ON THE UNIFORM COMPUTATIONAL CONTENT OF COMPUTABILITY THEORY

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Abstract. We demonstrate that the Weihrauch lattice can be used to study the uniform computational content of computability theoretic properties and theorems in one common setting. The properties that we study include diagonal non-computability, hyperimmunity, complete extensions of Peano arithmetic, 1-genericity, Martin-Löf randomness and cohesiveness. The theorems that we include in our case study are the Low Basis Theorem of Jockusch and Soare, the Kleene-Post Theorem and Friedman's Jump Inversion Theorem. It turns out that all the aforementioned properties and many theorems in computability theory, including all theorems that claim the existence of some Turing degree, have very little uniform computational content. They are all located outside of the upper cone of binary choice (also known as LLPO) and we call problems with this property *indiscriminative*. Since practically all theorems from classical analysis whose computational content has been classified are discriminative, our observation could yield an explanation for why theorems and results in computability theory typically have very little direct consequences in other disciplines such as analysis. A notable exception in our case study is the Low Basis Theorem which is discriminative, this is perhaps why it is considered to be one of the most applicable theorems in computability theory. In some cases a bridge between the indiscriminative world and the discriminative world of classical mathematics can be established via a suitable residual operation and we demonstrate this in case of the cohesiveness problem, which turns out to be the quotient of two discriminative problems, namely the limit operation and the jump of Weak Kőnig's Lemma.

Keywords: Computable analysis, Weihrauch lattice, computability theory, reverse mathematics.

Contents

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1. Introduction

In this paper we start a new line of research with the aim to analyze the uniform computational content of theorems and properties of computability theory itself in the Weihrauch lattice. This is very much in the spirit of previous research by Simpson [51], Downey et al. [24], Jockusch and Lewis [38] and others who have compared computability theoretic properties in the Medvedev lattice and, in fact, we can import results from there (see Lemma 7.14).

However, since we are using the higher degree of uniformity of the Weihrauch lattice, we can not only include computability properties in their fully relativized versions into our study, but also theorems of computability theory (in for all-exists form).

In this regard this paper can be seen as a continuation of the classification of the uniform computational content of theorems from analysis which has been started in [27, 47, 9, 8, 6, 12, 15, 11, 22]. Theorems that have been classified in this context include, among others:

(1) The Hahn-Banach Theorem [27],
(2) the Intermediate Value Theorem [8],
(3) the (functional analytic) Baire Category Theorem [8],
(4) the Banach Inverse Mapping Theorem [8],
(5) the Bolzano-Weierstraß Theorem [12],
(6) the Brouwer Fixed Point Theorem [15],
(7) the Nash Equilibria Existence Theorem [47],
(8) the Radon-Nykodim Theorem [33].

All these theorems have a certain combinatorial feature in common, they all compute binary choice, a property that we call *discriminative* in this paper. Mathematically speaking, this means that choice of the two point space C_2 or, equivalent LLPO, is Weihrauch reducible to the problem in question.

The most basic part of computability theory is fully constructive and computable, but as soon as the theory advances to non-trivial existence results it becomes increasingly non-constructive. The purpose of this study is to characterize the exact amount of non-constructivity, or the uniform computational content of this theory in terms of the Weihrauch lattice. In particular, we study the following theorems from computability theory:

(1) The Kleene-Post Theorem KPT,
(2) Friedberg's Jump Inversion Theorem JIT,
(3) The Low Basis Theorem of Jockusch and Soare LBT.

It turns out that almost all theorems of computability theory that we have studied are *indiscriminative* –they do not even compute binary choice– with the notable exception of the Low Basis Theorem. As a consequence of this, a large part of computability theory cannot have any direct implications on more classical mathematics, such as the part of analysis discussed above. More precisely, no indiscriminative theorem computes any discriminative theorem; that is, if A is discriminative and B is indiscriminative, then A is not Weihrauch reducible to B.

The exception in the part of computability theory that we have looked at is the Low Basis Theorem, and this may explain why this theorem has been considered as a particularly applicable theorem within computability theory. For instance, Barry Cooper writes of the Low Basis Theorem [21, Page 330]:

> "Here is our most useful basis, proved by Jockusch and Soare around 1972. You often come across applications of it in the most unexpected places."

We identify one common topological reason behind the fact that large parts of computability theory are indiscriminative: the corresponding theorems are densely realized in the sense that for-all-exists statements of the form

$$(\forall x \in X)(\exists y \in Y)\ P(x,y)$$

have multi-valued realizers (i.e., multi-valued Skolem functions) whose image is dense in Y. In Section 4 we give a more precise definition of this property, which we call *densely realized*. Sometimes this is a specific feature of P, but often a statement is densely realized because of the mere choice of Y. For instance, Turing degrees $\mathcal{D}$ are naturally represented by representatives $p \in \mathbb{N}^{\mathbb{N}}$ of the corresponding equivalence class with respect to $\equiv_{\mathrm{T}}$. Since Turing degrees are invariant under finite modifications of their representatives, we immediately obtain that *any* existence theorem of the form $(\forall x \in X)(\exists \mathbf{d} \in \mathcal{D})\ P(x,\mathbf{d})$, which claims the existence of a Turing degree, is automatically densely realized and hence indiscriminative. This explains why large parts of computability are indiscriminative.

The particular computability theoretic properties that we include in our study are the following:

(1) Diagonal non-computability DNC,
(2) Complete extensions of Peano arithmetic PA,
(3) hyperimmunity HYP,
(4) Martin-Löf randomness MLR,
(5) 1–genericity $\mathsf{1\text{-}GEN}$,
(6) cohesiveness COH.

In fact, one can also interpret all these properties as existence theorems of corresponding objects. For instance, Martin-Löf randomness MLR can be seen as the existence of a Martin-Löf random point q relative to some given p, so $\mathsf{MLR}(p)$ is the set of all Martin-Löf randoms relative to p. In this sense we consider fully relativized versions of these properties in the Weihrauch lattice.

Of these principles COH has already been studied form a uniform perspective in the context of combinatorial principles by Dorais et al. [22] and Hirschfeldt [32]. Martin-Löf randomness MLR has been looked at from the uniform perspective in [10] and [2]. Diagonal non-computability DNC has independently been studied by Higuchi and Kihara [30].

For a quick survey of our results the reader might wish to consult the diagram in Figure 1 on Page 19 and, in particular, the final diagram in Figure 2 on Page 38.

In Section 2 we present some basic facts on the Weihrauch lattice that are included in order to keep this paper self-contained. In Section 3 we include a preliminary discussion on some versions of choice and omniscience that we need throughout this paper. In Section 4 we introduce the notion of an indiscriminative problem or theorem and we prove some basic facts about them. In Section 5 we discuss diagonal non-computability DNC and in Section 6 the closely related problem of complete extensions of Peano arithmetic PA. In Section 7 we consider Martin-Löf randomness and different uniform versions of Weak Weak Kőnig's Lemma WWKL in relation to the previously mentioned problems. The relevant version of WWKL are taken from [11]. The results achieved in this context are summarized in the diagram in Figure 1 on Page 19. In Section 8 we classify the uniform computational content of the Low Basis Theorem LBT in relation to other known problems related to lowness and Weak Kőnig's Lemma. In Section 9 we continue with a study of the hyperimmunity problem HYP. In Sections 10 and 11 we study the Kleene-Post Theorem KPT and the Jump Inversion Theorem of Jockusch and Soare, respectively. The well-known fact that van Lambalgen's Theorem can be used to create a pair of incomparable Turing degrees yields a uniform reduction of the Kleene

Post Theorem to MLR. Likewise a theorem of Yu allows a reduction of KPT to the 1–genericity problem 1-GEN. The Jump Inversion Theorem JIT appears to be the only current example of a theorem that is continuous, but not computable. Sections 12 and 13 are devoted to the cohesiveness problem and the Section 14 to closely related weak versions of the Bolzano-Weierstraß Theorem. In this section we borrow some ideas of the proof theoretic study of cohesiveness COH and the Bolzano-Weierstraß Theorem in [41]. One of the main results on the cohesiveness problem is that it can be characterized with the help of the limit operation lim and the jump of Weak Weak Kőnig's Lemma WWKL' by

$$\mathsf{COH} \equiv_{\mathrm{W}} (\lim \to \mathsf{WWKL}').$$

The results of this paper are extended by further results on the uniform computational content of different versions of the Baire Category Theorem and 1–genericity that are presented in [13]. These results are also included in the diagram in Figure 2 on Page 38.

2. Preliminaries

In this section we give a brief introduction into the Weihrauch lattice and we provide some basic notions from probability theory.

Pairing Functions. We are going to use some standard pairing functions in the following that we briefly summarize. By $\mathbb{N} := \{0, 1, 2, ...\}$ we denote the set of *natural numbers*. As usual, we denote by $\langle n, k\rangle := \frac{1}{2}(n + k + 1)(n + k) + k$ the Cantor pair of two natural numbers $n, k \in \mathbb{N}$ and by $\langle p, q\rangle(n) := p(k)$ if $n = 2k$ and $\langle p, q\rangle(n) = q(k)$, if $n = 2k + 1$, the pairing of two sequences $p, q \in \mathbb{N}^{\mathbb{N}}$. By $\langle k, p\rangle(n) := kp$ we denote the natural pairing of a number $k \in \mathbb{N}$ with a sequence $p \in \mathbb{N}^{\mathbb{N}}$. We also define a pairing function $\langle p_0, p_1\rangle := \langle\langle p_0(0), p_1(0)\rangle, \langle\overline{p_0}, \overline{p_1}\rangle\rangle$, for $p_0, p_1 \in \mathbb{N} \times 2^{\mathbb{N}}$, where $\overline{p_i}(n) = p_i(n + 1)$. Finally, we use the pairing function $\langle p_0, p_1, p_2, ...\rangle\langle i, j\rangle := p_i(j)$ for $p_i \in \mathbb{N}^{\mathbb{N}}$.

The Weihrauch Lattice. The original definition of Weihrauch reducibility is due to Klaus Weihrauch and has been studied for many years (see [52, 55, 56, 29, 4, 5]). More recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of the computational content of mathematical theorems (see [27, 47, 48, 9, 8, 6, 12]). The basic reference for all notions from computable analysis is Weihrauch's textbook [57]. The Weihrauch lattice is a lattice of multi-valued functions on represented spaces.

A *representation* δ of a set X is just a surjective partial map $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. In this situation we call (X, δ) a *represented space*. In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. We work with partial multi-valued functions $f :\subseteq X \rightrightarrows Y$ where $f(x) \subseteq Y$ denotes the set of possible values upon input $x \in \mathrm{dom}(f)$. If f is single-valued, then for the sake of simplicity we identify $f(x)$ with its unique inhabitant. We denote the *composition* of two (multi-valued) functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Y \rightrightarrows Z$ either by $g \circ f$ or by gf. It is defined by

$$g \circ f(x) := \{z \in Z : (\exists y \in Y)(z \in g(y) \text{ and } y \in f(x))\},$$

where $\mathrm{dom}(g \circ f) := \{x \in X : f(x) \subseteq \mathrm{dom}(g)\}$. Using represented spaces we can define the concept of a realizer.

Definition 2.1 (Realizer)**.** Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called a *realizer* of f, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$.

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Now we can define Weihrauch reducibility.

Definition 2.2 (Weihrauch reducibility)**.** Let f, g be multi-valued functions on represented spaces. Then f is said to be *Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $H\langle \mathrm{id}, GK\rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be *strongly Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{sW}} g$, if an analogous condition holds, but with the property $HGK \vdash f$ in place of $H\langle \mathrm{id}, GK\rangle \vdash f$.

The difference between ordinary and strong Weihrauch reducibility is that the "output modifier" H has direct access to the original input in case of ordinary Weihrauch reducibility, but not in case of strong Weihrauch reducibility. There are algebraic and other reasons to consider ordinary Weihrauch reducibility as the more natural variant. For instance, one can characterize the reduction $f \leq_{\mathrm{W}} g$ as follows: $f \leq_{\mathrm{W}} g$ holds if and only if a Turing machine can compute f in such a way that it evaluates the "oracle" g exactly on one (usually infinite) input during the course of its computation (see [53, Theorem 7.2]). We will use the strong variant $\leq_{\mathrm{sW}}$ of Weihrauch reducibility mostly for technical purposes, for instance it is better suited to study jumps (since jumps are monotone with respect to strong reductions but in general not for ordinary reductions).

We note that the relations $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$ and $\vdash$ implicitly refer to the underlying representations, which we will only mention explicitly if necessary. It is known that these relations only depend on the underlying equivalence classes of representations and not on the specific representatives (see Lemma 2.11 in [9]). The relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ are reflexive and transitive, thus they induce corresponding partial orders on the sets of their equivalence classes (which we refer to as *Weihrauch degrees* and *strong Weihrauch degrees*, respectively). These partial orders will be denoted by $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ as well. The induced lattice and semi-lattice, respectively, are distributive (for details see [48] and [9]). We use $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{sW}}$ to denote the respective equivalences regarding $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$, by $<_{\mathrm{W}}$ and $<_{\mathrm{sW}}$ we denote strict reducibility and by $|_{\mathrm{W}}, |_{\mathrm{sW}}$ we denote incomparability in the respective sense.

The Algebraic Structure. The partially ordered structures induced by the two variants of Weihrauch reducibility are equipped with a number of useful algebraic operations that we summarize in the next definition. We use $X \times Y$ to denote the ordinary set-theoretic *product*, $X \sqcup Y := (\{0\} \times X) \cup (\{1\} \times Y)$ to denote *disjoint sums* or *coproducts*, by $\bigsqcup_{i=0}^{\infty} X_i := \bigcup_{i=0}^{\infty}(\{i\} \times X_i)$ we denote the *infinite coproduct*. By X^i we denote the i–fold product of a set X with itself, where $X^0 = \{()\}$ is some canonical singleton. By $X^* := \bigsqcup_{i=0}^{\infty} X^i$ we denote the set of all *finite sequences over* X and by $X^{\mathbb{N}}$ the set of all *infinite sequences over* X. All these constructions have parallel canonical constructions on representations and the corresponding representations are denoted by $[\delta_X, \delta_Y]$ for the product of (X, δ_X) and (Y, δ_Y), by δ_X^n for the n–fold product of (X, δ_X) with itself, where $n \in \mathbb{N}$ and δ_X^0 is a representation of the one-point set $\{()\} = \{\varepsilon\}$. By $\delta_X \sqcup \delta_Y$ we denote the representation of the coproduct, by δ_X^* the representation of X^* and by $\delta_X^{\mathbb{N}}$ the representation of $X^{\mathbb{N}}$. For instance, $(\delta_X \sqcup \delta_Y)$ can be defined by $(\delta_X \sqcup \delta_Y)\langle n, p\rangle := (0, \delta_X(p))$ if $n = 0$ and $(\delta_X \sqcup \delta_Y)\langle n, p\rangle := (1, \delta_Y(p))$ otherwise. Likewise, $\delta_X^*\langle n, p\rangle := (n, \delta_X^n(p))$. See [57] or [9, 48, 6] for details of the definitions of the other representations. We will always assume that these canonical representations are used, if not mentioned otherwise.

Definition 2.3 (Algebraic operations)**.** Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions. Then we define the following operations:

(1) $f \times g :\subseteq X \times Z \rightrightarrows Y \times W, (f \times g)(x,z) := f(x) \times g(z)$ (product)

(2) $f \sqcap g :\subseteq X \times Z \rightrightarrows Y \sqcup W, (f \sqcap g)(x,z) := f(x) \sqcup g(z)$ (sum)

(3) $f \sqcup g :\subseteq X \sqcup Z \rightrightarrows Y \sqcup W$, with $(f \sqcup g)(0,x) := \{0\} \times f(x)$ and $(f \sqcup g)(1,z) := \{1\} \times g(z)$ (coproduct)

(4) $f^* :\subseteq X^* \rightrightarrows Y^*, f^*(i,x) := \{i\} \times f^i(x)$ (finite parallelization)

(5) $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f}(x_n) := \mathop{\times}\limits_{i \in \mathbb{N}} f(x_i)$ (parallelization)

In this definition and in general we denote by $f^i :\subseteq X^i \rightrightarrows Y^i$ the i–th fold product of the multi-valued map f with itself (f^0 is the constant function on the canonical singleton). It is known that $f \sqcap g$ is the *infimum* of f and g with respect to both strong and ordinary Weihrauch reducibility (see [9], where this operation was denoted by $\oplus$). Correspondingly, $f \sqcup g$ is known to be the *supremum* of f and g with respect to ordinary Weihrauch reducibility $\leq_{\rm W}$ (see [48]). This turns the partially ordered structure of Weihrauch degrees (induced by $\leq_{\rm W}$) into a lattice, which we call the *Weihrauch lattice*. The two operations $f \mapsto \widehat{f}$ and $f \mapsto f^*$ are known to be closure operators in this lattice (see [9, 48]).

There is some useful terminology related to these algebraic operations. We say that f is a *a cylinder* if $f \equiv_{\rm sW} \mathrm{id} \times f$ where $\mathrm{id} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ always denotes the identity on Baire space, if not mentioned otherwise. For a cylinder f and any g we have that $g \leq_{\rm W} f$ is equivalent to $g \leq_{\rm sW} f$ (see [9]). We say that f is *idempotent* if $f \equiv_{\rm W} f \times f$ and *strongly idempotent*, if $f \equiv_{\rm sW} f \times f$. We say that a multi-valued function on represented spaces is *pointed*, if it has a computable point in its domain. For pointed f and g we obtain $f \sqcup g \leq_{\rm sW} f \times g$. The properties of pointedness and idempotency are both preserved under equivalence and hence they can be considered as properties of the respective degrees. For a pointed f the finite parallelization f^* can also be considered as *idempotent closure* since idempotency is equivalent to $f \equiv_{\rm W} f^*$ in this case. We call f *parallelizable* if $f \equiv_{\rm W} \widehat{f}$ and it is easy to see that $\widehat{f}$ is always idempotent. Analogously, we call f *strongly parallelizable* if $f \equiv_{\rm sW} \widehat{f}$.

More generally, we define *countable coproducts* $\bigsqcup_{i \in \mathbb{N}} f_i :\subseteq \bigsqcup_{i \in \mathbb{N}} X_i \rightrightarrows \bigsqcup_{i \in \mathbb{N}} Y_i$ for a sequence (f_i) of multi-valued functions $f_i :\subseteq X_i \rightrightarrows Y_i$ on represented spaces and then it denotes the operation given by $(\bigsqcup_{i \in \mathbb{N}} f_i)(i,u) := \{i\} \times f_i(u)$. Using this notation we obtain $f^* = \bigsqcup_{i \in \mathbb{N}} f^i$. In [6] a multi-valued function on represented spaces has been called *join-irreducible* if $f \equiv_{\rm W} \bigsqcup_{n \in \mathbb{N}} f_n$ implies that there is some n such that $f \equiv_{\rm W} f_n$. Analogously, we can define *strong join-irreducibility* using strong Weihrauch reducibility in both instances. We can also define a *countable sum* $\bigsqcap_{i \in \mathbb{N}} f_i :\subseteq \mathop{\times}_{i \in \mathbb{N}} X_i \rightrightarrows \bigsqcup_{i \in \mathbb{N}} Y_i$, defined by $\left(\bigsqcap_{i \in \mathbb{N}} f_i\right)(x_i)_i := \bigsqcup_{i \in \mathbb{N}} f_i(x_i)$.

One should note however, that $\bigsqcap$ and $\bigsqcup$ do not provide infima and suprema of sequences. By a result of Higuchi and Pauly [31, Proposition 3.15] the Weihrauch lattice has no non-trivial suprema (i.e., a sequence (s_n) has a supremum s if and only if s is already the supremum of a finite prefix of the sequence $(s_n)_n$) and likewise by [31, Corollary 3.18] the pointed Weihrauch degrees do not have non-trivial infima. In particular, the Weihrauch lattice is not complete.[1]

Compositional Products and Implications. While the Weihrauch lattice is not complete, some suprema and some infima exist in general. The following result was proved by the first author and Pauly in [17] and ensures the existence of certain important maxima and minima.

[1]We note, however, that for the continuous variant of Weihrauch reducibility the objects $\bigsqcup_{n \in \mathbb{N}} f_n$ and $\bigsqcap_{n \in \mathbb{N}} f_n$ are suprema and infima of the sequence $(f_n)_n$, respectively, and the corresponding continuous version of the Weihrauch lattice is actually countably complete (see [31]).

Proposition 2.4 (Compositional products and implication)**.** *Let f, g be multi-valued functions on represented spaces. Then the following Weihrauch degrees exist:*

(1) $f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f \text{ and } g_0 \leq_{\mathrm{W}} g\}$ *(compositional product)*

(2) $f \to g := \min\{h : g \leq_{\mathrm{W}} f * h\}$ *(implication)*

Here $f*g$ is defined over all $f_0 \leq_{\mathrm{W}} f$ and $g_0 \leq_{\mathrm{W}} g$ which can actually be composed (i.e., the target space of g_0 and the source space of f_0 have to coincide). In this way $f * g$ characterizes the most complicated Weihrauch degree that can be obtained by first performing a computation with the help of g and then another one with the help of f. Since $f * g$ is a maximum in the Weihrauch lattice, we can consider $f * g$ as some fixed representative of the corresponding degree. It is easy to see that $f \times g \leq_{\mathrm{W}} f * g$ holds. We can also define the *strong compositional product* by

$$f *_{\mathrm{s}} g := \sup\{f_0 \circ g_0 : f_0 \leq_{\mathrm{sW}} f \text{ and } g_0 \leq_{\mathrm{sW}} g\},$$

but we neither claim that it exists in general nor that it is a maximum. The compositional products were originally introduced in [12]. The implication $f \to g$ represents the weakest oracle which is needed in advance of f in order to compute g and it was introduced in [17].

