Almost Theorems of Hyperarithmetic Analysis

Richard A. Shore*
Department of Mathematics
Cornell University
Ithaca NY 14853

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Abstract

Theorems of hyperarithmetic analysis (THA) occupy an unusual neighborhood in the realms of reverse mathematics and recursion theoretic complexity. They lie above all the fixed (recursive) iterations of the Turing Jump but below ATR_0 (and so Π^1_1-CA_0 or the hyperjump). There is a long history of proof theoretic principles which are THA. Until Barnes, Goh and Shore [ta] revealed an array of theorems in graph theory living in this neighborhood, there was only one mathematical denizen. In this paper we introduce a new neighborhood of theorems which are almost theorems of hyperarithmetic analysis (ATHA). When combined with ACA_0 they are THA but on their own the are very weak. We generalize several conservativity classes (Π^1_1, r-Π^1_1 and Tanaka) and show that all our examples (and many others) are conservative over RCA_0 in all these senses and weak in other recursion theoretic ways as well. We provide denizens both mathematical and logical. These results answer a question raised by Hirschfeldt and reported in Montalbán [2001] by providing a long list of pairs of principles one of which is very weak over RCA_0 but over ACA_0 is equivalent to the other which may be strong (THA) or very strong going up a standard hierarchy and at the end being stronger than full second order arithmetic.

1 Introduction

The general project of calibrating the complexity of mathematical theorems and constructions has two important and interrelated classes of measuring rods. One, embodied in what is now called reverse mathematics, is proof theoretic and attempts to determine

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what axioms are sufficient and even necessary to prove a given theorem. The other is recursion theoretic and attempts to determine how hard (in terms of computational complexity) it is to construct a desired object or how complicated must an object be to satisfy given specifications. Each approach has its standard yardsticks of complexity. For reverse mathematics these are axiom systems in second order arithmetic. For the computational approach they are ones measured by specific constructions. Most prominently, they are calibrated in terms of the Turing jump and its iterates and generalizations. (Standard texts are Simpson [2009] and Hirschfeldt [2014] which emphasize the first and second approach, respectively.)

The early decades of this subject were marked by a large variety of results characterizing a wide array of theorems and constructions as being one of five or so specific levels of complexity. (These are Simpson's "big five" axioms systems and the corresponding recursion theoretic construction principles.) They begin with RCA_0 which is the standard weak base theory for reverse mathematics. It includes the usual simple first order axioms about $+, \times, \leq, \in, 0, 1$, extensionality with respect to $\in$, and induction for sets X as a free variable. The defining additional axiom scheme is Δ^0_1-CA, comprehension for sets defined by both Σ^0_1 and Π^0_1 formulas. This system corresponds to recursive constructions. We assume that it is included in every theory we consider. (For formal definitions and detailed information on all the systems see Simpson [2009].) In more recent decades, there has been a proliferation of results placing theorems and constructions outside the big five. Sometimes inserted linearly and sometimes with incomparabilities. They are now collectively often called the "zoo" of reverse mathematics. (For pictures, see https://rmzoo.math.uconn.edu/diagrams/.) The bulk of these interpolations have been at the lower end of these hierarchy. (Technically, this means below ACA_0, the proof theoretic system whose defining axiom scheme is comprehension for arithmetic formula. The recursion theoretic analog is the class of constructions which can be done effectively in finitely many iterations of the Turing jump.)

The next systems of reverse mathematics are ATR_0 and Π^1_1-CA_0 which are defined, respectively, by transfinite iterations of arithmetic comprehension and comprehension for formulas with one second order quantifier followed by an arithmetic formula. Recursion theoretically these roughly correspond to transfinite effective iterations of the Turing jump (hyperarithmetic sets) and Kleene's hyperjump. A fair number of mathematical theorems and constructions have turned up at precisely these levels but very few between them or above them.

As in Barnes, Goh and Shore [ta] (hereafter BGS) which led to this work, our concerns here are with a particularly unusual area of these hierarchies lying recursion theoretically above each fixed bounded countable iteration of the Turing jump but proof theoretically below the system ATR_0. It has a precise recursion theoretic definition (2.9) but, as the definition relies on using only the standard model of arithmetic and only true well orderings, it lacks a good proof theoretic definition (at least in first order logic). (See Van Wesep [1977, 2.2.2] and also Montalbán [2006, remarks after Definition 1.1].) Theorems and theories at this level are called ones of hyperarithmetic analysis (so THA).

There are quite a number of logical theories including ones about choice (Σ^1_1-AC) and comprehension (Δ^1_1-CA) that fall in this realm. Many were well studied in the 60s and 70s both before and after the introduction of the program of reverse mathematics (H. Friedman [1971], [1975]). Up until BGS, however, there was only one mathematical but not logical example, i.e. one not mentioning classes of first order formulas or their syntactic complexity. It was a result (INDEC) about indecomposability of linear orderings in Jullien's [1969] thesis (see Rosenstein [1982, Lemma 10.3]). It was shown to be a THA by Montalbán [2006] who investigated its place among the older systems as well as several other logical ones using variations of Steel forcing. In [2008] he included Π^1_1-Separation and new forcing variations. More analysis was provided by Neeman [2009], [2011].

This situation provoked the question as to whether there are other results from the mathematical literature that are THA. The issue was raised explicitly in Montalbán's "Open Questions in Reverse Mathematics" [2011, Q30]. It was answered by BGS who provide a host of such examples and study the relations among them and to previously known systems. (See also Goh [ta].) These were all variations and generalizations of a classical theorem of Halin [1965] in graph theory (Definition 2.5). (See also Halin [1970] and, for a contemporary treatment and references, Diestel [2017, Ch. 8].)

This paper grew out of a proof of what we would call a reduction (actually providing an equivalence) between two of these principles in Bowler, Carmesin and Pott [2015] (hereafter BCP). While BGS provides a number of such results in RCA_0, the reduction in BCP did not seem to fit the mold. While the proof sketch provided there appeared to be elementary, a closer look showed that underneath it seemed to use methods that were themselves THA and about as strong as the principles being proven equivalent. Our expectation was that these methods, like most of the ones studied in BGS, would also prove to be THA. That turned out not to be the case. Rather, the graph theoretic principle (Definition 3.5) that they used (that allowed one to restrict attention to locally finite graphs) implied (over ACA_0) some known THA. The unusual aspect of the situation was that we could prove that it was not possible to show that they implied any known THA in RCA_0. In particular they did not even imply ACA_0. We call such principles/theorems/theories almost theorems/theories of hyperarithmetic analysis ATHA (Definition 3).

Indeed, the one that was used in BCP and several variants are very weak over ACA_0. They are Π^1_1, r-Π^1_2 (Definitions 4.8-4.10) and Tanaka (Definition 7.1) conservative over RCA_0 (and more). We extend all our conservation results to what seem to be new classes of formulas by allowing, in addition to the basic formulas in these classes, closure under conjunctions, disjunctions, first order quantifiers and universal second order quantifiers (see Definitions 4.9, 4.10 and 7.1). These results also provide extended conservation results for many previously studied theories. In addition, given any model $\mathcal{N}$ of RCA_0, one can construct an extension $\mathcal{N}'$ with the same first order part satisfying these principles which adds no branch to any tree in $\mathcal{N}$ not already having one there. Moreover, one can construct such extensions $\mathcal{N}_0$ and $\mathcal{N}_1$ of $\mathcal{N}$ so that the intersection of their second order parts is that part of $\mathcal{N}$. Finally, given any countable collection C_i of subsets of the natural numbers N of a countable $\mathcal{N} \vDash$RCA$_0$ one can construct an extension $\mathcal{N}' \vDash$RCA$_0$ in which these theorems are true but which does not contain any of the C_i. (An interesting example here is the collection of all subsets of N definable over $\mathcal{N}$.)

We also show that many simple variations of known THA such as Σ^1_1-AC are also ATHAs with all these weakness properties over RCA$_0$. We prove these results by showing that all the principles studied here can be made true by iterating forcings from a quite general class of forcings that can be implemented to guarantee the conservation and branch or set omitting properties just described. On the other hand, when combined with ACA$_0$ each of these principles is equivalent to an already studied one known to be a THA. We then point out various known separation results for the old principles that also distinguishing among some of these new theories over RCA$_0$.

Finally, we extend our methods to prove similar results for hierarchies of variations of choice principles that are much stronger than Σ^1_1-AC and so well beyond THA. At the end of these hierarchies, we provide principles that have all the same conservation and branch/set omitting properties over RCA$_0$ but when combined with ACA$_0$ are strictly stronger than full second order arithmetic. We also discuss another type of conservation result for sentences of the form $\forall X \exists ! Y \Phi(X, Y)$ conjectured by Tanaka for WKL$_0$ and proved for it in Simpson, Tanaka and Yamazaki [2002] (hereafter STY) as well as strengthenings to include larger classes of sentences (see Definition 7.1.)

Thus, we view this paper as not only introducing a new interesting realm of the reverse mathematics/recursion theoretic universe but as also answering the question raised in Montalbán [2011, 6.1.1] immediately after the one about the existence of mathematical THA. Attributing the question to Hirschfeldt, Montalbán points out that there are very few examples where natural equivalences are known to hold over strong theories but not over RCA$_0$ particularly if one excludes the cases where the only additional axioms needed are forms of induction. Hirschfeldt asked for more. We would say that this paper provides a whole array of pairs of principles which are equivalent over ACA$_0$ but not over RCA$_0$ and so evidence that in some settings it would make sense to take ACA$_0$ as the base theory for reverse mathematical investigations rather than RCA$_0$.

1.1 Outline of Paper

We provide the needed basic graph theoretic notions and principles in §2. The next section (3) presents the principles used in BCP as mentioned above as well as some related graph theoretic principles and analyzes their strength over ACA$_0$. In particular, we show that, over ACA$_0$, each of them implies some known THA. In §4, we define a large class of forcings that include many well known ones such as Cohen, Laver, Mathias, Sacks and Silver forcing and many variations. We then show that generic extensions by any such forcing have all the preservation properties suggested above. Thus any principle that can be made true by iterating such forcings have the conservation and other weakness properties already mentioned. In particular, if the principles are Π^1_2 (see Definition 4.7), and for any instance of the principle there is a forcing in our class that adds a solution, a standard ω-length iteration of such forcings guarantees the truth of the principle in the limit model. This supplies all the conservation results for such Π^1_2 principles and includes many previously known theorems as well as strengthenings of conservation results to larger classes of formulas.

However, the ATHA in which we are mainly interested are not Π^1_2 principles so some additional twists are needed in addition to supplying the appropriate forcing notions. We analyze the argument of BCP mentioned above as the first (and in many ways the most interesting) of our examples. The definition of the forcing notion and the proof that it supplies solutions to the relevant principles are in Theorem 5.1. To get an iteration that solves all instances of the principle and so provides the desired conservation and weakness properties, we use one of length ω_1 (Theorem 5.2). We then turn our attention to various weaker versions or instances of Σ^1_1-AC for our next source of ATHA. Some of these (mathematical as well as logical) appeared naturally in BGS. Others are variations of well-studied classes of choice principles weaker than Σ^1_1-AC. They are all weak over RCA_0 but equivalent to one of the known THA over ACA_0. Examples here include unique and finite choice versions of Σ^1_1-AC. (The former is generally known as weak Σ^1_1-AC. The latter is a consequence of the Halin type theorems studied and proven to be THA in BGS and placed with respect to other studied versions of Σ^1_1-AC in Goh [ta].)

In §6 we move beyond Σ^1_1-AC and study weak version of higher order axioms of choice. The appropriate forcing notions in our class are not hard to come by. As the principles are of arbitrary syntactic complexity, it is not immediate, for example, that adding something that may look like a solution during the construction will actually be a solution at the end of even an ω_1 length iteration. In the strongest case, we modify what it means to provide solutions (Theorem 6.1). We then use the fact that there is a closed unbounded set of ordinals α such that the models $\mathcal{N}_\alpha$ are elementary submodels (in the second order language) of our limit model $\mathcal{N}_{\omega_1}$ to show that it has the desired properties (Theorem 6.2). A short argument (Theorem 6.3) shows that the whole hierarchies of weak principles are equivalent to the standard choice axioms (Σ^1_{n+1}-AC) over ACA_0. At the end, we have two principles with all our conservation and preservation properties which over ACA_0 are equivalent to the union of all the Σ^1_{n+1}-AC and so strictly stronger than full second order arithmetic.

The last section is devoted first to a description of about ten year old but unpublished work by Tanaka, Montalbán and primarily Yamazaki getting some of our conservation results for what they call the collection axioms Π^1_n and Π^1_∞ (in our terminology Σ^1_{n+1}-AC$^-$ and Σ^1_∞-AC$^-$). They also extend even earlier worked on WKL_0 in STY to get Tanaka conservation (Definition 7.1) for the collection axioms and a couple of other principles. Motivated by this work, we have proven the same and stronger conservation results for all the ATHA principles we consider in this paper for which we use forcing constructions to show that they are very weak over RCA_0 and, in particular, do not imply ACA_0. The basic conservation result (over RCA_0) they prove is for sentences of the form $\forall X \exists! Y \Phi(X,Y)$ with Φ arithmetic. We get the same results for all of our principles and most of theirs by what seems to be much simpler constructions. In addition, we extend the class of sentences covered by our methods analogously to the extensions made for Π^1_1 and r-Π^1_2 in previous sections. While handling the basic Tanaka conservativity requires some additional notions, the extensions are dealt with as in the Π^1_1 and r-Π^1_2 conservation results mentioned above.

2 Basic Notions

Formally, we are working in a model $\mathcal{N} = (N, S(\mathcal{N}), +, \times, \leq, \in, 0, 1)$ of second order arithmetic. (The first order quantifiers range over N. The second order ones over $S(\mathcal{N})$ which is a collection of subset of N.) We generally abbreviate the structures as $\mathcal{N} = (N, S(\mathcal{N}))$. We are interested in ones which are models of RCA_0. When we define semantics or forcing we expand the formal language to include constants for each element of N and $S(\mathcal{N})$ and possibly some recursive ($\mathbf{\Delta}^0_1$) predicates. (See Remark 4.4.) Informally, one can think of N as the standard natural numbers $\mathbb{N}$ with the usual operations and relations (and constants for every n and some class of subsets of $\mathbb{N}$ as well, perhaps, the predicate representing the universal Turing machine as in Remark 4.4). We use standard recursive codings of finite sequences, functions, relations and structures to represent all such objects as subsets of N and abuse notation by saying that such objects are in $\mathcal{N}$ or $S(\mathcal{N})$ to mean that the corresponding codes are in $S(\mathcal{N})$. Unless otherwise specified, all sets and structures we consider are countable.

Definition 2.1. A *graph* H is a pair $\langle V, E\rangle$ consisting of a set V (of *vertices*) and a set E of unordered pairs $\{u, v\}$ with $u \neq v$ from V (called *edges*). These structures are also called *undirected graphs (or here U-graphs)*. A structure H of the form $\langle V, E\rangle$ as above is a *directed graph (or here D-graphs)* if E consists of ordered pairs $\langle u, v\rangle$ of vertices with $u \neq v$. To handle both cases simultaneously, we often use X to stand for undirected (U) or directed (D). We then use (u, v) to stand for the appropriate kind of edge, i.e. $\{u, v\}$ or $\langle u.v\rangle$. Any such H is *locally finite* if, for each $u \in V$, the set $\{v | (u, v) \in E \vee (v, u) \in E\}$ of *neighbors of* u is finite.

An *X-subgraph* of the X-graph H is an X-graph $H' = \langle V', E'\rangle$ such that $V' \subseteq V$ and $E' \subseteq E$.

Definition 2.2. An *X-ray in H* is pair consisting of an X-subgraph $H' = \langle V', E'\rangle$ and an isomorphism $f_{H'}$ from N with edges $(n, n+1)$ for $n \in N$ to H'. We also describe this situation by saying that H contains the X-ray $\langle H', f_{H'}\rangle$. We use *finite X-ray* in the obvious way. We sometimes abuse notation by saying that the sequence $\langle f(n)\rangle$ of vertices is an X-ray in H.

H contains k many X-rays for $k \in N$ if there is a sequence $\langle H_i, f_i\rangle_{i<k}$ such that each $\langle H_i, f_i\rangle$ is an X-ray in H (with $H_i = \langle V_i, E_i\rangle$).

H contains k many disjoint (or vertex-disjoint) rays if the V_i are pairwise disjoint. *H contains k many edge-disjoint rays* if the E_i are pairwise disjoint. We often use Y to stand for either vertex (V) or edge (E) as in the following definitions.

