $\Sigma = \emptyset$, stop. If $\Sigma \neq \emptyset$, suppose $\Sigma = \{S_1, S_2, \ldots, S_r\}$. For any $1 \leq j \leq r$, replace the set S_j by two sets T_j and T'_j such that $S_j = T_j \cup T'_j$, $T_j \cap T'_j = \emptyset$, $v_i w \in H$ for any $w \in T_j$ and $v_i w' \notin H$ for any $w' \in T'_j$. Now we set

$$\Sigma := \{T_1, T_2, \ldots, T_r, T'_1, T'_2, \ldots, T'_r\}.$$

Remove all the empty sets from Σ. Go back to step 1.
Output: $v_1, \ldots, v_n$.

Remark 2.3. The above algorithm is a modification of an algorithm of Rose-Tarjan-Lueker. In section 5.2 of [RTL], they set

$$\Sigma := \{T_1, T'_1, T_2, T'_2, \ldots, T_r, T'_r\}.$$

The reason we difine Σ differently in Algorithm 2.2 is illustrated in Example 2.6 and Lemma 3.2.

Before proving Theorem 2.5, we make the following observation.

Lemma 2.4. *Let $v_1, \ldots, v_n$ be an output of Algorithm 2.2. If $v_i v_l \in H$, $v_j v_l \notin H$ and $i < j < l$, then there exists λ with $j < \lambda < l$ such that $v_i v_\lambda \notin H$ and $v_j v_\lambda \in H$.*

Proof. Since $v_i v_l \in H$, $v_j v_l \notin H$ and $i < j < l$, it follows from the algorithm that after v_l is taken from the first set of Σ, v_i and v_j will be in different sets of Σ and the set containing v_i is before the set containing v_j. If there does not exist $j < \lambda < l$ such that $v_i v_\lambda \notin H$ and $v_j v_\lambda \in H$, then after v_{j+1} is taken from the first set of Σ, the set containing v_i is still before the set containing v_j and in particular, v_j is not in the first set of the new Σ. So after removing v_{j+1} we need to remove a vertex different from v_j, which is a contradiction. So there must exist $j < \lambda < l$ such that $v_i v_\lambda \notin H$ and $v_j v_\lambda \in H$. □

Theorem 2.5. *The output of Algorithm 2.2 is a perfect elimination order of the chordal graph H.*

Proof. First, we see that $v_1, \ldots, v_n$ is a reordering of the vertices $x_1, \ldots, x_n$ of H. To show that $v_1, \ldots, v_n$ is a perfect elimination order, we need only show that for any $1 \leq i < j < l \leq n$, if $v_i v_j \in H$ and $v_i v_l \in H$, then $v_j v_l \in H$. Assume to the contrary that $v_j v_l \notin H$.

Since $v_i v_l \in H$, $v_j v_l \notin H$ and $i < j < l$, Lemma 2.4 implies that there exists $j < \lambda_1 < l$ such that $v_i v_{\lambda_1} \notin H$ and $v_j v_{\lambda_1} \in H$. And we choose the largest λ_1 which satisfies this property. If $v_{\lambda_1} v_l \in H$, then $(v_i v_j v_{\lambda_1} v_l)$ is a cycle of length 4 with no chord, which contradicts to the assumption that H is chordal. So $v_{\lambda_1} v_l \notin H$.

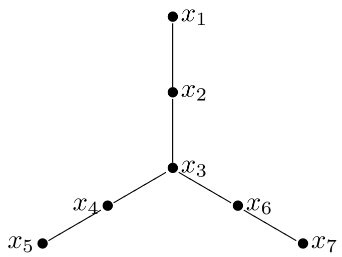
Since $v_i v_l \in H$, $v_{\lambda_1} v_l \notin H$ and $i < \lambda_1 < l$, Lemma 2.4 implies that there exists $\lambda_1 < \lambda_2 < l$ such that $v_i v_{\lambda_2} \notin H$ and $v_{\lambda_1} v_{\lambda_2} \in H$. And we choose the largest λ_2 which satisfies this property. Note that by the choice of λ_1, we have that $v_j v_{\lambda_2} \notin H$. If $v_{\lambda_2} v_l \in H$, then $(v_i v_j v_{\lambda_1} v_{\lambda_2} v_l)$ is a cycle of length 5 with no chord, which contradicts to the assumption that H is chordal. So $v_{\lambda_2} v_l \notin H$.

Since $v_i v_l \in H$, $v_{\lambda_2} v_l \notin H$ and $i < \lambda_2 < l$, Lemma 2.4 implies that there exists $\lambda_2 < \lambda_3 < l$ such that $v_i v_{\lambda_3} \notin H$ and $v_{\lambda_2} v_{\lambda_3} \in H$. And we choose the largest λ_3 which satisfies this property. Note that by the choices of λ_1 and λ_2, we have that $v_j v_{\lambda_3} \notin H$ and $v_{\lambda_1} v_{\lambda_3} \notin H$. If $v_{\lambda_3} v_l \in H$, then $(v_i v_j v_{\lambda_1} v_{\lambda_2} v_{\lambda_3} v_l)$ is a cycle of length 6 with no chord, which contradicts to the assumption that H is chordal. So $v_{\lambda_3}v_l \notin H$.

Proceeding in the same way, we get an infinite sequence of vertices $v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3}, \ldots$ such that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. This is a contradiction because there are only finitely many vertices. So $v_j v_l \in H$ and we are done. $\square$

The following example illustrates the difference among different perfect elimination orders.

Example 2.6. Let H be the following chordal graph.



Then $x_7, x_6, x_5, x_1, x_4, x_2, x_3$ is a perfect elimination order of H, but it can not be produced by Algorithm 2.2 or the algorithm in [RTL]; $x_7, x_5, x_6, x_4, x_3, x_2, x_1$ is a perfect elimination order which can be produced by the algorithm in [RTL] ; $x_7, x_6, x_5, x_4, x_3, x_2, x_1$ is a perfect elimination order which is produced by Algorithm 2.2.

If we compare these three perfect elimination orders, the third one looks nicer in the sense that there is no unnecessary "jump" in the perfect elimination order. Here, "jump" means going from one branch of the star-shaped graph H to another branch. For example, in the first perfect elimination order, x_5 is followed by x_1 instead of x_4; in the second perfect elimination order, x_7 is followed by x_5 instead of x_6. However, in the third perfect elimination order, this kind of "jump" does not happen unless it is necessary, say, x_6 is followed by x_5. This nice property of the perfect elimination orders produced by Algorithm 2.2 is reflected in Lemma 3.2 .

3. Construction of the Resolution

Let G be a simple graph with vertices $x_1, \ldots, x_n$. The *complement graph* $\overline{G}$ of G is the simple graph with the same vertex set whose edges are the non-edges of G. The *subgraph* of G *induced by vertices* $x_{i_1}, \ldots, x_{i_r}$ for some $1 \le i_1 < \cdots < i_r \le n$ is the simple graph with the vertices $x_{i_1}, \ldots, x_{i_r}$ and the edges that connect them in G. We define the *preneighborhood* of a vertex x_j in G to be the set

$$pnbhd(x_j) = \{x_i \mid i < j, x_i x_j \in G\}.$$

The following two lemmas will be important in section 3 and section 4.

Lemma 3.1. *Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$. For any $1 \le i < j < l \le n$, if $x_i x_j \in G$, then $x_i x_l \in G$ or $x_j x_l \in G$. In particular, if $pnbhd(x_i) \nsubseteq pnbhd(x_j)$ for some $1 \le i < j \le n$ then $x_i x_j \in G$.*

Proof. Assume to the contrary that $x_i x_l \notin G$ and $x_j x_l \notin G$, then $x_i x_l \in \overline{G}$ and $x_j x_l \in \overline{G}$. Since $x_1, \ldots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$, we have $x_i x_j \in \overline{G}$, and hence $x_i x_j \notin G$, which is a contradiction. $\square$

Lemma 3.2. *Let G be a simple graph with vertices $x_1, \dots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \dots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 2.2.*

(1) *If $x_i x_j \in \overline{G}$ for some $i < j$, then for any $i < t \leq j$ we have $pnbhd(x_i) \subseteq pnbhd(x_t)$ in G.*
(2) *If $pnbhd(x_i) \nsubseteq pnbhd(x_t)$ in G for some $i < t$, then $x_i x_j \in G$ for all $j \geq t$.*

Proof. Note that part (1) and part (2) are equivalent, so we only need to prove part (1). Assume to the contrary that there exists $i < t \leq j$ such that $\mathrm{pnbhd}(x_i) \nsubseteq \mathrm{pnbhd}(x_t)$ in G. We choose the minimal t which satisfies this property. Then there exists $l < i$ such that $x_l x_i \notin \overline{G}, x_l x_t \in \overline{G}$. Since $x_1, \dots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$, we must have that $x_i x_t \notin \overline{G}$ and in particular $t \neq j$. Now since $x_i x_t \notin \overline{G}, x_i x_j \in \overline{G}$ and $i < t < j$, Lemma 2.4 implies that there exists $i < m < t$ such that $x_m x_t \in \overline{G}, x_m x_j \notin \overline{G}$. However, $x_m x_t \in \overline{G}, x_l x_t \in \overline{G}$ and $l < m < t$ imply that $x_l x_m \in \overline{G}$, so that $\mathrm{pnbhd}(x_i) \nsubseteq \mathrm{pnbhd}(x_m)$ and $i < m < t < j$, which contradicts to the minimality of t. So for all $i < t \leq j$, $\mathrm{pnbhd}(x_i) \subseteq \mathrm{pnbhd}(x_t)$ in G. $\square$

Let G be a simple graph with vertices $x_1, \dots, x_n$. The *edge ideal* I_G of the graph G is the monomial ideal in the polynomial ring $S = k[x_1, \dots, x_n]$ with the minimal generating set $\{x_i x_j \mid x_i x_j \in G\}$. An important result about edge ideals was obtained by Fröberg in [Fr].