Jumps. In [12] *jumps* or *derivatives* f' of multi-valued functions f on represented spaces were introduced. The *jump* $f' :\subseteq (X, \delta_X') \rightrightarrows (Y, \delta_Y)$ of a multi-valued function $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ on represented spaces is obtained by replacing the input representation δ_X by its jump $\delta_X' := \delta_X \circ \lim$, where

$$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, ...\rangle \mapsto \lim_{n\to\infty} p_n$$

is the limit operation on Baire space $\mathbb{N}^{\mathbb{N}}$ with respect to the product topology on $\mathbb{N}^{\mathbb{N}}$. It follows that $f' \equiv_{\mathrm{sW}} f *_{\mathrm{s}} \lim$ (see [12, Corollary 5.16]). By $f^{(n)}$ we denote the n–fold jump. A δ_X'–name p of a point $x \in X$ is a sequence that converges to a δ_X–name of x. This means that a δ_X'–name typically contains significantly less accessible information on x than a δ_X–name. Hence, f' is typically harder to compute than f, since less input information is available for f'.

The jump operation $f \mapsto f'$ plays a similar role in the Weihrauch lattice as the Turing jump operation does in the Turing semi-lattice. In a certain sense f' is a version of f on the "next higher" level of complexity (which can be made precise using the Borel hierarchy [12]). It was proved in [12] that the jump operation $f \mapsto f'$ is monotone with respect to strong Weihrauch reducibility $\leq_{\mathrm{sW}}$, but not with respect to ordinary Weihrauch reducibility $\leq_{\mathrm{W}}$. This is another reason why it is beneficial to extend the study of the Weihrauch lattice to strong Weihrauch reducibility.

Closed Choice. A particularly useful multi-valued function in the Weihrauch lattice is closed choice (see [27, 9, 8, 6]) and it is known that many notions of computability can be calibrated using the right version of choice. We recall that a subset $U \subseteq X$ of a represented space X is open if its characteristic function

$$\chi_U : X \to \mathbb{S}, x \mapsto \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

is continuous, where $\mathbb{S} = \{0, 1\}$ is Sierpiński space (equipped with the topology $\{\emptyset, \{1\}, \mathbb{S}\}$). Analogously, U is *c.e. open* if χ_U is computable, where $\mathbb{S}$ is equipped with its Standard representation $\delta_{\mathbb{S}}$ defined by $\delta_{\mathbb{S}}(p) := 0$ if $p(n) = 0$ for all n and $\delta_{\mathbb{S}}(p) := 1$, otherwise. Closed and *co-c.e. closed* sets $A \subseteq X$ are sets whose complement $U := X \setminus U$ is open and c.e. open, respectively (see [6, 18] for more details). For sets A of natural numbers or subsets A of the natural numbers this leads to the usual notion of c.e. and co-c.e. sets. The co-c.e. closed subsets $A \subseteq 2^{\mathbb{N}}$ of Cantor space are exactly the usual Π^0_1–classes.

In general, if (X, δ_X) is a represented space then we denote by $\mathcal{A}_-(X)$ the set of closed subsets of X represented with respect to negative information. More precisely, we can define a representation ψ_- of $\mathcal{A}_-(X)$ by

$$\psi_-(p) = A :\iff [\delta_X \to \delta_{\mathbb{S}}](p) = \chi_{X\setminus A},$$

where $[\delta_X \to \delta_{\mathbb{S}}]$ is the canonical function space representation in the category of represented spaces (see [57]). This means that a ψ_-–name p of a closed set $A \subseteq X$ is a name for the characteristic function $\chi_{X\setminus A}$ of its complement.

We are mostly interested in closed choice for computable metric spaces X, which are separable metric spaces such that the distance function is computable on the given dense subset. We assume that computable metric spaces are represented via their Cauchy representations (see [57] for details). In this special case a computably equivalent definition of ψ_- can be obtained by

$$\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)},$$

where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius. Here a ψ_-–name p of a closed set $A \subseteq X$ is a list of sufficiently many open rational balls whose union exhausts exactly the complement of A. We are now prepared to define closed choice.

Definition 2.5 (Closed Choice)**.** Let X be a represented space. The *closed choice* problem of the space X is defined by

$$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

with $\mathrm{dom}(\mathsf{C}_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.

Intuitively, C_X takes as input a non-empty closed set in negative description (i.e., given by ψ_-) and it produces an arbitrary point of this set as output. Hence, $A \mapsto A$ means that the multi-valued map C_X maps the input $A \in \mathcal{A}_-(X)$ to the set $A \subseteq X$ as a set of possible outputs. We mention some classes of functions that can be characterized by closed choice. The following results have mostly been proved in [6]:

Proposition 2.6. *Let f be a multi-valued function on represented spaces. Then:*

(1) $f \leq_{\mathrm{W}} \mathsf{C}_1 \iff f$ *is computable,*
(2) $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \iff f$ *is computable with finitely many mind changes,*
(3) $f \leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} \iff f$ *is non-deterministically computable,*
(4) $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \iff f$ *is effectively Borel measurable.*

In case (4) we have to assume that $f : X \to Y$ is single-valued and defined on computable complete metric spaces X, Y.

Here and in general we identify each natural number $n \in \mathbb{N}$ with the corresponding finite subset $n = \{0, 1, ..., n-1\}$. The problem C_0, closed choice for the empty set $0 = \emptyset$, is the bottom element of the Weihrauch lattice. In [12] the jumps $\mathsf{C}'_X \equiv_{\mathrm{W}} \mathsf{CL}_X$ for computable metric spaces X were characterized using the cluster point problem CL_X of X, which is defined by

$$\mathsf{CL}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X, (x_n)_n \mapsto \{x \in X : x \text{ is a cluster point of } (x_n)_n\},$$

where $\mathrm{dom}(\mathsf{CL}_X)$ contains all sequences that have a cluster point.

3. Choice and Omniscience

In this paper we will make some essential use of a number of choice principles that we briefly discuss in this section. Firstly, C_2 is the problem of choosing a point in a non-empty subset $A \subseteq \{0, 1\}$, which is given by negative information. This information basically is a sequence that might contain no information at all or it might eventually contain the information that one of the points 0 or 1 is not included in A. It has been noticed [6, Example 3.2] that C_2 is equivalent to the *lesser limited principle of omniscience* LLPO as it is used in constructive analysis [3].

Fact 3.1. $\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{LLPO}$.

More generally, one obtains $\mathsf{C}_n \equiv_{\mathrm{sW}} \mathsf{MLPO}_n$ for all $n \geq 2$ for the generalizations MLPO_n of LLPO that have been introduced by Weihrauch in [56] (where "M" stands for *more* omniscient). Here we are rather interested in the following variant of choice, which we call *all or co-unique choice.*[2]

Definition 3.2 (All or co-unique choice)**.** Let X be a represented space. By $\mathsf{ACC}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ we denote the *all or co-unique choice operation* with $\mathrm{dom}(\mathsf{ACC}_X) := \{A \in \mathcal{A}_-(X) : |X \setminus A| \leq 1 \text{ and } |A| > 0\}$.

Here $|A|$ denotes the cardinality of the set A. In other words, all or co-unique choice ACC_X is the problem of choosing a point in a set $A \subseteq X$ in which at most one element of X is missing. It is easy to see that ACC_X is computable, if X has a non-isolated computable point: a non-isolated point cannot be the only point of X that is missing in a closed set $A \subseteq X$ and hence a computable realizer can just choose that computable point as a solution. In particular, ACC_X is computable for perfect computable metric spaces X. This is the reason that in contrast to C_X the problem ACC_X is mostly of interest for spaces $X \subseteq \mathbb{N}$.

Again, the principle ACC_n has been studied before in form of an omniscience principle LLPO_n by Weihrauch [56] and it can be defined as a multi-valued function by

$$\mathsf{LLPO}_n :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, \langle p_1, ..., p_n \rangle \mapsto \{i \in \mathbb{N} : p_i = \widehat{0}\}$$

where $\mathrm{dom}(\mathsf{LLPO}_n)$ is the set of those $\langle p_1, ..., p_n \rangle$ such that $p_i \neq \widehat{0}$ for at most one $i = 1, ..., k$ and $\widehat{n} \in \mathbb{N}^{\mathbb{N}}$ denotes the constant sequence with value $n \in \mathbb{N}$. With this definition we obtain $\mathsf{LLPO}_2 = \mathsf{LLPO}$ and the following fact is quite obvious.

Fact 3.3. $\mathsf{ACC}_n \equiv_{\mathrm{sW}} \mathsf{LLPO}_n$ *for all* $n \geq 2$.

The problem ACC_n also appeared under the name $\mathsf{C}_{n-1,n}$ in the proof of [11, Theorem 10.1] from which we can conclude $\mathsf{ACC}_{n+1} <_{\mathrm{W}} \mathsf{ACC}_n$ for all $n \geq 2$. However, this had already been proved by Weihrauch and with [56, Theorems 4.3 and 5.4] we obtain the following result.

Fact 3.4 (Weihrauch 1992)**.** *For all* $n > 2$ *we obtain*

$$\mathsf{ACC}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{ACC}_{n+1} <_{\mathrm{W}} \mathsf{ACC}_n <_{\mathrm{W}} \mathsf{ACC}_2 = \mathsf{C}_2 <_{\mathrm{W}} \mathsf{C}_n <_{\mathrm{W}} \mathsf{C}_{n+1} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}.$$

An analogous result holds true with $<_{\mathrm{sW}}$ *instead of* $<_{\mathrm{W}}$.

We note that $\mathsf{ACC}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{ACC}_{n+1} \leq_{\mathrm{sW}} \mathsf{ACC}_n$ holds obviously for all $n \geq 2$ and the fact that $\mathsf{ACC}_{n+1} <_{\mathrm{W}} \mathsf{ACC}_n$ for all $n \geq 2$ implies the strictness of the reduction $\mathsf{ACC}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{ACC}_{n+1}$. We note that the LLPO_n hierarchy has recently also been separated over IZF+DC [28].

[2]The name *all or co-unique choice* is motivated by *all or unique choice*, which is related to solving linear equations and Nash equilibrate, see [49] or [11].

4. Indiscriminative Theorems

The purpose of this section is to study a class of multi-valued functions or theorems with very little uniform content. We will call these functions or theorems *indiscriminative* since they cannot even be utilized to make binary choices. In the Weihrauch lattice binary choice C_2 represents the ability of making binary choices and the much weaker principle $\mathsf{ACC}_\mathbb{N}$ represents the ability of choosing one of countably many objects in a setting where at most one object is forbidden.

Definition 4.1 (Discriminative degrees)**.** A multi-valued function f on represented spaces is called *discriminative* if $\mathsf{C}_2 \leq_\mathrm{W} f$. Otherwise, it is called *indiscriminative*. We call f *ω–discriminative*, if $\mathsf{ACC}_\mathbb{N} \leq_\mathrm{W} f$ and *ω–indiscriminative* otherwise.

It follows directly from Fact 3.4 that every f which is discriminative is also ω–discriminative and every f which is ω–indiscriminative is also indiscriminative. All non-constructive theorems from classical analysis that have been classified in the Weihrauch lattice so far are discriminative (see for instance [8, 12]). An indiscriminative theorem is very weak in the sense that it is not even capable of binary choices and in all cases that we will encounter here this has a common reason. This reason is that the corresponding theorem has just so many solutions that any finite portion of a particular solution does not carry any useful information. This idea is made precise in the following definition.

Definition 4.2 (Dense realization)**.** Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued map on represented spaces (X, δ_X) and (Y, δ_Y). Then f is called *densely realized*, if

$$\{F(p) : F \vdash f\}$$

is dense in $\mathrm{dom}(\delta_Y)$ for all $p \in \mathrm{dom}(f\delta_X)$.

If we do not want to assume the Axiom of Choice for Baire space, then it is possible that some f do not have realizers $F \vdash f$. However, if f has a realizer, then f is densely realized if and only if $\delta_Y^{-1} \circ f \circ \delta_X(p)$ is dense in $\mathrm{dom}(\delta_Y)$ for every $p \in \mathrm{dom}(f\delta_X)$. This is is because if f has a realizer $F \vdash f$ at all, then the value $F(p)$ can be changed to any value in $\delta_Y^{-1} \circ f \circ \delta_X(p)$. It is relatively easy to see that every densely realized f is ω–indiscriminative. More than this, no such f can be strongly pointed or a cylinder. We formulate an even stronger property. We say that f *strongly bounds* g if $g \leq_\mathrm{sW} f$.

Proposition 4.3 (Dense realization)**.** *Let f be a multi-valued function on represented spaces that is densely realized. Then f is ω–indiscriminative and does not strongly bound any single-valued non-constant function on Baire space. In particular, f is not strongly pointed, does not strongly bound any cylinder and is not a cylinder itself.*

Proof. Let f be densely realized. Let us assume that $\mathsf{ACC}_\mathbb{N} \leq_\mathrm{W} f$. Then there are computable functions H, K such that $H\langle \mathrm{id}, FK\rangle \vdash \mathsf{ACC}_\mathbb{N}$ whenever $F \vdash f$. Consider a name p of $\mathbb{N}$ and let $G \vdash f$ be some realizer of f. Let us assume $H\langle p, GK(p)\rangle = i \in \mathbb{N}$. By the continuity of H there are some words $w \sqsubseteq p$ and $v \sqsubseteq GK(p)$ such that $H\langle w\mathbb{N}^\mathbb{N}, v\mathbb{N}^\mathbb{N}\rangle = \{i\}$. Then we consider some name q of $\mathbb{N}\setminus\{i\}$ with $w \sqsubseteq q$. Since f is densely realized, there is some realizer $F \vdash f$ such that $v \sqsubseteq FK(q)$. Now we obtain $H\langle q, FK(q)\rangle = i$, which contradicts $H\langle q, FK(q)\rangle \in \mathbb{N} \setminus \{i\} = \mathsf{ACC}_\mathbb{N}(\mathbb{N} \setminus \{i\})$. Hence $\mathsf{ACC}_\mathbb{N} \not\leq_\mathrm{W} f$ and f is ω–indiscriminative.

Let us now assume that $g \leq_\mathrm{sW} f$, where $g :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a non-constant function. Then there are computable H, K such that $HFK = g$ whenever $F \vdash f$. Let $F_0 \vdash f$ and let $p, q \in \mathrm{dom}(g)$ with $g(p) \neq g(q)$. Then there is some prefix $v \sqsubseteq g(p)$ such that $v \not\sqsubseteq g(q)$. Then $v \sqsubseteq HF_0K(p)$ and by continuity of H there is some finite prefix $w \sqsubseteq F_0K(p)$ such that $H(w\mathbb{N}^\mathbb{N}) \subseteq v\mathbb{N}^\mathbb{N}$. Since f is densely realized, there is some realizer $F \vdash f$ with $w \sqsubseteq FK(q)$ and hence $v \sqsubseteq HFK(q)$, which contradicts $HFK(q) = g(q)$. Hence $g \not\leq_{\mathrm{sW}} f$.

In particular, we obtain $\mathrm{id} \not\leq_{\mathrm{sW}} f$ and hence f is not strongly pointed. Since $\mathrm{id} \leq_{\mathrm{sW}} g$ holds for any cylinder g, it follows that $g \leq_{\mathrm{sW}} f$ cannot hold for a cylinder g. In particular, f is not a cylinder itself. □

Sometimes a multi-valued function is densely realized just due to the mere type of its output. This is the case if the output representation is densely realized itself in the following sense.

Definition 4.4 (Densely realized)**.** A represented space (Y, δ) is called *densely realized* if $\delta^{-1}(y)$ is dense in $\mathrm{dom}(\delta)$ for each $y \in Y$.

It is clear that the final topology induced by such a representation δ is always indiscrete, i.e., equal to $\{\emptyset, X\}$. If one does not assume that the Axiom of Choice holds for Baire space, then there might be multi-valued functions f on represented spaces without realizer. Any such f is a top element of the Weihrauch lattice and hence automatically discriminative. For f that admit a realizer we obtain the following sufficient condition for indiscrimination.

Proposition 4.5. *Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function on represented spaces and let Y be densely realized. If f has a realizer, then f is densely realized.*

Proof. Let (X, δ_X) and (Y, δ_Y) be represented spaces. If Y is densely realized, then $\delta_Y^{-1}(y)$ is dense in $\mathrm{dom}(\delta_Y)$ for all $y \in Y$. Hence, $\delta_Y^{-1} \circ f \circ \delta_X(p)$ is dense in $\mathrm{dom}(\delta_Y)$ for all $p \in \mathrm{dom}(f\delta_X)$. If f has a realizer, then this means that f is densely realized. □

Densely realized spaces occur quite naturally, for instance all derived spaces have this property.

Example 4.6 (Derived space)**.** If (X, δ) is a represented space, then the *derived space* (X, δ') with $\delta' := \delta \circ \lim$ is densely realized.

Derived spaces have been considered as output spaces for theorems, for instance the weak Bolzano-Weierstraß Theorem $\mathsf{WBWT}_{\mathbb{R}}$ studied in [42] has a derived space as output space.

Example 4.7 (Weak Bolzano-Weierstraß Theorem)**.** The Weak Bolzano-Weierstraß Theorem $\mathsf{WBWT}_{\mathbb{R}}$ is densely realized and hence ω–indiscriminative.

Another interesting case of a densely realized space is the space of Turing degrees. By $[p] := \{q \in \mathbb{N}^{\mathbb{N}} : p \equiv_{\mathrm{T}} q\}$ we denote the Turing degree of $p \in \mathbb{N}^{\mathbb{N}}$. In a natural way, p can be considered as representative of its degree $[p]$.

Definition 4.8 (Turing degrees)**.** Let $\mathcal{D} := \{[p] : p \in \mathbb{N}^{\mathbb{N}}\}$ be the set of Turing degrees with the representation $\delta_{\mathcal{D}} : \mathbb{N}^{\mathbb{N}} \to \mathcal{D}, p \mapsto [p]$.

In the following we understand all computability statements on $\mathcal{D}$ with respect to the representation $\delta_{\mathcal{D}}$. For every $p \in \mathbb{N}^{\mathbb{N}}$ and every word $w \in \mathbb{N}^*$ there is a $q \in \mathbb{N}^{\mathbb{N}}$ such that $w \sqsubseteq q \in [p]$. Hence, $(\mathcal{D}, \delta_{\mathcal{D}})$ is an example of a densely realized space and all multi-valued functions with output $\mathcal{D}$ are ω–indiscriminative.

Corollary 4.9 (Turing degrees)**.** *Let X be a represented space and $f :\subseteq X \rightrightarrows \mathcal{D}$ a multi-valued function. Then f is densely realized and hence ω–indiscriminative.*

Hence, every for-all-exists theorem that claims the existence of some Turing degree is automatically ω–indiscriminative. This means that large parts of computability theory have very little uniform computational content in terms of the Weihrauch lattice. As an example we consider the statement that for every $\mathbf{a} \in \mathcal{D}$ there exists a $\mathbf{b} \in \mathcal{D}$ such that $\mathbf{b} \not\leq \mathbf{a}$.

Example 4.10 (Non-computability problem)**.** The problem

$$\mathsf{NON} : \mathcal{D} \rightrightarrows \mathcal{D}, \mathbf{a} \mapsto \{\mathbf{b} : \mathbf{b} \not\leq \mathbf{a}\}$$

is ω–indiscriminative.

NON can be seen as one of the simplest non-computable existence statements of a Turing degree. It is clear that NON is not computable, we prove that it is also not continuous. In general, all ω–discriminative problems are automatically discontinuous (since $\mathsf{ACC}_{\mathbb{N}}$ is discontinuous), while for ω–indiscriminative problems discontinuity needs to be checked individually.

Proposition 4.11. NON *is discontinuous (even if restricted to an arbitrary cone* $\{\mathbf{a} \in \mathcal{D} : \mathbf{a}_0 \leq \mathbf{a}\}$ *with* $\mathbf{a}_0 \in \mathcal{D}$*).*

Proof. Let us assume that $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a continuous realizer of NON. Then F is computable with respect to some oracle $q \in \mathbb{N}^{\mathbb{N}}$. Let $p_0 \in \mathbb{N}^{\mathbb{N}}$ be arbitrary and $p := \langle p_0, q\rangle$. Then $p_0 \leq_{\mathrm{T}} p$ and $F(p) \leq_{\mathrm{T}} p$, in contrast to the assumption that F realizes NON. $\square$

5. Diagonally Non-Computable Functions

In this section we discuss diagonally non-computable functions. By $\varphi : \mathbb{N} \to \mathcal{P}$ we denote a standard Gödel numbering of the set $\mathcal{P}$ of partial computable functions $f :\subseteq \mathbb{N} \to \mathbb{N}$. We recall that a function $q : \mathbb{N} \to \mathbb{N}$ is called *diagonally non-computable* if $q(n) \neq \varphi_n(n)$ holds for all $n \in \mathbb{N}$. Here the inequality $q(n) \neq \varphi_n(n)$ can hold for two reasons: either $\varphi_n(n)$ is not defined or it is defined and has a value different from $q(n)$. We relativize this problem with respect to some oracle p and some set $X \subseteq \mathbb{N}$ of values and we use the relativized Gödel numbering φ^p for this purpose.