An X-graph H *contains arbitrarily many Y-disjoint X-rays* if it contains k many such rays for every $k \in N$.

An X-graph H *contains infinitely many Y-disjoint rays* if there is an X-subgraph $H' = \langle V', E' \rangle$ of H and a sequence $\langle H_i, f_i \rangle_{i \in N}$ such that each $\langle H_i, f_i \rangle$ is an X-ray in H (with $H_i = \langle V_i, E_i \rangle$) such that the V_i or E_i, respectively for $Y = V, E$, are pairwise disjoint and $V' = \cup V_i$ and $E' = \cup E_i$. (The analog of all the rays forming a subgraph is provable in RCA_0 for the finite case but not the infinite one so here so we make it explicit. It does follow even for the case of infinite sequences in ACA_0.)

Definition 2.3. An *X-path P* in an X-graph H is defined similarly to single rays except that the domain of f is a proper initial segment of $\mathbb{N}$ instead of $\mathbb{N}$ itself. Thus they are finite sequences of distinct vertices with edges between successive vertices in the sequence. If $P = \langle x_0, \dots, x_n \rangle$ is a path, we say it is a *path of length n* between x_0 and x_n.

Definition 2.4. A *tree* is a graph T with a designated element r called its *root* such that for each vertex $v \neq r$ there is a unique path from r to v. A *branch* in a tree T is a ray in T starting at its root. the set of all branches in T is denoted by $[T]$. Note, however, we are restricting ourselves to what would (in set theory) be called countable trees with all nodes of finite rank. Thus, we typically think of trees as *subtrees of* $\mathbb{N}^{<\mathbb{N}}$, i.e. the downward closed (under extension) sets of finite strings of numbers (as vertices) with an edge between σ and τ if and only if they differ by one being an extension of the other by one element, e.g. $\sigma\hat{\ }k = \tau$ and with root $\emptyset$. We call the *stem* of such a tree the longest σ which is comparable (under extension) with every element of the tree.

The starting point of the work in BGS and this paper is a theorem of Halin's [1965] that we call the infinite ray theorem as expressed in Diestel [2017, Theorem 8.2.5 (i)].

Definition 2.5 (Halin's Theorem)**.** *IRT, the infinite ray theorem,* is the principle that every graph H which contains arbitrarily many disjoint rays contains infinitely many.

The versions of Halin's theorem which we consider in this paper allow for H to be an undirected or a directed graph and for the disjointness requirement to be vertex or edge. They are labeled along the lines of BGS as IRT_{XY} to indicate whether the graphs are undirected or directed ($X = U$ or D) and whether the disjointness refers to the vertices or edges ($Y = V$ or E) in the obvious way. We often state a theorem for all XY and then in the proof use "graph", "edge" and "disjoint" unmodified with the intention that the proof can be read for any of the four cases.

Remark 2.6. We point out that unlike BGS (except in Remark 5.11) we do not consider the analogs of IRT for double rays (isomorphic to $\mathbb{Z}$ rather than $\mathbb{N}$). Halin [1970] proved the basic case here (UV) and BCP did the UE version. The other two (DE and DV) remain open. Some relevant results about the strength of special cases are in BGS. However, the local finiteness property that originally motivated this paper (Definition 3.5) fails for double rays in the DE and UE cases as can be seen by considering the star graph consisting of countably many copies of $\mathbb{Z}$ with one vertex common to all the copies. Thus these versions seemed less relevant to our concerns in this paper.

We now move on to the recursion theoretic notions needed to define THA. Here we are working with the usual set $\mathbb{N}$ of natural numbers and understand notions such as well-orderings in the usual way – there simply are no descending chains. (As opposed to thinking of some model of arithmetic $\mathcal{N}$ with perhaps a nonstandard first order part or even a standard model, i.e. $\mathcal{N} = \mathbb{N}$ but one in which the notion of well-foundednes for linear orderings as no descending chain in $S(\mathcal{N})$ is the not the same as no descending chain (possibly outside of $S(\mathcal{N})$.) A standard reference for hyperarithmetic theory is Sacks [1990]. We give a brief list of the notions we need.

Definition 2.7. We represent *ordinals* α as well-ordered relations on N. Typically such *ordinal notations* are endowed with various additional structure such as identifying 0, successor and limit ordinals and specifying cofinal ω-sequences for the limit ordinals. An ordinal is recursive (in a set X) if it has a recursive (in X) representation. For a set X and ordinal (notation) α recursive in X, we define the transfinite iterations X^{α} of the Turing jump of X by induction: $X^{(0)} = X$; $X^{(\alpha+1)} = (X^{\alpha})'$ and for a limit ordinal λ, $X^{(\lambda)} = \oplus\{X^{(\alpha)} | \alpha < \lambda\}$ (or as the sum over the $X^{(\alpha)}$ in the specified cofinal sequence).

Definition 2.8. $HYP(X)$, the collection of all sets *hyperarithmetic in* X consists of those sets recursive in some $X^{(\alpha)}$ for α an ordinal recursive in X. These are also the sets Δ^1_1 in X.

Definition 2.9. A sentence (theory) T is a *theorem (theory) of hyperarithmetic analysis (THA)* if

1. For every $X \subseteq \mathbb{N}$, $(\mathbb{N}, HYP(X)) \vDash T$ and
2. For every $S \subseteq 2^{\mathbb{N}}$, if $(\mathbb{N}, S) \vDash T$ and $X \in S$ then $HYP(X) \subseteq S$.

Definition 2.10. A theorem or theory T is an *almost theorem (theory) of hyperarithmetic analysis (ATHA)*, if $T \nvdash \mathrm{ACA}_0$ but $T + ACA_0$ is a THA.

We now turn to defining and analyzing some mathematical and logical theorems that turn out to be ATHA.

3 ATHA Principles

We wish to consider the argument in BCP [top of p. 2] that IRT_{UE} follows from IRT_{UV}. We fill in their sketch to bring out the use of instances of Σ^1_1-AC. They analyze only undirected graphs but the same arguments apply to directed ones so we present the two cases together.

Their proof can be presented as two Lemmas:

Lemma 3.1. *IRT_{XE} restricted to locally finite graphs implies IRT_{XE}.*

Lemma 3.2. *IRT_{XV} implies IRT_{XE} restricted to locally finite graphs. In fact, IRT_{XV} restricted to locally finite graphs implies IRT_{XE} restricted to locally finite graphs.*

The natural proof of Lemma 3.2 takes place in ACA_0.

Proof of Lemma 3.2 (ACA_0). Let $G = \langle V, E\rangle$ be a locally finite graph with arbitrarily many E-disjoint rays. Consider the line graph $L(G)$ of G, i.e. the graph whose vertices are the edges of G and whose edges are the $((x, y), (y, z))$ for $(x, y) \neq (y, z) \in E$. As G is locally finite so is $L(G)$. (The only way a given (u, v) can have an (x, y) as a neighbor is if they have a vertex in common. So if (u, v) had infinitely many neighbors, one of u or v would also have such in G.)

A set of k many E-disjoint rays $R_i = \langle V_i, E_i\rangle$ with isomorphisms f_i in G produces k many V-disjoint rays $\bar{R}_i =$ with vertices $(x_{i,n}, x_{i,n+1})$ in $L(G)$ where we write $x_{i,n}$ for $f_i(n)$. Applying the hypothesis of the Lemma gives us infinitely many V-disjoint rays T_i with vertices $(x_{i,j}, x_{i,j+1})$ in $L(G)$.

Now we use the local finiteness of G to construct the required infinitely many E-disjoint rays Q_i in G. Fix i and $T = T_i$ and $x_j = x_{i_j}$. By the local finiteness of G, for every $v \in V$ there are only finitely many n such that $v \in \{x_n, x_{n+1}\}$. (Otherwise, say v is x_n for infinitely many n and then (v, x_{n+1}) is a vertex in the $L(G)$ ray for all of these n. That means, however, that these edges are all distinct and so v has infinitely many neighbors in G for a contradiction.)

We build $Q = Q_i$ by recursion starting with $a_0 = x_0$ and let n_0 be the largest n such that $a_0 = x_n$. Let $a_1 = x_{n_0+1}$ so $(a_0, a_1) \in E$. Inductively, take n_{k+1} ($> n_k$) the largest n such that $a_k = x_n$ and set $a_{k+1} = x_{n_k+1}$. (We can find this n by ACA_0.) This recursion produces sequences n_k, a_{k+1} with $Q = \langle a_k\rangle$ a ray in G and $(a_n, a_{n+1}) \in T$ for every n. Let this Q be Q_i.

Claim: The Q_i are E-disjoint in G as required. If not, we have $(a_{i,n}, a_{i,n+1}) = (a_{j,m}, a_{j,m+1})$ for some $i \neq j$ and n and m. However, $(a_{i,n}, a_{i,n+1}) \in T_i$ and $(a_{j,m}, a_{j,m+1}) \in T_j$ contradicting the V-disjointness of the T_i in $L(G)$. □

On the other hand, while the proof of Lemma 3.1 seems to also take place in ACA_0, it, like that of IRT itself (BGS [ta, Theorem 4.1]), relies on a use of Σ^1_1-AC to get started.

Proof of Lemma 3.1 (Σ^1_1-AC_0). We are given a graph $G = \langle V, E\rangle$ with arbitrarily many E-disjoint rays. By Σ^1_1-AC_0 choose a sequence $S_k = \langle R_{k,1}, \ldots, R_{k,k}\rangle$ which consists, for each k, of k many E-disjoint rays in G. Now we construct the desired subgraph G' of G. It has the same set of vertices $V = \{v_i | i \in N\}$ as G. We specify its edges by providing a recursive construction of sets E_i of edges putting in a set of edges at each step. We guarantee that each E_i is a union of finitely many sets of E-disjoint rays in G and that after stage s no edge with a vertex v_i for $i < s$ is ever put into E after stage s.

Begin at stage 0 by putting all the edges in $R_{1,1}$ into E_1. Proceeding recursively at stage k we have E_k and consider $S_k = \langle R_{k,1}, \dots, R_{k,k} \rangle$. Each v_j for $j < k$ appears in $R_{k,i}$ at most once for each $i < k$ as $R_{k,i}$ is a ray. As we have the whole sequence of the S_k we can find (using only RCA_0) the i, j such that $v_j \in R_{k,i}$ and the location n in the sequence where it occurs, say as x_{k,i,n_j}. We now put into E_{k+1} all the edges appearing in any $R_{k,i}$ after all the x_{k,i,n_j} for $j < k$ which are defined. Let $E' = \cup E_k$. As, for every k, we have put in a tail of each $R_{k,i}$ for $i < k$ into E_k we have guaranteed that $G' = \langle V, E' \rangle$ contains arbitrarily many E-disjoint rays.

Thus we only need to show that G' is locally finite. Consider any vertex v_k. No edge containing v_k as a vertex is put in after stage k. On the other hand, E_k is the union of finitely many finite sets of E-disjoint rays (all of which have been computed uniformly). Each set of E-disjoint rays in this union has v_k appearing at most once in each of its rays. Thus at most two edges containing v_k appear in each of the finitely many rays in this set. Thus there are only finitely many edges containing v_k in each of the finite sets of E-disjoint making up E_k. All in all, this makes only finitely many edges containing v_k get put into G'. (In fact, we can compute the number of such edges in $\mathcal{N}$.) $\square$

Now we study the crucial Lemma 3.1 that reduces the problem to locally finite graphs. We first prove that the IRT_{XY} theorems for locally finite graphs are strictly weaker than the full theorems. Indeed, they are theorems of $\mathrm{ACA}_{0.}$

Proposition 3.3 (ACA_0)**.** *If G is a locally finite X-graph with arbitrarily many Y-disjoint rays then there is a sequence $\langle H_n \rangle$ of subgraphs of G with each H_n consisting of n many disjoint rays. (This statement is SCR_{XY} of Definition 5.5 for locally finite graphs.)*

Proof. Let $G = \langle V, E \rangle$ and $V = \{v_i | i \in N\}$. For each n and n-tuple $\left\langle v_{i_j} | j < n \right\rangle$ of distinct vertices of G consider the tree $T_{i,n}$ whose nodes are n-tuples of disjoint paths in G all of the length the height of the node in $T_{i,n}$. The root of $T_{i,n}$ is $\left\langle v_{i_j} | j < n \right\rangle$. If $\mu = \langle \sigma_j | j < n \rangle \in T_{i,n}$ then its immediate successors are all $\nu = \langle \tau_j | j < n \rangle$ such that for each j, τ_j is an extension of σ_j by one of the finitely many vertices v such that there is an edge from the last vertex in σ_j to v and the τ_j are disjoint rays in G. As G is locally finite, the $T_{i,n}$ are finitely branching trees (recursive in G'). Thus by ACA_0 (recursively in a few jumps of G) we can get the set of $\langle i, n \rangle$ such that $T_{i,n}$ has a branch and, indeed, a sequence $S_{i,n}$ each a branch in $T_{i,n}$. Every such branch provides a subgraph $G_{i,n}$ of G which consists of n disjoint rays. We can now just take the desired H_n to be $G_{i,n}$ for the least i such that $T_{i,n}$ has a branch. $\square$

Indeed we now have some equivalences.

Proposition 3.4. *ACA_0 is equivalent to each IRT_{XY} for locally finite graphs.*

Proof. To prove the implication from left to right, combine Proposition 3.3 and the fact that, by BGS [ta, Theorem 8.2], its conclusion for any graph implies (in ACA_0) that the graph has infinitely many disjoint rays. The graphs used in BGS [ta, Theorems 4.1, 4.2 and 5.9] to deduce ACA_0 from IRT_{XY} are disjoint unions of trees and so locally finite. □

Now we formulate a principle expressing the idea that one can reduce the problem of finding solutions to IRT to considering only the class of locally finite graphs. Of course, as the IRT_{XY} are THA and their restrictions to locally finite graphs are provable in ACA_0, this reduction must be strong.

Definition 3.5. *LF_{XY}* is the principle that every X-graph which contains arbitrarily many Y-disjoint rays contains a locally finite subgraph which also contains arbitrarily many Y-disjoint rays.

We now work towards analyzing the complexity of the LF_{XY}.

Proposition 3.6. *In ACA_0, LF_{XY} $\rightarrow$$IRT_{XY}$.*

Proof. Suppose we are given a graph H with arbitrarily many disjoint rays. Let H' be a locally finite subgraph with arbitrarily many rays. Again, Proposition 3.3 and Theorem 8.2 of BGS [ta] give us IRT_{XY}.

In fact, over ACA_0 we have equivalences □

Proposition 3.7. *IRT_{XY} $\rightarrow$$LF_{XY}$ and so they are equivalent over ACA_0.*

Proof. We are given a graph G with arbitrarily many disjoint rays and want to build a locally finite subgraph with the same property. We begin with the subgraph of the given graph consisting of infinitely many disjoint rays asserted to exist by IRT_{XY}. As IRT_{XY} $\rightarrow$$\mathrm{ACA}_0$ (BGS [ta, Theorem 5.1]), we can use ACA_0 to thin out this subgraph so that our new nth ray is simply the nth given ray above the last time any vertex less than n appears in it. (Any vertex appears at most once in any ray.) Thus every vertex less than n appears in at most n many of these new rays. In each one it has edges to at most two other vertices. Thus it is a locally finite subgraph of the original graph and also contains infinitely many disjoint rays. The equivalence now follows from Proposition 3.6. □

Proposition 3.8. *LF_{XY}+ACA_0 is a THA.*

Proof. Each IRT_{XY} is a THA by BGS [ta, Theorem 5.1] and so we have the result by Proposition 3.7 □

We will see in §5 that none of the LF_{XY} imply ACA_0 and so all are ATHA.

4 A Class of Forcings for Satisfying Π^1_2 Principles

We define a class $\mathcal{C}$ of notions of forcing $\mathcal{P}$ such that forcing with any one of them over a model $\mathcal{N} = (N, S(\mathcal{N}))$ of RCA_0 has several preservation type properties. (Our forcing language is flexible as to what else it might include for convenience but it does always include constants (as usual denoted by) n (or A) for each element of N (or $S(\mathcal{N})$). Note that these include class forcings in the sense that while each condition is (coded as) a set in $\mathcal{N}$, the collection of conditions need not be (coded as) a set in $\mathcal{N}$ nor even be definable over $\mathcal{N}$.