Theorem 3.3 (Fröberg)**.** *Let I_G be the edge ideal of a simple graph G. Then I_G has a linear free resolution if and only if $\overline{G}$ is chordal.*

By the above theorem, the edge ideal I_G of a simple graph G is called a *linear edge ideal* if $\overline{G}$ is chordal. The goal of this section is to construct the minimal free resolution of S/I_G where I_G is a linear edge ideal.

Construction 3.4. Let G be a simple graph with vertices $x_1, \dots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \dots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 2.2.

If $p \geq 1, q \geq 1, 1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq n$ and $\{x_{i_1}, \dots, x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_1})$, then the symbol $(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q})$ will be used to denote the generator of the free S-module $S(-x_{i_1} \cdots x_{i_p} x_{j_1} \cdots x_{j_q})$ in homological degree $p + q - 1$ and multidegree $x_{i_1} \cdots x_{i_p} x_{j_1} \cdots x_{j_q}$. We set

$$\mathcal{B} = \{1\} \cup \bigcup_{p \geq 1, q \geq 1} \left\{ (x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}) : \begin{array}{l} 1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq n \\ \{x_{i_1}, \dots, x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_1}) \end{array} \right\}.$$

We define the map d on the set $\mathcal{B}$ by $d(1) = 1$, $d(x_{i_1}|x_{j_1}) = x_{i_1}x_{j_1}$, and for $p+q \geq 3$,

$$
\begin{aligned}
&d(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q})\\
&=\sum_{s=1}^{p}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q})\\
&+\sum_{t=1}^{q}(-1)^{t+p}x_{j_t}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+\sum_{s=1}^{p}(-1)^{s+1+\beta}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_\beta},\dots,x_{j_q})\\
&+(-1)^{p}x_{j_\beta}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q}),
\end{aligned}
$$

where $\beta = \min\{t \mid 2 \leq t \leq q, \{x_{i_1},\dots,x_{i_p}\} \nsubseteq \operatorname{pnbhd}(x_{j_t})\}$.

Note that if $\{x_{i_1},\dots,x_{i_p}\} \subseteq \operatorname{pnbhd}(x_{j_t})$ for all $1 \leq t \leq q$, then β does not exist and there are no β terms in the above formula. Also, if $p+q \geq 3$, then the formula of d may yield symbols which are not in $\mathcal{B}$ and we will regard them as zeros. And Lemma 3.2 implies that for any $1 \leq t \leq \beta-1$ and $\beta \leq t' \leq q$, we have $x_{j_t}x_{j_{t'}} \in G$.

Example 3.5. The following are some examples for the formula of d.

(1). If $p \geq 2$ and $q = 1$, then just like the Koszul complex, we have that

$$d(x_{i_1},\dots,x_{i_p}|x_{j_1}) = \sum_{s=1}^{p}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1}).$$

(2). If $p \geq 2$, $q = 3$, $\{x_{i_1},\dots,x_{i_p}\}\backslash\operatorname{pnbhd}(x_{j_2}) = \{x_{i_1}\}$ and $\{x_{i_1},\dots,x_{i_p}\} \subseteq \operatorname{pnbhd}(x_{j_3})$, then $\beta = 2$ and a computation will reveal that

$$
\begin{aligned}
&d(x_{i_1},\dots,x_{i_p}|x_{j_1},x_{j_2},x_{j_3})\\
&=x_{i_1}[(x_{i_2},\dots,x_{i_p}|x_{j_1},x_{j_2},x_{j_3})+(x_{i_2},\dots,x_{i_p},x_{j_1}|x_{j_2},x_{j_3})]\\
&+\sum_{s=2}^{p}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},x_{j_2},x_{j_3})\\
&+(-1)^{2+p}x_{j_2}[(x_{i_1},\dots,x_{i_p}|x_{j_1},x_{j_3})+(x_{i_1},\dots,x_{i_p},x_{j_1}|x_{j_3})]\\
&+(-1)^{3+p}x_{j_3}(x_{i_1},\dots,x_{i_p}|x_{j_1},x_{j_2}).
\end{aligned}
$$

(3). If $p \geq 2$, $q \geq 4$, $\beta = 3$, $\{x_{i_1},\dots,x_{i_p}\}\backslash\operatorname{pnbhd}(x_{j_3}) = \{x_{i_1},x_{i_2}\}$ and $\{x_{i_1},\dots,x_{i_p}\} \nsubseteq \operatorname{pnbhd}(x_{j_4})$, then a computation will reveal that

$$
\begin{aligned}
d(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q}) =&\sum_{s=1}^{p}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q})\\
&+\sum_{t=1}^{q}(-1)^{t+p}x_{j_t}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q}).
\end{aligned}
$$

Lemma 3.6. *Let d be the map defined in Construction 3.4. Then $d^2 = 0$.*

The proof of the above lemma is very long and is given in section 4. The next theorem is the main result of this paper.

Theorem 3.7. *Let* $\mathbf{F}$ *be the multigraded complex of free* S*-modules with basis* $\mathcal{B}$ *and differential* d *as defined in Construction 3.4. Then* $\mathbf{F}$ *is the minimal free resolution of* S/I_G.

Proof. We prove by induction on the number of vertices of the graph G. If G has one or two vertices then it is clear. Now as in Construction 3.4, let G have vertices $x_1, \dots, x_n$ with $n \geq 3$.

If $\operatorname{pnbhd}(x_n) = \emptyset$ in G, then $x_i x_n \in \overline{G}$ for all $1 \leq i \leq n-1$. Since $x_1, \dots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$, it follows that $\overline{G}$ is a complete graph, so that G has no edges. Hence $I_G = (0)$ and there is nothing to prove. Next we will assume that $\operatorname{pnbhd}(x_n) = \{x_{\lambda_1}, \dots, x_{\lambda_r}\}$ for some $1 \leq \lambda_1 < \cdots < \lambda_r \leq n-1$.

Let G' be the graph obtained from G by deleting the edges $x_{\lambda_1}x_n, \dots, x_{\lambda_r}x_n$. Then I_G and $I_{G'}$ are both edge ideals in S. Note that $\overline{G'}$ is chordal. Indeed, it is easy to see that $x_n, x_1, x_2, \dots, x_{n-1}$ is in the reverse order of a perfect elimination order of $\overline{G'}$ produced by Algorithm 2.2. Setting $J = (x_{\lambda_1}, \dots, x_{\lambda_r}) \subseteq S$, we have $I_G = I_{G'} + x_n J$ and a natural short exat sequence

$$0 \longrightarrow \frac{I_{G'} + x_n J}{I_{G'}} \longrightarrow \frac{S}{I_{G'}} \longrightarrow \frac{S}{I_G} = \frac{S}{I_{G'} + x_n J} \longrightarrow 0.$$

Note that $x_n J \cap I_{G'} = x_n I_{G'}$: indeed, by Lemma 3.1 we see that $I_{G'} \subseteq J$ and hence $x_n I_{G'} \subseteq x_n J \cap I_{G'}$; on the other hand, if $x_n m \in I_{G'}$ for some monomial $m \in J$, then $m \in I_{G'}$, and hence $x_n J \cap I_{G'} \subseteq x_n I_{G'}$. Therefore,

$$\frac{I_{G'} + x_n J}{I_{G'}} \cong \frac{x_n J}{x_n J \cap I_{G'}} = \frac{x_n J}{x_n I_{G'}}.$$

Let G'' be the subgraph of G induced by the vertices $x_1, \dots, x_{n-1}$. Then $\overline{G''}$ is chordal and $x_1, \dots, x_{n-1}$ is in the reverse order of a perfect elimination order of $\overline{G''}$ produced by Algorithm 2.2. Let $S' = k[x_1, \dots, x_{n-1}] \subseteq S$. Then $I_{G''}$ is an edge ideal in the polynomial ring S' and $I_{G''}S = I_{G'}$. Set

$$\mathcal{B}' = \{1\} \cup \bigcup_{p \geq 1, q \geq 1} \left\{ (x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}) : \begin{array}{l} (x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}) \in \mathcal{B} \\ j_q \leq n-1 \end{array} \right\}.$$