Definition 5.1 (Diagonally non-computable functions)**.** Let $X \subseteq \mathbb{N}$. We define $\mathsf{DNC}_X :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ for all $p \in \mathbb{N}^{\mathbb{N}}$ by

$$\mathsf{DNC}_X(p) := \{q \in X^{\mathbb{N}} : (\forall n)\ \varphi_n^p(n) \neq q(n)\}.$$

It is clear that DNC_0 and DNC_1 are nowhere defined and hence computable. The latter follows since for every $p \in \mathbb{N}^{\mathbb{N}}$ there is some Gödel number n such that φ_n^p is a total function with constant value 1. For every set $X \subseteq \mathbb{N}$ with at least two values DNC_X is total. It is also clear that the degree of DNC_X for finite X only depends on the cardinality of X and that $X \subseteq Y$ implies $\mathsf{DNC}_Y \leq_{\mathrm{sW}} \mathsf{DNC}_X$.

It turns out that the problem of diagonally non-computable functions DNC_X is closely related to all or co-unique choice ACC_X. In the following result we show that DNC_X is the parallelization of ACC_X. Via Fact 3.3 an equivalent result has been proved independently by Higuchi and Kihara [30, Proposition 79].

Theorem 5.2 (Diagonally non-computable functions)**.** $\mathsf{DNC}_X \equiv_{\mathrm{sW}} \widehat{\mathsf{ACC}_X}$ *for all* $X \subseteq \mathbb{N}$ *with at least two elements.*

Proof. We use a representations of $\mathcal{A}_-(X)$ such that p is a name of a set $A \subseteq X$ if $\mathrm{range}(p) - 1 = X \setminus A$, i.e., $n + 1 \in \mathrm{range}(p) \iff n \notin A$.

We first prove $\widehat{\mathsf{ACC}_X} \leq_{\mathrm{sW}} \mathsf{DNC}_X$. Given a sequence $(A_i)_{i\in\mathbb{N}}$ of non-empty sets $A_i \subseteq X$ that miss at most one element, we need to find one element in each A_i. We can assume that each A_i is given by some $p_i \in \mathbb{N}^{\mathbb{N}}$ with $\mathrm{range}(p_i) - 1 = X \setminus A_i$. We let $p = \langle p_0, p_1, p_2, ...\rangle$. Then by the relativized smn-Theorem there is a computable function $s : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_{s(i)}^p(n) = \begin{cases} x_i & \text{if } X \setminus A_i = \{x_i\} \\ \uparrow & \text{if } X \setminus A_i = \emptyset \end{cases}.$$

The program with Gödel number $s(i)$ just has to scan the i–th projection p_i of the oracle p in order to find some non-zero element x_i+1 and halt with output x_i if there is such an element; otherwise it can search forever. Then every $q \in \mathsf{DNC}_X(p)$ has the property that $q(s(i)) \in A_i$. In other words, the computable function $H(q) := q \circ s$ satisfies $H \circ \mathsf{DNC}_X(p) \subseteq \widehat{\mathsf{ACC}_X}(p)$ and hence $\widehat{\mathsf{ACC}_X} \leq_{\mathrm{sW}} \mathsf{DNC}_X$.

Now we prove $\mathsf{DNC}_X \leq_{\mathrm{sW}} \widehat{\mathsf{ACC}_X}$. Let $p \in \mathbb{N}^{\mathbb{N}}$ be given. Given p we use a computable function K to compute a name $r = \langle r_0, r_1, r_2, ...\rangle = K(p)$ of a sequence $(A_i)_{i\in\mathbb{N}}$ of sets $A_i \subseteq X$ with

$$X \setminus A_n := \{\varphi_n^p(n)\} = \mathrm{range}(r_n) - 1$$

for all $n \in \mathbb{N}$. The function K works such that it starts to compute $\varphi_n^p(n)$ and as long as no result is available, it writes zeros into r_n; as soon as the computation $\varphi_n^p(n)$ halts, the function K switches to write the value $\varphi_n^p(n)+1$ into r_n. We obtain $\widehat{\mathsf{ACC}_X} \circ K(p) = A_0 \times A_1 \times A_2 \times ... \subseteq \mathsf{DNC}_X(p)$. This implies $\mathsf{DNC}_X \leq_{\mathrm{sW}} \widehat{\mathsf{ACC}_X}$. $\square$

It is clear that $\mathsf{ACC}_2 = \mathsf{C}_2$ and it is known that $\widehat{\mathsf{C}_2} \equiv_{\mathrm{sW}} \mathsf{WKL}$ (see [9, 6]). Hence, we obtain the following result.

Corollary 5.3 (Diagonally non-computable functions)**.** $\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{DNC}_2$.

We mention that the proof of $\mathsf{DNC}_2 \leq_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{WKL}$ also follows directly from the fact that the set

$$\mathsf{DNC}_2(p) = \{q \in 2^{\mathbb{N}} : (\forall n)\ \varphi_n^p(n) \neq q(n)\}$$

is closed and a name for it with respect to negative information can easily be computed from p.

We want to generalize this observation and for this purpose we introduce a generalization of WKL. Let $X \subseteq \mathbb{N}$. By Tr_X we denote the set of trees $T \subseteq X^*$ represented via their characteristic functions $\mathrm{cf}_T : 2^{(X^*)} \to \{0,1\}$. We call a tree $T \subseteq X^*$ *big*, if it has the following property: if w is a node of an infinite path of T, then all but at most one successor node of w are also on some infinite path of T.

Definition 5.4 (Weak Kőnig's Lemma for big trees)**.** Let $X \subseteq \mathbb{N}$. By WKL_X we denote the problem

$$\mathsf{WKL}_X :\subseteq \mathrm{Tr}_X \rightrightarrows X^{\mathbb{N}}$$

where $\mathsf{WKL}_X(T) = [T]$ is the set of infinite paths of T and $\mathrm{dom}(\mathsf{WKL}_X)$ is the set of big trees T with infinite paths.

Since all binary trees are automatically big, we have $\mathsf{WKL}_2 = \mathsf{WKL}$. In general we obtain the following.

Theorem 5.5 (Diagonally non-computable functions and big trees)**.** *If* $k \geq 2$*, then* $\mathsf{DNC}_k \equiv_{\mathrm{sW}} \mathsf{WKL}_k$.

Proof. For each k we can compute a tree $T \subseteq k^*$ with $\mathsf{DNC}_k(p) = [T]$ from p, where

$$\mathsf{DNC}_k(p) = \{q \in k^{\mathbb{N}} : (\forall n)\ \varphi_n^p(n) \neq q(n)\}.$$

This tree T is automatically big and has infinite paths (since $k \geq 2$). Hence $\mathsf{WKL}_k(T) = \mathsf{DNC}_k(p)$. This yields the reduction $\mathsf{DNC}_k \leq_{\mathrm{sW}} \mathsf{WKL}_k$.

For the other direction it suffices to prove $\mathsf{WKL}_k \leq_{\mathrm{sW}} \widehat{\mathsf{ACC}_k}$ by Theorem 5.2. We use some computable standard bijection $w : \mathbb{N} \to k^*$. Given a big tree $T \subseteq k^*$ with some infinite path, we need to find such a path. For each number n we consider the word $w_n = w(n)$ and we assign a set $A_n \subseteq k$ to it, where $A_n = k \setminus \{i\}$ for the first $i \in k$ for which $w_n i k^{\mathbb{N}} \cap [T] = \emptyset$ can be detected, if such an i exists and $A_n = k$ otherwise. This assignment is computable in the sense that negative information on A_n can be computed from T since the property $w_nik^{\mathbb{N}} \cap [T] = \emptyset$ is c.e. in T due to compactness of $w_nik^{\mathbb{N}}$. If $w_nk^{\mathbb{N}} \cap [T] \neq \emptyset$, then

$$A_n = \{i \in \{0,...,k-1\} : w_nik^{\mathbb{N}} \cap [T] \neq \emptyset\}.$$

The fact that T is big implies that at most one element of k is missing in each A_n, provided that $w_nk^{\mathbb{N}} \cap [T] \neq \emptyset$. If $w_nk^{\mathbb{N}} \cap [T] = \emptyset$, then $A_n \subseteq k$ is just some set in which exactly one element of k is missing. Now, given the sequence $(A_n)_n$ one can determine a sequence $p \in k^{\mathbb{N}}$ with $p(n) \in A_n$ for each $n \in \mathbb{N}$ with the help of $\widehat{\mathsf{ACC}_k}$. Starting from the number n_0 of the empty word w we can determine a sequence $q \in [T]$ inductively using p: We set $q(0) := p(n_0)$ and if $q|_i = q(0)...q(i-1)$ has already been determined and $n_i \in \mathbb{N}$ is such that $w_{n_i} = q|_i$, then $q(i) := p(n_i)$. Altogether, this yields an infinite path $q \in [T]$ and hence the reduction $\mathsf{WKL}_k \leq_{\mathrm{sW}} \widehat{\mathsf{ACC}_k}$. □

We use the compactness of the set $\mathsf{DNC}_X(p)$ for finite $X \subseteq \mathbb{N}$ also for the following result. Additionally, we need the following observation.

Lemma 5.6. *Let $n \geq 1$ and $p_0, ..., p_{n-1} \in \mathbb{N}^{\mathbb{N}}$. Then $\bigcap_{i=0}^{n-1} \mathsf{DNC}_{n+1}(p_i) \neq \emptyset$.*

The proof follows immediately with the help of countable choice.

Proposition 5.7. $\mathsf{ACC}_n \nleq_{\mathrm{W}} \mathsf{DNC}_{n+1}$ *for all $n \geq 2$.*

Proof. Let $n \geq 2$ and suppose that $\mathsf{ACC}_n \leq_{\mathrm{W}} \mathsf{DNC}_{n+1}$. Then there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $H\langle \mathrm{id}, GK\rangle$ is a realizer of ACC_n whenever G is a realizer of DNC_{n+1}. Let p be a name of the set $n = \{0, ..., n-1\}$. Since the set $\mathsf{DNC}_{n+1}K(p)$ is compact and H is continuous, it follows that H restricted to $\langle \{p\} \times \mathsf{DNC}_{n+1}K(p)\rangle$ is uniformly continuous. Hence there is a $k \in \mathbb{N}$ such that for every $q \in \mathsf{DNC}_{n+1}K(p)$ the value of $H\langle p, q\rangle$ is already determined by prefixes $p|_k$ and $q|_k$ of p and q, respectively, i.e., $H\langle p|_k\mathbb{N}^{\mathbb{N}} \times q|_k\{0, ..., n\}^{\mathbb{N}}\rangle = \{H\langle p, q\rangle\}$. The values $\varphi_n^{K(p)}(n)$ are defined for n from $D := \{n \in \mathbb{N} : n \in \mathrm{dom}(\varphi_n^{K(p)})$ and $n < k\}$. It follows from the Use Theorem that a prefix of length l of the oracle $K(p)$ is sufficient to guarantee that the corresponding computations halt, i.e. that $n \in \mathrm{dom}(\varphi_n^{K(p)|_l})$ for all $n \in D$. Due to the continuity of K there is a prefix $p|_m$ of p such that $K(p|_m\mathbb{N}^{\mathbb{N}}) \subseteq K(p)|_l\mathbb{N}^{\mathbb{N}}$, without loss of generality, we can assume $m \geq k$. There are $p_0, ..., p_{n-1}$ with $p_i|_m = p|_m$ and $\mathsf{ACC}_n(p_i) = \{0, ..., n-1\} \setminus \{i\}$ for all $i < n$. By Lemma 5.6 there is $r \in \bigcap_{i=0}^{n-1} \mathsf{DNC}_{n+1}K(p_i)$ and by our choices of m, l and p_i, we obtain $r|_k = q|_k$ for some $q \in \mathsf{DNC}_{n+1}K(p)$. Hence there is some realizer G of DNC_{n+1} such that $GK(p_i) = r$ for all $i < n$ and $H\langle p_i, GK(p_i)\rangle = H\langle p_i, r\rangle = H\langle p_i, q\rangle$ does not depend on i, due to our choice of k. This contradicts that $H\langle p_i, GK(p_i)\rangle \in \mathsf{ACC}_n(p_i) = \{0, ..., n-1\} \setminus \{i\}$ for all $i < n$. □

This yields an independent proof of $\mathsf{ACC}_n \nleq_{\mathrm{W}} \mathsf{ACC}_{n+1}$ (which is known by Fact 3.4) and it also yields the following corollary, which alternatively follows from [30, Corollary 80] or [37, Theorem 6].

Corollary 5.8. $\mathsf{DNC}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{DNC}_{n+1} <_{\mathrm{W}} \mathsf{DNC}_n$ *for all $n \geq 2$.*

In particular, DNC_n is indiscriminative for $n \geq 3$.

In [6, Proposition 9.5] it was proved that $f \nleq_{\mathrm{T}} p$ holds for every $f, p \in \mathbb{N}^{\mathbb{N}}$ such that f is diagonally non-computable and p is limit computable in the jump. Hence we obtain the following result, where $\lim_{\mathsf{J}}$ denotes the limit operation lim restricted to those $p = \langle p_0, p_1, ...\rangle$ such that the sequence of Turing jumps $(\mathsf{J}(p_i))_i$ converges.

Corollary 5.9. $\mathsf{DNC}_{\mathbb{N}} \nleq_{\mathrm{W}} \lim_{\mathsf{J}}$.

Since $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \lim_{\mathsf{J}}$ we also get the following corollary.

Corollary 5.10. $\mathsf{DNC}_{\mathbb{N}} \nleq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

6. Degrees of Complete Extensions of Peano Arithmetic

In this section we briefly want to discuss the problem of finding a degree that contains a complete extension of Peano arithmetic relative to some given input. We recall that a Turing degree $\mathbf{a}$ is called a *PA–degree relative to* another Turing degree $\mathbf{b}$, in symbols $\mathbf{a} \gg \mathbf{b}$, if every $\mathbf{b}$–computable infinite binary tree has an $\mathbf{a}$–computable path. The degrees $\mathbf{a} \gg \mathbf{0}$ are exactly the degrees of complete extensions of Peano arithmetic by results of Jockusch, Soare, and Solovay (see for example [37, Proposition 2]). We obtain the following well-known characterization of the relation $\mathbf{a} \gg \mathbf{b}$ that was formally introduced by Simpson [50].

Proposition 6.1. *Let $\mathbf{a}$ and $\mathbf{b}$ be Turing degrees and $n \geq 1$. Then the following conditions are equivalent to each other:*

1. $\mathbf{a} \gg \mathbf{b}$,
2. *every $\mathbf{b}$–computable infinite binary tree has an $\mathbf{a}$–computable path,*
3. *every $\mathbf{b}$–computable function $f :\subseteq \mathbb{N} \to \{0,1\}$ has a total $\mathbf{a}$–computable extension,*
4. *$\mathbf{a}$ is the degree of a function $f : \mathbb{N} \to \{0,1,...,n\}$ that is diagonally non-computable relative to $\mathbf{b}$.*

For the case $n = 1$ it is easy to see that (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2). The first implication follows since for every $\mathbf{b}$–computable function $f :\subseteq \mathbb{N} \to \{0,1\}$ the set $A \subseteq 2^{\mathbb{N}}$ of total extensions is co-c.e. closed in $\mathbf{b}$. The second implication (3) $\Longrightarrow$ (4) is obvious and the latter implication (4) $\Longrightarrow$ (2) is implicit in the proof of Theorem 5.5. The equivalence of (4) for $n = 2$ and numbers $n > 2$ follows from a relativized version of a Theorem of Jockusch (and Friedberg) [37, Theorem 5]. We now introduce the problem of Peano arithmetic as follows.

Definition 6.2 (Peano arithmetic)**.** We call $\mathsf{PA} : \mathcal{D} \rightrightarrows \mathcal{D}, \mathbf{b} \mapsto \{\mathbf{a} : \mathbf{a} \gg \mathbf{b}\}$ the *problem of Peano arithmetic*.

The following properties of the relation $\gg$ are easy to establish [50, Theorem 6.2].

Proposition 6.3. *Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be Turing degrees. Then*

1. $\mathbf{a} \gg \mathbf{b} \Longrightarrow \mathbf{a} > \mathbf{b}$,
2. $\mathbf{a}' \gg \mathbf{a}$,
3. $\mathbf{a} \gg \mathbf{b} \geq \mathbf{c} \Longrightarrow \mathbf{a} \gg \mathbf{c}$,
4. $\mathbf{c} \geq \mathbf{a} \gg \mathbf{b} \Longrightarrow \mathbf{c} \gg \mathbf{a}$.

The first condition implies $\mathsf{NON} \leq_{\mathrm{sW}} \mathsf{PA}$. It is also clear that $\mathsf{PA} \not\leq_{\mathrm{W}} \mathsf{NON}$ holds, since there are non-computable Turing degrees (for instance minimal ones [37, Corollary 2]), which are not degrees of complete extensions of Peano arithmetic. Since PA is densely realized, we also obtain $\mathsf{ACC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{PA}$.

If $f :\subseteq X \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is a multi-valued function, then we denote by

$$[f] :\subseteq X \rightrightarrows \mathcal{D}, x \mapsto \{[p] : p \in f(x)\}$$

the *degree version* of f. It is clear that $[f] \leq_{\mathrm{sW}} f$ holds in general and often $f \not\leq_{\mathrm{W}} [f]$ holds, since $[f]$ is always densely realized, while f might have stronger uniform computational content. This is the case for $f = \mathsf{DNC}_n$ and the characterization from Proposition 6.1(4) yields the following result.

Corollary 6.4 (Peano arithmetic)**.** $\mathsf{PA} \equiv_{\mathrm{sW}} [\mathsf{DNC}_n]$ *for all $n \geq 2$.*

By $\mathsf{J}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}, \mathbf{a} \mapsto \mathbf{a}'$ we denote the Turing jump operator on degrees (we have $[\mathsf{J}] \equiv_{\mathrm{sW}} \mathsf{J}_{\mathcal{D}}$). Then Proposition 6.3(2) yields the following uniform version.

Corollary 6.5. $\mathsf{PA} \leq_{\mathrm{sW}} \mathsf{J}_{\mathcal{D}}$ *and* $\mathsf{J}_{\mathcal{D}} \not\leq_{\mathrm{W}} \mathsf{PA}$.

The latter negative result holds since there are low PA–degrees (by an application of the Low Basis Theorem to the set $\mathsf{DNC}_2(\emptyset)$). Now we are going to provide an interesting characterization of PA as a an implication. As a preparation we prove the following result.

Proposition 6.6. $\mathsf{WKL} \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} * \mathsf{PA}$.

Proof. Given an infinite binary tree T by a name $t \in \mathbb{N}^{\mathbb{N}}$, with the help of PA we can obtain a $q \in \mathbb{N}^{\mathbb{N}}$ that is of PA degree relative to t. This q computes an infinite path $p \in 2^{\mathbb{N}}$ in T. We use an enumeration of all computable functions $\Phi_n :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. Then there must be some $n \in \mathbb{N}$ such that $p = \Phi_n(q)$ is a path in T. We test all numbers $n \in \mathbb{N}$ in parallel and we try to compute longer and longer prefixes of $\Phi_n(q)$. Whenever a longer prefix of $\Phi_n(q)$ than before lies completely in T, then we output the number n. Hence, any fixed number $n \in \mathbb{N}$ will be produced infinitely often if and only if $p = \Phi_n(q)$ is an infinite path in T. It was proved in [12, Theorem 9.4] that $\mathsf{C}'_{\mathbb{N}}$ is equivalent to the cluster point problem on the natural numbers, hence it can be used to find one number $n \in \mathbb{N}$ that has been produced infinitely often. Then $p = \Phi_n(q)$ is an infinite path in T, as desired. □

As a corollary we obtain the following characterization of PA.

Theorem 6.7 (Peano arithmetic)**.** $\mathsf{PA} \equiv_{\mathrm{W}} (\mathsf{C}'_{\mathbb{N}} \to \mathsf{WKL})$.

Proof. Proposition 6.6 implies $(\mathsf{C}'_{\mathbb{N}} \to \mathsf{WKL}) \leq_{\mathrm{W}} \mathsf{PA}$. By Corollary 5.3 we have $\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{DNC}_2$. Now let h be such that $\mathsf{DNC}_2 \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} * h$. Without loss of generality we can assume that h is of type $h :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. $\mathsf{C}'_{\mathbb{N}}$ only produces a natural number output. Now given some $p \in \mathbb{N}^{\mathbb{N}}$, the function h must be able (potentially after some additional computation) to produce an output $q \in \mathbb{N}^{\mathbb{N}}$ that (potentially after some further computation that uses the discrete output of $\mathsf{C}'_{\mathbb{N}}$) computes a diagonally non-computable function f relative to p. Such a function f is of PA degree relative to p. By Proposition 6.3 (4) we obtain that also q is of PA degree relative to p. Hence $q \in \mathsf{PA}(p)$. This proves $\mathsf{PA} \leq_{\mathrm{W}} h$ and hence $\mathsf{PA} \leq_{\mathrm{W}} (\mathsf{C}'_{\mathbb{N}} * \mathsf{WKL})$. □

We note that for the direction $\mathsf{PA} \leq_{\mathrm{W}} (\mathsf{C}'_{\mathbb{N}} * \mathsf{WKL})$ we have not used any property of $\mathsf{C}'_{\mathbb{N}}$ other than that it produces a natural number output. Hence, the same proof shows $\mathsf{PA} \equiv_{\mathrm{W}} (\mathsf{C}^{(n)}_{\mathbb{N}} \to \mathsf{WKL})$ for every $n \geq 1$. It is unclear what happens in case of $n = 0$. Using Proposition 6.6 we can also obtain the following result.

Corollary 6.8. $\mathsf{DNC}_2 \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} * \mathsf{DNC}_k$ *for all* $k \geq 2$.