Definition 4.1. A notion of forcing $\mathcal{P} = \langle P, \leq \rangle$ is a *tree forcing (t-forcing)* if the following hold:

1. Conditions in $\mathcal{P}$ are of the form $\langle \tau, T \rangle$ where $T \in S(N)$ is a subtree of $N^{<N}$ (i.e. a subset of $N^{<N}$ in $\mathcal{N}$ closed under initial segments with respect to $\subseteq$) and τ is comparable with every $\sigma \in T$.
2. If $\langle \tau', T' \rangle \leq \langle \tau, T \rangle$ then $\tau' \supseteq \tau$ and $T' \subseteq T$,
3. For every $n \in N$ the class $\{\langle \tau, T \rangle \,||\tau| \geq n\}$ is dense in $\mathcal{P}$, i.e. $(\forall \langle \tau, T \rangle \in \mathcal{P})(\exists \langle \tau', T' \rangle)(\langle \tau', T' \rangle \leq \langle \tau, T \rangle \ \& \ |\tau'| \geq n)$.

Definition 4.2. A tree notion of forcing $\mathcal{P}$ is an *effective tree forcing (et-forcing)* if for every $\langle \tau, T \rangle \in \mathcal{P}$ the class $Ext(\langle \tau, T \rangle) = \{\tau' | (\exists T')(\langle \tau', T' \rangle \leq \langle \tau, T \rangle)\}$ is $\mathbf{\Sigma}^0_1$, i.e. there is an $A \in S(N)$ such that $Ext(\langle \tau, T \rangle)$ is $\Sigma^0_1(A)$ (over N).

Notation 4.3. *If $\mathcal{G}$ is a filter on a t-forcing $\mathcal{P}$ which is generic for a class $\mathcal{D}$ of dense sets containing at least the $D_n = \{\langle \tau, T \rangle \,||\tau| \geq n\}$, then the generic function $G : N \to N$ associated with $\mathcal{G}$ is $\cup\{\tau | \exists T(\langle \tau, T \rangle \in \mathcal{G}\}$. We then say that G is $\mathcal{D}$-generic on $\mathcal{P}$. (Note that this G is always a function from N to N by the definitions of $\mathcal{P}$ being a t-forcing and of $\mathcal{G}$ being a $\mathcal{D}$-generic filter on $\mathcal{P}$.) We also say that a $G : N \to N$ is on $\langle \tau, T \rangle$ if $G \in [T]$, i.e. $\forall n(G \restriction n \in T)$. So if G is $\{D_n\}$-generic it is on every $\langle \tau, T \rangle \in \mathcal{G}$.*

We denote by $\mathcal{N}[G]$ the structure for second-order arithmetic with first order part the same as $\mathcal{N}$ (i.e. N) and second order part the closure of $S(\mathcal{N}) \cup \{G\}$ under Δ^0_1-CA.

For $\sigma \in T$, $T^\sigma = \{\rho \in T | \ \sigma \supseteq \rho \vee \rho \supseteq \sigma\}$ is what we call the tree T above σ*.*

Remark 4.4. There are a variety of ways to define forcing for models of second order arithmetic. Until the very last section of this paper we only need to consider forcing sentences of the form $\exists k \Phi(k, \bar{n}, A, G)$ where Φ is formula with, if, one wants, bounded but certainly no unbounded quantifiers, $\bar{n} \in N$, $A \in S(\mathcal{N})$ (and $\exists k$ as its only unbounded quantifier) and perhaps some additional fixed recursive predicates. As usual we say $\langle \tau, T \rangle \Vdash \exists k \Phi(k, \bar{n}, A, G)$ if and only if there is a $k < |\tau|$ such that τ contains all the information about G to guarantee the truth of $\Phi(k, \bar{n}, A, G)$ (even from the viewpoint of $\mathcal{N}$). Guarantee here means that $\mathcal{N}[G] \vDash \Phi(k, \bar{n}, A, G)$ for every $G \supseteq \tau$. Thus we also write this as $\tau \Vdash \exists k \Phi(k, \bar{n}, A, G)$. Note that if $\mathcal{P}$ is an et-forcing then $\{\langle \tau', \bar{n} \rangle \,| \tau' \in EXT(\langle \tau, T \rangle)\ \&\ \tau' \Vdash \exists k \Phi(k, \bar{n}, A, G)\}$ is $\mathbf{\Sigma}_1^0$ over $\mathcal{N}$ and all $\mathbf{\Sigma}_1^0$ formulas over $\mathcal{N}[G]$ (with free variables) are equivalent to formulas of this form (with added free variables).

Personally, we like the recursion theoretic view that includes in the language a recursive predicates (with Δ_1^0 definitions independent of $\mathcal{N}$) for the pairing functions and a coding of finite strings σ with their length, the relation $\sigma \subseteq X$ and the universal Turing functional $\Phi(e, \sigma, x, y)$. Intuitively $\Phi(e, \sigma, x, y)$ says that machine e with input x and using as an oracle only the finite sequence σ converges with output y in at most $|\sigma|$ many steps. This, or any other similar coding procedure, provides a universal Σ_1^0 predicate, i.e. every Σ_1^0 predicate with set variable $X_0, \ldots X_k$ and number variables $n_0, \ldots n_l$ is equivalent to $\exists \sigma_0, \ldots, \sigma_k (\sigma_0 \subseteq X_0 \wedge \ldots \wedge \sigma_k \subseteq X_k \ \wedge\ \Phi(e, \langle \sigma_o, \ldots, \sigma_k \rangle, \langle n_0, \ldots n_l \rangle, 1)$ where $e \in \mathbb{N}$ can be calculated recursively from the the given Σ_1^0 formula. As is common we often write this this as $\Phi_e^{X_0, \ldots, X_k}(n_0, \ldots n_l) = 1$. In this notation, for example, every set in $\mathcal{N}[G]$ has its characteristic function of the form $\Phi_e^{A,G}$ for an $A \in S(\mathcal{N})$. Similarly, τ forcing a Σ_1^0 sentence of the forcing language (with, e.g. set parameter $A \in S(\mathcal{N})$ and number parameter $n \in N$) is equivalent to $\mathcal{N} \vDash \Phi(e, A \upharpoonright |\tau|, \tau, n, 1)$. These are essentially the only types of sentences we deal with until §7.

We note that many common notions of forcing used to produce reals are et-forcings or easily seen to be equivalent to such. These include Cohen, Laver, Mathias, Sacks and Silver forcing and many variations. In this paper we use some of these as well as more specialized et-forcings in §5 to prove the conservation results that show, in particular, that the principles considered in §3 do not imply ACA_0.

Theorem 4.5. *If $\mathcal{P}$ is an et-forcing over a countable model $\mathcal{N}$ of RCA_0 there is a countable collection $\mathcal{D}$ of dense sets (including the ones specified in Definition 4.1) such that*

1. *If G is $\mathcal{P}$-generic for $\mathcal{D}$, then $\mathcal{N}[G] \vDash \mathrm{RCA}_0$.*
2. *If R is a subtree of $N^{<N}$ (not necessarily in $S(\mathcal{N})$) with no branch in $S(\mathcal{N})$, then there is a countable collection $\mathcal{D}' \supseteq \mathcal{D}$ of dense sets such that if G is $\mathcal{P}$-generic for $\mathcal{D}'$, then there is no branch of R in $\mathcal{N}[G]$.*
3. *Thus for any countable collection R_i of trees as in 2 (such as all those in $S(\mathcal{N})$) there is a single $\mathcal{D}'$ as in 2 which works for every R_i. In particular, for a set $\{C_i | i \in \omega\}$ with $C_i \subseteq N$ and $C_i \notin S(\mathcal{N})$ for every $i \in \omega$, there is a $\mathcal{D}' \supseteq \mathcal{D}$ such that, for any $\mathcal{D}'$-generic G, no $C_i \in \mathcal{N}[G]$.*

Proof. The third clause follows immediately from the second by the countability of $S(\mathcal{N})$ and then by taking $R_i = \{\rho \in N^{<N} | \rho \subset C_i\}$. We prove each of the first two assertions by specifying the appropriate collections of dense sets.

1. By a classic result of H. Friedman [1976], it suffices to show that for any Σ_1^0 formula $\exists k\Phi(k,m,A,G)$ with $A \in S(\mathcal{N})$ such that $\mathcal{N}[G] \vDash \exists k\Phi(k,m,A,G)$ there is an $\mathcal{N}$-least m' such that $\mathcal{N}[G] \vDash \exists k\Phi(k,m',A,G)$. By the definition of G there is a condition $\langle\tau,T\rangle \in \mathcal{G}$ such that $\tau \Vdash \exists k\Phi(k,m,A,G)$. We show that the conditions that guarantee that there is an $\mathcal{N}$-minimal such m are dense below $\langle\tau,T\rangle$ and so we can extend $\mathcal{D}$ to guarantee that $\mathcal{N}[G] \vDash$RCA$_0$ as there are only countably many Σ_1^0 formulas $\exists k\Phi(k,m,A,G)$ and conditions $\langle\tau,T\rangle$. As $\mathcal{P}$ is an et-forcing the set $\{m' \leq m|(\exists\tau' \in Ext(\langle\tau,T\rangle))(\tau' \Vdash \exists k\Phi(k,m',A,G))\}$ is $\mathbf{\Sigma}_1^0$ in $\mathcal{N}$ and so as $\mathcal{N} \vDash$RCA$_0$, there is an $\mathcal{N}$-least such m' with an associated $\langle\tau',T'\rangle$. If $\mathcal{D}$ includes the corresponding dense set consisting of such $\langle\tau',T'\rangle$ for each $\langle\tau,T\rangle$, it is clear that m' is the least m such that $\mathcal{N}[G'] \vDash \exists k\Phi(k,m,A,G')$ for any $\mathcal{D}$-generic G' as desired. (Otherwise, there would be a $\langle\tau'',T''\rangle \leq \langle\tau',T'\rangle$ a $k' \in N$ and an $m'' <_{\mathcal{N}} m'$ such that $\tau'' \Vdash \Phi(k',m,A,G)$.)

2. Consider any R as in the claim. We again want to specify the additional dense sets needed. Consider an arbitrary function in some $\mathcal{N}[G]$. By the definition of $\mathcal{N}[G]$ (and basic facts of about recursive functions true in RCA$_0$), it is of the form $\Phi_e^{A,G}$ for some $e \in N$ and $A \in S(\mathcal{N})$.

   If for every $\langle\tau,T\rangle \in \mathcal{G}$ there is a $\rho \notin R$ ($\rho \in N^{<N}$) and a $\tau' \in EXT(\langle\tau,T\rangle)$ such that $\tau' \Vdash \Phi_e^{A,G}(l) = \rho(l)$ for every $l < |\rho|$ then the set of conditions guaranteeing that $\Phi_e^{A,G}$ is not on R is dense. Thus we may assume that we have a $\langle\tau,T\rangle \in \mathcal{G}$ such that for every $\tau' \in Ext(\langle\tau,T\rangle)$ and every ρ, if $\tau' \Vdash \Phi_e^{A,G}(l) = \rho(l)$ for every $l < |\rho|$, then $\rho \in R$.

   Next, if there is a $\tau' \in Ext(\langle\tau,T\rangle)$ and an $l \in N$ such that there is no $m \in N$ and $\tau'' \in Ext(\langle\tau,T\rangle)$ with $\tau'' \supseteq \tau'$ such that $\tau'' \Vdash \Phi_e^{A,G}(l) = m$ then the associated $\langle\tau',T'\rangle \leq \langle\tau,T\rangle$ guarantees that $\Phi_e^{A,G}(l)\uparrow$ and so $\Phi_e^{A,G}$ is again not a branch on R. Thus we may assume that for every $l \in N$ and $\tau' \in Ext(\langle\tau,T\rangle)$ there is an $m \in N$ and $\tau'' \in Ext(\langle\tau,T\rangle)$ with $\tau'' \supseteq \tau'$ such that $\tau'' \Vdash \Phi_e^{A,G}(l) = m$.

   We now prove that there is a branch f on R which is in $S(\mathcal{N})$ for a contraction: By our last assumption on $\langle\tau,T\rangle$, we can define an $f : N \to N$ by recursion in $\mathcal{N}$ starting with our $\langle\tau,T\rangle$ and $\tau_{-1} = \tau$: We build sequences of $\tau_l \in Ext(\langle\tau,T\rangle)$ and m_l such that $\tau_l \Vdash \Phi_e^{A,G}(l) = m_l$ and $\tau_l \subseteq \tau_{l+1}$. This is a recursive procedure in $\mathcal{N}$ as $\mathcal{P}$ is an et-forcing and so we can search for the next witnesses (and find them) effectively in $\mathcal{N}$. By our first assumption on $\langle\tau,T\rangle$, the sequence $\langle m_l|l < n\rangle$ is in R for every $n \in N$ and so f is the desired branch on R in $\mathcal{N}$.

□

Another property of extensions of theories expressing weakness is having a minimal pair of extensions. We note that it follows from the cone avoiding property in an even stronger form.

Corollary 4.6. *If $\mathcal{N}_0$ and $\mathcal{N}_1$ are countable models of RCA_0 with the same first order part N, $\mathcal{P}_0$ and $\mathcal{P}_1$ are et-forcings with classes $\mathcal{D}_0$ and $\mathcal{D}_1$ of dense sets as above (over $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively), then there are G_i for $i = 0, 1$ which are $\mathcal{D}_i$-generic for $\mathcal{P}_i$ such that $\mathcal{N}_0[G_0], \mathcal{N}_1[G_1] \vDash RCA_0$ and $\mathrm{N}_0[G_0] \cap \mathrm{N}_1[G_1] = N$. In fact, for any countable models $\mathcal{N} \subseteq \mathcal{N}_0$ of RCA_0 with the same first order part and $\mathcal{P}$ an et-forcing over $\mathcal{N}$ there is a countable collection $\mathcal{D}$ of dense sets in $\mathcal{P}$ such that for any $\mathcal{D}$-generic G, $\mathcal{N}_1 = \mathcal{N}[G] \vDash RCA_0$ and $\mathcal{N}_0 \cap \mathcal{N}_1 = \mathcal{N}$. (Note that $\mathcal{N}_0 \cap \mathcal{N}_1$ denotes the second order structure whose first order part is N and second order part is $S(\mathcal{N}_0) \cap S(\mathcal{N}_1)$.)*

Proof. Let C_i list the subsets of N which are in $\mathcal{N}_0$ but not in $\mathcal{N}$. Apply Theorem 4.5.3 to get the desired collection of dense sets. □

From now on, in all the cases where we establish or use the cone avoiding property similar definitions and conclusions can be made for minimal pairs of extensions. In particular, this applies to Definition 4.7, Theorem 4.13 and Theorem 6.2.

It is now standard to prove that any Π^1_2 principle $Q \equiv \forall X(\Phi(X) \rightarrow \exists Y \Psi(X, Y)$ with Φ and Ψ arithmetic for which solutions can be provided by an et-forcing $\mathcal{P}$ are Π^1_1 and r-Π^1_2 conservative over RCA_0. We formulate some relevant notions more generally for later use.

Definition 4.7. Let $Q \equiv \forall X(\Phi(X) \rightarrow \exists Y \Psi(X, Y)$ be a *principle. Instances* of Q are specified by an X such that $\Phi(X)$ holds. A set Y is a *solution for the instance of Q specified by X* if $\Psi(X, Y)$ holds. If Φ and Ψ are arithmetic we say Q is a Π^1_2 *principle.*

1. We say that *solutions to Q can be provided by forcing* if, for every countable $\mathcal{N} \vDash \mathrm{RCA}_0$ and instance of Q specified by an $X \in N$, there is a notion of forcing $\mathcal{P}$ over $\mathcal{N}$ and a countable collection $\mathcal{D}$ of dense subclasses of $\mathcal{P}$ such that for every $\mathcal{D}$-generic G for $\mathcal{P}$, $\mathcal{N}[G] \vDash \mathrm{RCA}_0$ & $\exists Y \Psi(X, Y)$.

2. We say that *solutions can be added without adding branches to trees* if the forcing $\mathcal{P}$ and dense sets $\mathcal{D}$ can always be chosen so that, in addition, any tree T in $\mathcal{N}$ that has no branch in $\mathcal{N}$ has no branch in $\mathcal{N}[G]$.