Suppose that $\mathbf{L}$ is the multigraded complex of free S'-modules with basis $\mathcal{B}'$ and differential $d_{\mathbf{L}} = d$ as defined in Construction 3.4, then by the induction hypothesis, $\mathbf{L}$ is the minimal free resolution of $S'/I_{G''}$. Let $\mathbf{F}' = \mathbf{L} \bigotimes S$. Since $S = S'[x_n]$ is a flat S'-module, it follows that $\mathbf{F}'$ is the multigraded minimal free resolution of the S-module $S'/I_{G''} \bigotimes S = S/(I_{G''}S) = S/I_{G'}$, and $\mathbf{F}'$ has basis $\mathcal{B}'$ and differential $d' = d_{\mathbf{L}} = d$ as in Construction 3.4. Setting

$$\mathcal{A} = \{(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) : (x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}) \in \mathcal{B}'\},$$
$$\mathcal{T} = \{(x_{i_1}, \dots, x_{i_p} | x_n) : p \geq 1, \{x_{i_1}, \dots, x_{i_p}\} \subseteq \operatorname{pnbhd}(x_n)\},$$

we have the disjoint union

$$\mathcal{B} = \mathcal{B}' \cup \mathcal{A} \cup \mathcal{T}.$$

Let $\mathbf{E} : \cdots \to E_1 \to E_0 \to x_n I_{G'}$ be the multigraded minimal free resolution of $x_n I_{G'}$ induced naturally by the minimal free resolution $\mathbf{F}'$ of $S/I_{G'}$. Then $\mathbf{E}$ has basis $\mathcal{A}$ and the basis element $(x_{i_1}, \dots, x_{i_p} \mid x_{j_1}, \dots, x_{j_q}, x_n)$ is in homological degree $p+q-2$ in $\mathbf{E}$. We denote the differential of $\mathbf{E}$ by $d_{\mathbf{E}}$. Note that $d_{\mathbf{E}}(x_{i_1} \mid x_{j_1}, x_n) = x_{i_1} x_{j_1} x_n$. Let $\mathbf{K}$ be the multigraded complex of free S-modules with basis $\mathcal{T}$ and differential $-\partial = -d$ where d is as in Construction 3.4. Note that the basis element $(x_{i_1},\dots,x_{i_p}\mid x_n)$ is in homological degree $p-1$ in $\mathbf{K}$. And it is easy to see that $\mathbf{K}$ is the minimal free resolution of x_nJ.

For any $(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)\in\mathcal{A}$, we have that

$$\begin{aligned} d(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n) &= \mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)\\ &\quad+\mu_2(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)\\ &\quad+\mu_3(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n),\end{aligned}$$

where $\mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)$ is the sum of the terms of $d(x_{i_1},\dots,x_{i_p}\mid x_{j_1},\dots,x_{j_q},x_n)$ that contain basis elements in $\mathcal{A}$, $\mu_2(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)$ is the sum of the terms that contain basis elements in $\mathcal{T}$ and $\mu_3(x_{i_1},\dots,x_{i_p}\mid x_{j_1},\dots,x_{j_q},x_n)$ is the sum of the terms that contain basis elements in $\mathcal{B}'$. Note that $\mu_3(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)=(-1)^{q+1+p}x_n(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q})$. And by the definition of d, we can check that if $p+q\geq 3$, then

$$\mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)=d_{\mathbf{E}}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n).$$

We claim that $-\mu_2:\mathbf{E}\to\mathbf{K}$ is a multigraded complex map of degree 0 lifting the inclusion map $\phi: x_nI_{G'}\to x_nJ$. Indeed, $\phi d_{\mathbf{E}}(x_{i_1}|x_{j_1},x_n)=x_{i_1}x_{j_1}x_n$, and

$$\begin{aligned}(-\partial)(-\mu_2)(x_{i_1}|x_{j_1},x_n) &= \begin{cases}\partial(x_{j_1}(x_{i_1}|x_n)), & \text{if } x_{i_1}x_n\in G\\ \partial(x_{i_1}(x_{j_1}|x_n)), & \text{if } x_{i_1}x_n\notin G\end{cases}\\ &= x_{i_1}x_{j_1}x_n.\end{aligned}$$

Hence, $\phi d_{\mathbf{E}}(x_{i_1}|x_{j_1},x_n)=(-\partial)(-\mu_2)(x_{i_1}|x_{j_1},x_n)$. Then we need to show that for $p+q\geq 3$,

$$(-\mu_2)d_{\mathbf{E}}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)=(-\partial)(-\mu_2)(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n).$$

By Lemma 3.6, we have that

$$\begin{aligned}(1)\qquad 0 &= d^2(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)\\ &= \mu_1\mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)+\mu_2\mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)\\ &\quad+\mu_3\mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)+\partial\mu_2(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)\\ &\quad+d\mu_3(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n).\end{aligned}$$

In the above formula, collecting the terms which contain basis elements in $\mathcal{T}$, we get

$$\mu_2\mu_1(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)+\partial\mu_2(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)=0.$$

Since $\mu_1=d_{\mathbf{E}}$ for $p+q\geq 3$, it follows that

$$(-\mu_2)d_{\mathbf{E}}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n)=(-\partial)(-\mu_2)(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q},x_n),$$

and the claim is proved.

Let $\mathbf{F}''$ be the mapping cone $\mathrm{MC}(-\mu_2)$. Then $\mathbf{F}'':\cdots\to F_1''\to F_0''\to x_nJ/x_nI_{G'}$ is a multigraded free resolution of $x_nJ/x_nI_{G'}$. Note that $F_0''=K_0$ and $F_i''=E_{i-1}\bigoplus K_i$ for $i\geq 1$. If we denote the differential of $\mathbf{F}''$ by d'', then $d_0''(x_{i_1}|x_n)=-\partial(x_{i_1}|x_n)=-x_{i_1}x_n$, $d_1''(x_{i_1}|x_{j_1},x_n)=-\mu_2(x_{i_1}|x_{j_1},x_n)$, $d_1''(x_{i_1},x_{i_2}|x_n)=-\partial(x_{i_1},x_{i_2}|x_n)$, that is, $d_1''=(-\mu_2,-\partial)$, and for $i\geq 2$,

$$d_i''=\begin{pmatrix}-d_{\mathbf{E}} & 0\\ -\mu_2 & -\partial\end{pmatrix}=\begin{pmatrix}-\mu_1 & 0\\ -\mu_2 & -\partial\end{pmatrix}.$$

Since the differential matrices of $\mathbf{F}''$ have monomial entries, $\mathbf{F}''$ is the minimal free resolution of $x_nJ/x_nI_{G'}\cong(I_{G'}+x_nJ)/I_{G'}$.

Next we define a map $\mu : \mathbf{F}'' \to \mathbf{F}'$ such that $\mu : F_0'' = K_0 \to F_0' = S$ is given by $\mu(x_{i_1}|x_n) = x_{i_1}x_n$ and for $i \geq 1$, $\mu : F_i'' = E_{i-1} \bigoplus K_i \to F_i'$ is given by $\mu = (\mu_3, 0)$. We claim that $-\mu$ is a multigraded complex map of degree 0 lifting the inclusion map $\psi : (I_{G'} + x_nJ)/I_{G'} \to S/I_{G'}$. Indeed, if $i = 0$ then $\psi d_0''(x_{i_1}|x_n) = -x_{i_1}x_n$, $d_0'(-\mu)(x_{i_1}|x_n) = -x_{i_1}x_n$, and hence $\psi d_0'' = d_0'(-\mu)$. If $i = 1$ then

$$\begin{aligned}
(-\mu)d_1''(x_{i_1}|x_{j_1}, x_n) &= (-\mu)(-\mu_2)(x_{i_1}|x_{j_1}, x_n) \\
&= \begin{cases} \mu(x_{j_1}(x_{i_1}|x_n)), & \text{if } x_{i_1}x_n \in G \\ \mu(x_{i_1}(x_{j_1}|x_n)), & \text{if } x_{i_1}x_n \notin G \end{cases} \\
&= x_{i_1}x_{j_1}x_n, \\
d_1'(-\mu)(x_{i_1}|x_{j_1}, x_n) &= d_1'(x_n(x_{i_1}|x_{j_1})) \\
&= x_{i_1}x_{j_1}x_n, \\
(-\mu)d_1''(x_{i_1}, x_{i_2}|x_n) &= (-\mu)(-\partial)(x_{i_1}, x_{i_2}|x_n) \\
&= \mu(x_{i_1}(x_{i_2}|x_n) - x_{i_2}(x_{i_1}|x_n)) \\
&= x_{i_1}x_{i_2}x_n - x_{i_2}x_{i_1}x_n = 0, \\
d_1'(-\mu)(x_{i_1}, x_{i_2}|x_n) &= d_1'(0) = 0,
\end{aligned}$$

and hence $(-\mu)d_1'' = d_1'(-\mu)$. If $i \geq 2$ then it is easy to see that for $p \geq 3$,

$$(-\mu)d_i''(x_{i_1}, \ldots, x_{i_p}|x_n) = d_i'(-\mu)(x_{i_1}, \ldots, x_{i_p}|x_n) = 0,$$

so we need only to prove that for $p + q = i + 1 \geq 3$,

$$(-\mu)d_i''(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = d_i'(-\mu)(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n),$$

that is,

$$\mu(-\mu_1 - \mu_2)(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = d\mu_3(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n).$$

Since $\mu\mu_2(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = 0$, it suffices to prove that

$$-\mu_3\mu_1(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = d\mu_3(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n).$$

However, in formula (1), collecting the terms which contain basis elements in $\mathcal{B}'$, we see that

$$\mu_3\mu_1(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) + d\mu_3(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = 0,$$

and the claim is proved. So $\mu : \mathbf{F}'' \to \mathbf{F}'$ is a complex map lifting $-\psi : (I_{G'} + x_nJ)/I_{G'} \to S/I_{G'}$, and it is eay to see that μ is multigraded of degree 0.