R. Friedberg proved that the Turing degrees of DNC_2–functions coincide with the Turing degrees of DNC_k–functions for all $k \geq 2$ [37, Theorem 5]. Dorais, Hirst and Shafer analyzed the uniform content of the equivalence in reverse mathematics under the presence of $\mathbf{\Sigma}^0_2$–induction [23, Theorem 2.7]. Since $\mathsf{C}'_{\mathbb{N}}$ is the counterpart of $\mathbf{\Sigma}^0_2$–induction in the Weihrauch lattice, Corollary 6.8 can be seen as a uniform version of their result. Again it remains unclear whether we can replace $\mathsf{C}'_{\mathbb{N}}$ by $\mathsf{C}_{\mathbb{N}}$ here.

7. Martin-Löf Randomness and Weak Weak Kőnig's Lemma

Another problem that is located in the neighborhood of diagonally non-computable functions in the Weihrauch lattice is Martin-Löf randomness. By $\mathsf{MLR} : \mathbb{N}^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ we denote the problem such that $\mathsf{MLR}(p)$ contains all q which are Martin-Löf random relative to p (see [45, 25] for definitions). If $p \leq_{\mathrm{T}} q$, then $\mathsf{MLR}(q) \subseteq \mathsf{MLR}(p)$. Since any finite modification of a $q \in \mathsf{MLR}(p)$ is also in $\mathsf{MLR}(p)$ we immediately get the following corollary of Proposition 4.3.

Lemma 7.1 (Martin-Löf Randomness)**.** MLR *is densely realized and hence ω–indiscriminative.*

In particular, this implies $\mathsf{DNC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{MLR}$. This is in sharp contrast to the non-uniform situation, where Kučera [43] proved that each Martin-Löf random computes a diagonally non-computable function (see also [25, Theorem 8.8.1]). Like in Corollary 5.9 the result of Kučera yields the following corollary with the help of [6, Proposition 9.5].

Corollary 7.2. $\mathsf{MLR} \not\leq_{\mathrm{W}} \lim_{\mathrm{J}}$.

The relation between diagonally non-computable functions and Martin-Löf randomness has also been studied in the non-uniform sense of reverse mathematics, for instance by Ambos-Spies et al. in [1]. We utilize these results in order to show that MLR and $\mathsf{DNC}_{\mathbb{N}}$ are actually incomparable in the uniform sense of the Weihrauch lattice.

Proposition 7.3. $\mathsf{DNC}_{\mathbb{N}} \,|_{\mathrm{W}}\, \mathsf{MLR}$.

Proof. As mentioned above, $\mathsf{DNC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{MLR}$ follows from Lemma 7.1 and Theorem 5.2. Let us now assume $\mathsf{MLR} \leq_{\mathrm{W}} \mathsf{DNC}_{\mathbb{N}}$ for a contradiction. Then there are computable functions H, K such that $H\langle p, GK(p)\rangle$ is a realizer of MLR for any realizer G of $\mathsf{DNC}_{\mathbb{N}}$. By [1, Theorem 1.8] there exists a diagonally non-computable function $g : \mathbb{N} \to \mathbb{N}$ such that for every computable $h : \mathbb{N} \to \mathbb{N}$ and every diagonally non-computable function $f : \mathbb{N} \to \mathbb{N}$ that is bounded by h one obtains $f \not\leq_{\mathrm{T}} g$. Let now G be a realizer of $\mathsf{DNC}_{\mathbb{N}}$ that assigns to every computable p the function $G(p) = g$ and let p be computable and fixed. Then $GK(p) = g$ and $q := H\langle p, g\rangle = H\langle p, GK(p)\rangle \in \mathsf{MLR}(p)$. Now by [1, Theorem 1.4] there is a computable $h : \mathbb{N} \to \mathbb{N}$ and a diagonally non-computable function f, which is bounded by h and such that $f \leq_{\mathrm{T}} q$. Since $q = H\langle p, g\rangle$, this implies $f \leq_{\mathrm{T}} q \leq_{\mathrm{T}} g$ in contradiction to the conclusion above. Hence $\mathsf{MLR} \not\leq_{\mathrm{W}} \mathsf{DNC}_{\mathbb{N}}$. $\square$

Dorais et al. [22] introduced a quantitative version ε-WWKL of Weak Weak Kőnig's Lemma that was studied further in [11]. Here ε-$\mathsf{WWKL} :\subseteq \mathrm{Tr}_2 \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$ is the same problem as WKL, but restricted to the set $\mathrm{dom}(\varepsilon\text{-}\mathsf{WWKL})$ of trees T with measure $\mu([T]) > \varepsilon$, where μ denotes the uniform measure on Cantor space $2^{\mathbb{N}}$. Moreover, $\mathsf{WWKL} := 0\text{-}\mathsf{WWKL}$. We also study the problem $(1-*)$-$\mathsf{WWKL} :\subseteq \mathrm{Tr}_2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$, introduced in [11], which is defined by

$$(1-*)\text{-}\mathsf{WWKL}((T_n)_n) := \bigsqcup_{n\in\mathbb{N}} (1-2^{-n})\text{-}\mathsf{WWKL}(T_n).$$

Intuitively, this problem can be described such that given a sequence $(T_n)_n$ of trees with $\mu([T_n]) > 1 - 2^{-n}$, one needs to find *one* path $p \in [T_n]$ (i.e., for one arbitrary n). Logically, this corresponds to a uniform existential quantification $(\exists n)(1-2^{-n})$-WWKL. It is clear that $(1-*)$-$\mathsf{WWKL} \leq_{\mathrm{W}} \varepsilon$-$\mathsf{WWKL}$ for all $\varepsilon < 1$. The following result is also immediate.

Lemma 7.4. $\mathsf{MLR} <_{\mathrm{W}} (1-*)$-$\mathsf{WWKL}$ *and* $\mathsf{MLR} <_{\mathrm{sW}} (1-*)$-$\mathsf{WWKL}$.

The positive part of the proof follows using a universal Martin-Löf test $(U_n)_n$ in p, which can be computably converted into a sequence $(T_n)_n$ of trees with $\mu([T_n]) > 1 - 2^{-n}$ and any such tree has only infinite paths, which are Martin-Löf random in p (the proof is included in a follow-up draft of [11]). The negative part of the statement in Lemma 7.4 follows (besides other known proofs) from the following observation together with Lemma 7.1.

Lemma 7.5. $\mathsf{ACC}_{\mathbb{N}} <_{\mathrm{W}} (1-*)$-$\mathsf{WWKL}$ *and* $\mathsf{ACC}_{\mathbb{N}} <_{\mathrm{sW}} (1-*)$-$\mathsf{WWKL}$.

Proof. Given a set $A \subseteq \mathbb{N}$ in which at most one element of $\mathbb{N}$ is missing and a number n, we consider the subtrees $w_i 2^*$, where $\{w_0, ..., w_{2^{n+1}-1}\} = \{0,1\}^{n+1}$ is the set of all binary words of length $n+1$. If one of the numbers $i = 0, ..., 2^{n+1}-2$ is missing in A, then we remove the corresponding subtree $w_i 2^*$ from the full tree in order to get a tree T_n, if a number $i \geq 2^{n+1}-1$ is missing in A, then we remove $w_{2^{n+1}-1}2^*$ in order to obtain T_n. If no number i is missing in A, then T_n is the full tree 2^*. The map that maps (n, A) (where A is given by negative information) to T_n is computable and T_n satisfies $\mu([T_n]) \geq 1 - 2^{-n-1} > 1 - 2^n$. Any infinite path $p \in [T_n]$ can be used to identify in a computable way a number $i \in A$. Hence, $\mathsf{ACC}_\mathbb{N} \leq_{\mathrm{sW}} (1-*)\text{-}\mathsf{WWKL}$. It is clear that the reductions are strict, since $\mathsf{ACC}_\mathbb{N}$ only produces computable values on computable inputs (in fact, natural numbers), while there is a computable sequence $(T_n)_n$ of trees T_n with $\mu([T_n]) > 1 - 2^{-n}$ and such that no T_n has has an infinite computable path (such a sequence of trees can be obtained, for instance, by a universal Martin-Löf test, as explained above). $\square$

In particular, $(1-*)\text{-}\mathsf{WWKL}$ is ω–discriminative. Next we want to show that also $\mathsf{DNC}_\mathbb{N} \leq_{\mathrm{W}} (1-*)\text{-}\mathsf{WWKL}$. This follows from Lemma 7.5 and the following result.

Proposition 7.6. $(1-*)\text{-}\mathsf{WWKL}$ *is strongly parallelizable.*

Proof. We need to prove $\widehat{(1-*)\text{-}\mathsf{WWKL}} \leq_{\mathrm{sW}} (1-*)\text{-}\mathsf{WWKL}$. Given a double sequence $(T_{k,n})$ of binary trees with $\mu([T_{k,n}]) > 1 - 2^{-n}$ for all k, n we need to compute one sequence (T_n) of trees with $\mu([T_n]) > 1 - 2^{-n}$ such that from an infinite path $p \in [T_n]$ for an arbitrary n, we can compute one infinite path $p_k \in [T_{k,n_k}]$ for some arbitrary n_k for each $k \in \mathbb{N}$. Given the double sequence $(T_{k,n})$ we can compute a sequence (T_n) of trees such that

$$[T_n] = \bigcap_{k=0}^{\infty} [T_{k,n+k+1}]$$

for all n. Then $\mu[T_n] > 1 - \sum_{k=0}^{\infty} 2^{-n-k-1} = 1 - 2^{-n}$. Moreover, an infinite path $p \in [T_n]$ for some n is also an infinite path $p \in [T_{k,n+k+1}]$ for all k, which completes the desired reduction. $\square$

Lemma 7.5, Theorem 5.2 and Proposition 7.6 yield the desired corollary. The fact that the reduction in the following corollary is strict, follows from Lemma 7.4 and Proposition 7.3.

Corollary 7.7. $\mathsf{DNC}_\mathbb{N} <_{\mathrm{W}} (1-*)\text{-}\mathsf{WWKL}$ *and* $\mathsf{DNC}_\mathbb{N} <_{\mathrm{sW}} (1-*)\text{-}\mathsf{WWKL}$.

We note that this result shows that $\mathsf{DNC}_\mathbb{N}$ admits a Las Vegas algorithm in the sense of [11], even one of any success probability arbitrarily close to 1, while $\mathsf{DNC}_\mathbb{N}$ cannot be reduced to MLR by Lemma 7.1.

We next want to prove that PA is not reducible to any jump of WWKL. This can be established using a theorem of Jockusch and Soare [39, Corollary 5.4].

Lemma 7.8 (Jockusch and Soare 1972)**.** *Let $A \subseteq 2^\mathbb{N}$ be a set such that for each $r \in A$ there exists a PA–degree $\mathbf{a}$ with $\mathbf{a} \leq [r]$. Then $\mu(A) = 0$.*

We recall (see [16] and [11]) that a function $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ on represented spaces is called *probabilistic*, if there is a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and a family $(A_p)_{p \in D}$ of measurable sets $A_p \subseteq 2^\mathbb{N}$ with $D := \mathrm{dom}(f\delta_X)$ such that $\mu(A_p) > 0$ for all $p \in D$ and $\delta_Y F(p, r) \in f\delta_X(p)$ for all $p \in D$ and $r \in A_p$. Roughly speaking, a function is probabilistic if it can be computed with help of a piece of random advice originating from some set of positive measure that can depend non-uniformly and non-effectively on the input. We now transfer the proof of [16, Theorem 20] into our setting.

Proposition 7.9. PA *is not probabilistic.*

Proof. Let us assume that $\mathsf{PA} : \mathcal{D} \rightrightarrows \mathcal{D}$ is probabilistic. Then there exists a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and a family $(A_p)_{p\in\mathbb{N}^\mathbb{N}}$ of measurable sets $A_p \in 2^\mathbb{N}$ such that $\mu(A_p) > 0$ for all $p \in \mathbb{N}^\mathbb{N}$ and such that $[F(p,r)] \in \mathsf{PA}([p])$ for all $p \in \mathbb{N}^\mathbb{N}$ and $r \in A_p$. We fix the computable zero sequence p. Then we obtain for each $r \in A_p$ that $\mathbf{a} := [F(p,r)] \in \mathsf{PA}(p)$ is a PA–degree and $\mathbf{a} \leq [r]$. Hence $\mu(A_p) = 0$ by Lemma 7.8, which is a contradiction. $\square$

Since probabilistic Weihrauch degrees are closed downwards [11, Proposition 14.3] we obtain the following corollary.

Corollary 7.10. DNC_n *is not probabilistic for all* $n \geq 2$.

We note that this contrasts the situation for $\mathsf{DNC}_\mathbb{N}$, which is probabilistic by Corollary 7.7 and [11, Corollary 14.7]. Proposition 7.9 also yields the following conclusion.

Corollary 7.11. $\mathsf{PA} \not\leq_\mathrm{W} \mathsf{WWKL}^{(k)}$ *for all* $k \in \mathbb{N}$.

We note that the proof of [11, Theorem 10.1] yields the following result.

Lemma 7.12. $\mathsf{ACC}_{n+1} \leq_\mathrm{sW} \frac{n-1}{n}\text{-}\mathsf{WWKL}$ *and* $\mathsf{ACC}_n \not\leq_\mathrm{W} \frac{n-1}{n}\text{-}\mathsf{WWKL}$ *for all* $n \geq 2$.

As a side result we obtain the following conclusion from Lemma 7.12, Corollary 7.11 and Theorem 5.2, which contrasts Proposition 7.6.

Corollary 7.13. $\varepsilon\text{-}\mathsf{WWKL}$ *is not parallelizable for all* $\varepsilon \in [0,1)$.

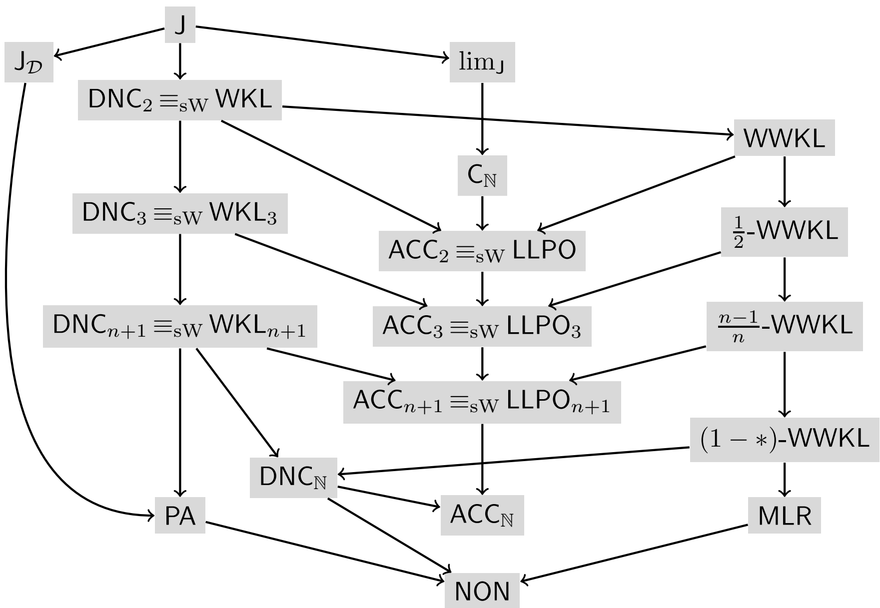

Figure 1. Diagonally non-computable functions in the Weihrauch lattice.

In order to import further knowledge on diagonally non-computable functions into our lattice, it is useful to mention the following relation between Weihrauch reducibility and Medvedev reducibility. We recall that for two sets $A, B \subseteq \mathbb{N}^\mathbb{N}$ we define $A \leq_\mathrm{M} B$, A is *Medvedev reducible to* B, if there exists a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $F(B) \subseteq A$. For the following result we need to assume that g has a realizer (or we need to assume the Axiom of Choice for Baire space, which implies this property).

Lemma 7.14 (Weihrauch and Medvedev reducibility). *For $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ (such that g has a realizer) we obtain*

$$f \leq_\mathrm{W} g \Longrightarrow (\forall \textit{ computable } p \in \mathrm{dom}(f))(\exists \textit{ computable } q \in \mathrm{dom}(g))\ f(p) \leq_\mathrm{M} g(q).$$

Proof. If $f \leq_\mathrm{W} g$, then there are computable H, K such that $H\langle \mathrm{id}, GK\rangle$ is a realizer of f whenever G is a realizer of g. Let $p \in \mathrm{dom}(f)$ be computable. Then $q := K(p)$ is computable and $GK(p) \in g(q)$ for every realizer G of g. In fact, for every $r \in g(q)$ there is a realizer G of g such that $GK(p) = r$ and hence $H\langle p, r\rangle \in f(p)$ for all $r \in g(q)$. In other words, the function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with $F(r) := H\langle p, r\rangle$ is computable and satisfies $F(g(q)) \subseteq f(p)$. This means $f(p) \leq_\mathrm{M} g(q)$. □

In [24, Theorem 5.4] Downey et al. proved that the Martin-Löf random points are not Medvedev reducible to the diagonally non-computable functions with three values. We note that the Medvedev degrees of $\mathsf{DNC}_3(q)$ are identical for all computable q. This implies $\mathsf{MLR} \not\leq_\mathrm{W} \mathsf{DNC}_3$ by Lemma 7.14 and hence together with Proposition 7.3 we obtain the following result.

Corollary 7.15. $\mathsf{MLR} \,|_\mathrm{W}\, \mathsf{DNC}_3$.

In the diagram in Figure 1 we collect the results on diagonally non-computable functions, Weak Weak Kőnig's Lemma and Martin-Löf randomness. All lines indicate strong Weihrauch reductions $\leq_\mathrm{sW}$ against the direction of the arrow, i.e., if $f \leq_\mathrm{sW} g$, then the arrow points from g to f. We note that by Corollary 7.11 and Corollary 7.15 we also get the following separation.

Corollary 7.16. $\mathsf{WKL}_n \,|_\mathrm{W}\, \frac{k-1}{k}$-$\mathsf{WWKL}$ *for all $n \geq 3$ and $k \geq 2$.*

8. The Low Basis Theorem

The purpose of this section is to classify the computational content of the Low Basis Theorem of Jockusch and Soare [39, Theorem 2.1]. It states that every computable infinite binary tree has a low path. We consider the natural relativized version that states that every computable infinite binary tree has a path that is low relative to the tree. This version has a straightforward interpretation in the Weihrauch lattice.

Definition 8.1 (Low Basis Theorem). By $\mathsf{LBT} :\subseteq \mathrm{Tr}_2 \rightrightarrows \{0,1\}^\mathbb{N}$ we denote the multi-valued function with

$$\mathsf{LBT}(T) := \{p \in [T] : p' \leq_\mathrm{T} T'\},$$

where $\mathrm{dom}(\mathsf{LBT}) := \{T \in \mathrm{Tr}_2 : T \text{ infinite}\}$.

The (relativized version of the) classical Low Basis Theorem guarantees that LBT is actually well-defined, i.e., $\mathsf{LBT}(T)$ is non-empty whenever T is an infinite binary tree. The Uniform Low Basis Theorem [6, Theorem 8.3] can be used in order to derive a rough classification of LBT. Here $\mathsf{L} := \mathsf{J}^{-1} \circ \lim$ denotes the *low map* introduced in [6], which is the composition of the Turing jump operator $\mathsf{J} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with the limit map lim (the image of the set of computable points under L is exactly the set of low points).

Proposition 8.2. $\mathsf{WKL} \leq_\mathrm{sW} \mathsf{LBT} \leq_\mathrm{sW} \mathsf{L}$.

Proof. Firstly, it is clear that $\mathsf{WKL} \leq_\mathrm{sW} \mathsf{LBT}$ holds, since LBT is a restriction of WKL in the image. Secondly, the Uniform Low Basis Theorem [6, Theorem 8.3] implies that $\mathsf{LBT} \leq_\mathrm{sW} \mathsf{L}$ holds, since it shows that L computes a realizer of WKL that produces outputs which are low relative to the input and LBT is the restriction of WKL in the image to exactly such outputs. Hence L computes a realizer of LBT. □

In order to separate the Low Basis Theorem LBT from Weak Kőnig's Lemma WKL, we can use the Hyperimmune-Free Basis Theorem of Jockusch and Soare [39, Theorem 2.4], which states that every computable infinite binary tree has a path that is of hyperimmune-free degree (i.e., each function computable from the degree is dominated by a computable function, see Section 9 for precise definitions). We reformulate this theorem in a way, which is directly applicable for our purposes.

Theorem 8.3 (Hyperimmune-free Basis Theorem)**.** *If f is a multi-valued function on represented spaces with $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$, then f has a realizer, which maps computable inputs to outputs of hyperimmune-free degree.*

Proof. Firstly, WKL has a realizer which maps computable inputs to outputs of hyperimmune-free degree, which is a direct consequence of the Hyperimmune-Free Basis Theorem of Jockusch and Soare. Since $\mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \times \mathsf{WKL}$ by [6, Corollary 4.9] the same holds for $\mathsf{C}_{\mathbb{R}}$, as np is of hyperimmune-free degree if $n \in \mathbb{N}$ and p is of hyperimmune-free degree. Since hyperimmune-free Turing degrees are closed downwards with respect to Turing reducibility, it follows that every $f \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{R}}$ has a realizer, which maps computable inputs to output of hyperimmune-free degree. Since $\mathsf{C}_{\mathbb{R}}$ is a cylinder by [6, Proposition 8.11], it follows that $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$ implies $f \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{R}}$, which proves the result. □

Since there are computable infinite binary trees without computable paths and low points that are non-computable are not of hyperimmune-free degree (see [45, Proposition 1.5.12]), we can conclude that LBT does not have a realizer, which maps computable inputs to outputs of hyperimmune-free degree. This implies that $\mathsf{LBT} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$ by Theorem 8.3. Now we also prove a separation in the reverse direction. We recall that $\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to \{0, 1\}$ is the characteristic function of $\{\widehat{0}\}$.