3. We say that *solutions can be added with cone avoiding* if the forcing $\mathcal{P}$ can be chosen such that for any $\{C_i | i \in \omega\}$ with $C_i \subseteq S(\mathcal{N})$ and $C_i \notin S(N)$ for each $i \in \omega$, there is a $\mathcal{D}$ such that for any $\mathcal{D}$-generic G on $\mathcal{P}$, $C_i \notin \mathcal{N}[G]$ for every $i \in \omega$.

One can now prove by fairly standard methods that providing solutions by the various types of forcing insures specific conservation results and other evidences of the weakness of the given principle. We extend the usual arguments for conservation results to cover larger classes of formulas that we now describe.

Definition 4.8. If Γ is a class of sentences and T a theory of second order arithmetic, we say T is Γ conservative (over RCA_0), if for every $\Lambda \in \Gamma$ such that $T \vdash \Lambda$, $\mathrm{RCA}_0 \vdash \Lambda$.

Definition 4.9. A theory T is Π^1_1 *conservative* if it is conservative for the class of sentences Λ of the form $\forall X \Phi(X)$ with Φ arithmetic. We extend this to G-Π^1_1, *generalized* Π^1_1, *conservative* by including all sentences Λ in the G-Π^1_1 *class of formulas* defined by closing the quantifier free formulas under conjunction ($\wedge$), disjunction ($\vee$), first order quantification ($\forall x$ and $\exists x$ for number variables) and universal second order quantification ($\forall X$ for set variables).

Definition 4.10. Hirschfeldt and Shore [2009, Corollary 3.15] define r-Π^1_2 *conservativity* by the class of sentences Λ of the form $\forall X(\Phi(X) \rightarrow \exists Y \Theta(X, Y)$ where Φ is arithmetic and Θ is Σ^0_3. We extend this to G-r-Π^1_2 *conservativity* by including all sentences in the G-r-Π^1_2 *class of formulas* defined by closing all formulas which are either quantifier free or of the form $\exists Y \Theta(Y)$ where Θ is Σ^0_3 under the same operations as in the definition of $G - \Pi^1_1$ ($\wedge$, $\wedge$, $\forall x$, $\exists x$ and $\forall X$).

We introduce other classes of conservativity results related to sentences of the form $\forall X \exists! Y \Phi(X, Y)$ for Φ arithmetic in §7. They require additional uniformity type conditions on our et-forcings.

All of our proofs of conservation results have the same general format. We have a class Γ of formulas and a theory T of second order arithmetic. We want to prove T is conservative (over RCA_0) for sentences in Γ. For the sake of a contradiction, we assume that there is a sentence $\Lambda \in \Gamma$ such that $T \vdash \Lambda$ and a countable model $\mathcal{N}$ of RCA_0 such that $\mathcal{N} \vDash \neg\Lambda$. We then construct, by iterated forcing, a model $\mathcal{N}_\infty$ of T. In each case, we have a notion of forcing that adds solutions for all the sentences of T, e.g. of Π^1_2 principles Q. We construct a limit ordinal length iteration of forcings producing witnesses for the solutions for these principle. For Π^1_2 principles these have length ω but other lengths will be used and we denote the length ambiguously by ∞. This gives us a sequence of models $\mathcal{N}_{i+i} = \mathcal{N}_i[G_i]$ and $\mathcal{N}_\lambda = (N, \cup\{S(\mathcal{N}_\alpha)|\alpha < \lambda\}$ of RCA_0 (all with the same first order part N) such that $\mathcal{N}_\infty = (N, \cup\{S(\mathcal{N}_i)|i < \infty\})$ is a model of $\mathrm{RCA}_0 + T$ by arranging that every instance of the principles Q of T specified by an X in some $\mathcal{N}_i$ is given a solution Y in some later $\mathcal{N}_j$.

Next, we argue that $\mathcal{N}_\infty \vDash \neg\Lambda$ as well for a contradiction. We prove that the truth of $\neg\Lambda$ is preserved for all sentences $\Lambda \in \Gamma$ with constants from N and $S(N)$ by an induction on the complexity of Λ. The argument can be seen as playing a game between the two models to eliminate number quantifiers or universal set quantifiers as well as the positive connectives. This will then complete each proof of conservativity that we provide.

Theorem 4.11. *If solutions to a Π^1_2 principle Q can be provided by forcing, then $RCA_0 +$ Q is G-Π^1_1 conservative over RCA_0.*

Proof. We begin the plan outlined above with quantifier free G-Π^1_1 sentences Λ with constants from N and $S(\mathcal{N})$. Here the truth of both Λ and $\neg\Lambda$ are preserved from $\mathcal{N}$ to $\mathcal{N}_\infty$ as $N = N_\infty$ and $S(\mathcal{N}) \subseteq S(\mathcal{N}_\infty)$. Suppose next that $\Lambda = \Delta_0 \wedge \Delta_1$. As $\mathcal{N} \vDash \neg\Lambda$, $\mathcal{N} \vDash \neg\Delta_i$ for at least one $i \in \{0, 1\}$. By induction then $\mathcal{N}_\infty \vDash \neg\Delta_i$ as well as required. If

$\Lambda = \Delta_0 \vee \Delta_1$ and $\mathcal{N} \vDash \neg\Lambda$ then $\mathcal{N} \vDash \neg\Delta_0 \wedge \neg\Delta_1$ and so by induction $\mathcal{N}_\infty \vDash \neg\Delta_0 \wedge \neg\Delta_1$ and $\mathcal{N}_\infty \vDash \neg(\Delta_0 \vee \Delta_1)$ as required. Next, suppose $\Lambda = \forall x\Delta(x)$ and $\mathcal{N} \vDash \neg\Lambda$. Choose an $n \in N$ such that $\mathcal{N} \vDash \neg\Delta(n)$. By induction, $\mathcal{N}_\infty \vDash \neg\Delta(n)$ and so $\mathcal{N}_\infty \vDash \neg\forall x\Delta(x)$. Suppose $\Lambda = \exists x\Delta(x)$ and $\mathcal{N} \vDash \neg\Lambda$. If $\mathcal{N}_\infty \vDash \exists x\Delta(x)$ choose a witness $n \in N_\infty = N$ so that $\mathcal{N}_\infty \vDash \Delta(n)$. As $\Delta(n)$ is also a sentence in Γ, we have that $\mathcal{N} \vDash \Delta(n)$ by induction for the desired contradiction. Finally, if $\Lambda = \forall X\Delta(X)$ and $\mathcal{N} \vDash \neg\Lambda$, choose a $W \in S(\mathcal{N})$ such that $\mathcal{N} \vDash \neg\Delta(W)$. As $\Delta(W)$ is a sentence in Γ, we again have $\mathcal{N}_\infty \vDash \neg\Delta(W)$ as required. □

Similarly, we can prove a G-r-Π^1_2 conservation result for such Q when solutions are provided by et-forcings using Theorem 4.5.3.

Theorem 4.12. *If solutions to a Π^1_2 principle Q can be provided without adding branches through trees, then $RCA_0 + Q$ is G-r-Π^1_2 conservative over RCA_0.*

Proof. The argument for quantifier free sentences is as in Theorem 4.11 as are the inductive cases for $\wedge$, $\vee$, $\exists x$, $\forall x$ and $\forall X$. Here we also have to begin with sentences Λ of the form $\exists Y\Theta(\bar{x}, \bar{y}, Y)$ with Θ a Σ^0_3 formula with constants from N and $S(\mathcal{N})$ and suppose that $\mathcal{N} \vDash \neg\Lambda$. As in Hirschfeldt, Shore and Slaman [2009, last paragraph of p. 5818], the point here is that for any model $\mathcal{N}$ of RCA_0 the failure of a sentence $\exists Y\Theta(Y)$ with Θ being Σ^0_3 (with set constants $\bar{W}$) is equivalent to their being, for each $k \in N$, a specifically defined tree T_k (recursive in $\bar{W}$) which has no branch in the model. Thus none of these trees has a branch in $\mathcal{N}$ and so by our assumptions on the forcings none in $\mathcal{N}_\infty$ either. Thus $\mathcal{N}_\omega \vDash \neg\exists Y\Theta(\bar{x}, \bar{y}.Y)$ as required. □

We next note the analog of these conservation results for cone avoiding forcings.

Theorem 4.13. *If Q is a Π^1_2 principle such that solutions can be added by cone avoiding forcings, $\mathcal{N} \vDash RCA_0$ is countable, $\{C_j | j \in \omega\} \subseteq S(\mathcal{N})$ and $\forall j \in \omega(C_j \notin S(\mathcal{N}))$, then there is an extension $\mathcal{N}'$ of $\mathcal{N}$ with the same first order part such that $\mathcal{N}' \vDash Q + RCA_0$ and no $C_j \in S(\mathcal{N}')$.*

Proof. As there are here no conservation results to verify the proof is simply the basic argument given above for the construction. Then one simply notes that by the choice of forcings no C_j is added on at any successor step and so none enter at a limit level either. □

We now note that by interspersing the appropriate forcings in the iterations, the class of problems described in each of the four clauses of Definition 4.7 are closed under conjunction. Indeed, if they hold for each Q_i for $i \in \omega$ they hold for the theory $T = \{Q_i | i \in \omega\}$. So to then do the conservation results for each of the first two theorems associated with each class of Qs and the cone avoiding theorem for the third. Thus we can add on any principle with solutions give by et-forcings such as COH (Cholak, Jockusch and Slaman [2001, Theorem 9.1] and Hirschfeldt and Shore [2007, Theorem 2.21]) –

Mathias forcing); AMT (Hirschfeldt, Shore and Slaman [2009, Corollary 3.15]) as well as BCT–II and RCA_0^+ of Brown and Simpson [1997, §4 and Corollary 6.5] and, $\Pi^0_\infty G = \cup \Pi^0_n G$ in the terminology of Hirschfeldt, Lange and Shore [2017, p. 89] $\Pi^0_\infty = \cup\Pi^0_n$ – Cohen forcing; the existence of minimal covers for Turing reducibility ([as mentioned in Shore [2010, p. 395] – Sacks forcing.

Remark 4.14. Moving outside of et-forcings, we can, for example, extend the G-Π^1_1 conservation results from RCA_0 to WKL_0 as WKL is a Π^1_2 principle for which solutions can be provided by forcing (Harrington, see Simpson [2009, IX.2]). So, in particular, if solutions to a Π^1_2 principle Q can be provided by forcing, then $Q + WKL_0 \nvdash ACA_0$ as ACA_0 is not Π^1_1 conservative over RCA_0. (Indeed, Simpson [2009, VIII.1.8] shows that ACA_0 is not conservative over RCA_0 even for Π^0_1 sentences.) Note that as WKL is itself an r-Π^1_2 formula, it is not r-Π^1_2 conservative over RCA_0 and so solutions for it cannot be produced by et-forcings. (The are, however, produced in the usual proof by tree forcings, just not effective ones.)

The proof in Simpson [2009, VIII.1.8] is an application of Gödel's second incompleteness theorem. A semantic and more dramatic demonstration that, in this setting, $Q + WKL_0 \nvdash ACA_0$ is provided by Theorem 4.5.3 when solutions for Q can be provided by cone avoiding forcings: If $\mathcal{N}$ is a model of RCA_0 but not ACA_0, i.e. there is an $X \in S(\mathcal{N})$ such that there is no $Y \in S(\mathcal{N})$ satisfying the definition of X' then there is an extension $\mathcal{N}'$ of $\mathcal{N}$ with the same first order part which also has no such Y. Indeed, we can even omit every subset of N which is definable over $\mathcal{N}$ but not in $S(\mathcal{N})$. These remarks also apply when we add on WKL as the forcing that provides solutions for it has the cone avoiding property by using the standard arguments for cone avoiding for Π^0_1 forcing in recursion theory (Jockusch and Soare [1972])

We would like to apply all these results to the principles LF_{XY} of Definition 3.5 as well as others to show that they do not imply ACA_0 and indeed are highly conservative over RCA_0. As we already showed that each LF_{XY} becomes a THA when added to ACA_0, this will show that all of them (even when combined with each other as well as WKL, COH and more) are ATHA. Our plan is to first show that they all have solutions provided by et-forcings. The problem will then be that they are not Π^1_2 principles and so we will also have to extend the theorems above to a larger class of principles.

5 Extending the Class of Principles

We want to prove that the principles asserted to be ATHA in §3 do not imply ACA_0 by showing that solutions can be provided by et-forcings and that we can extend the conservation/preservation results of §4 to a wider class of principles than Π^1_2 that include all of the ones that we claimed to be ATHA and more. We begin with showing how solutions for all of them can be provided by et-forcings. The most interesting ones are the ones about finding locally finite subgraphs with various properties that began our study of ATHA: LF_{XY}.

Theorem 5.1. *Given an X-graph H in a countable $\mathcal{N} \models RCA_0$ which contains arbitrarily many Y-disjoint rays we can define an et-forcing $\mathcal{P}$ and a countable collection of dense sets $\mathcal{D}$ such that any $\mathcal{D}$-generic G provides a locally finite X-subgraph H' of H which also contains arbitrarily many Y-disjoint rays.*

Proof. The set of vertices of H' is just that of H which, without loss of generality, we may take to be N. Thus we only need to specify the edges of H'. We use a pairing function to view the numbers n as all possible edges. We write $M(v, n)$ to mean that the vertex v is an element of the edge n.

Our conditions $\langle \tau, T \rangle$ will satisfy various requirements in addition to the ones common to all et-forcings. The first is that, the trees are binary (i.e. subsets of $2^{<N}$) and for any $\sigma \in T$, if (the pair) n is not an edge in H then $\sigma(n) = 0$. Thus each generic $G : N \to N$ can be seen as a subgraph of H in $\mathcal{N}[G]$. (G supplies the characteristic function of the set of edges. The set of vertices we have already set to be N.)

The intuition behind the rest of the definition of the notion of forcing is that we want to be able, on one hand, to specify that the set of edges from some sequence of disjoint rays of length m can be added to the final graph. On the other, for an arbitrary vertex v we want to be able to specify that no additional edges containing v can be added to the graph and to guarantee that only finitely many of the edges already guaranteed to be in the graph have v as a vertex.

To these ends, for each potential condition $\langle \tau, T \rangle$ we first specify various sets of edges n based on what the condition says about their membership in G. First we have the $n > |\tau|$ such that all branches G in T have $G(n) = 1$. We denote the set of such n as $Y^{\langle \tau, T \rangle} = \{n > |\tau| | (\forall \sigma \in T)(\sigma(n) = 1)\}$. Next we have the ones with $G(n) = 0$ for all G on T, $N^{\langle \tau, T \rangle} = \{n > |\tau| | (\forall \sigma \in T)(\sigma(n) = 0)\}$. Finally, we have the n at which G can go either way, $U^{\langle \tau, T \rangle} = \{n > |\tau| | (\forall \sigma \in T)(|\sigma| = n \to \sigma\hat{\ }0 \in T \ \wedge \ \sigma\hat{\ }1 \in T\}$. Note that as T is binary all these sets are in $\mathcal{N}$ (even uniformly in $\langle \tau, T \rangle$). We require that every $n > |\tau|$ is in one of these (clearly disjoint) sets.

We also impose some requirements on the nature of the first two of these three sets. For $Y^{\langle \tau, T \rangle}$ there is a function $f_{\langle \tau, T \rangle} = f$ such that $\forall v (f(v) = |\{n \in Y^{\langle \tau, T \rangle} | M(v, n)\}|)$. So not only are there only finitely many $n \in Y^{\langle \tau, T \rangle}$ such that v is a member of the edge n but we know how many and so we can uniformly determine which they are. Finally, for $N^{\langle \tau, T \rangle}$ there is a finite set $A_{\langle \tau, T \rangle} = A$ such that $(\forall n \in N^{\langle \tau, T \rangle})(\exists v \in A) M(n, v)$ and $(\forall n > |\tau|)(\forall v \in A)(M(n, v) \to n \notin U^{\langle \tau, T \rangle})$. We call such f and A *witnesses* that a $\langle \tau, T \rangle$ satisfying the other requirements on $Y^{\langle \tau, T \rangle}$, $N^{\langle \tau, T \rangle}$ and $U^{\langle \tau, T \rangle}$ is a condition. Note that f is uniquely determined but A need not be.

Given this set of conditions, the forcing partial order is just the one defined by the basic requirement for et-forcings in Definition 4.1.2.

Note that if $\langle\tau, T\rangle$ is a condition with witnesses f and A and $\sigma \in T$ with $|\sigma| > |\tau|$ then $\langle\sigma, T^\sigma\rangle$ (as defined in Notation 4.3) is also a condition with the same A and a slightly modified f' as witnesses: let $f'(v) = f(v) - |\{n \in Y^{\langle\tau,T\rangle}|\ |\tau| \leq n < |\sigma|\ \wedge M(v,n)\}|$. Thus we have defined an et-forcing.