Let $\mathbf{F}^*$ be the mapping cone $\mathrm{MC}(\mu)$. Then $\mathbf{F}^* : \cdots \to F_1^* \to F_0^* \to \mathrm{coker}(-\psi)$ gives a multigraded free resolution of $\mathrm{coker}(-\psi) = S/I_G$. Note that $F_0^* = S$, $F_1^* = F_0'' \bigoplus F_1' = K_0 \bigoplus F_1'$ and for $i \geq 2$, $F_i^* = F_{i-1}'' \bigoplus F_i' = E_{i-2} \bigoplus K_{i-1} \bigoplus F_i'$. If we denote the differential of $\mathbf{F}^*$ by d^*, then $d_0^*(1) = 1$, $d_1^* = (\mu, d_1')$,

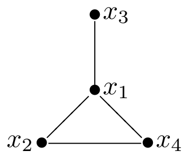
$$d_2^* = \begin{pmatrix} -d_1'' & 0 \\ \mu & d_2' \end{pmatrix} = \begin{pmatrix} \mu_2 & \partial & 0 \\ \mu_3 & 0 & d \end{pmatrix},$$

and for $i \geq 3$,

$$d_i^* = \begin{pmatrix} -d_{i-1}'' & 0 \\ \mu & d_i' \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 & 0 \\ \mu_2 & \partial & 0 \\ \mu_3 & 0 & d \end{pmatrix}.$$

Note that $\mathbf{F}^*$ and $\mathbf{F}$ have the same basis and the same differential. So $\mathbf{F}^* = \mathbf{F}$, and then $\mathbf{F}$ is a multigraded free resolution of S/I_G. Since $d_i(F_i) \subseteq (x_1, \ldots, x_n)F_{i-1}$ for all $i \geq 1$, the resolution $\mathbf{F}$ is minimal, and we are done. $\square$

Example 3.8. Let G be the following graph.



Then $\overline{G}$ is chordal and x_1, x_2, x_3, x_4 is in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 2.2. Note that

$$S = k[x_1, x_2, x_3, x_4], \quad I_G = (x_1x_2, x_1x_3, x_1x_4, x_2x_4),$$

$$\text{pnbhd}(x_1) = \emptyset, \ \text{pnbhd}(x_2) = \{x_1\}, \ \text{pnbhd}(x_3) = \{x_1\}, \ \text{pnbhd}(x_4) = \{x_1, x_2\}.$$

By Construction 3.4, the minimal free resolution of S/I_G has basis

$$1; (x_1|x_2, x_3, x_4), (x_1|x_2, x_3), (x_1|x_2, x_4), (x_1|x_2);$$
$$(x_1|x_3, x_4), (x_1|x_3); (x_1, x_2|x_4), (x_1|x_4), (x_2|x_4).$$

And we have the map d such that

$$\begin{aligned}
&d(x_1|x_2) = x_1x_2, \quad d(x_1|x_3) = x_1x_3,\\
&d(x_1|x_4) = x_1x_4, \quad d(x_2|x_4) = x_2x_4,\\
&d(x_1|x_2, x_3) = x_2(x_1|x_3) - x_3(x_1|x_2),\\
&d(x_1|x_2, x_4) = x_2(x_1|x_4) - x_4(x_1|x_2),\\
&d(x_1|x_3, x_4) = x_3(x_1|x_4) - x_4(x_1|x_3),\\
&d(x_1, x_2|x_4) = x_1(x_2|x_4) - x_2(x_1|x_4),\\
&d(x_1|x_2, x_3, x_4) = x_2(x_1|x_3, x_4) - x_3(x_1|x_2, x_4) + x_4(x_1|x_2, x_3).
\end{aligned}$$

Therefore, the minimal free resolution of S/I_G is

$$\begin{aligned}
0 \to S(-x_1x_2x_3x_4) &\xrightarrow{d_3} S(-x_1x_2x_3) \oplus S(-x_1x_2x_4) \oplus S(-x_1x_3x_4) \oplus S(-x_1x_2x_4)\\
&\xrightarrow{d_2} S(-x_1x_2) \oplus S(-x_1x_3) \oplus S(-x_1x_4) \oplus S(-x_2x_4) \xrightarrow{d_1} S \to S/I_G,
\end{aligned}$$

where

$$d_3 = \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ 0 \end{pmatrix}, d_2 = \begin{pmatrix} -x_3 & -x_4 & 0 & 0 \\ x_2 & 0 & -x_4 & 0 \\ 0 & x_2 & x_3 & -x_2 \\ 0 & 0 & 0 & x_1 \end{pmatrix}, d_1 = \begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_2x_4 \end{pmatrix}.$$

Remark 3.9. In the above example, we have that $\text{pnbhd}(x_1) \subseteq \text{pnbhd}(x_2) \subseteq \text{pnbhd}(x_3) \subseteq \text{pnbhd}(x_4)$. But in general, given a linear edge ideal I_G, there may not exist a perfect elimination order of $\overline{G}$ such that its reverse order $x_1, \ldots, x_n$ satisfies $\text{pnbhd}(x_i) \subseteq \text{pnbhd}(x_{i+1})$ in G for $i = 1, \ldots, n-1$. For example, if $\overline{G}$ is the star-shaped chordal graph in Example 2.6, then we can check that $\overline{G}$ has no perfect elimination order satisfying the above property. However, the following proposition says that if the above property is satisfied then the perfect elimination order of $\overline{G}$ can be produced by Algorithm 2.2.

Proposition 3.10. *Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ such that $\text{pnbhd}(x_i) \subseteq \text{pnbhd}(x_{i+1})$ in G for $i = 1, \ldots, n-1$. Then the perfect elimination order $x_n, \ldots, x_1$ of $\overline{G}$ can be produced by Algorithm 2.2.*

Proof. First we choose $v_n = x_1$ in Algorithm 2.2. Since $\mathrm{pnbhd}(x_2) \subseteq \mathrm{pnbhd}(x_j)$ in G for any $2 < j \leq n$, it follows that if $x_1x_2 \notin \overline{G}$ then $x_1x_j \notin \overline{G}$ for all $2 < j \leq n$, so that in Algorithm 2.2 we can choose $v_{n-1} = x_2$. Now suppose that we have chosen $v_n = x_1, v_{n-1} = x_2, \ldots, v_{n-(i-2)} = x_{i-1}$ for some $3 \leq i \leq n$. Since $\mathrm{pnbhd}(x_i) \subseteq \mathrm{pnbhd}(x_j)$ in G for any $i < j \leq n$, it follows that for any $1 \leq l \leq i-1$, if $x_lx_i \notin \overline{G}$ then $x_lx_j \notin \overline{G}$ for all $i < j \leq n$, so that in Algorithm 2.2 we can choose $v_{n-(i-1)} = x_i$. So by using induction we see that $x_n, \ldots, x_1$ can be the output of Algorithm 2.2 and we are done. □

Remark 3.11. If the conditions in the above proposition are satisfied, then there will be no β terms in the differential formula. However, as we have seen in Remark 3.9, the conditions in the above proposition can not always be satisfied, especially when $\overline{G}$ is a complicated chordal graph. So in general, the β terms in the differential formula can not be avoided.

Remark 3.12. Let $G = K_n$ be the complete graph with n vertices $x_1, \ldots, x_n$. Then we have the Eliahou-Kervaire resolution of S/I_G. It is easy to see that the basis element $(x_ix_j; i_1, \ldots, i_p, j_1, \ldots, j_q)$ with $i_1 < \cdots < i_p < i < j_1 < \cdots < j_q < j$ in the Eliahou-Kervaire resolution corresponds naturally to the basis element $(x_{i_1}, \ldots, x_{i_p}, x_i | x_{j_1}, \ldots, x_{j_q}, x_j)$ in Construction 3.4. But the differential maps defined on them are different. For example, if $G = K_3$, then $d(x_2x_3; 1) = x_1(x_2x_3; \emptyset) - x_3(x_1x_2; \emptyset)$, but $d(x_1, x_2|x_3) = x_1(x_2|x_3) - x_2(x_1|x_3)$. So in the case of complete graphs, the resolution defined in Construction 3.4 does not recover the Eliahou-Kervaire resolution. By contrast, the resolution in [Ho] recovers the Eliahou-Kervaire resolution in the case of complete graphs.

4. The Proof of $d^2 = 0$

Before proving Lemma 3.6, we look at the following example.