Proposition 8.4. $\mathsf{LPO} \not\leq_{\mathrm{W}} \mathsf{LBT}$.

Proof. Let $\mathsf{LBT}_{\mathrm{c}}$ denote the restriction of LBT to computable infinite binary trees and let $\mathsf{LPO}_{\mathrm{c}}$ denote the restriction of LPO to computable inputs. It is easy to see that $\mathsf{LPO}_{\mathrm{c}} \equiv_{\mathrm{sW}} \mathsf{LPO}$. We claim that $\mathsf{LBT}_{\mathrm{c}}$ is parallelizable, i.e., $\widehat{\mathsf{LBT}_{\mathrm{c}}} \equiv_{\mathrm{W}} \mathsf{LBT}_{\mathrm{c}}$. Given a sequence of infinite binary trees $(T_i)_i$ we can compute a sequence of closed sets $(A_i)_i$ with $A_i = [T_i]$ for all i (with respect to negative information) and we can compute the product $A := \langle A_0 \times A_1 \times A_2 \times ...\rangle$ with respect to the computable infinite tupling function on Cantor space. Given A we can also compute an infinite binary tree T such that $A = [T]$ and $p = \langle p_0, p_1, ...\rangle$ is an infinite path in T if and only if p_i is an infinite path in T_i for all $i \in \mathbb{N}$. If the sequence $(T_i)_i$ is computable and $p \in [T]$ is low relative to T, then T is computable and we obtain for the Turing jumps

$$\langle p_0', p_1', p_2', ...\rangle \leq_{\mathrm{T}} \langle p_0, p_1, p_2, ...\rangle' = p' \leq_{\mathrm{T}} T' \equiv_{\mathrm{T}} \emptyset'$$

and hence $p_i' \leq_{\mathrm{T}} \emptyset' \equiv_{\mathrm{T}} T_i'$ for all i. This proves $\widehat{\mathsf{LBT}_{\mathrm{c}}} \leq_{\mathrm{sW}} \mathsf{LBT}_{\mathrm{c}}$.

Let us now assume that $\mathsf{LPO} \leq_{\mathrm{W}} \mathsf{LBT}$ holds. It is known that $\widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \lim$ holds (see [9, Corollary 6.4] and [5, Proposition 9.1]). We obtain $\mathsf{LPO} \equiv_{\mathrm{W}} \mathsf{LPO}_{\mathrm{c}} \leq_{\mathrm{W}} \mathsf{LBT}_{\mathrm{c}}$ and hence $\lim \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}} \leq_{\mathrm{W}} \widehat{\mathsf{LBT}_{\mathrm{c}}}$ follows. This is a contradiction to the reduction $\mathsf{LBT}_{\mathrm{c}} \leq_{\mathrm{W}} \mathsf{LBT} \leq_{\mathrm{W}} \mathsf{L} <_{\mathrm{W}} \lim$, which holds by Proposition 8.2 (here $\lim \not\leq_{\mathrm{W}} \mathsf{L}$ holds, since there are limit computable p which are not low). □

Since $\mathsf{LPO} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$, we can conclude that $\mathsf{C}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{LBT}$ holds. Since also $\mathsf{WKL} <_{\mathrm{W}} \mathsf{C}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{L}$ is known [6, Theorem 8.7], we also obtain $\mathsf{L} \not\leq_{\mathrm{W}} \mathsf{LBT}$. Altogether, the results of this section can be summarized as follows.

Corollary 8.5 (Low Basis Theorem)**.** $\mathsf{WKL} <_{\mathrm{W}} \mathsf{LBT} <_{\mathrm{W}} \mathsf{L}$ *and* $\mathsf{LBT} \,|_{\mathrm{W}}\, \mathsf{C}_{\mathbb{R}}$.

This characterizes the position of LBT in the Weihrauch lattice relative to its immediate neighborhood.

9. The Hyperimmunity Problem

In this section we want to study the hyperimmunity problem. The statement behind this problem is that for every function $p : \mathbb{N} \to \mathbb{N}$ there exists a function $q : \mathbb{N} \to \mathbb{N}$ such that no function r that is computable in p dominates q. Here we say that r *dominates* q, if it satisfies $(\forall n)\ q(n) \leq r(n)$. Functions that are not dominated by any computable function are called *hyperimmune* and hence the principle can also be stated such that for every function p there is a function q, which is hyperimmune relative to p.

Definition 9.1 (Hyperimmunity)**.** We call $\mathsf{HYP} : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ with

$$\mathsf{HYP}(p) := \{q \in \mathbb{N}^\mathbb{N} : (\forall r \leq_{\mathrm{T}} p)(\exists n)\ r(n) < q(n)\}$$

the *hyperimmunity problem*.

It is clear that $\mathsf{HYP}(p)$ contains exactly all hyperimmune q if p is computable. We will implicitly prove below that HYP is actually total.

Now we want to compare the hyperimmunity problem with the weak 1–genericty problem. We recall that $p \in 2^\mathbb{N}$ is called *weakly* 1*–generic* in $q \in 2^\mathbb{N}$ if $p \in U$ for each dense set $U \subseteq 2^\mathbb{N}$ that is c.e. open in q (see [45, Definition 1.8.47]).

Definition 9.2 (Weak 1-genericity)**.** By $\text{1-}\mathsf{WGEN} : 2^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ we denote the problem

$$\text{1-}\mathsf{WGEN}(q) := \{p : p \text{ is weakly 1–generic in } q\}.$$

It is a well-known result due to Kurtz that every weakly 1–generic point $p \in 2^\mathbb{N}$ is hyperimmune [44] (see also [45, Proposition 1.8.49] and [25, Theorem 2.24.12]). We follow this idea and show that it also holds uniformly.

Proposition 9.3. $\mathsf{HYP} \leq_{\mathrm{sW}} \text{1-}\mathsf{WGEN}$.

Proof. Given some $p \in \mathbb{N}^\mathbb{N}$ we can use $\text{1-}\mathsf{WGEN}$ to find some point $s \in \text{1-}\mathsf{WGEN}(p)$ that is weakly 1–generic in p. We consider $s \in 2^\mathbb{N}$ as the characteristic function of the set $A := s^{-1}\{1\}$ and we compute $q := p_A$ be the principal function of A. Let $r : \mathbb{N} \to \mathbb{N}$ be an arbitrary function that is computable from p. For every word $w \in \{0,1\}^*$ we define $n_{r,w} := r(|w|) + 1$ and

$$U_r := \bigcup_{w \in \{0,1\}^*} w0^{n_{r,w}}\mathbb{N}^\mathbb{N}.$$

Then $U_r \subseteq 2^\mathbb{N}$ is a dense set, which is c.e. open in r and hence in p. Since s is weakly 1–generic in p, it follows that $s \in U_r$ and hence there is some $w \in \{0,1\}^+$ such that $w0^{n_{r,w}} \sqsubseteq s$. Let $n := |w|$. Then $q(n-1) = p_A(n-1) \geq n-1$ and hence $q(n) = p_A(n) \geq n_{r,w} = r(n) + 1 > r(n)$. Hence q is hyperimmune relative to p, i.e., $q \in \mathsf{HYP}(p)$. □

We note that the proof implicitly includes a proof that HYP is actually total. We prove that we get at least an ordinary Weihrauch reduction in the opposite direction. The result that hyperimmune and weakly 1–generic degrees coincide is due to Kurtz and the following proof is essentially a uniformized (and simplified) version of the proof of [25, Theorem 2.24.14].

Proposition 9.4. $\text{1-}\mathsf{WGEN} \leq_{\mathrm{W}} \mathsf{HYP}$.

Proof. Let $p \in 2^\mathbb{N}$. Then with the help of HYP we can find a $q \in \mathbb{N}^\mathbb{N}$ that is hyperimmune relative to p, i.e., no function computable from p dominates q. We now describe a computable function H that can compute from p and any such q an $r \in 2^\mathbb{N}$ that is weakly 1–generic relative to p. With this function we obtain that $H\langle \mathrm{id}, G\rangle$ is a realizer of $\text{1-}\mathsf{WGEN}$ whenever G is a realizer of HYP.

Without loss of generality, we can assume that q is increasing. From p we can compute a sequence $(f_i)_i$ of functions $f_i : \mathbb{N} \to \{0,1\}^*$ such that $(S_i)_i$ with $S_i := \mathrm{range}(f_i)$ is an enumeration of all subsets of $\{0,1\}^*$ that are c.e. relative to p. We let

$$S_i[s] := \{\sigma \in \{0,1\}^* : (\exists j \leq q(s))\ f_i(j) = \sigma\}.$$

Now we compute a sequence $\sigma_s \in \{0,1\}^*$ of words that will converge to r. In order to ensure that r is weakly 1–generic relative to p, it is sufficient to satisfy the requirements

$$R_i : \text{ If } S_i\{0,1\}^{\mathbb{N}} \text{ is dense in } \{0,1\}^{\mathbb{N}}, \text{ then } (\exists \sigma \in S_i)\ \sigma \sqsubseteq r.$$

We say that R_i requires attention at stage s if $(\forall \sigma \in S_i[s])\ \sigma \not\sqsubseteq \sigma_s$, but there is a τ with $\sigma_s \sqsubseteq \tau$ and $\sigma \in S_i[s]$ with $\sigma \sqsubseteq \tau$. With the help of p, we can decide whether a requirement R_i requires attention at a certain stage.

Now we describe the algorithm that computes r in stages $s = 0, 1, 2, ...$ At Stage 0 we set $\sigma_0 := q(0)$. At Stage $s+1$, if no requirement R_i with $i \leq s+1$ requires attention, then we let $\sigma_{s+1} := \sigma_s$. Otherwise, let R_i be the strongest requirement that requires attention and let m be the least value such that $\sigma_s \sqsubseteq f_i(m)$. If $|f_i(m)| > s+1$, then we let $\sigma_{s+1} := \sigma_s$ and otherwise we let $\sigma_{s+1} := f_i(m)$.

We need to prove that r is weakly 1–generic. Assume that S_i is dense. Let $g(s)$ be the least k such that for each $\sigma \in \{0,1\}^{s+1}$ there is a $j \leq k$ with $\sigma \sqsubseteq f_i(j)$. Then $g \leq_{\mathrm{T}} p$. Let s be a stage after which no requirement stronger than R_i ever requires attention. Since q is not dominated by any function computable from p, there is a $t > s$ such that $q(t) > g(t)$. At Stage t either R_i is already satisfied or it requires attention and continues to require attention until it is met. Thus for some $t' \geqslant t$ there exists $\sigma \in S_i[t']$, and hence in S_i, such that $\sigma \sqsubseteq r$. □

We obtain the following corollary.

Corollary 9.5. $\mathsf{HYP} \equiv_{\mathrm{W}} \mathsf{1\text{-}WGEN}$.

This immediately raises the following question.

Question 9.6. $\mathsf{HYP} \equiv_{\mathrm{sW}} \mathsf{1\text{-}WGEN}$?

By the Hyperimmune-Free Basis Theorem 8.3 it is clear that we obtain the following corollary.

Corollary 9.7. $\mathsf{HYP} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$.

In the following we also use the problem of 1–genericity. For each $q \in 2^{\mathbb{N}}$ we consider some fixed enumeration $(U_i^q)_{i\in\mathbb{N}}$ of all sets $U_i^q \subseteq 2^{\mathbb{N}}$ that are c.e. open in q. A point $p \in 2^{\mathbb{N}}$ is called *1–generic in* $q \in 2^{\mathbb{N}}$, if for all $i \in \mathbb{N}$ there exists some $w \sqsubseteq p$ such that $w2^{\mathbb{N}} \subseteq U_i^q$ or $w2^{\mathbb{N}} \cap U_i^q = \emptyset$. It follows directly from this definition that every point $p \in 2^{\mathbb{N}}$ which is 1–generic in q is also weakly 1–generic in q. We call p just *1–generic*, if it is 1–generic in some computable $q \in 2^{\mathbb{N}}$. We use the concept of 1–genericity in order to define the problem $\mathsf{1\text{-}GEN}$ of *1–genericity*.

Definition 9.8 (Genericity)**.** We define $\mathsf{1\text{-}GEN} : 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ by

$$\mathsf{1\text{-}GEN}(q) := \{p : p \text{ is 1–generic in } q\}$$

for all $p \in 2^{\mathbb{N}}$.

Since $\mathsf{MLR} \leq_{\mathrm{W}} \mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$ by Lemma 7.4 and $\mathsf{HYP} \leq_{\mathrm{W}} \mathsf{1\text{-}GEN}$ by Proposition 9.3, we obtain that $\mathsf{1\text{-}GEN} \not\leq_{\mathrm{W}} \mathsf{MLR}$. By Corollary 7.2 we have $\mathsf{MLR} \not\leq_{\mathrm{W}} \lim_{\mathrm{J}}$ and since $\mathsf{1\text{-}GEN} \leq_{\mathrm{W}} \lim_{\mathrm{J}}$ by a corollary proved in [13] we also obtain $\mathsf{MLR} \not\leq_{\mathrm{W}} \mathsf{1\text{-}GEN}$. Altogether, we have the following corollary.

Corollary 9.9. $\mathsf{MLR} \,|_{\mathrm{W}}\, \mathsf{1\text{-}GEN}$.

10. The Kleene-Post Theorem

A basic theorem in computability theory is the Kleene-Post Theorem, which shows that Turing reducibility does not generate a linear order, i.e., there are Turing incomparable degrees (see [40] or [46, Theorem V.2.2]):

Theorem 10.1 (Kleene and Post 1954). *There exist $p, q \in \mathbb{N}^{\mathbb{N}}$ such that $p \,|_{\mathrm{T}}\, q$.*

We interpret this result such that for any given r we want to find two incomparable degrees above it (alternatively, one could also impose an upper bound here, for instance r', but that would yield a slightly different problem):

Definition 10.2 (Theorem of Kleene and Post). By $\mathsf{KPT} : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ we denote the function with

$$\mathsf{KPT}(r) := \{\langle p, q\rangle \in \mathbb{N}^{\mathbb{N}} : r \leq_{\mathrm{T}} \langle p, q\rangle \text{ and } p \,|_{\mathrm{T}}\, q\}$$

for all $p \in \mathbb{N}^{\mathbb{N}}$.

In order to prove that KPT is reducible to MLR we use the following version of van Lambalgen's Theorem [54]:

Theorem 10.3 (van Lambalgen's Theorem 1990). *Let $p, q, r \in 2^{\mathbb{N}}$. If $\langle p, q\rangle$ is Martin-Löf random in r, then q is Martin-Löf random in r and p is Martin-Löf random in $\langle q, r\rangle$.*

The proofs given in [25, Theorem 6.9.1] or [45, Theorem 3.4.6] relativize in the stated sense. It is well-known in computability theory that any Martin-Löf random yields an incomparable pair of degrees due to van Lambalgen Theorem (see [25, Corollary 6.9.4]). We just translate this observation into our setting.

Corollary 10.4. $\mathsf{KPT} \leq_{\mathrm{sW}} \mathsf{MLR}$.

Likewise, we can use the analogue of van Lambalgen's Theorem for genericity, the following theorem of Yu, to prove a similar statement for 1–genericity. We first formulate a suitable relativized version of the theorem of Yu (see [58] and [25, Theorem 8.20.1]):

Theorem 10.5 (Yu's Theorem 2006). *Let $p, q, r \in 2^{\mathbb{N}}$. If $\langle p, q\rangle$ is 1–generic in r, then q is 1–generic in r and p is 1–generic in $\langle q, r\rangle$.*

The proof is a direct relativization of the proof given in [25, Theorem 8.20.1]. We obtain the following corollary.

Corollary 10.6. $\mathsf{KPT} \leq_{\mathrm{sW}} 1\text{-}\mathsf{GEN}$.

Since MLR and $1\text{-}\mathsf{GEN}$ are incomparable by Corollary 9.9 it follows that both reductions in Corollaries 10.4 and 10.6 are strict. We note that the Theorem of Yu 10.5 has another interesting consequence, it implies that $1\text{-}\mathsf{GEN}$ is closed under composition.[3]

Proposition 10.7. $1\text{-}\mathsf{GEN} * 1\text{-}\mathsf{GEN} \equiv_{\mathrm{W}} 1\text{-}\mathsf{GEN}$.

Proof. It is clear that $1\text{-}\mathsf{GEN} \leq_{\mathrm{W}} 1\text{-}\mathsf{GEN} * 1\text{-}\mathsf{GEN}$. Let $f \leq_{\mathrm{W}} 1\text{-}\mathsf{GEN} * 1\text{-}\mathsf{GEN}$. Without loss of generality, we can assume that f is of type $f :\subseteq 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$. Hence, there are computable single-valued functions $F, G, H :\subseteq 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that any $q \in 1\text{-}\mathsf{GEN}(G(r))$ has the property that any $p \in 1\text{-}\mathsf{GEN}(H\langle q, r\rangle)$ yields some $F\langle p, q, r\rangle \in f(r)$. Hence, if $\langle p, q\rangle \in 1\text{-}\mathsf{GEN}(r)$, then by Yu's Theorem 10.5 it follows that $q \in 1\text{-}\mathsf{GEN}(r)$ and $p \in 1\text{-}\mathsf{GEN}(\langle q, r\rangle)$. Now, $G(r) \leq_{\mathrm{T}} r$ and $H\langle q, r\rangle \leq_{\mathrm{T}} \langle q, r\rangle$ and hence $1\text{-}\mathsf{GEN}(r) \subseteq 1\text{-}\mathsf{GEN}(G(r))$ and $1\text{-}\mathsf{GEN}(\langle q, r\rangle) \subseteq 1\text{-}\mathsf{GEN}(H\langle q, r\rangle)$. This implies $F\langle p, q, r\rangle \in f(r)$. Thus, the function F yields a reduction $f \leq_{\mathrm{W}} 1\text{-}\mathsf{GEN}$. $\square$

[3] Likewise it was observed by Brattka, Gherardi and Hölzl (unpublished notes) that van Lambalgen's Theorem implies closure of Martin-Löf randomness under composition; that is, $\mathsf{MLR} * \mathsf{MLR} \equiv_{\mathrm{W}} \mathsf{MLR}$.

11. The Jump Inversion Theorem

In this section we study the uniform computational content of Friedberg's Jump Inversion Theorem, which is an interesting example since it is continuous but not computable. For two Turing degrees $\mathbf{a}, \mathbf{b}$ we denote the extension of Turing reducibility to degrees by $\mathbf{a} \leq \mathbf{b}$ and by $\mathbf{a}'$ we denote the jump of $\mathbf{a}$. Then Friedberg's Jump Inversion Theorem in its original formulation reads as follows (see [26]).

Theorem 11.1 (Friedberg's Jump Inversion Theorem)**.** *For every degree* $\mathbf{a} \in \mathcal{D}$ *there exists a degree* $\mathbf{b} \in \mathcal{D}$ *with* $\mathbf{b}' = \mathbf{a} \cup 0'$.

In particular, the theorem implies that the Turing jump operator on degrees $\mathsf{J}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}, \mathbf{a} \mapsto \mathbf{a}'$ is surjective onto the upper cone $\{\mathbf{a} \in \mathcal{D} : 0' \leq \mathbf{a}\}$. We can formalize the Jump Inversion Theorem as follows.

Definition 11.2 (Jump Inversion Theorem)**.** We call

$$\mathsf{JIT} : \mathcal{D} \rightrightarrows \mathcal{D}, \mathbf{a} \mapsto \{\mathbf{b} \in \mathcal{D} : \mathbf{b}' = \mathbf{a} \cup 0'\}$$

the *Jump Inversion Theorem.*

The inverse $\mathsf{J}_{\mathcal{D}}^{-1}$ of the Turing jump operator is then a restriction of JIT to the upper cone $\{\mathbf{a} \in \mathcal{D} : 0' \leq \mathbf{a}\}$ and hence it is clear that $\mathsf{J}_{\mathcal{D}}^{-1} <_{\mathrm{sW}} \mathsf{JIT}$ holds (the reduction is strict, since $\mathsf{J}_{\mathcal{D}}^{-1}$ has no computable points in its domain). It follows directly from Proposition 4.5 that JIT, $\mathsf{J}_{\mathcal{D}}$ and $\mathsf{J}_{\mathcal{D}}^{-1}$ are indiscriminative. We will see that JIT is also indiscriminative for a different reason, namely it is even continuous. This follows from the following result that is a direct consequence of the classical proof of the Friedberg Jump Inversion Theorem.

As usual we denote by $\varphi_i^p(n)$ the i–th partial computable function relative to $p \in \mathbb{N}^{\mathbb{N}}$ on input $n \in \mathbb{N}$, and $\varphi_i^\sigma(n)$, for $\sigma \in \mathbb{N}^*$, is undefined if the computation consults the oracle beyond the $|\sigma|$ bit and equals the output otherwise. For simplicity, we assume that the Turing degrees are represented by $2^{\mathbb{N}}$ in this section. As usual, for two words $\sigma, \tau \in \{0,1\}^*$ the concatenation is denoted by $\sigma\frown\tau$ and similarly $\sigma\frown p$ denotes the concatenation of σ with $p \in \{0,1\}^{\mathbb{N}}$. We denote by $\mathrm{c}_{\emptyset'} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ the constant function with the value of the halting problem.

Proposition 11.3. $\mathsf{JIT} <_{\mathrm{sW}} \mathrm{c}_{\emptyset'} \times \mathrm{id}$.