We now argue that we can describe a collection $\mathcal{D}$ of dense sets that guarantee that any $\mathcal{D}$-generic G determines (as described above) a locally finite subgraph H' of H which contains arbitrarily many disjoint rays.

First, we claim that for each vertex v and condition $\langle\tau, T\rangle$ with witnesses A and f as above there is a $\langle\tau', T'\rangle \leq \langle\tau, T\rangle$ with witnesses f' and A' such that $v \in A'$. In fact, it is easy to see from the definition of the allowed conditions that for any $v \notin A_{\langle\tau,T\rangle}$, there is a $T' \subseteq T$ such that $\langle\tau, T'\rangle$ is a condition with witness f and $A' = A \cup \{v\}$. We refine T by removing any $\sigma \supset \tau$ such that there is an $n \geq |\tau|$ which is an edge containing v as a vertex such that both $\sigma \restriction n\hat{}0$ and $\sigma \restriction n\hat{}1$ are in T and $\sigma(n) = 1$. This moves n from $U^{\langle\tau,T\rangle}$ to $N^{\langle\tau,T\rangle}$ and so does not affect the calculation of the required f.

So by genericity (for these dense sets) for any vertex v there is a condition $\langle\tau, T\rangle$ in the generic filter such that it has a witness $A_{\langle\tau,T\rangle}$ containing v.

As for this enforcing local finiteness, consider any condition $\langle\tau, T\rangle$ with witnesses such that $v \in A_{\langle\tau,T\rangle}$. We claim that for any G on T there are only finitely many n with $G(n) = 1$ and $M(v,n)$ Of course, there are at most $|\tau|$ many edges $n < |\tau|$ such that $M(v,n)$. There are only $f(v)$ many $n \in Y^{\langle\tau,T\rangle} = Y^{\langle\tau,T'\rangle}$ such that $M(v,n)$ and by our second condition about $v \in A$ there are no $n \in U^{\langle\tau,T\rangle}$ such that $M(v,n)$. Of course, there are no $n \in N^{\langle\tau,T'\rangle}$ such that $G(n) = 1$. (We have, in fact, uniformly over all G on T calculated the number of n such that $M(v,n)$.)

Finally, we show that for each $m \in N$ and condition $\langle\tau, T\rangle$ with witnesses as above, there is an extension $\langle\tau, T'\rangle$ such that for any G on $\langle\tau, T'\rangle$ the associated graph contains m many disjoint rays.

By assumption there is a sequence $\langle V_i, E_i, f_i\rangle_{i<m+|\tau|}$ of disjoint rays in H. Obviously, there are at most $|\tau|$ many n such that $\tau(n) = 0$. As each edge n can appear in at most one E_i we can thin out the given sequence to one $\langle V_i', E_i', f_i'\rangle_{i<m}$ of disjoint rays none of which contains an edge n for which $\tau(n) = 0$.

Next we deal with the conditions imposed by $A_{\langle\tau,T\rangle}$. For each $i < m$ we let $g(i) = 0$ if no $v \in A_{\langle\tau,T\rangle}$ is in V_i'. Otherwise we let $g(i) = \max\{n | f_i'(n) \in A_{\langle\tau,T\rangle}$ and $f_i'(n) \in V_i'\} + 1$. Our construction so far guarantees that $g : m \to N$ is a member of $\mathcal{N}$. Thus we can thin out $\langle V_i', E_i', f_i'\rangle_{i<m}$ by taking the tail of each V_i' beyond $g(i)$ to get $V_i'' = V_i' - \{f_i'(n) | n < g(i)\}$, $E_i'' = E_i' - \{\langle f_i'(n), f_i'(n+1)\rangle\, | n < g(i)\}$ and $f_i''(n) = f_i'(n + g(i))$. Let $E'' = \cup\{E_i'' | i < m\}$. This sequence clearly provides m many disjoint rays in H. Thus (by genericity) it suffices to define a condition $\langle\tau, T'\rangle \leq \langle\tau, T\rangle$ such that $E'' \subseteq Y^{\langle\tau,T'\rangle}$ as then for any G on $\langle\tau, T'\rangle$, $G(n) = 1$ for all the edges $n \in E_i''$ for every $i < m$ and so $\langle V_i'', E_i'', f_i''\rangle_{i<m}$ shows in $\mathcal{N}[G]$ that the subgraph H' of H associated with G has m many disjoint rays.

The crucial point is that we have designed $\left\langle V_i'', E_i'', f_i''\right\rangle_{i<m}$ so that $E'' \cap N^{\langle\tau,T\rangle} = \emptyset$.

Thus we may define a tree $T' \subseteq T$ by simply removing all σ such that $\sigma(n) = 0$ for some $n \in E''$. This change at most moves some edges $n \in U^{\langle \tau,T \rangle}$ to $Y^{\langle \tau,T' \rangle}$ and makes $Y^{\langle \tau,T' \rangle} = Y = Y^{\langle \tau,T \rangle} \cup E''$ as desired while keeping $N^{\langle \tau,T' \rangle} = N^{\langle \tau,T \rangle}$. If $\langle \tau, T' \rangle$ is a condition, it clearly extends $\langle \tau, T \rangle$ and so we only have to verify that it is one. As we already have the required facts about $Y^{\langle \tau,T' \rangle}$, $N^{\langle \tau,T \rangle}$ and $U^{\langle \tau,T' \rangle}$, we only need to supply witnesses f' and A'. As we have the witness f for $\langle \tau, T \rangle$, to get f' it clearly suffices to compute for each v the number of $n > |\tau|$ such that $n \in E'' - Y^{\langle \tau,T \rangle}$ and $M(v, n)$. We begin with the sequence $\langle V_i'', E_i'', f_i'' \rangle_{i<m}$ of disjoint rays such that $E'' = \cup\{E_i''|i < m\}$. We first determine those i for which $v \in V_i''$. (For the other j there are no $n \in E_j''$ with $M(v, n)$.) As v appears in each of these V_i'' only once, there are at most two such edges in each of these E_i''. Using this information we can determine for which edges $n \in E_i'' - Y^{\langle \tau,T \rangle}$ we have $M(v, n)$. As the E_i'' are disjoint, we can now simply add up the contributions (of one or two) from each $E_i'' - Y^{\langle \tau,T \rangle}$ with $v \in V_i''$ to get amount we need to add to $f(v)$ to get the desired $f'(v)$. Finally, as $N^{\langle \tau,T \rangle} = N^{\langle \tau,T' \rangle}$ and $U^{\langle \tau,T' \rangle} \subseteq U^{\langle \tau,T \rangle}$, we can take the witness A for $\langle \tau, T \rangle$ to also be the desired witness A' for $\langle \tau, T' \rangle$. □

So solutions to all the LF principles can be provided by et-forcings. We would like to draw the conclusions that give the conservation and preservation results of Theorems 4.11-4.13 for all the LF principles. The problem is that they are not Π_2^1-principles. We actually needed two properties of principles to get the applications we proved for Π_2^1 principles.

One was that once a solution was provided in $\mathcal{N}[G]$ by forcing, it remained a solution in each later extension and so at the end. This was immediate from the fact that the properties of interest were arithmetic and that the extensions preserved the first order part of the model. This property still holds for the conclusions of the LF principles as the conditions required of the constructed subgraph H' (local finiteness) and the finite sequences $\langle V_i, E_i, f_i \rangle_{i<m}$ (they are finite sequences of rays) are also arithmetic.

The other property was that every instance of the problem that is in the final limit model already appears in one of the models along the way. (This allows us to handle all the instances that there are at end as we go along.) This property is not obvious for the LF principles as the condition for H to be an instance is that H is a graph that contains arbitrarily many rays and so of the form $\forall m \exists W (\langle W^{[i]}|i < m \rangle$ is a sequence of m many disjoint rays in H). It could be that for some graph H constructed along the way the required Ws for each $m \in N$ get constructed cofinally in the sequence of extensions and so are instances in the limit model never solved along the way. (There are special situations for which this cannot happen. One, in particular, is that the failure of X to be an instance is equivalent to some trees T in the model with H do not have branches in the model. Examples of this is situation is TAC and its variants in Proposition 5.13 and Definition 5.15. As et-forcings preserve this fact, H would not be an instance in the limit stage model.) Here we provide a simpler more generally applicable solution: extend the iteration to ω_1. (One can also get by with a countable iteration albeit longer than ω by a look ahead procedure to make sure possible instances with parameters that we have now but witnesses that may occur later are handled now. The ω_1 iteration is, however, simpler and useful later.)

Theorem 5.2. *The conclusions of Theorems 4.11-4.13 hold for each the LF_{XY} principles as Q.*

Proof. Continue the iteration by the et-forcings that provide solutions to an instance of one of the LF_{XY} principles through ω_1 many steps in such a way that every H that appears as an instance at any $\mathcal{N}_\alpha$ gets a solution in some $\mathcal{N}_{\beta+1}$ for $\beta \geq \alpha$. At limit levels we still act continuously: $\mathcal{N}_\lambda$ is the union of the $\mathcal{N}_\alpha$ for $\alpha < \lambda$. Now if $H \in \mathcal{N}_{\omega_1}$ is an instance of the LF_{XY} principle then, not only does H appear in some $\mathcal{N}_\alpha$, but so do all the witnesses for H containing m many disjoint rays for every m as there are only countably many $m \in \mathcal{N}$ and the full sequence is of length ω_1. □

Theorem 5.3. *All of the LF_{XY} principles are ATHA. Indeed, all of them together are ATHA. Moreover, one can add on all the other et-forcings mentioned here as well as WKL while maintaining the preservation and conservation theorems and so still not proving ACA_0 and being and ATHA.*

Proof. Each of these four principles together with ACA_0 is a THA by Proposition 3.6. By Theorem 5.2, none of them imply ACA_0. Combining them and any other principles for which solutions can be added by et-forcings and even WKL by ω_1 iterations is routine following the route indicated in Remark 4.14 and the comments proceeding it for ω length iterations. □

We now turn to other examples from the work on Halin type theorems as well as direct variations of choice principles. The guiding idea here is that when a principle calls for a solution which is a sequence X_i of sets each satisfying some property $\Psi(i, X)$ we are willing to accept some variations. One is that we accept a sequence Y_i such that each Y_i differs from an X_i as required by a finite set. The other basic variation is that we allow the desired witnesses to be arbitrarily distributed among the Y_i. That is for each i there is a j such that $\Psi(i, Y_j)$. We designate these modifications of a principle P by P^* and P^-, respectively. Of course, we could also consider allowing both changes: the list contains a finite variant of each X_i. The proof of the following implication shows that nothing new appears with this combination.

Proposition 5.4 (RCA_0). *For any principle P whose conclusion asks for a sequence X_i such that $\forall i \Psi(i, X_i)$, $P^* \to P^-$.*

Proof. Take the solutions Y_i given by P^* and construct the sequence $Y_{i,\sigma}$ for each finite (binary) string σ with $Y_{i,\sigma}(n) = \sigma(n)$ for $n < |\sigma|$ and $Y_{i,\sigma}(n) = Y_i(n)$ for $n \geq |\sigma|$. □

We begin with a principle from BGS [ta] that we examined for locally finite graphs in Proposition 3.3. It extracts the use of Σ^1_1-AC needed to prove the Halin-type theorems IRT_{XY} in ACA_0 as in BGS [ta, Theorem 8.2].

Definition 5.5. (SCR_{XY}): If an X-graph G has arbitrarily many Y-disjoint X-rays then there is a Z such that, for each k, $Z^{[k]}$ is a sequence $\langle Z_{k,i} | 1 \leq i \leq k \rangle$ of pairwise Y-disjoint X-rays in G.

We note a couple of facts about SCR_{XY} from BGS [ta, Proposition 7.3 and Corollary 7.4] that include its being a THA and then a couple of variations along the lines described above that produce ATHAs.

Proposition 5.6. *(RCA_0) $SCR_{XY} \rightarrow ACA_0$.*

Corollary 5.7. *(RCA_0) $SCR_{XY} \Leftrightarrow IRT_{XY}$.*

Definition 5.8. A sequence $\langle x_n \rangle$ of vertices in an X-graph G is almost an X-ray in G if, for some k, $\langle x_{k+n} \rangle$ is an X-ray in G. A sequence $\langle X_n \rangle$ of almost X-rays $\langle x_{n.i} \rangle$ is almost Y-disjoint if for every $n \neq m$ the there are only finitely many i, j such that $x_{n,i} = x_{m,j}$ (for $Y = V$). For $Y = E$ we require that there are only finitely many i, j such that $(x_{n,i}, x_{n,i+1}) = (x_{m,j}, x_{m,j+1})$.

Definition 5.9. (SCR^*_{XY}): If an X-graph G has arbitrarily many pairwise almost Y-disjoint almost X-rays then there is a Z such that, for each k, $Z^{[k]}$ is a sequence $\langle Z_{k,i} | 1 \leq i \leq k \rangle$ of pairwise almost Y-disjoint almost X-rays in G. (Note the use of $*$ here and in later such principles is suggested by the common usage of $=^*$ to mean equal up to finite difference and is not related to the induction axioms characterizing, e.g. ACA^*_0 and related principles in [BGS].)

Definition 5.10. (SCR^-_{XY}): If an X-graph G has arbitrarily many pairwise almost Y-disjoint almost X-rays then there is a Z such that, for each k, there is an l such that $Z^{[l]}$ is a sequence $\langle Z_{k,i} | 1 \leq i \leq k \rangle$ of pairwise almost Y-disjoint almost X-rays in G.

Clearly each of ACA_0+SCR^*_{XY} and ACA_0+SCR^-_{XY} imply SCR_{XY}, a THA. Thus to show that SCR^*_{XY} and SCR^-_{XY} are ATHA, we only have to show that they do not imply ACA_0. As should be expected, all eight variants have solutions provided by et-forcings and so satisfy all the conclusions of Theorems 4.11-4.13. However, instead of presenting specific forcing notions for them we turn to Σ^1_1-AC. It is clear that it implies SCR_{XY}. The * and $^-$ analogs for it also imply those of SCR_{XY} and we provide the notions of forcing for them instead. This then shows that all of these principles are ATHA. We will then consider other well studied weakenings of Σ^1_1-AC which are THA but whose * and $^-$ analogs will also be ATHA. In the next section we turn to stronger versions of choice which are too strong to be THA but whose * and $^-$ variants also have all of the same weakness properties over RCA_0.

Remark 5.11. We make an brief exception to Remark 2.6 to sketch one consideration of double rays in directed graphs because we can get ATHAs which yield equivalences with a standard theory. We say a sequence $\langle x_n | n \in \mathbb{Z} \rangle$ of vertices in a D-graph G is *almost a double directed ray in* G if changing finitely many of the x_n to a different vertex or removing it from the list (and reindexing) produces a double directed ray. From the natural analogs SCR_{DYD} for double Y-disjoint directed rays we form the analogous SCR^*_{DYD} and SCR^-_{DYD} where, as we have allowed removing vertices in the definition of almost rays, we use full Y-disjointness. As for SCR_{XY}, these are also consequences of $\Sigma^1_1\text{-AC}^*_0$ and $\Sigma^1_1\text{-AC}^-_0$, respectively and so weak over RCA_0. On the other hand, it is easy to see from the proof of BGS [Theorem 6.13] that, as for SCR_{XY}, SCR^*_{DYD} and SCR^-_{DYD} restricted to directed forests (i.e. directed graphs whose underlying graph gotten by symmetrizing the edge relation is a disjoint union of trees) plus ACA_0 is equivalent to $\Sigma^1_1\text{-AC}_0$ over $\mathrm{I}\Sigma^1_1$. Thus we have two mathematical ATHAs which with the addition of ACA_0 are equivalent to $\Sigma^1_1\text{-AC}_0$ over $\mathrm{I}\Sigma^1_1$.

As there is some variation in the formulations of these principles in the literature, we want to make the versions and the relations among them explicit. We begin with Σ^1_1-AC itself which is a THA.

Definition 5.12. Σ^1_1-AC is the principle $\forall A[\forall n \exists X \Phi(n, X) \rightarrow \exists Y \forall n \Phi(n, Y^{[n]})]$. Here Φ is an arithmetic formula possibly with free set variables A and X but not Y. (We take these restrictions on the free set variables for granted in all future similar situations.) Equivalently (over RCA_0), we may allow Φ to be Σ^1_1.