Example 4.1. Let G be the graph such that $\overline{G}$ is the chordal graph given in Example 2.6. Then $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ is in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 2.2. Note that in G,

$$\mathrm{pnbhd}(x_5) = \{x_1, x_2, x_3\} \not\subseteq \mathrm{pnbhd}(x_6) = \{x_1, x_2, x_4, x_5\}.$$

Next we check that $d^2(x_1, x_2, x_3|x_5, x_6) = 0$. In fact, by the definition of d, we have that

$$\begin{aligned}
d(x_1, x_2, x_3|x_5, x_6) =& x_1(x_2, x_3|x_5, x_6) - x_2(x_1, x_3|x_5, x_6) \\
&+ x_3[(x_1, x_2|x_5, x_6) + (x_1, x_2, x_5|x_6)] - x_6(x_1, x_2, x_3|x_5), \\
d(x_1(x_2, x_3|x_5, x_6)) =& x_1x_2(x_3|x_5, x_6) - x_1x_3[(x_2|x_5, x_6) + (x_2, x_5|x_6)] \\
&+ x_1x_6(x_2, x_3|x_5), \\
d(-x_2(x_1, x_3|x_5, x_6)) =& -x_2x_1(x_3|x_5, x_6) + x_2x_3[(x_1|x_5, x_6) + (x_1, x_5|x_6)] \\
&- x_2x_6(x_1, x_3|x_5), \\
d(x_3(x_1, x_2|x_5, x_6)) =& x_3x_1(x_2|x_5, x_6) - x_3x_2(x_1|x_5, x_6) \\
&- x_3x_5(x_1, x_2|x_6) + x_3x_6(x_1, x_2|x_5), \\
d(x_3(x_1, x_2, x_5|x_6)) =& x_3x_1(x_2, x_5|x_6) - x_3x_2(x_1, x_5|x_6) + x_3x_5(x_1, x_2|x_6), \\
d(-x_6(x_1, x_2, x_3|x_5)) =& -x_6x_1(x_2, x_3|x_5) + x_6x_2(x_1, x_3|x_5) - x_6x_3(x_1, x_2|x_5).
\end{aligned}$$

So the sum of the terms in $d^2(x_1,x_2,x_3|x_5,x_6)$ containing x_1x_2 is

$$x_1x_2(x_3|x_5,x_6) - x_2x_1(x_3|x_5,x_6) = 0;$$

the sum of the terms in $d^2(x_1,x_2,x_3|x_5,x_6)$ containing x_1x_3 is

$$-x_1x_3[(x_2|x_5,x_6)+(x_2,x_5|x_6)] + x_3x_1(x_2|x_5,x_6) + x_3x_1(x_2,x_5|x_6) = 0;$$

and similarly, we have

$$\begin{aligned} x_2x_3[(x_1|x_5,x_6)+(x_1,x_5|x_6)] - x_3x_2(x_1|x_5,x_6) - x_3x_2(x_1,x_5|x_6) = 0,\\ -x_3x_5(x_1,x_2|x_6) + x_3x_5(x_1,x_2|x_6) = 0,\\ x_1x_6(x_2,x_3|x_5) - x_6x_1(x_2,x_3|x_5) = 0,\\ -x_2x_6(x_1,x_3|x_5) + x_6x_2(x_1,x_3|x_5) = 0,\\ x_3x_6(x_1,x_2|x_5) - x_6x_3(x_1,x_2|x_5) = 0. \end{aligned}$$

Therefore, $d^2(x_1,x_2,x_3|x_5,x_6) = 0$.

Proof of Lemma 3.6. First we have that

$$\begin{aligned} d^2(x_{i_1}|x_{j_1}) &= d(x_{i_1}x_{j_1}) = x_{i_1}x_{j_1} = 0 \quad \text{in } S/I_G,\\ d^2(x_{i_1},x_{i_2}|x_{j_1}) &= d(x_{i_1}(x_{i_2}|x_{j_1}) - x_{i_2}(x_{i_1}|x_{j_1}))\\ &= x_{i_1}x_{i_2}x_{j_1} - x_{i_2}x_{i_1}x_{j_1} = 0,\\ d^2(x_{i_1}|x_{j_1},x_{j_2}) &= \begin{cases} d(x_{j_1}(x_{i_1}|x_{j_2}) - x_{j_2}(x_{i_1}|x_{j_1})), & \text{if } x_{i_1}x_{j_2} \in G\\ d(x_{i_1}(x_{j_1}|x_{j_2}) - x_{j_2}(x_{i_1}|x_{j_1})), & \text{if } x_{i_1}x_{j_2} \notin G \end{cases}\\ &= \begin{cases} x_{j_1}x_{i_1}x_{j_2} - x_{j_2}x_{i_1}x_{j_1}, & \text{if } x_{i_1}x_{j_2} \in G\\ x_{i_1}x_{j_1}x_{j_2} - x_{j_2}x_{i_1}x_{j_1}, & \text{if } x_{i_1}x_{j_2} \notin G \end{cases}\\ &= 0. \end{aligned}$$

Next we need only to prove that $d^2(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q}) = 0$ for $p+q \geq 4$. Just as in Example 4.1, it suffices to prove that if we write out all the terms of $d^2(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})$, then given any $\lambda,\lambda' \in \{i_1,\ldots,i_p,j_1,\ldots,j_q\}$, the sum of the terms containing $x_\lambda x_{\lambda'}$ is zero, that is all the terms containing $x_\lambda x_{\lambda'}$ cancel. Hence, a computation will reveal that if β does not exist, that is $\{x_{i_1},\ldots,x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_t})$ for all $1 \leq t \leq q$, then $d^2(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q}) = 0$. So we will assume that $q \geq 2$ and β exists. The proof is case by case and there are five main cases.

[**Case A**]: $\lambda,\lambda' \in \{i_1,\ldots,i_p\}$.

[**Case A-a**]: if $1 \leq s < s' \leq p$ such that $x_{i_s}x_{j_\beta} \in G$ and $x_{i_{s'}}x_{j_\beta} \in G$, then the sum of the terms containing $x_{i_s}x_{i_{s'}}$ is

$$\begin{aligned} &(-1)^{s+1}x_{i_s}(-1)^{s'}x_{i_{s'}}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_{s'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\ &+(-1)^{s'+1}x_{i_{s'}}(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_{s'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q}) = 0. \end{aligned}$$

[**Case A-b**]: suppose that there is a term containing $x_{i_s}x_{i_\alpha}$ for some $1 \leq s,\alpha \leq p$ such that $x_{i_s}x_{j_\beta} \in G$ and $x_{i_\alpha}x_{j_\beta} \notin G$. Without the loss of generality, we assume $s < \alpha$.

Subcase (i): if $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_s}x_{i_\alpha}$ is

$$\begin{aligned} &(-1)^{s+1}x_{i_s}(-1)^{\alpha}x_{i_\alpha}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\ &+(-1)^{\alpha+1}x_{i_\alpha}(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q}) = 0. \end{aligned}$$

Subcase (ii): if $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_\beta})$, then we set

$$\beta'=\min\{t\mid \beta<t\le q,\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Lemma 3.2 implies that for any $\beta\le t\le q$, $x_{j_1}x_{j_t},\ldots,x_{j_{\beta-1}}x_{j_t}\in G$, so we have

$$\beta'=\min\{t\mid \beta<t\le q,\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}\}\not\subseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Subsubcase (ii)(a): if one of the following conditions is satisfied:

1) β' does not exist,
2) $x_{i_s}x_{j_{\beta'}}\in G$,
3) $x_{i_s}x_{j_{\beta'}}\notin G$ and $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_{\beta'}})$,

then the sum of the terms containing $x_{i_s}x_{i_\alpha}$ is

$$\begin{aligned}&(-1)^{s+1}x_{i_s}(-1)^{\alpha}x_{i_\alpha}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,x_{j_q})]\\&+(-1)^{\alpha+1}x_{i_\alpha}[(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\beta}(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,x_{j_q})]=0.\end{aligned}$$

Subsubcase (ii)(b): if $x_{i_s}x_{j_{\beta'}}\notin G$, $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta'}})$, then the sum of the terms containing $x_{i_s}x_{i_\alpha}$ is

$$\begin{aligned}&(-1)^{s+1}x_{i_s}(-1)^{\alpha}x_{i_\alpha}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,x_{j_q})]\\&+(-1)^{\alpha+1}x_{i_\alpha}\{(-1)^{s+1}x_{i_s}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\beta'}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta'-1}}|x_{j_{\beta'}},\ldots,x_{j_q})]\\&+(-1)^{\beta}(-1)^{s+1}x_{i_s}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,x_{j_q})\\&+(-1)^{\beta'-\beta+1}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta'-1}}|x_{j_{\beta'}},\ldots,x_{j_q})]\}=0.\end{aligned}$$

Note that in the above two subsubcases, if $s=1$ and $\alpha=p=2$ then the terms containing $(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})$ are zeros.

[Case A-c]: suppose that there is a term containing $x_{i_\alpha}x_{i_{\alpha'}}$ for some $1\le\alpha<\alpha'\le p$ such that $x_{i_\alpha}x_{j_\beta}\notin G$ and $x_{i_{\alpha'}}x_{j_\beta}\notin G$.

Subcase (i): if $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{i_{\alpha'}}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\alpha'}x_{i_{\alpha'}}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\alpha'+1}x_{i_{\alpha'}}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})=0.\end{aligned}$$

Subcase (ii): if $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{i_{\alpha'}}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\alpha'}x_{i_{\alpha'}}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,x_{j_q})]\\&+(-1)^{\alpha'+1}x_{i_{\alpha'}}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,x_{j_q})]=0.\end{aligned}$$

Note that if $\alpha=1$ and $\alpha'=p=2$, then in the above formula, the two terms containing $(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,\widehat{x_{i_{\alpha'}}},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})$ are zeros.

[Case B]: $\lambda\in\{i_1,\ldots,i_p\}$ and $\lambda'=j_1$.