Proof. We briefly recall the proof of Theorem 11.1 in [46, Theorem V.2.24]. Given $A \subseteq \mathbb{N}$ we construct the characteristic function χ_B of $B \subseteq \mathbb{N}$ using the finite extension method. That is, we define a monotone sequence $(\sigma_n)_n$ of words $\sigma_s \in \{0,1\}^*$ with $\chi_B = \sup_n \sigma_n$ inductively in stages $s = 0, 1, ...$ as follows. We let σ_0 be the empty sequence. If σ_s is already given, then we continue as follows:

- If $s = 2i$ and there is a $\sigma \in \{0,1\}^*$ with $\sigma_s \sqsubseteq \sigma$ such that $\varphi_i^\sigma(i)$ is defined, then we let $\sigma_{s+1} := \sigma$ for the smallest such σ and otherwise $\sigma_{s+1} := \sigma_s$.
- If $s = 2i + 1$ then we define $\sigma_{s+1} := \sigma_s\frown\chi_A(i)$.

The first condition is computable in $\emptyset'$ and the second one in A. Hence the construction of $(\sigma_n)_n$ and hence of B is computable in $\emptyset' \oplus A$. This shows that the function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ that maps A to B is strongly reducible to $\mathrm{c}_{\emptyset'} \times \mathrm{id}$. We still need to show that F realizes JIT. The first condition guarantees $i \in B' \iff \varphi_i^{\sigma_{2i+1}}(i)\downarrow$ and hence $B' \leq_{\mathrm{T}} \emptyset' \oplus A$. On the other hand, $i \in \mathsf{A} \iff \sigma_{2i+1}(|\sigma_{2i+1}|) = 1$ and hence $A \leq_{\mathrm{T}} B$ and consequently $\emptyset' \oplus A \leq_{\mathrm{T}} B'$, which completes the proof of $\mathsf{JIT} \leq_{\mathrm{sW}} \mathrm{c}_{\emptyset'} \times \mathrm{id}$. The reduction is strict by Proposition 4.3 since JIT is densely realized (alternatively it is strict, because $\mathrm{c}_{\emptyset'} \times \mathrm{id}$ does not map computable inputs to computable outputs, but JIT does). $\square$

The proof of Proposition 11.3 shows that JIT has a realizer whose range only contains 1–generic points. Hence we also obtain the following corollary by a proposition proved in [13] (which states that every f that has some limit computable realizer whose range only contains 1–generic points satisfies $f \leq_{\mathrm{sW}} \lim_{\mathrm{J}}$).

Corollary 11.4. $\mathsf{JIT} \leq_{\mathrm{sW}} \lim_{\mathrm{J}}$.

The reduction is strict by Proposition 4.3 since JIT is indiscriminative. An obvious question is now whether JIT is perhaps even computable? We prove that this is not the case. For the proof we introduce some notation for $a = a_0...a_n \in \mathbb{N}^*$ we denote by $2a := a_0a_0...a_na_n$ the word, where each symbol is doubled. Analogously, we define 2α for $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Proposition 11.5. $\mathsf{J}_{\mathcal{D}}^{-1}$ *is not computable.*

Proof. For this proof we represent $\mathcal{D}$ by $2^{\mathbb{N}}$ in some effective way. Suppose there is a computable Turing functional $\varphi = \varphi_i :\subseteq 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ that realizes $\mathsf{J}_{\mathcal{D}}^{-1}$. Then $\alpha \not\equiv_{\mathrm{T}} \beta$ implies $\varphi^\alpha \neq \varphi^\beta$ for all $\alpha, \beta \in 2^{\mathbb{N}}$ which compute the halting problem, since $(\varphi^\alpha)' \equiv_{\mathrm{T}} \alpha$ and $(\varphi^\beta)' \equiv_{\mathrm{T}} \beta$. Hence, for all words $\sigma \in \{0,1\}^*$ there exist $a, b \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$ such that $\varphi^{\sigma\frown 2a}(n) \neq \varphi^{\sigma\frown 2b}(n)$ and such that both values exist. We use some fixed effective enumeration of $\{0,1\}^* \times \{0,1\}^* \times \mathbb{N}$. We inductively construct a computable perfect splitting tree $f : \{0,1\}^* \to \{0,1\}^*$ for φ with branches of arbitrarily high Turing degree. To begin with, f takes the empty word to the empty word. Now suppose we have defined f on all words of length n and let $\sigma \in \{0,1\}^n$. We define

$$f(\sigma\frown d) := \begin{cases} \sigma\frown 2a\frown 01 & \text{if } d = 0 \\ \sigma\frown 2b\frown 10 & \text{if } d = 1 \end{cases},$$

for $d \in \{0,1\}$, where $(a, b, s) \in \{0,1\}^* \times \{0,1\}^* \times \mathbb{N}$ is minimal in our fixed enumeration such that there exists some n with $\varphi^{\sigma\frown 2a}(n)[s] \neq \varphi^{\sigma\frown 2b}(n)[s]$ and such that both values exist. This completes the construction of f. The prefix closure $T \subseteq \{0,1\}^*$ of the image $f(\{0,1\}^*)$ is a computable binary tree such that for all incompatible $a, b \in T$ there exists some $n \in \mathbb{N}$ with $\varphi^a(n) \neq \varphi^b(n)$ (and such that both values exist). Moreover, we can embed the Turing semi-lattice into $[T]$ (the set of infinite paths of T) via the pairs $01, 10$ that have been added in the construction of f.

Now let $\alpha \in [T]$ be such that $\alpha \equiv_{\mathrm{T}} \emptyset'$. This guarantees that $\alpha \in \mathrm{dom}(\varphi)$. We show that α is computable from φ^α, which contradicts the assumption that φ realizes $\mathsf{J}_{\mathcal{D}}^{-1}$ (i.e., the fact that $(\varphi^\alpha)' \equiv_{\mathrm{T}} \alpha$). We compute a monotone increasing sequence $(\sigma_n)_n$ of words $\sigma \in \{0,1\}^n$ such that $\alpha = \sup_n \sigma_n$. That is, σ_0 is the empty word. Since $\alpha \in [T]$, one of $f(0)$ and $f(1)$ is compatible with α. By construction, we can find some $n \in \mathbb{N}$ such that $\varphi^{f(0)}(n) \neq \varphi^{f(1)}(n)$ (and such that both values exist) and so by comparing $\varphi^\alpha(n)$ with $\varphi^{f(0)}(n)$ and $\varphi^{f(1)}(n)$ we can decide which of $f(0)$ and $f(1)$ is a prefix of α. We let $\sigma_1 := i$ for the corresponding i with $\varphi^\alpha(n) = \varphi^{f(i)}(n)$. An analogue construction works on all levels: having constructed σ_n, we have that $\sigma_n\frown i$ is an initial segment of α for the i such that $\varphi^\alpha(n) = \varphi^{f(i)}(n) \neq \varphi^{f(1-i)}(n)$ for some n (where all these values exist). Since $\alpha = \sup_n f(\sigma_n)$, we obtain $\alpha \leq_{\mathrm{T}} \varphi^\alpha$. $\square$

As a direct consequence of Propositions 11.3 and 11.5 we obtain the following corollary.

Corollary 11.6. JIT *is limit computable and continuous, but not computable.*

We note that $\mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{J}_{\mathcal{D}}^{-1} = 0$ (where 0 denotes the nowhere defined function), since by a version of the Jump Inversion Theorem due to Cooper (see [20] and also

[25, Theorem 2.18.7]) for every $a \geq 0'$ there is some minimal b with $b' = a$ and hence $b \not\geq 0'$. In fact, we can even formulate the following stronger observation as a corollary of Cooper's Theorem.

Corollary 11.7. $\mathsf{J}_{\mathcal{D}}^{-1} *_{\mathrm{s}} \mathsf{J}_{\mathcal{D}}^{-1} \equiv_{\mathrm{sW}} 0$.

Arno Pauly (personal communication) noted that the following corollary follows from Proposition 11.3, which answers an open question from an earlier version of this article.

Corollary 11.8. $\mathsf{JIT} * \mathsf{JIT} \leq_{\mathrm{W}} \mathrm{c}_{\emptyset'} \leq_{\mathrm{W}} \lim$.

12. The Cohesiveness Problem

In this section we want to discuss some properties of the cohesiveness problem. A set $X \subseteq \mathbb{N}$ is called *cohesive* for a sequence $(R_i)_i$ of sets $R_i \subseteq \mathbb{N}$ if it is infinite and for each $i \in \mathbb{N}$ we have $X \subseteq^* R_i$ or $X \subseteq^* (R_i)^{\mathrm{c}}$. Here we write $X \subseteq^* Y$ if $X \setminus Y$ is finite, i.e., if X is included in Y up to at most finitely many exceptions. One can prove that for every sequence of $(R_i)_i$ of sets there is always a cohesive set X.

Definition 12.1 (Cohesiveness Problem)**.** By $\mathsf{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ with

$$\mathsf{COH}(R_i)_i := \{X \subseteq \mathbb{N} : X \text{ cohesive for } (R_i)_i\}$$

we denote the *cohesiveness problem*.

The cohesiveness problem has been introduced into reverse mathematics by [19] and the Weihrauch degree of COH has already been studied in [22]. In computability theory a set is called *r-cohesive*, if it is cohesive for the sequence of all computable sets, *p-cohesive* if it is cohesive for the sequence of all primitive recursive sets and just *cohesive* if it is cohesive for the sequence of all c.e. sets [34]. We say that a Turing degree has property P if it has a member with property P and we extend the different notions of cohesiveness to degrees in this way. By [34, Corollary 2.4] the r-cohesive Turing degrees coincide with the cohesive ones. In order to capture the notion of cohesiveness, we introduce the following variant of COH.

Definition 12.2. By COH_+ we denote the map $\mathsf{COH}_+ : \mathcal{A}_+(\mathbb{N})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ with $\mathsf{COH}_+((R_i)_i) := \mathsf{COH}((R_i)_i)$.

While $2^{\mathbb{N}}$ can be identified with $\mathcal{A}(\mathbb{N})$, the set of closed subsets $A \subseteq \mathbb{N}$ represented by full information (i.e., by characteristic functions), the set $\mathcal{A}_+(\mathbb{N})$ captures the set of subsets $A \subseteq \mathbb{N}$ represented by positive information (i.e., enumerations). We will see that COH can be seen as the uniform version of the notion of p–cohesiveness, whereas COH_+ rather captures cohesiveness (or r–cohesiveness). It is also easy to see that COH_+ is located in between COH and COH'.

Proposition 12.3. $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{COH}_+ \leq_{\mathrm{sW}} \mathsf{COH}'$.

Proof. Since $\mathrm{id} : \mathcal{A}(\mathbb{N}) \to \mathcal{A}_+(\mathbb{N})$ is computable and $\mathrm{id} : \mathcal{A}_+(\mathbb{N}) \to \mathcal{A}(\mathbb{N})$ is limit computable by [7, Proposition 4.2] we obtain $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{COH}_+ \leq_{\mathrm{sW}} \mathsf{COH}'$. ☐

Next we want to apply a very useful characterization of p–cohesive degrees of Jockusch and Stephan [34][4] in order to separate COH from COH_+. We recall that a function is called *primitive recursive relative to* $q \in \mathbb{N}^{\mathbb{N}}$, if it can be generated from the primitive recursive basic functions and q using the closure schemes of substitution and primitive recursion. We call a set *p–cohesive relative to* q, if it is cohesive for the sequence of all sets that are primitive recursive relative to q.

[4] We note that none of the results from [34] that we use here are affected by the correction [35].

Using this terminology we get the following version of [34, Theorem 2.1], where we can eliminate any reference to primitive recursiveness in the formulation due to uniformity.

Theorem 12.4. *Let* $\mathbf{a}, \mathbf{b}$ *be Turing degrees. Then the following are equivalent:*

(1) $\mathbf{a}$ *is cohesive for every* $\mathbf{b}$*–computable sequence* $(A_i)_i$ *of sets* $A_i \subseteq 2^{\mathbb{N}}$,
(2) $\mathbf{a}' \gg \mathbf{b}'$.

Proof. We are actually going to prove that the following conditions are pairwise equivalent to each other:

(1) $\mathbf{a}$ is cohesive for every $\mathbf{b}$–computable sequence $(A_i)_i$ of sets $A_i \subseteq 2^{\mathbb{N}}$,
(2) $\mathbf{a}$ is cohesive for every $\mathbf{b}$–computable sequence $(A_i)_i$ of sets $A_i \subseteq 2^{\mathbb{N}}$ that includes all sets that are primitive recursive relative to $\mathbf{b}$,
(3) $\mathbf{a}$ is p–cohesive relative to $\mathbf{b}$,
(4) $\mathbf{a}' \gg \mathbf{b}'$.

We obviously obtain (1)$\Longrightarrow$(2)$\Longrightarrow$(3). The proofs of (3)$\Longrightarrow$(4) and (4)$\Longrightarrow$(2) are immediate relativizations of the proof of [34, Theorem 2.1]. We still need to prove (2)$\Longrightarrow$(1). If $(A_i)_i$ is a $\mathbf{b}$–computable sequence of sets, then we can add all sets that are primitive recursive relative to $\mathbf{b}$ to this sequence and we obtain a $\mathbf{b}$–computable sequence $(B_i)_i$ in this way. By (2) the degree $\mathbf{a}$ is cohesive for $(B_i)_i$ and hence, in particular, for the subsequence $(A_i)_i$. ☐

By Theorem 12.4 and by using ideas from the proof of [34, Theorem 2.9(ii)] we obtain the following separation result.

Proposition 12.5. $\mathsf{COH}_+ \not\leq_{\mathrm{W}} \mathsf{COH}$.

Proof. Suppose that $\mathsf{COH}_+ \leq_{\mathrm{W}} \mathsf{COH}$. Then there are computable functions H, K such that $H\langle \mathrm{id}, GK\rangle$ is a realizer of COH_+ whenever G is a realizer of COH. Let now $p \in \mathbb{N}^{\mathbb{N}}$ be a computable name for the sequence of all c.e. subsets $B \subseteq \mathbb{N}$ and consider the sequence $(A_i)_i$ of computable sets given by $K(p)$. By the proof of [34, Theorem 2.9(ii)] there is a Turing degree $\mathbf{a}$ that is not cohesive, but such that $\mathbf{a}' \gg \mathbf{0}'$. Hence, by Theorem 12.4 the degree $\mathbf{a}$ is cohesive for $(A_i)_i$. We choose a realizer G of COH that yields a set $r := GK(p) \in 2^{\mathbb{N}}$ that is cohesive for $(A_i)_i$ and of degree $\mathbf{a}$ and hence not cohesive. Since cohesive degrees are closed upwards with respect to Turing reducibility by [36, Corollary 1], it follows that $H\langle p, GK(p)\rangle \leq_{\mathrm{T}} r$ is not of cohesive degree either, which is absurd. ☐

Next we prove that HYP is reducible to COH. The proof is based on a relativized version of the proof of [34, Theorem 3.1].

Theorem 12.6. $\mathsf{HYP} \leq_{\mathrm{sW}} \mathsf{COH}$.

Proof. Given $p \in \mathbb{N}^{\mathbb{N}}$ we compute a sequence $(R_i)_i$ of all primitive recursive sets relative to p. Given some $A \in \mathsf{COH}((R_i)_i)$ we can compute the principal function p_A of A (i.e., $A = \{p_A(0) < p_A(1) < p_A(2) < ...\}$). We claim that $p_A \in \mathsf{HYP}(p)$. This yields the reduction $\mathsf{HYP} \leq_{\mathrm{sW}} \mathsf{COH}$.

We prove the claim using a relativized version of the proof of [34, Theorem 3.1]. Let us assume for a contradiction that $p_A \notin \mathsf{HYP}(p)$, so there is some $r \leq_{\mathrm{T}} p$ that dominates p_A in the sense that $p_A(n) \leq r(n)$ for all $n \in \mathbb{N}$. If we can prove that every partial p'–computable function $\gamma :\subseteq \mathbb{N} \to \{0,1\}$ can be extended to a p'–computable function $h : \mathbb{N} \to \{0,1\}$, then we obtain $[p'] \gg [p']$ by Proposition 6.1, which is a contradiction to Proposition 6.3(1). Let now $\gamma :\subseteq \mathbb{N} \to \{0,1\}$ be p'–computable. Then by the Limit Lemma there is a function $g : \mathbb{N}^2 \to \{0,1\}$, which is primitive-recursive in p and such that $\gamma(e) = \lim_{s\to\infty} g(e,s)$ for all $e \in \mathrm{dom}(\gamma)$. We note that $p_A(n) \in B_n := \{n, n+1, ..., r(n)\}$ since $r(n) \geq p_A(n) \geq n$ for all

n. Let S be the set of all pairs $(e,y) \in \mathbb{N} \times \{0,1\}$ such that there are only finitely many n with $g(e,s) = y$ for all $s \in B_n$. Then S is c.e. in $\langle p', r\rangle \leq_{\mathrm{T}} p'$. We prove that for each e there exists $y \in \{0,1\}$ with $(e,y) \in S$. Let us assume the contrary. Then there is some e such that $(e,y) \notin S$ for both $y \in \{0,1\}$. Then the function $g_e : \mathbb{N} \to \mathbb{N}, s \mapsto g(e,s)$ assumes each value $y \in \{0,1\}$ infinitely often on A and hence the sets $S_e := \{s : g(e,s) = 1\}$ form a sequence of sets that are primitive-recursive in p and such that A is not cohesive for $(S_e)_e$. This is a contradiction to $A \in \mathsf{COH}((R_i)_i)$ and hence for each e there exists $y \in \{0,1\}$ with $(e,y) \in S$. Let f be a function that selects the first y with $(e,y) \in S$ in a p'–computable enumeration of S. Then $f(e) \neq \gamma(e)$ for all e and hence $h := 1 - f$ is a total p'–computable extension of γ. □

The previous result implies $1\text{-}\mathsf{WGEN} \leq_{\mathrm{W}} \mathsf{COH}$ by Corollary 9.5. This result cannot be strengthened to $1\text{-}\mathsf{GEN} \leq_{\mathrm{W}} \mathsf{COH}$ by the following observation.[5]

Proposition 12.7. $1\text{-}\mathsf{GEN} \not\leq_{\mathrm{W}} \mathsf{COH}^+$.

Proof. By a result of Jockusch [36, Corollary 2] every degree a that is high (in the sense that $a' \geq 0''$) contains a cohesive set. By Cooper's Jump Inversion Theorem [20, Theorem 1] there is a minimal high degree a. From a minimal degree a one cannot compute a 1–generic (which follows, for instance, from Corollary 10.6). Let us now assume that $1\text{-}\mathsf{GEN} \leq_{\mathrm{W}} \mathsf{COH}^+$ and let p be a computable input to $1\text{-}\mathsf{GEN}$. Then from this input we can compute a sequence $(R_i)_i$ of c.e. sets $R_i \subseteq \mathbb{N}$ and $\mathsf{COH}^+(R_i)_i$ contains an X of minimal high degree, from which together with p we cannot compute a 1–generic. Contradiction! □

It is clear that COH_+ is densely realized. Hence we obtain $\mathsf{ACC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{COH}_+$ and, in particular, the following.

Corollary 12.8. $\mathsf{DNC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{COH}$.

By the next result COH is also not above MLR.

Proposition 12.9. $\mathsf{MLR} \not\leq_{\mathrm{W}} \mathsf{COH}$.

Proof. Let us assume that $\mathsf{MLR} \leq_{\mathrm{W}} \mathsf{COH}$. Then there are computable H, K such that $H\langle \mathrm{id}, GK\rangle$ is a realizer of MLR for any realizer G of COH. Let now p be computable. We consider the computable sequence $(R_i)_i$ of computable sets given by $K(p)$. Let $(U_i)_i$ be a universal Martin-Löf test. Then $A := 2^{\mathbb{N}} \setminus U_0$ is a co-c.e. closed set that contains only random points and, in particular, A contains no computable points. Hence, by [19, Lemma 9.16] there is a cohesive set $q \in 2^{\mathbb{N}}$ for the sequence $(R_i)_i$ that does not compute any point in A. We choose a realizer G of COH such that $GK(p) = q$. Then $r := H\langle p, q\rangle \in \mathsf{MLR}(p)$ is Martin-Löf random and by the Lemma of Kučera (see [43, Lemma 3] or [25, Lemma 6.10.1]) r computes a point $x \in A$. Hence $x \leq_{\mathrm{T}} r \leq q$, which is a contradiction! Hence $\mathsf{MLR} \not\leq_{\mathrm{W}} \mathsf{COH}$. □

Finally, we prove that cohesiveness is limit computable.

Proposition 12.10. $\mathsf{COH} \leq_{\mathrm{sW}} \lim$.

Proof. Given a sequence $(R_i)_i$ of sets $R_i \subseteq \mathbb{N}$ we need to compute a set $X \in \mathsf{COH}((R_i)_i)$ in the limit. We use the notation $1 \cdot R := R$ and $(-1) \cdot R := \mathbb{N} \setminus R$ for sets $R \subseteq \mathbb{N}$ and for every word $y \in \{0,1\}^*$ we define

$$R^y := \bigcap_{i<|y|} (-1)^{y(i)} \cdot R_i.$$

[5]We thank Bjørn Kjos-Hanssen for pointing out the proof of Proposition 12.7, see `http://mathoverflow.net/questions/188596/is-below-every-cohesive-set-a-1-generic`.