One direction of this equivalence in RCA_0 is immediate as all arithmetic formulas are trivially equivalent to Σ^1_1 formulas. For the other direction consider $\Phi = \exists Z \Psi(A, n, X, Z)$ (Ψ arithmetic). One simply considers the instance $\forall n \exists X \Psi(A, n, X^{[0]}, X^{[1]})$. Clearly one can recursively recover the Y required for Φ from the one given by Σ^1_1-AC for $\Psi(A, n, X^{[0]}, X^{[1]})$. We choose the version with Φ arithmetic to match the common terminology for weak Σ^1_1-AC (U-Σ^1_1-AC below). On the other hand, there is another common a priori weaker version for which the proof of the equivalence uses ACA_0. This is not an issue for Σ^1_1-AC as even this "weaker" version implies ACA_0. This will no longer be true of our * and $^-$ variants.

Proposition 5.13. *Σ^1_1-AC is equivalent to the principle TAC: For every sequence T_i of trees, if $\forall n \exists f (f \in [T_i])$, then $\exists f \forall n (f^{[n]} \in [T_n])$.*

This Proposition is well known and follows easily from the normal form theorem proved in ACA_0 as Lemma V.5.4 of Simpson [2009].

We now define our variants of Σ^1_1-AC.

Notation 5.14. *For a function f, finite string μ (or set X) and σ a finite (binary) string we write f_σ, μ_σ (or X_σ) to mean the function, finite string (or set) gotten by using σ to define its initial segment of length $|\sigma|$: $f_\sigma(i) = \sigma(i)$ for $i < |\sigma|$, $f_\sigma(i) = f(i)$ for $i \geq |\sigma|$ and similarly for μ_σ and X_σ. We write $f^{[n]}_\sigma$ for $(f^{[n]})_\sigma$ and $X^{[n]}_\sigma$ for $(X^{[n]})_\sigma$. Similarly, for a tree T we write $T_\sigma = \{\mu_\sigma | \mu \in T\}$. We write T^σ_ρ for $(T^\sigma)_\rho$ where T^σ is defined in Notation 4.3.*

Definition 5.15. For Φ arithmetic

Σ_1^1-AC*: $\forall A[\forall n\exists X\Phi(A,n,X)\to\exists Y\forall n\exists\sigma\Phi(A,n,Y_\sigma^{[n]})]$ and

Σ_1^1-AC$^-$: $\forall A[\forall n\exists X\Phi(A,n,X)\to\exists Y\forall n\exists m\Phi(A,n,Y^{[m]})]$.

TAC*: For every sequence $\langle T_n\rangle$ of trees, if $\forall n\exists f(f\in[T_n])$ then $\exists f\forall n\exists\sigma(f_\sigma^{[n]}\in[T_n]$.

TAC$^-$: For every sequence $\langle T_n\rangle$ of trees, if $\forall n\exists f(f\in[T_n])$ then $\exists f\forall n\exists m(f^{[m]}\in[T_n])$.

As with Σ_1^1-AC, Σ_1^1-AC* and Σ_1^1-AC$^-$ are each equivalent (over RCA$_0$) to the analogous principle with Φ being Σ_1^1.

Proposition 5.16. *In RCA$_0$, Σ_1^1-AC$\to$ Σ_1^1-AC* $\to$ Σ_1^1-AC$^-$ & TAC*; Σ_1^1-AC$^-$ $\to$ TAC$^-$; TAC* $\to$TAC$^-$ and TAC$^-$ $\to$TAC*. In ACA$_0$ all of these principles are equivalent to Σ_1^1-AC.*

Proof. The implication Σ_1^1-AC$\to$ Σ_1^1-AC* is immediate as solutions to instances of the former are also solutions to the same instance of the latter. The implications from a * version to the corresponding $^-$ one in RCA$_0$ are essentially instances of Proposition 5.4. Of course, the versions of TAC are simply special cases of the corresponding one for Σ_1^1-AC (The $\langle T_i\rangle$ is absorbed into the set parameters.). For the equivalences in ACA$_0$ it thus suffices to show that TAC$^-$ $\to$ Σ_1^1-AC$_0$. Given an instance of Σ_1^1-AC specified by A and Φ, construct the trees T_n such that $\forall n\exists X\Phi(A,n,X)$ if and only if T_n has a branch and any such branch uniformly computes a witness X such that $\Phi(A,n,X)$. (This is again essentially Lemma V.5.4 of Simpson [2009] and the previously mentioned remarks on Π_2^0 formulas and branches through trees.) Now apply TAC$^-$ to get its sequence f. As the question of whether $f^{[m]}$ is a branch on T_n is arithmetic (Π_1^0), we have by ACA$_0$ a function g such that $f^{[g(n)]}$ uniformly computes an X such that $\Phi(A,n,X)$. Thus we have an X such that $\forall n\Phi(A,n,X^{[n]})$ as required.

Finally that RCA$_0\vdash$ TAC$^-$ $\to$TAC* requires an argument that does not work for the Σ_1^1-AC analogs. We are given a sequence $\langle T_n\rangle$ of trees such that $\forall n\exists f(f\in[T_n])$ and an f such that $\forall n\exists m(f^{[m]}\in[T_n])$ and must produce a g such that $\forall n\exists\sigma(g_\sigma^{[n]}\in[T_n]$. To construct $g^{[n]}$ we start by copying $f^{[0]}$ until we have an s_0 such that $f^{[0]}\restriction s_0+1\notin T_n$. If we never find such an s we have computed a branch on T_n and so $g^{[n]}=g_\emptyset^{[n]}$ is as required. If we find such an s_0 we switch to copying $f^{[1]}$ for inputs from s_0 onward. We continue until we once again fall off T_n, i.e. $f^{[1]}\restriction s_1+1\notin T_n$. By the conclusion of TAC$^-$ from some point onward there is a fixed m such that we are copying $f^{[m]}$ and $f^{[m]}\in[T_n]$. We have thus constructed a g such that $g_\sigma\in[T_n]$ with $\sigma=f^{[m]}\restriction s_{m-1}$. $\square$

We do not know if RCA$_0\vdash\Sigma_1^1$-AC$^-$ $\to$ Σ_1^1-AC*. We also note other views of Σ_1^1-AC* and Σ_1^1-AC$^-$.

Proposition 5.17. *In RCA$_0$, Σ_1^1-AC* is equivalent to Σ_1^1-AC restricted to predicates $\Phi(A,n,X)$ that are invariant under finite changes in X, i.e. $\forall A\forall n\forall X\forall\sigma(\Phi(A,n,X)\Leftrightarrow\Phi(A,n,X_\sigma))$.*

Proof. If a given instance $\Phi(A,n,X)$ of Σ^1_1-AC is invariant under finite changes then a solution for the same instance of Σ^1_1-AC* is also one for Σ^1_1-AC. In the other direction, given an instance $\Phi(A,n,X)$ of Σ^1_1-AC*, consider the one $\Psi(A,n,X) \equiv \exists\sigma\Phi(A,n,X_\sigma)$ for Σ^1_1-AC. Clearly $\Psi(A,n,X)$ is closed under finite changes and any Σ^1_1-AC solution for Ψ is also a Σ^1_1-AC* solution for the Σ^1_1-AC* instance $\Phi(A,n,X)$. $\square$

As for Σ^1_1-AC$^-$, when we told some people about some of the results in this paper both Antonio Montalbán and Keita Yokoyama informed us of some early work by Tanaka, Yamazaki and Montalbán on variations of choice principles. In particular, they considered Σ^1_1-AC$^-$ under the natural name Σ^1_1-collection as well as the natural generalizations we call Σ^1_n-AC$^-$ and Σ^1_∞-AC$^-$ under the names Π^1_n-collection and Π^1_∞-collection and proved several conservation results. We discuss those results in §7.

Theorem 5.18. *For each of the* * *and* $^-$ *versions of* Σ^1_1*-AC in Definition 5.15 solutions can be provided by et-forcings. Thus the conclusions of Theorems 4.11-4.13 hold for each of them as well.*

Proof. By Proposition 5.16 it suffices to prove the Theorem for Σ^1_1-AC*. Given a countable model $\mathcal{N}$ of RCA$_0$ and an arithmetic Φ such that $\mathcal{N} \vDash \forall n\exists X\,\Phi(A,n,X)$ we define a forcing with conditions $\langle\tau,T\rangle$ such that, in $\mathcal{N}$, there is a finite set F and a sequence $\langle X_i|i\in F\rangle$ such that $\forall i\in F\Phi(A,i,X_i)$ and for all $\sigma\in T$, $|\sigma|>n\geq|\tau|$ with $n=\langle i,m\rangle$ for some $i\in F$, $\sigma(n)=X_i(m)$. (Otherwise there are no restrictions on σ.)

It is now easy to see that the associated notion of forcing is et: If $\langle\tau,T\rangle$ is a condition and $\sigma\in T$ then $\langle\sigma,T^\sigma\rangle$ is a condition with the same F and X_i which extends $\langle\tau,T\rangle$. It is also clear the the sets D_i of conditions such that i is a member of the associated F are dense. (Just thin out a given $\langle\tau,T\rangle$ by choosing an X_i such that $\Phi(A,i,X_i)$ and keeping only those $\sigma\in T$ which satisfy the condition that $\sigma(\langle i,m\rangle)=X_i(m)$ for $\langle i,m\rangle\geq|\tau|$.) Moreover, for any G on this thinned out tree, $G^{[i]} =^* X_i$. Thus for any G generic for these D_i, G is the desired witness for this instance of Σ^1_1-AC*.

The argument in the proof of Theorem 5.2 now shows that the conclusions of Theorems 4.11-4.13 hold for all of these choice principles as well. $\square$

We now turn to Σ^1_1-AC itself and some of its choice like consequences. Each of them has versions with the property $\Psi(i,Y)$ required of the X_i being arithmetic or equivalently Σ^1_1 and versions with it being Π^0_2 or equivalently asking for a branch on a tree T_i in a uniform sequence of trees. These are easily seen to be equivalent over ACA$_0$ but not over RCA$_0$. In particular, we want to consider the versions where we restrict the principles to Φ such that, for each n, there are only finitely many X for which $\Phi(A,n,X)$ holds (F-Σ^1_1-AC) or exactly one such X (U-Σ^1_1-AC) which is generally called weak Σ^1_1-AC. These can also be phrased in terms of sequences $\langle T_i\rangle$ of trees as in TAC by restricting to T_i with only finitely many or exactly one branch. We will explicitly just consider the Σ^1_1-AC versions. We are here interested in the * and $^-$ versions.

Theorem 5.19. *$RCA_0 \vdash \Sigma^1_1$-$AC^* \to F$-Σ^1_1-$AC^* :\to U$-Σ^1_1-AC^*. None of these implications can be reversed and all of these principles are ATHA. The same holds for the $^-$ versions.*

Proof. We consider the * versions but all the arguments apply to the $^-$ ones as well.

It is obvious that Σ^1_1-$\mathrm{AC}_0 \to \Sigma^1_1$-$\mathrm{AC}^* \to$F-$\Sigma^1_1$-$\mathrm{AC}^* \to$U-$\Sigma^1_1$-$\mathrm{AC}^*$ in RCA_0. As in Proposition 5.16, it is easy to see that with the addition of ACA_0 each of the * principles is equivalent to the standard unstarred version. Each of the standard principles are THA (see Montalbán [2008, p. 564], the references there and Goh [ta].). Theorem 5.18 shows that none of them imply ACA_0 (over RCA_0) and so are all ATHA. The known separations of all of the unstarred versions provide witnesses that are even standard models of much more than ACA_0 (see Steel [1978], Van Wesep [1977] and Goh [ta]). They then are also witnesses for the nonimplications among the * versions. □

6 Higher Choice Principles

In this section we want to study the * and $^-$ variations choice principles that replace the arithmetic formulas Φ and Ψ in Σ^1_1-AC by arbitrary formulas. The usual terminology has Σ^1_{n+1}-AC being the principle $\forall A[\forall n \exists X \Phi(A, n, X) \to \exists Y \forall n \Phi(A, n, Y^{[n]})]$ for Φ Σ^1_{n+1}. As with Σ^1_1 these and their * and $^-$ versions are equivalent to the ones with Φ being Π^1_n. We take the Π^1_n versions to be our official definitions for notational convenience. As usual Σ^1_∞-AC is the union of all the Σ^1_{n+1}-AC and so for the * and $^-$ versions. The variations on these principles supply us with another collection of principles that are very weak over RCA_0 but very strong over ACA_0. At the end we have Σ^1_∞-AC^* (and so Σ^1_∞-AC^-) which have solutions produced (in a new sense) by et-forcings and for which we argue for all the properties guaranteed for Π^1_2 principles by Theorems 4.11-4.13. The forcing notions are quite straightforward. The argument that the property holds at the limit of even an ω_1 length iteration, however, needs a new twist.

For Σ^1_∞-AC^- actually, there is a very simple known et-forcing that does more and requires no new ideas. The small trick is that one adds to a given countable $\mathcal{N} \vDash \mathrm{RCA}_0$ a generic G such that $\forall A \in S(\mathcal{N})(\exists i)(A = G^{[i]})$. The conditions are just $\langle \tau, T \rangle$ with T a binary tree such that, in $\mathcal{N}$, there is a finite set F and a sequence $\langle A_i | i \in F \rangle$ such that for $\sigma \in T$ with $|\sigma| > \langle i, n \rangle \geq |\tau|$ ($i \in F$ and $n \in N$), $\sigma(\langle i, n \rangle) = A_i(n)$. Otherwise for $\sigma \supseteq \tau$ there are no restrictions on $\sigma \in T$. Clearly this forcing adds a G as required. Iterating this forcing ω_1 many times gives a model $\mathcal{N}_{\omega_1}$ of Σ^1_∞-AC^-: For any instance specified by $\forall n \exists X \Phi(A, n, X)$ with $A \in \mathcal{N}_{\omega_1}$ and so $A \in \mathcal{N}_\alpha$ for α countable, witness X_n that $\exists X \Phi(n, X)$ for each n appear in some $\mathcal{N}_\beta$ for a countable β. Thus each is a $G^{[k]}_{\beta+1}$ for some k as required by Σ^1_∞-AC^-.

Our proof for Σ^1_∞-AC^* requires more interesting twists and we present it in detail. As Σ^1_∞-AC^* implies all of the other principles in RCA_0, proving the conservation results for it implies them for the others. Thus we need not expand on the sketch just given for Σ^1_∞-AC^-.

Theorem 6.1. *For $\mathcal{N}$ a countable model of RCA_0, any any second order Φ such that $\mathcal{N} \vDash \forall k \exists X \Phi(A, k, X)$ and $\mathcal{N}'$ a countable extension of $\mathcal{N}$ satisfying RCA_0 with the same first order part as $\mathcal{N}$, there is an et-forcing $\mathcal{P}$ with an appropriate collection $\mathcal{D}$ of dense sets such that for any $\mathcal{D}$-generic G over $\mathcal{N}'$, $G^{[k]} \in S(\mathcal{N})$ for every $k \in N$ and for every $k \in N$, $\mathcal{N} \vDash \exists \sigma \Phi(A, k, G_\sigma^{[k]})$.*

Proof. Forcing conditions are like those described above for Σ^1_1-AC$^-$but tied to Φ and A: $\langle \tau, T \rangle$ with T a binary tree such that there is, in $\mathcal{N}$, a finite set F and a sequence $\langle X_i | i \in F \rangle$ such $\mathcal{N} \vDash \Phi(A, i, X_i)$ for every $i \in F$ and for $\sigma \supseteq \tau$ and $|\sigma| = \langle i, n \rangle \geq |\tau|$, if $i \in F$, $\sigma \hat{} j \in T \Leftrightarrow j = X_i(n)$. For other $\langle i, n \rangle$, both $\sigma \hat{} 0$ and $\sigma \hat{} 1$ are in T. While this forcing is not in general definable over $\mathcal{N}'$ as it refers to membership in $\mathcal{N}$, is is clearly an et-forcing over $\mathcal{N}'$ as each condition is in $\mathcal{N} \subseteq \mathcal{N}'$ and $\sigma \in Ext(\langle \tau, T \rangle) \Leftrightarrow \sigma \in T$ as then $\langle \sigma, T_\sigma \rangle \leq \langle \tau, T \rangle$. Moreover, the sets $\{\langle \tau, T \rangle | i \in F_{\langle \tau, T \rangle}\}$ are clearly dense for each i (again even if not definable over $\mathcal{N}'$). Thus, for any $\mathcal{D}$-generic G where $\mathcal{D}$ includes these sets, it is clear that $G^{[i]} =^* X$ for some X such that $\mathcal{N} \vDash \Phi(A, i, X)$ as required for G to satisfy the desired property. □

All we need to do now is prove that $\mathcal{N}_{\omega_1} \vDash \exists Y \forall n \Phi(A, n, Y^{[n]})$.