[Case B-a]: suppose that there is a term containing $x_{i_s}x_{j_1}$ for some $1 \le s \le p$ such that $x_{i_s}x_{j_\beta} \in G$, then it is easy to see that $\beta \ne 2$ and the sum of the terms containing $x_{i_s}x_{j_1}$ is

$$\begin{aligned}&(-1)^{s+1}x_{i_s}(-1)^{1+(p-1)}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_2},\dots,x_{j_q})\\ &+(-1)^{p+1}x_{j_1}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_2},\dots,x_{j_q})=0.\end{aligned}$$

[Case B-b]: suppose that there is a term containing $x_{i_\alpha}x_{j_1}$ for some $1 \le \alpha \le p$ such that $x_{i_\alpha}x_{j_\beta} \notin G$.

Subcase (i): $\beta = 2$. If we have $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then it is easy to see that there is no term containing $x_{i_\alpha}x_{j_1}$, hence we must have $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_\beta})$ and the sum of the terms containing $x_{i_\alpha}x_{j_1}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}[(-1)^{p}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})\\ &+(-1)^{\beta}(-1)^{p+1}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},\widehat{x_{j_1}}|x_{j_2},\dots,x_{j_q})]=0.\end{aligned}$$

Subcase (ii): if $\beta > 2$ and $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{j_1}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{p}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})\\ &+(-1)^{p+1}x_{j_1}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})=0.\end{aligned}$$

Subcase (iii): if $\beta > 2$ and $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{j_1}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}[(-1)^{p}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})\\ &+(-1)^{\beta}(-1)^{p+1}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},\widehat{x_{j_1}},x_{j_2},\dots,x_{j_{\beta-1}}|x_{j_\beta},\dots,x_{j_q})]\\ &+(-1)^{p+1}x_{j_1}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})\\ &+(-1)^{\beta-1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},\widehat{x_{j_1}},x_{j_2},\dots,x_{j_{\beta-1}}|x_{j_\beta},\dots,x_{j_q})]=0.\end{aligned}$$

[Case C]: $\lambda \in \{i_1,\dots,i_p\}$ and $\lambda' \in \{j_2,\dots,j_q\}$.

[Case C-a]: if $1 \le s \le p$, $2 \le t \le q$ such that $x_{i_s}x_{j_\beta} \in G$ and $t \ne \beta$, then the sum of the terms containing $x_{i_s}x_{j_t}$ is

$$\begin{aligned}&(-1)^{s+1}x_{i_s}(-1)^{t+(p-1)}x_{j_t}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\ &+(-1)^{t+p}x_{j_t}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})=0.\end{aligned}$$

[Case C-b]: suppose that there is a term containing $x_{i_\alpha}x_{j_t}$ for some $1 \le \alpha \le p$, $2 \le t \le q$ such that $x_{i_\alpha}x_{j_\beta} \notin G$ and $t \ne \beta$.

Subcase (i): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{j_t}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{t+(p-1)}x_{j_t}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\ &+(-1)^{t+p}x_{j_t}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})=0.\end{aligned}$$

Subcase (ii): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_\beta})$, then as in subcase (ii) of [Case A-b], we set

$$\begin{aligned}\beta' &= \min\{t \mid \beta < t \le q, \{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_t})\}\\ &= \min\{t \mid \beta < t \le q, \{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}\} \nsubseteq \mathrm{pnbhd}(x_{j_t})\}.\end{aligned}$$

Subsubcase (ii)(a): if $t < \beta$, then the sum of the terms containing $x_{i_\alpha} x_{j_t}$ is

$$(-1)^{\alpha+1} x_{i_\alpha}[(-1)^{t+(p-1)} x_{j_t}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$+ (-1)^{\beta}(-1)^{t+(p-1)+1} x_{j_t}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_{\beta-1}} | x_{j_\beta}, \dots, x_{j_q})]$$
$$+ (-1)^{t+p} x_{j_t}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$+ (-1)^{\beta-1}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_{\beta-1}} | x_{j_\beta}, \dots, x_{j_q})] = 0.$$

Subsubcase (ii)(b): if one of the following conditions is satisfied:

1) $t > \beta$ and β' does not exist,
2) $t > \beta$ and $t \neq \beta'$,
3) $t = \beta' = q$,
4) $t = \beta'$ and $\{x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}\} \not\subseteq \mathrm{pnbhd}(x_{j_{\beta'+1}})$,

then the sum of the terms containing $x_{i_\alpha} x_{j_t}$ is

$$(-1)^{\alpha+1} x_{i_\alpha}[(-1)^{t+(p-1)} x_{j_t}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$+ (-1)^{\beta}(-1)^{t+p-1} x_{j_t}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_\beta}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})]$$
$$+ (-1)^{t+p} x_{j_t}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$+ (-1)^{\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_\beta}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})] = 0.$$

Note that in the above two subsubcases, if $\alpha = p = 1$ then the terms containing $(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$ are zeros and β' does not exist.

Subsubcase (ii)(c): if $t = \beta'$ and $\{x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_{\beta'+1}})$, then the sum of the terms containing $x_{i_\alpha} x_{j_t}$ is

$$(-1)^{\alpha+1} x_{i_\alpha}\{(-1)^{t+(p-1)} x_{j_t}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$+ (-1)^{t}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{t-1}} | x_{j_{t+1}}, \dots, x_{j_q})]$$
$$+ (-1)^{\beta}(-1)^{t+p-1} x_{j_t}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_\beta}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$(-1)^{t-\beta+1}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{t-1}} | x_{j_{t+1}}, \dots, x_{j_q})]\}$$
$$+ (-1)^{t+p} x_{j_t}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})$$
$$+ (-1)^{\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_\beta}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q})] = 0.$$

[Case C-c]: suppose that there is a term containing $x_{i_s} x_{j_\beta}$ for some $1 \le s \le p$ such that $x_{i_s} x_{j_\beta} \in G$. We set

$$\beta'' = \min\{t \mid \beta < t \le q, \{x_{i_1}, \dots, x_{i_p}\} \not\subseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Lemma 3.2 implies that for any $\beta \le t \le q$, $x_{j_1} x_{j_t}, \dots, x_{j_{\beta-1}} x_{j_t} \in G$, so we have

$$\beta'' = \min\{t \mid \beta < t \le q, \{x_{i_1}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}}\} \not\subseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Subcase (i): if $\beta = q$ or $\{x_{i_1}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p}\} \not\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{i_s} x_{j_\beta}$ is

$$(-1)^{s+1} x_{i_s}(-1)^{\beta+(p-1)} x_{j_\beta}(x_{i_1}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q})$$
$$+ (-1)^{\beta+p} x_{j_\beta}(-1)^{s+1} x_{i_s}(x_{i_1}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) = 0.$$

Subcase (ii): if $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$ and $x_{i_s}x_{j_{\beta+1}}\notin G$, then $\beta''=\beta+1$ and the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]\\
&+(-1)^{\beta+p}x_{j_\beta}(-1)^{s+1}x_{i_s}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]=0.
\end{aligned}$$

Subcase (iii): if one of the following conditions is satisfied:

1) $\beta<q$ and β'' does not exist,
2) $\beta''>\beta+1$ and $x_{i_s}x_{j_{\beta''}}\in G$,
3) $\beta''>\beta+1$, $x_{i_s}x_{j_{\beta''}}\notin G$ and $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_{\beta''}})$

then the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]\\
&+(-1)^{\beta+p}x_{j_\beta}[(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta}(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]=0.
\end{aligned}$$

Subcase (iv): if $\beta''>\beta+1$, $x_{i_s}x_{j_{\beta''}}\notin G$, $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta''}})$, then the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]\\
&+(-1)^{\beta+p}x_{j_\beta}\{(-1)^{s+1}x_{i_s}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta''-1}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_{\beta''-1}}|x_{j_{\beta''}},\ldots,x_{j_q})]\\
&+(-1)^{\beta}(-1)^{s+1}x_{i_s}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})\\
&+(-1)^{\beta''-\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_{\beta''-1}}|x_{j_{\beta''}},\ldots,x_{j_q})]\}=0.
\end{aligned}$$

[Case C-d]: suppose that there is a term containing $x_{i_\alpha}x_{j_\beta}$ for some $1\le\alpha\le p$ such that $x_{i_\alpha}x_{j_\beta}\notin G$. As in [Case C-c], we set

$$\begin{aligned}
\beta''&=\min\{t\mid \beta<t\le q,\{x_{i_1},\ldots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_t})\}\\
&=\min\{t\mid \beta<t\le q,\{x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}\}\not\subseteq \mathrm{pnbhd}(x_{j_t})\}.
\end{aligned}$$

Subcase (i): if $\beta=q$ or $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\beta+(p-1)}x_{j_\beta}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\
&+(-1)^{\beta+p}x_{j_\beta}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})=0.
\end{aligned}$$

Subcase (ii): if $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_\beta})$, then we have the following three subsubcases.