Now we can compute a tree $T := \{y \in \{0,1\}^* : |R^y| > |y|\}$ in the limit, where $|R^y|$ denotes the cardinality of R^y and $|y|$ the length of y. This tree will have an infinite path $p \in [T]$, since R infinite implies that $R \cap R_{i+1}$ is infinite or $R \cap R_{i+1}^{\mathrm{c}}$ is infinite. However, we cannot compute such an infinite path p from T in general, but by backtracking we can at least limit compute the left-most path p in $[T]$, which enables us to compute a cohesive set $X \subseteq \mathbb{N}$ for $(R_i)_i$ from T. In detail, the algorithm can be described as follows: we follow the left-most path of T as far as possible. If we reach level k, i.e., a node $y \in T$ with $|y| = k$ then we enumerate a k–th element into X, which is chosen from R^y, which is possible since $|R^y| > |y| = k$. If we arrive in a dead end of the tree T on level k, then we backtrack to the last node where we can choose an alternative branch and we follow again the left-most path until we reach level $k+1$, from where we continue enumerating elements into X as before. Whenever necessary, we repeat the backtracking. This backtracking algorithm computes an infinite path in the limit since $[T] \neq \emptyset$. In doing so, it needs to return to any fixed level k at most finitely many times and from then on the prefix y of the infinite path $p \in [T]$ of length $|y| = k$ that we limit compute is fixed and hence all elements enumerated into X will be in R^y up to at most finitely many exceptions. This guarantees that X is cohesive for $(R_i)_i$. We note that the enumeration of X is computable in T even though the path p is only limit computable in T. Altogether, X is limit computable in $(R_i)_i$. $\square$

With the help of Proposition 12.3 we obtain the following corollary.

Corollary 12.11. $\mathsf{COH}_+ \leq_{\mathrm{sW}} \lim'$.

13. Cohesive Degrees

Theorem 12.4 highlights the relation of cohesive degrees to PA–degrees (in light of Proposition 6.1). A uniform version of this relation can be expressed as follows.

Proposition 13.1. $[\mathsf{COH}] \equiv_{\mathrm{sW}} \mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}$.

Proof. Firstly, we note that $\mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}(\mathbf{b}) = \{\mathbf{a} \in \mathcal{D} : \mathbf{a}' \gg \mathbf{b}'\}$. Hence, we obtain $[\mathsf{COH}] \leq_{\mathrm{sW}} \mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}$ by Theorem 12.4. For the other direction of the proof we note that given a degree $\mathbf{b} \in \mathcal{D}$, we can compute a sequence $(P_i)_i$ of all sets $P_i \subseteq \mathbb{N}$ which are primitive recursive relative to $\mathbf{b}$. By the equivalence given in the proof of Theorem 12.4 we obtain that every $\mathbf{a} \in [\mathsf{COH}](P_i)_i$ satisfies $\mathbf{a}' \gg \mathbf{b}'$ and hence we obtain $\mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}} \leq_{\mathrm{sW}} [\mathsf{COH}]$. $\square$

We note that any antitone jumps can be characterized as follows.

Lemma 13.2 (Antitone jumps)**.** *Let $f :\subseteq \mathcal{D} \rightrightarrows \mathcal{D}$ be antitone in the sense that*

$$\mathbf{a} \leq \mathbf{b} \Longrightarrow \emptyset \neq f(\mathbf{b}) \subseteq f(\mathbf{a})$$

for all $\mathbf{a} \in \mathrm{dom}(f)$ and $\mathbf{b} \in \mathcal{D}$. Then $f' \equiv_{\mathrm{sW}} f \circ \mathsf{J}_{\mathcal{D}}$.

Proof. Since the Turing jump J is limit computable, there is a computable function $G :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\lim \circ G = \mathsf{J}$ and hence $f \circ [\lim] \circ G = f \circ \mathsf{J}_{\mathcal{D}} \circ \delta_{\mathcal{D}}$, which implies $f \circ \mathsf{J}_{\mathcal{D}} \leq_{\mathrm{sW}} f *_{\mathrm{s}} \lim \equiv_{\mathrm{sW}} f'$. For the other direction let us assume that $g \leq_{\mathrm{sW}} f' \equiv_{\mathrm{sW}} f *_{\mathrm{s}} \lim$. Without loss of generality we can assume that g is of type $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Then there are computable $H, K : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $M :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{D}$ such that $H \circ f \circ M \circ \lim(p) \subseteq g(p)$ for all $p \in \mathrm{dom}(g)$. Moreover, $M \circ \lim(p) \leq \mathsf{J}_{\mathcal{D}}(p)$ and since f is antitone, we obtain $\emptyset \neq f \circ \mathsf{J}_{\mathcal{D}}(p) \subseteq f \circ M \circ \lim(p)$. This implies

$$H \circ f \circ \mathsf{J}_{\mathcal{D}}(p) \subseteq H \circ f \circ M \circ \lim(p) \subseteq g(p)$$

for all $p \in \mathrm{dom}(g)$, which implies $g \leq_{\mathrm{sW}} f \circ \mathsf{J}_{\mathcal{D}}$. Since this holds for all g, we obtain $f' \leq_{\mathrm{sW}} f \circ \mathsf{J}_{\mathcal{D}}$. $\square$

By Proposition 6.3(3) PA is antitone. This implies that $\mathsf{PA}' \equiv_{\mathrm{sW}} \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}$. Using Proposition 6.3(4) we obtain the following conclusion.

Corollary 13.3. $[\mathsf{COH}] \leq_{\mathrm{sW}} \mathsf{PA}'$.

Using the previous observations we derive the following purely algebraic characterization of COH on degrees in terms of the jump of Peano arithmetic PA.

Theorem 13.4 (Cohesive degrees)**.** $[\mathsf{COH}] \equiv_{\mathrm{W}} (\lim \to \mathsf{PA}') \equiv_{\mathrm{W}} (\mathsf{J}_{\mathcal{D}} \to \mathsf{PA}')$.

Proof. Since $\mathsf{J}_{\mathcal{D}} \leq_{\mathrm{sW}} \mathsf{J} \equiv_{\mathrm{sW}} \lim$ and $\mathsf{PA}' \equiv_{\mathrm{sW}} \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}$ by Lemma 13.2, it suffices to prove $(\mathsf{J}_{\mathcal{D}} \to \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}) \leq_{\mathrm{W}} [\mathsf{COH}] \leq_{\mathrm{W}} (\mathsf{J} \to \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}})$. It is clear that $\mathsf{PA} \circ \mathsf{J}_{\mathcal{D}} \leq_{\mathrm{W}} \mathsf{J}_{\mathcal{D}} * (\mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}) \equiv_{\mathrm{W}} \mathsf{J}_{\mathcal{D}} * [\mathsf{COH}]$ by Proposition 13.1. Hence $(\mathsf{J}_{\mathcal{D}} \to \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}) \leq_{\mathrm{W}} [\mathsf{COH}]$. For the second reduction we assume that h is such that $\mathsf{PA} \circ \mathsf{J}_{\mathcal{D}} \leq_{\mathrm{W}} \mathsf{J} * h$. Without loss of generality, we can assume that h is of type $h :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and that h is a cylinder (since the cylindrification $\mathrm{id} \times h$ satisfies $\mathrm{id} \times h \equiv_{\mathrm{W}} h$). Hence, there are computable F, G, K such that $\mathsf{J} * h \equiv_{\mathrm{W}} F \circ \mathsf{J} \circ G \circ h \circ K$. Since $F \circ \mathsf{J} \circ G$ is limit computable, there is a computable H such that $F \circ \mathsf{J} \circ G = H \circ \mathsf{J}$. This implies that $\mathsf{PA} \circ \mathsf{J}_{\mathcal{D}} \leq_{\mathrm{W}} H \circ \mathsf{J} \circ h \circ K$. Hence, there is a $q \leq_{\mathrm{T}} H \circ \mathsf{J} \circ h \circ K(p) \leq_{\mathrm{T}} \mathsf{J} \circ h \circ K(p)$ such that $[q] \in \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}([p])$, which implies $[q] \gg [p'] = [p]'$. Hence, $[h \circ K(p)]' \gg [p]'$ by Proposition 6.3(4). This implies by Proposition 13.1 that $[\mathsf{COH}] \equiv_{\mathrm{sW}} \mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}} \leq_{\mathrm{W}} h \circ K \leq_{\mathrm{W}} h$ and hence we obtain $[\mathsf{COH}] \leq_{\mathrm{W}} (\mathsf{J} \to \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}})$. $\square$

Since $\mathsf{PA} \equiv_{\mathrm{sW}} [\mathsf{WKL}]$ by Corollaries 6.4 and 5.3, we can also express Theorem 13.4 as follows.

Corollary 13.5. $[\mathsf{COH}] \equiv_{\mathrm{W}} (\lim \to [\mathsf{WKL}]')$.

In the next section we will prove a corresponding characterization of COH.

14. The Weak Bolzano-Weierstrass Theorem

In this section we briefly discuss the relation of the Bolzano-Weierstraß Theorem to the cohesiveness problem, which was already subject of [41]. We recall that the Bolzano-Weierstraß Theorem can be formalized as follows (see [12, 42]).

Definition 14.1 (Bolzano-Weierstraß Theorem)**.** Let X be a computable metric space and let

$$\mathsf{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X, (x_i)_i \mapsto \{x : x \text{ is a cluster point of } (x_i)_i\}$$

where $\mathrm{dom}(\mathsf{BWT}_X)$ is the set of all sequences $(x_i)_i$ in X such that $\{x_i : i \in \mathbb{N}\}$ has a compact closure.

We note the following [12, Corollaries 11.6 and 11.17].

Fact 14.2. $\mathsf{BWT}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{BWT}_{[0,1]^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{WKL}'$.

$\mathsf{BWT}_{\mathbb{R}}$ determines a cluster point of a given bounded sequence $(x_n)_n$. One could also consider the problem of finding a convergent subsequence of $(x_n)_n$. Equivalently, we define the following weakening of the Bolzano-Weierstraß Theorem.

Definition 14.3 (Weak Bolzano-Weierstraß Theorem)**.** Let X be a computable metric space with Cauchy representation δ_X. By $\mathsf{WBWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X'$ we denote the same problem as BWT_X, but with the jump δ_X' of the Cauchy representation as representation on the output side.

This means that the output of a realizer of WBWT_X on some input $(x_n)_n$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ that converges to a Cauchy name of a cluster point of $(x_n)_n$. The following result expresses the relation between the Bolzano-Weierstraß Theorem and the Weak Bolzano-Weierstraß Theorem.

Theorem 14.4 (Weak Bolzano-Weierstraß Theorem). $\mathsf{WBWT}_X \equiv_{\mathrm{W}} (\lim \to \mathsf{BWT}_X)$ *for every computable metric space X.*

Proof. It follows immediately from the definitions that $\mathsf{BWT}_X \leq_{\mathrm{W}} \lim * \mathsf{WBWT}_X$, which implies $(\lim \to \mathsf{BWT}_X) \leq_{\mathrm{W}} \mathsf{WBWT}_X$. Let now h be a multi-valued function such that $\mathsf{BWT}_X \leq_{\mathrm{W}} \lim * h$. Without loss of generality we can assume that h is of type $h :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and that h is a cylinder. Hence there are computable functions H, K such that $\lim \circ H \circ h \circ K \equiv_{\mathrm{W}} \lim * h$. Since $\mathsf{BWT}_X \leq_{\mathrm{W}} \lim * h \equiv_{\mathrm{W}} \lim \circ H \circ h \circ K$ it follows that $\mathsf{WBWT}_X \leq_{\mathrm{W}} H \circ h \circ K \leq_{\mathrm{W}} h$, which implies $\mathsf{WBWT}_X \leq_{\mathrm{W}} (\lim \to \mathsf{BWT}_X)$. □

We obtain $\mathsf{WBWT}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{WBWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{WBWT}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{WBWT}_{[0,1]^{\mathbb{N}}}$ with Fact 14.2 and Theorem 14.4. In Proposition 14.7 we are going to prove a slightly stronger result.

While the Weak Bolzano-Weierstraß Theorem determines a sequence that converges to a cluster point, one could also consider a variant where the result is a function that selects a converging subsequence.

Definition 14.5 (Subsequential Bolzano-Weierstraß Theorem). Let X be a computable metric space and let $\mathsf{SBWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be defined by

$$\mathsf{SBWT}_X((x_i)_i) := \{s \in \mathbb{N}^{\mathbb{N}} : (x_{s(n)})_n \text{ converges and } s \text{ is strictly monotone}\}$$

where $\mathrm{dom}(\mathsf{SBWT}_X)$ is the set of all sequences $(x_i)_i$ in X such that $\{x_i : i \in \mathbb{N}\}$ has a compact closure.

This version comes closest to the weak Bolzano-Weierstraß Theorem that has been introduced in [41]. The following result is easy to prove. However, we note that one only obtains equivalence and not strong equivalence as the access to the input is crucial in both directions.

Proposition 14.6. $\mathsf{WBWT}_X \equiv_{\mathrm{W}} \mathsf{SBWT}_X$ *for every computable metric space X.*

As for the other versions of the Bolzano-Weierstraß Theorem, SBWT_X yields one and the same equivalence class for many different computable metric spaces X.

Proposition 14.7. $\mathsf{SBWT}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{SBWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]^{\mathbb{N}}}$.

Proof. The reduction $\mathsf{SBWT}_{2^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathsf{SBWT}_{\mathbb{R}}$ can be established using the function $f : 2^{\mathbb{N}} \to \mathbb{R}, p \mapsto \sum_{i=0}^{\infty} 2p(i)3^{-i-1}$ that maps $2^{\mathbb{N}}$ computably and injectively to the Cantor discontinuum. This map has a partial continuous inverse and hence $(x_{s(n)})_n$ converges in $2^{\mathbb{N}}$ whenever $(f(x_{s(n)}))_n$ converges in $\mathbb{R}$. The reduction $\mathsf{SBWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]}$ follows similarly using the function $f : \mathbb{R} \to [0,1], x \mapsto \pi \arctan(x) + \frac{1}{2}$. The reduction $\mathsf{SBWT}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]^{\mathbb{N}}}$ follows using the canonical injection $f : [0,1] \to [0,1]^{\mathbb{N}}$. Finally, for the reduction $\mathsf{SBWT}_{[0,1]^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathsf{SBWT}_{2^{\mathbb{N}}}$ we note that there is a representation $\rho : 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ of $[0,1]^{\mathbb{N}}$ that is effectively equivalent to the Cauchy representation (as for any other compact metric space). Given a sequence $(x_n)_n$ in $[0,1]^{\mathbb{N}}$ by a sequence of names $p_n \in 2^{\mathbb{N}}$ with $x_n = \rho(p_n)$, we obtain that $(x_{s(n)})_n$ converges if $(p_{s(n)})_n$ converges. □

We note that basically the same proof shows that we could also get strong equivalences in case of WBWT. Now essentially the same proof as that of [41, Theorem 3.2] yields the following result.

Theorem 14.8 (Subsequential Bolzano-Weierstraß Theorem). $\mathsf{SBWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{COH}$.

Proof. Firstly, we note that $\mathsf{SBWT}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]}|_{\mathbb{Q}^{\mathbb{N}}}$. This is because given a sequence $(x_i)_i$ in $[0,1]$, we can compute a sequence $(y_i)_i$ in $\mathbb{Q} \cap [0,1]$ such that $|x_i - y_i| < 2^{-i}$ and hence the sets of cluster points of both sequences coincide. By Proposition 14.7 it suffices to prove $\mathsf{SBWT}_{[0,1]}|_{\mathbb{Q}^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathsf{COH} \leq_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]^{\mathbb{N}}}$.

Now we prove $\mathsf{SBWT}_{[0,1]}|_{\mathbb{Q}^\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{COH}$. Given a sequence $(x_i)_i$ of rational numbers in $[0,1]$ we compute the sequence $(R_i)_i$ of sets with

$$R_i := \left\{ j \in \mathbb{N} : x_j \in \bigcup_{k \leq 2^{i-1}} \left[\frac{2k}{2^i}, \frac{2k+1}{2^i} \right] \right\}$$

for all $i \in \mathbb{N}$. Now given a cohesive set $A \in \mathsf{COH}((R_i)_i)$ for $(R_i)_i$ we can compute the principal function p_A of A, which is strictly monotone. It suffices to show that $(x_{p_A(n)})_n$ is a Cauchy sequence. For every set $R \subseteq \mathbb{N}$ we use the notation $1 \cdot R := R$ and $(-1) \cdot R := \mathbb{N} \setminus R$ and we define for every word $y \in \{0,1\}^*$

$$R^y := \bigcap_{i<|y|} (-1)^{y(i)} \cdot R_i.$$

By definition of the sets R_i it follows that $i, j \in R^y$ implies $|x_i - x_j| \leq 2^{-|y|+1}$. Since A is cohesive for $(R_i)_i$ it follows that for every $k \in \mathbb{N}$ there is some $m \in \mathbb{N}$ and some $y \in \{0,1\}^{k+2}$ such that $p_A(n) \in R^y$ for all $n > m$. In particular, for every $k \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that $|x_{p_A(n)} - x_{p_A(m)}| < 2^{-k}$ for all $n > m$. This means that $(x_{p_A(n)})_n$ is a Cauchy sequence.

Now we prove that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{SBWT}_{[0,1]^\mathbb{N}}$. Given a sequence $(R_i)_i$ of sets $R_i \in \mathbb{N}$ we compute a sequence $(x_i)_i$ in $[0,1]^\mathbb{N}$ by

$$x_i(n) := \begin{cases} 1 & \text{if } i \in R_n \\ 0 & \text{otherwise} \end{cases}$$

Now $\mathsf{SBWT}_{[0,1]^\mathbb{N}}((x_i)_i)$ yields a strictly increasing function $s : \mathbb{N} \to \mathbb{N}$ such that $(x_{s(i)})_i$ is a convergent subsequence of $(x_i)_i$. Given s we can compute its range $A := \{s(n) : n \in \mathbb{N}\}$ and it suffices to show that A is cohesive for $(R_i)_i$. We equip $[0,1]^\mathbb{N}$ with the usual product norm given by $||y|| := \sum_{n=0}^{\infty} 2^{-n-1}|y(n)|$. The fact that $(x_{s(i)})_i$ is convergent and hence a Cauchy sequence implies that for all $k \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that $||x_{s(j)} - x_{s(m)}|| < 2^{-k}$ for all $j \geq m$, i.e., such that $s(j) \in R_n \iff s(m) \in R_n$ for all $j \geq m$ and $n < k$. This proves that A is cohesive for $(R_i)_i$. $\square$

Using Theorems 14.4 and 14.8, Proposition 14.6 and Fact 14.2 we obtain the following purely algebraic characterization of cohesiveness in terms of the jump of Weak Kőnig's Lemma, which is the counterpart of Corollary 13.5 that expresses a similar relation on the corresponding problems on Turing degrees.

Corollary 14.9 (Cohesiveness)**.** $\mathsf{COH} \equiv_{\mathrm{W}} (\lim \to \mathsf{WKL}')$.

We can derive the following interesting consequence of this corollary.

Proposition 14.10. $\mathsf{COH} \equiv_{\mathrm{W}} \widehat{\mathsf{WBWT}_2}$.

Proof. It is clear that $\mathsf{WBWT}_2 \leq_{\mathrm{W}} \mathsf{WBWT}_{2^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{COH}$ by Theorem 14.8 and Proposition 14.6. Since COH is parallelizable by definition, it follows that $\widehat{\mathsf{WBWT}_2} \leq_{\mathrm{W}} \mathsf{COH}$. For the other direction of the reduction we note that $\mathsf{BWT}_{2^\mathbb{N}} \equiv_{\mathrm{W}} \widehat{\mathsf{BWT}_2}$ by [12, Corollary 11.12] and hence we obtain by Fact 14.2 that

$$\mathsf{WKL}' \equiv_{\mathrm{W}} \mathsf{BWT}_{2^\mathbb{N}} \leq_{\mathrm{W}} \widehat{\mathsf{BWT}_2} \leq_{\mathrm{W}} \widehat{\lim_2 * \mathsf{WBWT}_2} \leq_{\mathrm{W}} \widehat{\lim_2} * \widehat{\mathsf{WBWT}_2} \leq_{\mathrm{W}} \lim * \widehat{\mathsf{WBWT}_2}.$$

Hence, we obtain $\mathsf{COH} \leq_{\mathrm{W}} \widehat{\mathsf{WBWT}_2}$ by Corollary 14.9. $\square$

From Corollary 14.9 it follows that $\mathsf{WKL}' \leq_{\mathrm{W}} \lim * \mathsf{COH}$. In Corollary 14.15 we are going to prove that even equivalence holds. As a preparation, we first prove a theorem that shows that computable functions that map converging sequences to converging sequences can be mimicked on the limits by a function that is computable in the halting problem.

We recall that we use the coding $\langle\ \rangle : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, defined by

$$\langle p_0, p_1, p_2, ...\rangle\langle n, k\rangle := p_n(k)$$

for all $p_i \in \mathbb{N}^{\mathbb{N}}$ and $n, k \in \mathbb{N}$. Given words $v_0, ..., v_t \in \mathbb{N}^*$, we define analogously $\langle v_0, ..., v_t\rangle$ to be the longest word $w \in \mathbb{N}^*$ such that $w\langle n, k\rangle = v_n(k)$ is defined for all $n, k \in \mathbb{N}$ with $\langle n, k\rangle < |w|$.

Theorem 14.11 (Double Limit). *For every computable function $G :\subseteq \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ there exists a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ that is computable in $\emptyset'$ and such that*

$$F(p) \in \lim \circ G \circ \lim^{-1}(p)$$

for all $p \in \mathrm{dom}(\lim \circ G \circ \lim^{-1})$.