Theorem 6.2. *If $\mathcal{N}$ is a countable model of RCA_0 then there are extensions $\mathcal{N}_\alpha$ for $\alpha < \omega_1$ with the same first order part a $\mathcal{N}$ such that $\cup\{\mathcal{N}_\alpha | \alpha < \omega_1\} = \mathcal{N}_{\omega_1} \vDash RCA_0 + \Sigma^1_\infty$-AC* and these extensions have all the properties needed to guarantee the conclusions of Theorems 4.11-4.13.*

Proof. We define a sequence $\mathcal{N}_\alpha$, $\alpha < \omega_1$ of countable models of RCA$_0$ with the same first order part. We begin with $\mathcal{N}_0 = \mathcal{N}$. Given $\mathcal{N}_\alpha$ we list the countably many instances given by A_j and Φ_j for $j \in \omega$ such that $\mathcal{N}_\alpha \vDash \forall k \exists X \Phi_j(A_j, k, X)$. We then define an ω length iteration to construct $\mathcal{N}_{\alpha,l}$ starting at $\mathcal{N}_{a,0} = \mathcal{N}_\alpha$ and taking $\mathcal{N}_{\alpha,l+1}$ to be an extension of $\mathcal{N}_{\alpha,l}$ by a generic for the forcing described above for the lth instance of $\forall n \exists X \Phi(A, n, X)$ in $\mathcal{N}_\alpha$. We set $\mathcal{N}_{\alpha+1} = \cup\{\mathcal{N}_{\alpha,l} | l \in \omega\}$. As we use et-forcings, all the conditions needed for Theorems 4.11-4.13 are met along the way and at $\mathcal{N}_{\omega_1} = \cup\{\mathcal{N}_\alpha | \alpha < \omega_1\}$.

All that needs to be verified here beyond what was done for Σ^1_1-AC* is that $\mathcal{N}_{\omega_1} \vDash \Sigma^1_\infty$-AC*. The crucial fact here is that the α such that $\mathcal{N}_\alpha$ is an elementary submodel of $\mathcal{N}_{\omega_1}$ (in the full second order language) include a closed unbounded set. With this in mind, consider any $A \in \mathcal{N}_{\omega_1}$ and $\Phi(A, n, X)$ such that $\mathcal{N}_{\omega_1} \vDash \forall n \exists X \Phi(A, n, X)$. Take an α such that $A \in \mathcal{N}_\alpha$ which is an elementary submodel of $\mathcal{N}_{\omega_1}$ and so also satisfies $\forall n \exists X \Phi(A, n, X)$. Our construction therefore guarantees that there is an l and a $Z \in \mathcal{N}_{\alpha,l}$ so that for every $n \in N$, $Z^{[n]} \in \mathcal{N}_\alpha$ and $\mathcal{N}_\alpha \vDash \exists \sigma \Phi(A, n, Z_\sigma^{[n]})$. As $A, Z \in \mathcal{N}_{\omega_1}$ which is an elementary extension of $\mathcal{N}_\alpha$, we now have that $\mathcal{N}_{\omega_1} \vDash \exists \sigma \Phi(A, n, Z_\sigma^{[n]})$ for every $n \in N$ and so $\mathcal{N}_{\omega_1} \vDash \exists Z \forall n \exists \sigma \Phi(A, n, Z_\sigma^{[n]})$ as required. □

Theorem 6.3. *For each $n \in \omega$, $ACA_0 \vdash \Sigma^1_{n+2}$-AC$^* \to \Sigma^1_{n+2}$-AC$^- \to \Sigma^1_{n+1}$-CA$_0$ and so $ACA_0 \vdash \Sigma^1_\infty$-AC$^* \to \Sigma^1_\infty$-AC$^- \to \Sigma^1_{n+1}$-CA$_0$.*

Proof. The implication Σ^1_{n+2}-AC$^* \rightarrow \Sigma^1_{n+2}$-AC$^-$ is Proposition 5.4. We prove Σ^1_{n+2}-AC$^- \rightarrow \Sigma^1_{n+1}$-CA$_0$ in ACA$_0$ by induction on n. Consider $\exists X\Psi(k,X)$ for a Π^1_n formula Ψ (Π^1_0 is arithmetic). We want to prove that there is a set $R = \{k|\exists X\Psi(k,X)\}$. Define $\Phi(k,X)$ as $\Psi(k,X) \ \vee \ (X = \emptyset \wedge \neg\exists Y\psi(k,Y))$. Clearly Φ is Π^1_{n+1} and $\forall k\exists X\Phi(k,X)$. Thus we may apply Σ^1_{n+2}-AC$^-$ to get a set Z such that $\forall k\exists m\Phi(k,Z^{[m]})$. Thus $k \in R \Leftrightarrow \exists X\Psi(k,X) \Leftrightarrow \exists m\Psi(k,Z^{[m]})$. Now $S = \{\langle k,m\rangle\,|\Psi(k,Z^{[m]})\}$ is Π^1_n and so exists by induction. (For $n = 0$ this is ACA$_0$.) and so $R = \{k|\exists m(\langle k,m\rangle \in S\}$ exists by ACA$_0$. $\square$

Thus we have whole hierarchies of principles that are very weak over RCA$_0$ but very strong and indeed equivalent to a hierarchy of standard systems. At the end, $\Sigma^1_\infty - AC^*$ and Σ^1_∞-AC$^-$ satisfy all the conservation and preservation principles of Theorems 4.11-4.13 over RCA$_0$ but, over ACA$_0$, are both equivalent to $\Sigma^1_\infty - AC_\infty$ and so strictly stronger than full second order arithmetic (Feferman and Levy; see Simpson [2009, Remark VII.6.3]). We view these results and the ones on ATHA that are equivalent to known THA over ACA$_0$ as supplying answers to the question raised by Hirschfeldt and repeated in Montalbán [2011] by providing an ample list of many pairs of principles that are very different over RCA$_0$ but equivalent over ACA$_0$. It could well be argued that these weak ones should really be seen as the same as their strong counterparts in an analysis that works over ACA$_0$ rather than RCA$_0$.

7 Tanaka Conservativity

We close with some words about earlier work on the collection axioms (Σ^1_n-AC$^-$ and Σ^1_∞-AC$^-$) and another type of conservation result that was brought to our attention by this work which applies to all the principles we have investigated here.

The work by Tanaka, Montalbán and Yamazaki on conservativity of Σ^1_∞-AC$^-$ (or as they call it, Π^1_∞ collection) over RCA$_0$ has, far as we have determined, never been published. The only source I have access to is a set of slides from a talk by Yamazaki [2009] sent to me by Keita Yokoyama, Based on those slides, the methods used seem considerably more complicated than the ones presented here. In particular, to prove Π^1_1 conservativity they seem to restrict attention to principle models of RCA$_0$ (ones with a single set such that every set in the model is recursive in it) and use both Π^0_1 forcing (i.e. infinite binary trees recursive in a single set) and forcing with uniformly pointed perfect trees along with ω_1 iterations. (Of course, the slides just outline proofs at best.)

Our proofs, certainly for Σ^1_∞-AC$^-$ and, I would say, even for Σ^1_∞-AC*, are much simpler. Yamazaki does get more by including WKL$_0$ as well (and so the use of Π^0_1 forcing makes sense). We have already pointed out that for Π^1_1 conservativity, we can easily add WKL$_0$ to our constructions with its own forcing notion to get this Π^1_1 conservativity result and the same one for Σ^1_∞-AC*. Yamakazi does not consider r-Π^1_2 conservativity although he does present an analog of minimal pairs for models of WKL$_0$ from Simpson, Tanaka and Yamakazi [2002]. They are are used in STY even in their proof of Π^1_1 conservativity.

I expect this analysis was motivated in the same way as in STY by the desire to prove a different kind of conservativity over RCA_0 conjectured for WKL_0 by Tanaka and proved in STY. We establish this conservation result for Σ^1_1-AC* and indeed for all principles we have analyzed using forcing. Moreover, we prove both a generalization of Tanaka conservativity analogous to our generalization for Π^1_1 conservativity (Definition 4.9 and Theorem 4.11) and a more inclusive variation analogous to our generalization of r-Π^1_2 conservativity (Definition 4.10 and Theorem 4.12). (This one fails for WKL as for the previously mentioned r-Π^1_2 conservativity as it is more general.)

Definition 7.1. *Tanaka conservativity* means conservativity for all sentences of the form $\forall X \exists! Y \Phi(X, Y)$ for arithmetic Φ. We define *G-Tanaka conservativity to* be conservativity for sentences in the class of *G-Tanaka formulas* defined by closing the quantifier free formulas and those of the form $\exists! Y \Phi(Y)$ for arithmetic Φ under the same operations as for $G - \Pi^1_1$ in Definition 4.9 ($\wedge$, $\wedge$, $\forall x$, $\exists x$ and $\forall X$). We define *G-r-Tanaka formulas and conservativity* by adding in to the base case of the previous definition formulas of the form $\exists! Y \exists Z \Psi(\bar{x}, Y, Z)$ with Ψ a Σ^0_3 formula.

Remark 7.2. Tanaka conservativity was called $\mathfrak{Uniq}$ conservativity in Yokoyama [2009]. He studied it primarily for Π^1_2 theories including WKL, COH, RCA^+ (or Π^1_∞G) over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ and he cites earlier work of Kihara [2009] on COH and Yamazaki [2009] on RCA^+ and unpublished work on COH. We thank Yokoyama for these references as well.

It is clear that G-r-Tanaka conservativity includes all the other versions defined here as well as those in §4. We prove these conservativity results by isolating one extra property of et-forcings needed to carry out the proof and note that the et-forcings used for our $\Sigma^1_\infty - AC^*$ results as well as all the others in this paper have this property. The idea is that for any condition $\langle \tau, T \rangle$ the subtrees above any two $\rho, \sigma \in T$ of the same length look the same. Although stronger or simpler restrictions can be given that fit most of our examples, we formulate "look the same" in a fairly general way that matches our overall approach to et-forcings yet is strong enough to eliminate some technical problems.

Definition 7.3. An et-forcing $\mathcal{P}$ is *uniform (a uet-forcing)* if, for every condition $\langle \tau, T \rangle$, every $\rho, \sigma \in Ext(\langle \tau, T \rangle)$ with $|\rho| = |\sigma|$, and every $\langle \rho'', R'' \rangle \leq \langle \rho', R' \rangle \leq \langle \tau, T \rangle$ with $\rho \subseteq \rho'$, $\langle \rho''_\sigma, R''_\sigma \rangle \leq \langle \rho'_\sigma, R'_\sigma \rangle \leq \langle \tau, T \rangle$. (Of course, then $\sigma \subseteq \rho'_\sigma$.) As a technical convenience we add on another condition that clearly cannot change the results of a forcing construction: If $\langle \tau, T \rangle \in \mathcal{P}$ and the stem of T is some $\sigma \supset \tau$ then $\langle \rho, T \rangle \leq \langle \tau, T \rangle$ whenever $\sigma \supseteq \rho \supseteq \tau$.

The crucial Lemma that we need about uet-forcings is the following:

Lemma 7.4. *Suppose $\mathcal{P}$ is a uet-forcing (over a countable $\mathcal{N} \vDash RCA_0$) and $\mathcal{D}$ is a countable collection of dense sets. We can extend $\mathcal{D}$ to another countable collection of dense sets $\mathcal{D}'$ such that for any $\langle \tau, T \rangle \in \mathcal{P}$, $\rho, \sigma \in Ext(\langle \tau, T \rangle)$ of the same length and $\mathcal{D}'$-generic $G \supseteq \rho$ whose generic filter contains $\langle \tau, T \rangle$, G_σ is $\mathcal{D}$-generic.*

Proof. We want to guarantee that for any $G \supseteq \rho$ with a $\mathcal{D}'$*-generic sequence* $\langle \rho^i, R^i \rangle$ (i.e. for each $D' \in \mathcal{D}'$, there is an i such that $\langle \rho^i, R^i \rangle \in D'$) all extending $\langle \tau, T \rangle$ and wlog $\rho^i \supseteq \rho$, $\langle \rho^i_\sigma, R^i_\sigma \rangle$ is a $\mathcal{D}$-generic sequence. It is a decreasing sequence of conditions all extending $\langle \tau, T \rangle$ by uniformity. Of course, its generic $(\cup \rho^i_\sigma)$ is G_σ as required. Consider then any $D \in \mathcal{D}$. We want an m such that $\langle \rho^m_\sigma, R^m_\sigma \rangle \in D$. We define a D' by removing all $\langle \tau', T' \rangle \leq \langle \rho^0, R^0 \rangle$ from D and adding in $\left\langle \tau'_\rho, T'_\rho \right\rangle$ for each $\langle \tau', T' \rangle \leq \langle \rho^0_\sigma, R^0_\sigma \rangle$ in D.

Now if D' is dense we can put it into $\mathcal{D}'$. On this assumption, we have an m such that $\langle \rho^m, R^m \rangle = \left\langle \tau'_\rho, T'_\rho \right\rangle$ for some $\langle \tau', T' \rangle \leq \langle \rho^0_\sigma, R^0_\sigma \rangle$ in D. Now $\langle \rho^m_\sigma, R^m_\sigma \rangle = \left\langle (\tau'_\rho)_\sigma, (T'_\rho)_\sigma \right\rangle = \langle \tau', T' \rangle \in D$ as required. (Note that $\tau' \supseteq \rho^0_\sigma \supseteq \sigma$ so $(\tau'_\rho)_\sigma = \tau'$ as $|\sigma| = |\rho|$.)

All that remains is to prove that each such D' is dense. Consider any $\langle \tau, T \rangle$. If it is incompatible with $\langle \rho^0, R^0 \rangle$ then any extension in D is in D'. Otherwise we have a $\langle \tau', T' \rangle$ extending both. By uniformity, $\langle \tau'_\sigma, T'_\sigma \rangle \leq \langle \rho^0_\sigma, R^0_\sigma \rangle$. By the density of D we have a $\langle \tau'', T'' \rangle \leq \langle \tau'_\sigma, T'_\sigma \rangle$ in D. So by uniformity again, $\left\langle \tau''_\rho, T''_\rho \right\rangle \leq \langle (\tau'_\sigma)_\rho, (T'_\sigma)_\rho \rangle = \langle \tau', T' \rangle$ as required. $\square$

Up until now we have not needed more about forcing than the starting level for Σ^0_1 sentences. For Theorem 7.5 we need to be able to handle all arithmetic sentences. Rather than try to give formal definitions for models $\mathcal{N}$ of RCA_0 (of which there are several in the literature of reverse mathematics) we just note the relevant properties. Typically one considers forcings which are at least definable. While most of the ones we have used are definable, the one for Σ^1_1-AC* was not. Now one can make due with definable forcings there by at every even stage or limit stage λ first using the forcing defined for Σ^1_1-AC$^-$ that makes every $A \in \mathcal{N}_\lambda$ a column of $G_{\lambda+1}$. Then one can define the forcing we wanted to use at each successor stage for Σ^1_1-AC* for the $G_{\lambda+n+2}$ and defining the forcing over $\mathcal{N}_{\lambda+n+1}$ where one can quantify over sets in $\mathcal{N}_\lambda$ by using $G_{\lambda+1}$ as a parameter. However, we do claim that this is not really necessary to get the required properties of forcing. What we want to know is that we can define the forcing relation $\langle \tau, T \rangle \Vdash \Theta$ starting at the Σ^0_1 level as before so that there it depends only on the τ in our conditions $\langle \tau, T \rangle$ in $\mathcal{N}$ and implies truth for all extensions of τ. (This level includes what we have already assumed about members of $\mathcal{N}[G]$ being of a form that we think of as $\Phi^{A,G}_e$ for $e \in N$ and $A \in S(\mathcal{N})$ in a way that only relies on the initial segments τ of G in the conditions in the generic filter.) We can then continue on up the arithmetic hierarchy so as to guarantee the density of conditions deciding each sentence and forcing equals truth for all sufficiently generic sets. (So for G generic, $\mathcal{N}[G] \vDash \Theta$ if and only if there is a condition $\langle \tau, T \rangle$ in the generic filter such that $\langle \tau, T \rangle \Vdash \Theta$.) We are not concerned with the level of the definability of the forcing relation (or even with the notions being definable over $\mathcal{N}$ at all).