Subsubcase (ii)(a): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$ and $x_{i_\alpha}x_{j_{\beta+1}}\notin G$, then $\beta''=\beta+1$ and the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}[(-1)^{\beta+(p-1)}x_{j_\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(-1)^{\beta+p-1}x_{j_\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]=0.\end{aligned}$$

Subsubcase (ii)(b): if $\{x_{i_1},\dots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$ and one of the following conditions is satisfied:

1) β'' does not exist,
2) $x_{i_\alpha}x_{j_{\beta''}}\in G$,
3) $x_{i_\alpha}x_{j_{\beta''}}\notin G$ and $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_{\beta''}})$,

then the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}[(-1)^{\beta+(p-1)}x_{j_\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(-1)^{\beta+p-1}x_{j_\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}[(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]=0.\end{aligned}$$

Subsubcase (ii)(c): if $\beta''\geq\beta+2$, $x_{i_\alpha}x_{j_{\beta''}}\notin G$ and $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta''}})$, then the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}[(-1)^{\beta+(p-1)}x_{j_\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(-1)^{\beta+p-1}x_{j_\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}\{(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta''-1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_{\beta''-1}}|x_{j_{\beta''}},\dots,x_{j_q})]\\&+(-1)^{\beta}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})\\&+(-1)^{\beta''-\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_{\beta''-1}}|x_{j_{\beta''}},\dots,x_{j_q})]\}=0.\end{aligned}$$

Subcase (iii): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_\beta})$, then just as in subcase (ii), we have the following three subsubcases.

Subsubcase (iii)(a): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$ and $x_{i_\alpha}x_{j_{\beta+1}}\notin G$, then $\beta''=\beta+1$ and the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,x_{j_q})]=0.\end{aligned}$$

Subsubcase (iii)(b): if $\{x_{i_1},\dots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$ and one of the following conditions is satisfied:

1) β'' does not exist,
2) $x_{i_\alpha}x_{j_{\beta''}}\in G$,
3) $x_{i_\alpha}x_{j_{\beta''}}\notin G$ and $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\}\not\subseteq \mathrm{pnbhd}(x_{j_{\beta''}})$,

then the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}[(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]=0.\end{aligned}$$

Subsubcase (iii)(c): if $\beta''\geq\beta+2$, $x_{i_\alpha}x_{j_{\beta''}}\notin G$ and $\{x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}\}\subseteq$ pnbhd$(x_{j_{\beta''}})$, then the sum of the terms containing $x_{i_\alpha}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}\{(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta''-1}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_{\beta''-1}}|x_{j_{\beta''}},\ldots,x_{j_q})]\\&+(-1)^{\beta}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})\\&+(-1)^{\beta''-\beta}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_{\beta''-1}}|x_{j_{\beta''}},\ldots,x_{j_q})]\}=0.\end{aligned}$$

[Case D]: $\lambda=j_1$ and $\lambda'\in\{j_2,\ldots,j_q\}$.

[Case D-a]: suppose that there is a term containing $x_{j_1}x_{j_t}$ for some $2\leq t\leq q$ such that $t\neq\beta$, then $\beta\neq 2$ and if $t=2$ then $\beta\neq 3$. Hence, the sum of the terms containing $x_{j_1}x_{j_t}$ is

$$\begin{aligned}&(-1)^{1+p}x_{j_1}(-1)^{(t-1)+p}x_{j_t}(x_{i_1},\ldots,x_{i_p}|\widehat{x_{j_1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})\\&+(-1)^{t+p}x_{j_t}(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|\widehat{x_{j_1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})=0.\end{aligned}$$

[Case D-b]: suppose that there is a term containing $x_{j_1}x_{j_\beta}$.

Subcase (i): $\beta=2$. Assume that $\{x_{i_1},\ldots,x_{i_p}\}\not\subseteq$ pnbhd(x_{j_3}), then there is no term containing $x_{j_1}x_{j_\beta}$, hence we must have $\{x_{i_1},\ldots,x_{i_p}\}\subseteq$ pnbhd(x_{j_3}) and the sum of the terms containing $x_{j_1}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{\beta+p}x_{j_\beta}[(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|\widehat{x_{j_1}},x_{j_3},\ldots,x_{j_q})\\&+(-1)^{\beta}(-1)^{p+2}x_{j_1}(x_{i_1},\ldots,x_{i_p},\widehat{x_{j_1}}|x_{j_3},\ldots,x_{j_q})]=0.\end{aligned}$$

Subcase (ii): if $\beta>2$ such that $\beta=q$ or $\{x_{i_1},\ldots,x_{i_p}\}\not\subseteq$ pnbhd$(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{j_1}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{1+p}x_{j_1}(-1)^{(\beta-1)+p}x_{j_\beta}(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta+p}x_{j_\beta}(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})=0.\end{aligned}$$

Subcase (iii): if $\beta>2$ and $\{x_{i_1},\ldots,x_{i_p}\}\subseteq$ pnbhd$(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{j_1}x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{1+p}x_{j_1}(-1)^{(\beta-1)+p}x_{j_\beta}[(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta-1}(x_{i_1},\ldots,x_{i_p},x_{j_2},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]\\&+(-1)^{\beta+p}x_{j_\beta}[(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})\\&+(-1)^{\beta}(-1)^{p+2}x_{j_1}(x_{i_1},\ldots,x_{i_p},x_{j_2},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]=0.\end{aligned}$$

[Case E]: $\lambda,\lambda'\in\{j_2,\ldots,j_q\}$.

[Case E-a]: if $2 \le t < t' \le q$ such that $t \ne \beta$ and $t' \ne \beta$, then the sum of the terms containing $x_{j_t} x_{j_{t'}}$ is

$$\begin{aligned}&(-1)^{t+p} x_{j_t} (-1)^{(t'-1)+p} x_{j_{t'}} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_{t'}}}, \ldots, x_{j_q})\\ &+ (-1)^{t'+p} x_{j_{t'}} (-1)^{t+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_{t'}}}, \ldots, x_{j_q}) = 0.\end{aligned}$$

[Case E-b]: suppose that there is a term containing $x_{j_t} x_{j_\beta}$ for some $2 \le t \le q$ with $t \ne \beta$. As in [Case C-c], we set

$$\begin{aligned}\beta'' &= \min\{t \mid \beta < t \le q, \{x_{i_1}, \ldots, x_{i_p}\} \not\subseteq \operatorname{pnbhd}(x_{j_t})\}\\ &= \min\{t \mid \beta < t \le q, \{x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}\} \not\subseteq \operatorname{pnbhd}(x_{j_t})\}.\end{aligned}$$

Subcase (i): if one of the following conditions is satisfied:

1) $\beta = q$,
2) $\beta = q-1$ and $t = q$,
3) $\beta'' = \beta + 1$ and $t \ne \beta''$,
4) $\beta'' = \beta + 1$, $t = \beta''$ and $\{x_{i_1}, \ldots, x_{i_p}\} \not\subseteq \operatorname{pnbhd}(x_{j_{\beta+2}})$,
5) $\beta'' = \beta + 2$ and $t = \beta + 1$,

then the sum of the terms containing $x_{j_t} x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{t+p} x_{j_t} (-1)^{(\beta-1)+p} x_{j_\beta} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})\\ &+ (-1)^{\beta+p} x_{j_\beta} (-1)^{t+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})\\ &= 0, \quad \text{for } t < \beta;\\ &(-1)^{t+p} x_{j_t} (-1)^{\beta+p} x_{j_\beta} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})\\ &+ (-1)^{\beta+p} x_{j_\beta} (-1)^{(t-1)+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})\\ &= 0, \quad \text{for } t > \beta.\end{aligned}$$

Subcase (ii): if $\beta'' = \beta + 1$, $t = \beta''$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \operatorname{pnbhd}(x_{j_{\beta+2}})$, then the sum of the terms containing $x_{j_t} x_{j_\beta}$ is

$$\begin{aligned}&(-1)^{t+p} x_{j_t} (-1)^{\beta+p} x_{j_\beta} [(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \widehat{x_{j_t}}, \ldots, x_{j_q})\\ &+ (-1)^{\beta} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+2}}, \ldots, x_{j_q})]\\ &+ (-1)^{\beta+p} x_{j_\beta} (-1)^{(t-1)+p} x_{j_t} [(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \widehat{x_{j_t}}, \ldots, x_{j_q})\\ &+ (-1)^{\beta} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+2}}, \ldots, x_{j_q})] = 0.\end{aligned}$$

Subcase (iii): if one of the following conditions is satisfied:

1) $\beta = q-1$, $t < \beta$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \operatorname{pnbhd}(x_{j_q})$,
2) $\beta \le q-2$ and β'' does not exist,
3) $\beta'' > \beta + 1$, $t \ne \beta''$ such that $t \ne \beta + 1$ or $\beta'' \ne \beta + 2$,
4) $\beta'' > \beta + 1$ and $t = \beta'' = q$,
5) $\beta'' > \beta + 1$, $t = \beta''$ and $\{x_{i_1}, \ldots, x_{i_p}\} \not\subseteq \operatorname{pnbhd}(x_{j_{\beta''+1}})$,

then the sum of the terms containing $x_{j_t}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{t+p}x_{j_t}(-1)^{(\beta-1)+p}x_{j_\beta}[(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\
&+(-1)^{\beta-1}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,\widehat{x_{j_t}},\dots,\widehat{x_{j_\beta}}|x_{j_{\beta+1}},\dots,x_{j_q})]\\
&+(-1)^{\beta+p}x_{j_\beta}[(-1)^{t+p}x_{j_t}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,\widehat{x_{j_\beta}},\dots,x_{j_q})\\
&+(-1)^{\beta}(-1)^{t+p+1}x_{j_t}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,\widehat{x_{j_t}},\dots,\widehat{x_{j_\beta}}|x_{j_{\beta+1}},\dots,x_{j_q})]\\
&=0,\quad \text{for } t<\beta;\\
&(-1)^{t+p}x_{j_t}(-1)^{\beta+p}x_{j_\beta}[(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+(-1)^{\beta}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})]\\
&+(-1)^{\beta+p}x_{j_\beta}[(-1)^{(t-1)+p}x_{j_t}(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+(-1)^{\beta}(-1)^{t-1+p}x_{j_t}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})]\\
&=0,\quad \text{for } t>\beta.
\end{aligned}$$