Proof. Let $G :\subseteq \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ be computable. Then there exists a computable monotone function $g : \mathbb{N}^* \to 2^*$ that approximates G in the sense that $G(q) = \sup_{w \sqsubseteq q} g(w)$ for all $q \in \mathrm{dom}(G)$. Let $g_i : \mathbb{N}^* \to 2^*$ be the part of g that contributes to $(\lim \circ G(q))(i)$: for $w \in \mathbb{N}^*$ and $i \in \mathbb{N}$ we define $g_i(w) := g(w)\langle 0, i\rangle ... g(w)\langle n, i\rangle$ for the largest $n \in \mathbb{N}$ such that all the values are defined. Let $A \subseteq \{0, 1\} \times \mathbb{N}^3 \times (\mathbb{N}^*)^2$ be the set of values $(b, k, i, s, w, \langle u_0, ..., u_s\rangle)$ such that there exist $t > s$ and $u_{s+1}, ..., u_t \in \mathbb{N}^*$ with

(a) $w \sqsubseteq u_\iota$ for all $\iota = s + 1, ..., t$,
(b) $g_i(\langle u_0 0^t, ..., u_s 0^t, u_{s+1}, ..., u_t\rangle)$ contains at least k–times the bit b.

The set A is c.e. and hence computable in $\emptyset'$.

Now we describe how we can compute a suitable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ with the help of A. Given some input $p \in \mathrm{dom}(\lim \circ G \circ \lim^{-1})$ we compute $F(p)(i)$ with the help of a sequence $(u_s)_s$ of words $u_s \in \mathbb{N}^*$.

For each fixed $i \in \mathbb{N}$ we determine this sequence $(u_s)_s$ inductively in stages $j = 0, 1, 2, ...$ We start with the empty sequence $(u_s)_s$ and $s_0 := -1$. At stage $j = 2k + b$ we assume that $u_0, ..., u_{s_j}$ are already determined and then we check whether

$$(b, k, i, s_j, p|_j, \langle u_0, ..., u_{s_j}\rangle) \quad \in \quad A. \tag{1}$$

If so, then we compute corresponding words $u_{s_j+1}, ..., u_t$ that satisfy the conditions (a) and (b) given above and we extend $u_0, ..., u_{s_j}$ by these words, so $s_{j+1} := t$. Otherwise, we leave the sequence as it is and set $s_{j+1} := s_j$.

For each $k \in \mathbb{N}$ the Test (1) above is positive for at least one $b \in \{0, 1\}$: the word $\langle u_0, ..., u_{s_j}\rangle$ can be extended to $q := \langle u_0\widehat{0}, ..., u_{s_j}\widehat{0}, p, p, p, ...\rangle$ and $\lim(q) = p$ and hence $q \in \mathrm{dom}(G)$ and $\lim \circ G(q)$ exists, which means that $(\lim \circ G(q))(i) = b$ for some $b \in \{0, 1\}$ and for this b a suitable t and an extension $u_{s_j+1}, ..., u_t$ can be found by continuity of G. In particular, the sequence $(u_s)_s$ is actually infinite and $u := \langle u_0\widehat{0}, u_1\widehat{0}, u_2\widehat{0}, ...\rangle$ satisfies $\lim(u) = p$ by construction.

Since $\lim(u) = p$ it follows that $u \in \mathrm{dom}(G)$ and $\lim \circ G(u)$ exists, which means that $(\lim \circ G(u))(i) = d$ for some $d \in \{0, 1\}$. Hence for our fixed $(u_s)_s$ and $b = 1 - d$ the Test (1) above is positive for only finitely many k and $j = 2k + b$. If the test is negative for some fixed k, b, then it is also negative for all larger k and the corresponding j. Hence, there is a minimal $k \in \mathbb{N}$ such that the Test (1) is positive for $j = 2k + b$ with a unique $b \in \{0, 1\}$ and for this unique b we obtain $b = d$.

Hence, in order to compute $F(p)(i)$ given u, we just search for the minimal $k \in \mathbb{N}$ with the property that the Test (1) is satisfied for $j = 2k + b$ with a unique b and we let $F(p)(i) = b$, which guarantees

$$F \circ \lim(u) = F(p) = \lim \circ G(u)$$

and hence $F(p) \in \lim \circ G \circ \lim^{-1}(p)$. Since A is computable in the halting problem $\emptyset'$, it follows that F is so. $\square$

If the function G is extensional in the sense that the limit of its output only depends on the limit of its input, then the function F is uniquely determined. This unique version generalizes [14, Theorem 22 (2)$\Longrightarrow$(1)]. Already this unique version shows that F cannot be computable in general (a function G that is constant and computes a sequence that converges to the halting problem is a counterexample). The technique used to prove Theorem 14.11 is somewhat reminiscent of the first jump control technique [19, Section 4]. We obtain that following corollary.

Corollary 14.12. $\lim *_{\rm s} \lim^{-1} \equiv_{\rm sW} {\rm c}_{\emptyset'} \times {\rm id}$ *and* $\lim * \lim^{-1} \equiv_{\rm W} \lim$.

The second equivalence is strictly speaking not a corollary, but it simply holds since $\lim^{-1}$ is computable. We note that $f \leq_{\rm W} {\rm c}_{\emptyset'} \times {\rm id}$ holds if and only if f is computable with respect to the halting problem. The proof of Theorem 14.11 works analogously for the following parameterized version. We just need to replace the c.e. set A in the proof by a suitable set A_q that depends on the additional parameter q. In this case A_q is c.e. in q and hence it can be computed with the help of the Turing jump $\mathsf{J}(q) = q'$ of q.

Theorem 14.13 (Parameterized Double Limit)**.** *For every computable function* $G :\subseteq \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ *there exists a computable function* $F :\subseteq \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ *such that*

$$F\langle \mathsf{J}(q), p\rangle \in \lim \circ G \circ \langle {\rm id} \times \lim^{-1}\rangle(q,p)$$

for all $(q,p) \in {\rm dom}(\lim \circ G \circ \langle {\rm id} \times \lim^{-1}\rangle)$.

Now we obtain the following characterization of the Bolzano-Weierstraß Theorem with the help of the Parameterized Double Limit Theorem 14.13.

Theorem 14.14. $\mathsf{BWT}_\mathbb{R} \equiv_{\rm W} \lim * \mathsf{WBWT}_\mathbb{R}$ *and* $\mathsf{BWT}_\mathbb{R} \equiv_{\rm sW} \lim *_{\rm s} \mathsf{WBWT}_\mathbb{R}$.

Proof. It is clear that $\mathsf{BWT}_\mathbb{R} \leq_{\rm W} \lim * \mathsf{WBWT}_\mathbb{R}$ and $\mathsf{BWT}_\mathbb{R} \leq_{\rm sW} \lim *_{\rm s} \mathsf{WBWT}_\mathbb{R}$. For the other reduction of the first claim it suffices to prove $\lim_{2^\mathbb{N}} * \mathsf{WBWT}_{2^\mathbb{N}} \leq_{\rm W} \mathsf{BWT}_{2^\mathbb{N}}$ by Fact 14.2, Propositions 14.7, 14.6 and [5, Proposition 9.1]. To this end, let $g \leq_{\rm W} \lim_{2^\mathbb{N}}$ and $h \leq_{\rm W} \mathsf{WBWT}_{2^\mathbb{N}}$ be such that $g \circ h$ exists. Since $\lim_{2^\mathbb{N}}$ is a cylinder, we can even assume $g \leq_{\rm sW} \lim_{2^\mathbb{N}}$. Without loss of generality, we can also assume that g, h are of type $g, h :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$. We need to prove $g \circ h \leq_{\rm W} \mathsf{BWT}_{2^\mathbb{N}}$. There are computable $H, G, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that

$$H \circ \lim_{2^\mathbb{N}} \circ G\langle p, \mathsf{WBWT}_{2^\mathbb{N}} K(p)\rangle \subseteq g \circ h(p)$$

for all $p \in {\rm dom}(h)$. Since $\mathsf{WBWT}_{2^\mathbb{N}} K(p)$ yields arbitrary converging sequences $q \in 2^\mathbb{N}$ as output that converge to some cluster point of the input $K(p)$, G needs to be defined on all corresponding pairs $\langle p, q\rangle$ for each fixed $p \in {\rm dom}(h)$. By the Parameterized Double Limit Theorem 14.13 we obtain that there is a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ such that

$$H \circ F\langle \mathsf{J}(p), \mathsf{BWT}_{2^\mathbb{N}} K(p)\rangle \subseteq H \circ \lim_{2^\mathbb{N}} \circ G\langle p, \lim^{-1} \circ \mathsf{BWT}_{2^\mathbb{N}} K(p)\rangle \subseteq g \circ h(p)$$

for all $p \in {\rm dom}(h)$, which implies $g \circ h \leq_{\rm W} \mathsf{J} \times \mathsf{BWT}_{2^\mathbb{N}} \equiv_{\rm W} \lim \times \mathsf{BWT}_{2^\mathbb{N}}$, since $\mathsf{J} \equiv_{\rm W} \lim$. Since $\mathsf{BWT}_{2^\mathbb{N}}$ is idempotent [12, Corollary 11.13] and $\lim \leq_{\rm W} \mathsf{BWT}_{2^\mathbb{N}}$ [12, Corollary 11.22], we obtain $\lim \times \mathsf{BWT}_{2^\mathbb{N}} \leq_{\rm W} \mathsf{BWT}_{2^\mathbb{N}}$ and hence $g \circ h \leq_{\rm W} \mathsf{BWT}_{2^\mathbb{N}}$, as desired.

The second reduction of the second claim follows from

$$\lim *_{\rm s} \mathsf{WBWT}_\mathbb{R} \leq_{\rm W} \lim * \mathsf{WBWT}_\mathbb{R} \leq_{\rm W} \mathsf{BWT}_\mathbb{R}$$

together with the fact that $\mathsf{BWT}_\mathbb{R}$ is a cylinder [12, Corollary 11.13], which implies $\lim *_{\rm s} \mathsf{WBWT}_\mathbb{R} \leq_{\rm sW} \mathsf{BWT}_\mathbb{R}$. ☐

By Fact 14.2, Proposition 14.6, Theorem 14.8 and 14.14 we obtain the following corollary.

Corollary 14.15. $\mathsf{WKL}' \equiv_{\mathrm{W}} \lim * \mathsf{COH}$.

We continue this section with some lowness properties of the weak version of the Bolzano-Weierstraß Theorem $\mathsf{WBWT}_{\mathbb{R}}$ and cohesiveness COH. In [12] a multi-valued function f is called *low*, if $f \leq_{\mathrm{sW}} \mathsf{L}$ and it is called *low*$_2$, if $f \leq_{\mathrm{sW}} \mathsf{L}_2$, where $\mathsf{L}_2 := \mathsf{J}^{-1} \circ \mathsf{J}^{-1} \circ \lim \circ \lim$.

Theorem 14.16. $\mathsf{WBWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{L}_2$ *and* $\mathsf{WBWT}_{\mathbb{R}} \not\leq_{\mathrm{sW}} \mathsf{L}$*, i.e.,* $\mathsf{WBWT}_{\mathbb{R}}$ *is low*$_2$*, but not low.*

Proof. By [12, Corollary 11.15] we have $\mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{L}' \equiv_{\mathrm{sW}} \mathsf{L} *_{\mathrm{s}} \lim$ and by [12, Corollary 8.8] we have $\lim *_{\mathrm{s}} \mathsf{L} \equiv_{\mathrm{sW}} \lim$. With the help of Theorem 14.14 we obtain

$$\lim *_{\mathrm{s}} \lim *_{\mathrm{s}} \mathsf{WBWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathsf{L} *_{\mathrm{s}} \lim \leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \lim .$$

This implies $\mathsf{WBWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{L}_2$ by [12, Theorem 8.6].

Let us now assume that $\mathsf{WBWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{L}$. With the help of Theorem 14.14 we obtain

$$\mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathsf{WBWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathsf{L} \leq_{\mathrm{sW}} \lim$$

in contradiction to [12, Theorem 12.7]. □

The first statement of this result implies the following non-uniform version, which was already proved in [41, Theorem 3.5(1)].

Corollary 14.17. *For every bounded sequence* $(x_n)_n$ *of real numbers there exists a low*$_2$ *sequence of reals that converges to a cluster point of* $(x_n)_n$.

The proof idea of the second part of Theorem 14.16 can be used to show that low$_2$ cannot be replaced by low in this corollary.

We note that Proposition 14.6 is only formulated for ordinary Weihrauch reducibility and not for strong reducibility. Hence, we cannot directly transfer Theorem 14.16 to $\mathsf{SBWT}_{\mathbb{R}}$. However, we can easily derive a corresponding result for $\mathsf{SBWT}_{\mathbb{R}}$ or equivalently (by Theorem 14.8) for COH.

Theorem 14.18. $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{L}_2$ *and* $\mathsf{COH} \not\leq_{\mathrm{sW}} \mathsf{L}$*, i.e.,* COH *is low*$_2$*, but not low.*

Proof. By [12, Theorem 8.6] $\lim * \mathsf{WKL} \leq_{\mathrm{W}} \lim$ since WKL is low by [6, Corollary 8.5] and hence $\lim * \mathsf{WKL}' \leq_{\mathrm{W}} \lim'$. With Corollary 14.15 we obtain

$$\lim' *_{\mathrm{s}} \mathsf{COH} \leq_{\mathrm{W}} \lim * \lim * \mathsf{COH} \leq_{\mathrm{W}} \lim * \mathsf{WKL}' \leq_{\mathrm{W}} \leq_{\mathrm{W}} \lim * \lim \leq_{\mathrm{W}} \lim' .$$

Since $\lim'$ is a cylinder this implies $\lim' *_{\mathrm{s}} \mathsf{COH} \leq_{\mathrm{sW}} \lim'$. Hence COH is low$_2$ by [12, Theorem 8.6].

Let us now assume that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{L}$. With the help of Theorems 14.14, 14.8 and Proposition 14.6 we obtain

$$\mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{W}} \lim * \mathsf{WBWT}_{\mathbb{R}} \leq_{\mathrm{W}} \lim * \mathsf{COH} \leq_{\mathrm{W}} \lim * \mathsf{L} \leq_{\mathrm{W}} \lim$$

in contradiction to [12, Theorem 12.7]. □

Finally, we want to show that COH is not probabilistic. For this purpose, we need the following technical lemma.

Lemma 14.19. *The set* $C = \{n \in \mathbb{N} : \mu\{p \in 2^{\mathbb{N}} : F(p)(n) = j\} > r\}$ *is a* $\mathbf{\Sigma}^0_2$*–set for every limit computable* $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $r \in \mathbb{Q}$ *and* $j \in \mathbb{N}$.

Proof. By $V_n := \{q \in 2^{\mathbb{N}} : q(n) = j\}$ we define a computable sequence of c.e. open sets and hence $A_n := F^{-1}(V_n)$ is a computable sequence of $\mathbf{\Sigma}^0_2$–sets with respect to the effective Borel hierarchy [5]. For every such $\mathbf{\Sigma}^0_2$–set there is a computable sequence $(U_i)_i$ of open sets such that $A_n = (\bigcap_{i=0}^{\infty} U_i)^{\mathrm{c}}$. By $U_i[s]$ we denote the enumeration of U_i up to stage $s \in \mathbb{N}$ (which is a finite union of basic open balls). Then we obtain by continuity of the probability measure μ

$$\mu(A_n) \leq r \iff \mu\left(\bigcap_{i=0}^{\infty} U_i\right) \geq 1-r \iff (\forall t)(\exists s)\, \mu\left(\bigcap_{i=0}^{t} U_i[s]\right) > 1-r-2^{-t},$$

which is a $\mathbf{\Pi}^0_2$–property. Hence, the complementary property $\mu(A_n) > r$ is a $\mathbf{\Sigma}^0_2$–property. $\square$

In order to express the next result we need a generalization of the concept of being probabilistic. We call a function $f :\subseteq (X,\delta_X) \rightrightarrows (Y,\delta_Y)$ on represented spaces *limit probabilistic*, if there is a limit computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and a family $(A_p)_{p\in D}$ of measurable sets $A_p \subseteq 2^{\mathbb{N}}$ with $D := \mathrm{dom}(f\delta_X)$ such that $\mu(A_p) > 0$ for all $p \in D$ and $\delta_Y F(p,r) \in f\delta_X(p)$ for all $p \in D$ and $r \in A_p$. This concept is obviously weaker than being probabilistic, WKL is for instance not probabilistic by [11, Proposition 14.8], but it is limit computable and hence limit probabilistic. However, we can prove the following.

Lemma 14.20. WKL' *is not limit probabilistic.*

Proof. Firstly, by [27, Theorem 6.7] $\mathsf{SEP} \equiv_{\mathrm{W}} \mathsf{WKL}$ holds for the separation problem SEP for enumerated sets $A, B \subseteq \mathbb{N}$. Since WKL is a cylinder, we even obtain $\mathsf{SEP} \leq_{\mathrm{sW}} \mathsf{WKL}$ and, in particular $\mathsf{SEP}' \leq_{\mathrm{sW}} \mathsf{WKL}'$. Secondly, the proof of [39, Theorem 5.3] relativizes and yields the following statement: if $A, B \subseteq \mathbb{N}$ cannot be separated by a Δ^0_2–set, then

$$\mu\{D \in 2^{\mathbb{N}} : (\exists C \leq_{\mathrm{T}} D')\ C \text{ separates } A, B\} = 0.$$

The proof follows exactly along the lines of the proof given in [39], with the additional observation that the sets C_i defined analogously to those in that proof are $\mathbf{\Sigma}^0_2$–sets by Lemma 14.19. Finally, there are $\mathbf{\Sigma}^0_2$–sets $A, B \subseteq \mathbb{N}$ that cannot be separated by a Δ^0_2–set. Altogether, this yields that SEP' and hence WKL' are not limit probabilistic. $\square$

Now we can derived the following corollary.

Corollary 14.21. COH *is not probabilistic.*

Proof. Let us assume that COH is probabilistic. Then $\lim * \mathsf{COH}$ is limit probabilistic and hence WKL' too by Corollary 14.15. This contradicts Lemma 14.20. $\square$

15. Conclusion

In this paper we have started to classify the uniform computational content of computability theory with the help of the Weihrauch lattice. While we have studied a number of relevant computability theoretic properties, we have only started to look at some of the most basic (non-constructive) theorems of computability theory. It is a promising new research programme to continue along these lines and to study more advanced theorems that require priority constructions and other techniques whose uniform computational content has not yet been investigated.

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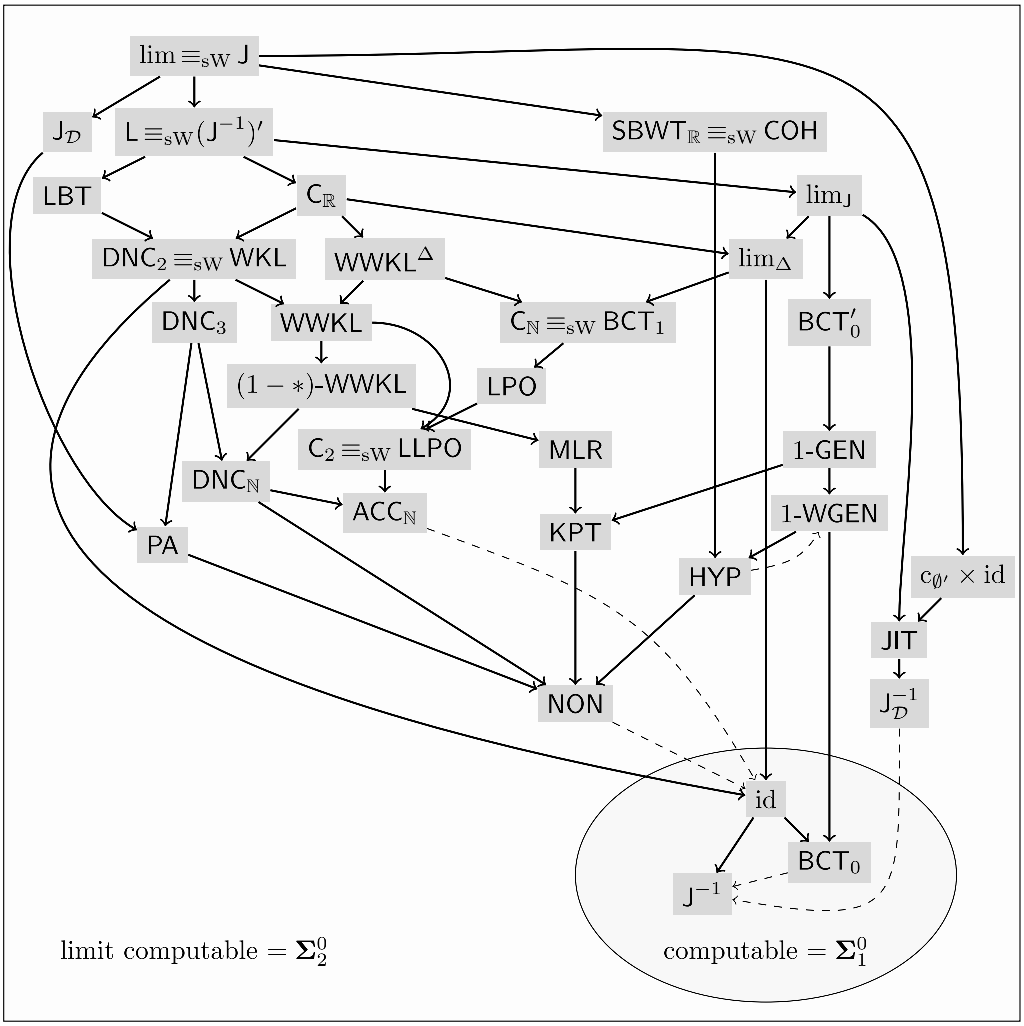

FIGURE 2. The computability theory zoo in the Weihrauch lattice. The solid arrows indicate strong Weihrauch reductions in the opposite direction and the dashed arrows indicate ordinary Weihrauch reductions.

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