If one wanted to be more specific, we would deal only with prenex normal sentences and take negation to be a shorthand for the prenex normal equivalent of the negation of the given sentence. At the Π^0_1 level we explicitly define $\langle \tau, T \rangle \Vdash \forall x \Phi(x, G)$ as there being no $\sigma \in Ext(\langle \tau, T \rangle)$ and no n such that $\sigma \Vdash \neg \Phi(n, G)$. We then proceed by induction on the number of quantifiers in our prenex normal Φ as usual: $\langle \tau, T \rangle \Vdash \exists x \Psi(x, G)$ if

$\langle \tau, T \rangle \Vdash \Psi(n, G)$ for some n; $\langle \tau, T \rangle \Vdash \forall x \Psi(x, G)$ if there is no $\langle \tau', T' \rangle \leq \langle \tau, T \rangle$ and no n such that $\langle \tau', T' \rangle \Vdash \neg \Psi(n, G)$.

We can now state the main property need for our conservation results.

Theorem 7.5. *For any countable model $\mathcal{N}$ of RCA_0 and any extension $\mathcal{N}_\infty$ constructed via forcings from any class $\mathcal{C}$ of uet-forcings $\mathcal{P}$ (defined uniformly over any $\mathcal{N}'$ extending $\mathcal{N}$ with the same first order part) and any G-r-Tanaka sentence Λ, if $\mathcal{N} \vDash \neg \Lambda$ then $\mathcal{N}_\infty \vDash \neg \Lambda$.*

Proof. The argument for quantifier free sentences is as in Theorem 4.11 as are the inductive cases for $\wedge$, $\vee$, $\exists x$, $\forall x$ and $\forall X$. We need to verify the claim for sentences of the form $\exists! Y \Phi(Y)$ for arithmetic Φ and $\exists! Y \exists Z \Psi(Y, Z)$ with Ψ a Σ_3^0 formula (each with constants for elements of N and $S(\mathcal{N})$).

Consider first one of the form $\exists! Y \Phi(Y)$ for arithmetic Φ. If there are two Y in $\mathcal{N}$ such that $\mathcal{N} \vDash \Phi(Y)$ then they also satisfy the same sentence in $\mathcal{N}_\infty$ for the desired contradiction. So we assume there is no such Y in $\mathcal{N}$ while there is (exactly) one, say V, in $\mathcal{N}_\infty$. So there is a least α such that $V \in \mathcal{N}_{\alpha+1} = \mathcal{N}_\alpha[G]$. (We write G for G_α for notational convenience.) Let $V = \Phi_e^{A,G}$ for some $A \in S(\mathcal{N}_\alpha)$ and $v_n = V(n)$.

If Φ is arithmetic, we take a $\langle \tau, T \rangle$ in the generic filter for G (for the forcing used over $\mathcal{N}_\alpha$) such that $\langle \tau, T \rangle \Vdash \forall x (\Phi_e^{A,G}(x) = 0 \vee \Phi_e^{A,G}(x) = 1) \wedge \Phi(\Phi_e^{A,G})$. For each n we have a $\langle \tau_n, T_n \rangle$ in the generic filter for G (and so wlog extending $\langle \tau, T \rangle$) such that $\tau_n \Vdash \Phi_e^{A,G}(n) = v_n$. Much as in the proof of Theorem 4.5, if there is no $\tau' \in Ext(\langle \tau, T \rangle)$ and n such that $\tau' \Vdash \Phi_e^{A,G}(n) = 1 - v_n$ then $V = \Phi_e^{A,G} \in \mathcal{N}_\alpha$ for a contradiction. By the definition of uet-forcings we may choose $\langle \tau', T' \rangle \leq \langle \tau, T \rangle$ and $m \geq n$ such that $|\tau_m| = |\tau'| = k > |\tau|$. As $G \supseteq \tau_m$ is generic, $G_{\tau'}$ is generic for the dense sets deciding all arithmetic formulas and so as $\langle \tau, T \rangle$ is also in the filter for $G_{\tau'}$, $\mathcal{N}_\alpha[G_{\tau'}] \vDash \forall x (\Phi_e^{A,G_{\tau'}}(x) = 0 \vee \Phi_e^{A,G_{\tau'}}(x) = 1) \wedge \Phi(\Phi_e^{A,G_{\tau'}})$. As $G_{\tau'}$ differs from G by a finite set, it is also in $\mathcal{N}_\alpha[G] = \mathcal{N}_\alpha[G_{\tau'}] = \mathcal{N}_{\alpha+1}$. Thus in $\mathcal{N}_{\alpha+1}$, $\Phi_e^{A,G_{\tau'}} = V'$ and $\Phi_e^{A,G} = V$ both exist, are different at n and are witnesses for $\Phi(Y)$. Thus they remain such in $\mathcal{N}_\infty$ for a contradiction.

We now turn to the case for Λ of the form $\exists! Y \exists Z \Psi(Y, Z)$ for a Σ_3^0 formula Ψ. As in the previous case we have a least α such that there is witness V for Y in $\mathcal{N}_{\alpha+1}$. If there is also a witness U for Z in $\mathcal{N}_{\alpha+1}$ then essentially the same argument as above (writing U as $\Phi_i^{B,G}$) shows that there are two distinct witnesses V and V' in $\mathcal{N}_{\alpha+1}$ which have witness U and U' such that $\mathcal{N}_{\alpha+1} \vDash \Psi(X, V, U) \wedge \Psi(X, V', U')$ which again provides the contradiction to there being only one witness for Y in $\mathcal{N}_\infty$. On the other hand, as in the proof of Theorem 4.12, if there is no such U in $\mathcal{N}_{\alpha+1}$ then, for every k, the tree in $\mathcal{N}_{\alpha+1}$ (recursive in V) associated with k being a witness for the Σ_3^0 formula Ψ has no branch in $\mathcal{N}_{\alpha+1}$. By Theorem 4.5.3 the nonexistence of a branch in any of these trees is propagated through the iteration and so V has no witness for Z in $\mathcal{N}_\infty$ for another contradiction. $\square$

By our usual arguments, this Theorem provides the further conservation results for all our principles as well as many others as Corollaries.

Corollary 7.6. *The following principles are all G-r-Tanaka (and hence G-Tanaka, G-r-Π^1_2 and G-Π^1_1) conservative over RCA_0: $\Sigma^1_\infty - AC^*$ (and all its consequences such as $\Sigma^1_\infty - AC^-$, $\Sigma^1_n - AC^*$ and $\Sigma^1_n - AC^-$, SCR^*_{XY} and SCR^-_{XY}), LF_{XY}, COH, AMT, $\Pi^0_\infty G$ (RCA_0^+), the existence of minimal covers for Turing reducibility and related theorems.*

Proof. We need only check that the et-forcings used or mentioned so far are actually uet-forcings. These checks are basically straightforward except for Sacks forcing. The version of Sacks forcing typically used to construct, for example, minimal covers is not uniform in our sense. This application as well as other similar results can, as is well known, be proven using uniform recursive (in a specific A) trees as conditions (as in Lerman [1983, Ch. VI]. That construction is then easily seen to be one with a uet-forcing. □

The proof of the case for $\exists! Y \Phi(Y)$ in Theorem 7.5 essentially shows that if there is a $Y \in S(\mathcal{N}[G]) - S(\mathcal{N})$ for a sufficiently generic G over a uet-forcing such that $\mathcal{N}[G] \vDash \Phi(Y)$ then there are at least two $Z \in S(\mathcal{N}[G])$ such that $\mathcal{N}[G] \vDash \Phi(Z)$. We improve this to there being infinitely many. We then mention some applications of this improvement as well as that of going to G-Tanaka formulas that do not seem to follow from standard Tanaka conservativity.

Proposition 7.7. *If $\mathcal{N} \vDash RCA_0$ is countable, $\mathcal{P}$ is a uet-forcing over $\mathcal{N}$, $\Theta(Y)$ is arithmetic, G is sufficiently generic over $\mathcal{P}$, $\Theta(Y)$ an arithmetic formula over $\mathcal{N}$ such that there is a $Y \in S(\mathcal{N}[G]) - S(\mathcal{N})$ for which $\mathcal{N}[G] \vDash \Theta(Y)$ then there is a sequence $\langle Y_i | i \in N\rangle \in S(\mathcal{N}[G])$ with the Y_i pairwise distinct such that $\mathcal{N}[G] \vDash (\forall i)(\Theta(Y_i))$. (Actually if one defines the forcing relation for all second order sentences, essentially the same argument will work for arbitrary Θ.)*

Proof. Suppose that the Y given by the Proposition is $\Phi_e^{A\oplus G}$ for some $e \in N$ and $A \in S(\mathcal{N})$ and $\langle \tau, T\rangle \Vdash \Theta(Y)$. The argument in the proof of Theorem 7.5 shows that for any $\sigma \subseteq G$ (and so $\sigma \in Ext(\langle \tau, T\rangle)$ there are $\sigma', \tau' \supseteq \sigma$ in $Ext(\langle \tau, T\rangle)$, $j \neq k$ and x such that $\sigma' \subseteq G$, $\sigma' \Vdash \Phi_e^{A\oplus G}(x) = j$ and $\tau' \Vdash \Phi_e^{A\oplus G}(x) = k$. In $\mathcal{N}[G]$ we can then recursively in G construct sequences $\sigma_i, \tau_i, x_i, j_i$ and k_i such that $\sigma_i \subseteq \sigma_{i+1} \subseteq G$, $\sigma_i \subseteq \tau_i \in Ext(\langle \tau, T\rangle)$, $\sigma_i \Vdash \Phi_e^{A\oplus G}(x_i) = j_i$ and $\tau_i \Vdash \Phi_e^{A\oplus G}(x_i) = k_i$. We now claim that the sequence $\Phi_e^{A\oplus G_{\tau_i}}$ is as desired. The argument in Theorem 7.5 shows that each of these is in $\mathcal{N}[G]$ and satisfies Θ there. The construction guarantees that, for each i, $\Phi_e^{A\oplus G_{\tau_i}}(x_i) = k_i \neq j_i = \Phi_e^{A\oplus G_{\sigma_i}}(x_i) = \Phi_e^{A\oplus G}(x_i) = \Phi_e^{A\oplus G_{\tau_l}}(x_i)$ for every $l > i$. Thus the $\Phi_e^{A\oplus G_{\tau_i}}$ are pairwise distinct. □

We now consider some generalizations of unique existence assertions to other cardinality quantifiers and applications to show that for such assertions we can also derive information about the existence of recursive solutions. (This is done in [STY, Theorem 4.18] for unique existence for WKL_0.)

We begin by formalizing the notions of "there are exactly", "at least" or "at most" m many Y such that $\Phi(Y)$ holds. In general, Φ can be arbitrary but we will restrict our attention to the arithmetic case for our applications. We are formalizing the definition of cardinality m that asserts the existence of a one-one correspondence with the natural numbers less than m in a way that works well in RCA_0.

Definition 7.8. We say that there are *exactly m many Y such that* $\Phi(Y)$, $(\exists^{=m}Y)\Phi(Y)$ if there is a pairwise distinct sequence $\langle Y_i | i < m\rangle$ such that $(\forall i < m)\Phi(Y)$ and $\forall W(\Phi(W) \to \exists i < m)((W = Y_i))$. Note that, in RCA_0, this is equivalent to the existence of a unique such sequence where the Y_i are in strict ascending lexicographic order. It is also worth pointing out that for $m \in \mathbb{N}$ we can express this by a single formula not mentioning m. On the other, hand we can view m as a variable over the numbers N in any model of RCA_0. This allows to to express the quantifier *there are finitely many Y such that* $\Phi(Y)$ as $\exists m(\exists^{=m}Y)(\Phi(Y))$ which we write as $(\exists^{Fin}Y)(\Phi(Y))$. Similarly, we say $(\exists^{\geq m}Y)\Phi(Y)$ or $(\exists^{\leq m}Y)(\phi(Y))$, if there is a pairwise distinct sequence $\langle Y_i | i < m\rangle$ such that $(\forall i < m)\Phi(Y)$ or, respectively, if there is a pairwise distinct sequence $\langle Y_i | i < m\rangle$ such that $\forall W(\Phi(W) \to \exists i < m)((W = Y_i))$. Of course, $(\exists^{=m}Y)\Phi(Y) \Leftrightarrow (\exists^{\geq m}Y)\Phi(Y)$ & $(\exists^{\leq m}Y)(\phi(Y))$.

We now give some applications.

Theorem 7.9. *Let Q be any of the theories mentioned this section (or combinations of them) which can be guaranteed to hold by iterating uet-forcings over any countable model $\mathcal{N}$ of RCA_0 to produce a model $\mathcal{N}_\infty$ of Q with the same first order part as $\mathcal{N}$. Let $\Phi(Y)$ be any arithmetic formula with its only free variable being Y and $k \in \mathbb{N}$ be a standard number.*

1. *If $Q \vdash (\exists^{Fin}Y)\Phi(Y)$ then $RCA_0 \vdash (\exists^{Fin}Y)\Phi(Y)$.*
2. *If $Q \vdash (\exists^{Fin}Y)\Phi(Y)$ & $(\exists^{\geq k}Y\Phi(Y)$ then $RCA_0 \vdash (\exists^{Fin}Y)\Phi(Y)$ & $(\exists^{\geq k}Y)(Y$ is recursive and $\Phi(Y))$.*
3. *If $Q \vdash (\exists^{\leq k}Y)\Phi(Y)$ then $RCA_0 \vdash (\exists^{\leq k}Y)\Phi(Y)$.*
4. *If $Q \vdash (\exists^{=k}Y)\Phi(Y)$ then $RCA_0 \vdash (\exists^{=k}Y)\Phi(Y)$ & $(\forall Y)(\Phi(Y) \to Y$ is recursive).*

Proof. For each assertion, suppose we have a countable $\mathcal{N} \vDash RCA_0$ which provides a counterexample to the desired conclusion. We argue for a contradiction to the associated hypothesis.

1. Note that $(\exists^{Fin}Y)\Phi(Y)$ is equivalent (in RCA_0) to $\exists m\exists! Z(Z$ is a sequence of sets $\langle Z_i | i < m\rangle$ in strictly ascending lexicographical order such that $(\forall i < m)(\Phi(Z_i))$. As the formula in parentheses here is arithmetical (in m and Z), the whole assertion is a G-Tanaka sentence. Theorem 7.5 then guarantees that it and so $(\exists^{Fin}Y)\Phi(Y)$ is a theorem of RCA_0 as desired.

2. We already have that $\mathrm{RCA}_0 \vdash (\exists^{Fin} Y)\Phi(Y)$ and so we assume that we have a countable model $\mathcal{N}$ of RCA_0 not containing k many recursive solutions to Φ. Let $\mathcal{N}'$ have the same first order part as $\mathcal{N}$ and second order part $R(\mathcal{N})$ the collection of subsets of N which are recursive in $\mathcal{N}$. Of course, $\mathcal{N}' \vDash RCA_0$ as well but $\mathcal{N}' \vDash \neg\exists^{\geq k} Y(\Phi(Y))$ as every set in it is recursive in it and in $\mathcal{N}$. Now construct $\mathcal{N}'_\infty \vDash Q$ by an iteration beginning with $\mathcal{N}'$. By our assumption there are at least k many solutions Y for Φ in $\mathcal{N}'_\infty$ and so one not in $\mathcal{N}'$ must appear for the first time at some $\mathcal{N}_{\alpha+1}$. Proposition 7.7 then guarantees that there is, in $\mathcal{N}_{\alpha+1}$ and so in $\mathcal{N}_\infty$, an infinite sequence $\langle Z_i \rangle$ of solutions to Φ for the desired contradiction to the assumption that $Q \vdash (\exists^{Fin} Y)\Phi(Y)$.

3. This one is simple. If we have a countable model $\mathcal{N}$ of RCA_0 with more than k many solutions Y for Φ then all of them are solutions in $\mathcal{N}_\infty$ for a contradiction.

4. This follows directly from the previous cases.

□

If in the previous theorem and proof we consider formulas $\Phi(X, Y)$ with a free set variable X and assume that the hypotheses hold for the universal closure with respect to X, then so do the conclusions where we replace recursive by recursive in X.

8 Bibliography

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