Subcase (iv): if $\beta''>\beta+1$, $t=\beta''$ and $\{x_{i_1},\dots,x_{i_p}\}\subseteq \text{pnbhd}(x_{j_{\beta''+1}})$, then the sum of the terms containing $x_{j_t}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{t+p}x_{j_t}(-1)^{\beta+p}x_{j_\beta}[(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+(-1)^{\beta}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})]\\
&+(-1)^{\beta+p}x_{j_\beta}\{(-1)^{(t-1)+p}x_{j_t}[(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+(-1)^{t-1}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_{t-1}}|x_{j_{t+1}},\dots,x_{j_q})]\\
&+(-1)^{\beta}(-1)^{t-1+p}x_{j_t}[(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&(-1)^{t-\beta}(x_{i_1},\dots,x_{i_p},x_{j_1},\dots,\widehat{x_{j_\beta}},\dots,x_{j_{t-1}}|x_{j_{t+1}},\dots,x_{j_q})]\}=0.
\end{aligned}$$

Since the above five main cases have included all the possible terms, it follows that $d^2(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q})=0$ and we are done. $\square$

5. Betti Numbers

In Section 3, to construct the differential maps of the minimal free resolution of S/I_G, we need to assume that $x_n,\dots,x_1$ is a perfect elimination order of $\overline{G}$ produced by Algorithm 2.2. However, to get a nice formula for Betti numbers (Corollary 5.2), we only need to know a basis for the minimal free resolution. Therefore, we have the following theorem which does not require that the perfect elimination order $x_n,\dots,x_1$ of $\overline{G}$ is produced by Algorithm 2.2.

Theorem 5.1. *Let G be a simple graph with vertices $x_1,\dots,x_n$ such that $\overline{G}$ is chordal and $x_1,\dots,x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$. Then in the polynomial ring $S=k[x_1,\dots,x_n]$ we have the linear edge ideal I_G of the graph G. Let the symbol $(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q})$ be as defined in Construction 3.4. And we set*

$$\mathcal{B}=\{1\}\cup\bigcup_{p\ge1,q\ge1}\left\{(x_{i_1},\dots,x_{i_p}|x_{j_1},\dots,x_{j_q}):\ \begin{array}{l}1\le i_1<\dots<i_p<j_1<\dots<j_q\le n\\ \{x_{i_1},\dots,x_{i_p}\}\subseteq pnbhd(x_{j_1})\end{array}\right\}.$$

Then there exists a multigraded minimal free resolution $\mathbf{F}$ of S/I_G such that $\mathbf{F}$ has basis $\mathcal{B}$.

We will not prove Theorem 5.1 because the proof is very similar to the proof of Theorem 3.7. The only difference is that in the proof of Theorem 3.7 we know the complex maps $-\mu_2 : \mathbf{E} \to \mathbf{K}$ and $\mu : \mathbf{F}'' \to \mathbf{F}'$ explicitly, while in the proof of Theorem 5.1 we only know their existence. However, we can still use the mapping cones to show the existence of the multigraded minimal free resolution with the desired basis $\mathcal{B}$.

Now Theorem 5.1 imply immediately the following corollary about Betti numbers and the projective dimension of S/I_G.

Corollary 5.2. *Let I_G be a linear edge ideal as defined in Theorem 5.1. For $2 \le i \le n$, we set $\lambda_i = |pnbhd(x_i)|$. Then for $i \ge 1$, the Betti numbers of S/I_G are*

$$b_{i,j}(S/I_G) = \begin{cases} \displaystyle\sum_{l=2}^{n} \left(\sum_{p=1}^{\lambda_l} \binom{\lambda_l}{p} \binom{n-l}{i-p} \right), & \text{if } j = i+1, \\ 0, & \text{if } j \ne i+1, \end{cases}$$

and the projective dimension of S/I_G is

$$projdim(S/I_G) = n - \min\{i - \lambda_i : 2 \le i \le n \text{ and } \lambda_i \ne 0\} \le n-1.$$

Proof. The formula for Betti numbers follows from counting the number of basis elements of homological degree i and degree $i+1$ in $\mathcal{B}$. The projective dimension formula also follows easily by looking at the basis elements in $\mathcal{B}$. Since $\lambda_i \le i-1$ for $2 \le i \le n$, it follows that $\operatorname{projdim}(S/I_G) \le n-1$. $\square$

Example 5.3. Let G be the graph such that $\overline{G}$ is the chordal graph given in Example 2.6. Then $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ is in the reverse order of a perfect elimination order of $\overline{G}$ and we have that

$$\lambda_2 = 0,\ \lambda_3 = 1,\ \lambda_4 = 2,\ \lambda_5 = 3,\ \lambda_6 = 4,\ \lambda_7 = 5.$$

Therefore, by Corollary 5.2, we have $\operatorname{projdim}(S/I_G) = 5$ and a computation will reveal that the Betti numbers of S/I_G are

$$\mathrm{b}_{1,2} = 15,\ \mathrm{b}_{2,3} = 40,\ \mathrm{b}_{3,4} = 45,\ \mathrm{b}_{4,5} = 24,\ \mathrm{b}_{5,6} = 5.$$

In [RV] and [HV], the following formula for the Betti numbers is proved by using Hochster's formula. Now we prove the formula by using Theorem 5.1.

Corollary 5.4. *Let I_G be the linear edge ideal of a graph G with vertices $x_1, \ldots, x_n$. For any nonempty subset σ of $\{x_1, \ldots, x_n\}$, let $\overline{G}_\sigma$ be the subgraph of $\overline{G}$ induced by σ and let $\#(\overline{G}_\sigma)$ be the number of connected components of $\overline{G}_\sigma$. Then for $i \ge 1$, we have*

$$b_{i,j}(S/I_G) = \begin{cases} \displaystyle\sum_{\sigma \subseteq \{x_1,\ldots,x_n\}, |\sigma| = i+1} \left(\#(\overline{G}_\sigma) - 1 \right), & \text{if } j = i+1, \\ 0, & \text{if } j \ne i+1. \end{cases}$$

Proof. Without the loss of generality, we can assume that $x_n, \ldots, x_1$ is a perfect elimination order of the chordal graph $\overline{G}$. Let $\mathcal{B}$ be as defined in Theorem 5.1. We say that the vertex x_s is smaller than the vertex x_t if $s < t$. For any $i \ge 1$, let $\sigma = \{x_{\alpha_1}, \ldots, x_{\alpha_{i+1}}\}$ be a subset of $\{x_1, \ldots, x_n\}$ for some $1 \le \alpha_1 < \cdots < \alpha_{i+1} \le n$. We claim that $(x_{\alpha_1}, \ldots, x_{\alpha_{p-1}} | x_{\alpha_p}, \ldots, x_{\alpha_{i+1}}) \in \mathcal{B}$ if and only if $p \ne 1$ and x_{α_p} is the smallest vertex in the connected component of $\overline{G}_\sigma$ containing x_{α_p}. Indeed, if $p \ge 2$ and x_{α_p} is the smallest vertex in the connected component of $\overline{G}_\sigma$ containing x_{α_p}, then $x_{\alpha_s}x_{\alpha_p} \in G$ for all $1 \leq s \leq p-1$, so that $(x_{\alpha_1}, \ldots, x_{\alpha_{p-1}} | x_{\alpha_p}, \ldots, x_{\alpha_{i+1}}) \in \mathcal{B}$. On the other hand, assume that $p \geq 2$ and there exists $1 \leq s \leq p-1$ such that x_{α_s} and x_{α_p} are in the same connected component of $\overline{G}_\sigma$. Set $\sigma' = \{x_{\alpha_1}, \ldots, x_{\alpha_p}\} \subseteq \sigma$. Since $x_{\alpha_{i+1}}, \ldots, x_{\alpha_1}$ is a perfect elimination order of $\overline{G}_\sigma$, it is easy to see that x_{α_s} and x_{α_p} are still in the same connected component of $\overline{G}_{\sigma'}$. Therefore, there exists $1 \leq s' \leq p-1$ such that $x_{\alpha_{s'}}x_{\alpha_p} \in \overline{G}_{\sigma'}$, and hence $x_{\alpha_{s'}}x_{\alpha_p} \notin G$, which implies $(x_{\alpha_1}, \ldots, x_{\alpha_{p-1}} | x_{\alpha_p}, \ldots, x_{\alpha_{i+1}}) \notin \mathcal{B}$. So the claim is proved. It follows that there are $\#(\overline{G}_\sigma) - 1$ basis elements in $\mathcal{B}$ with multidegree $x_{\alpha_1} \cdots x_{\alpha_{i+1}}$ and we are done. $\square$

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