Biinterpretability up to double jump in the degrees below $\mathbf{0}'$

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Abstract

We prove that, for every $\mathbf{z} \leq \mathbf{0}'$ with $\mathbf{z}'' > \mathbf{0}''$ (i.e. $\mathbf{z} \in \bar{\mathbf{L}}_2$), the structure $\mathcal{D}(\leq \mathbf{z})$ of the Turing degrees below $\mathbf{x}$ is biinterpretable with first order arithmetic up to double jump. As a corollary, every relation on $\mathcal{D}(\leq \mathbf{z})$ which is invariant under double jump is definable in $\mathcal{D}(\leq \mathbf{z})$ if and only if it is definable in arithmetic.

1 Introduction

A major issue in the analysis of relative complexity of computation over the past few decades has been the determination of the complexity of various structures that capture some reducibility, i.e. a notion of relative computability, and a class of sets ($A \subseteq \mathbb{N}$) (or equivalently functions $f : \mathbb{N} \to \mathbb{N}$) whose relative complexities are to be compared. The basic notion of relative complexity is that of Turing. We say that $A \leq_T B$ if there is a Turing machine Φ_e, equipped with an oracle for B, i.e. access to membership information about B, which can compute membership in A. The trend has been to show that such structures are as complicated as possible: first, in the sense that their theories are as complicated as possible given the definability of the structure itself in some (often second order) version of arithmetic; second, in the sense that the class of relations on the structures definable in them is as rich as possible.

Considering only Turing reducibility, the class of sets to consider first is that of all subsets of $\mathbb{N}$. As usual, we move to the equivalence classes under $\leq_T$, i.e. the (Turing)

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degrees), to define the basic structure $\mathcal{D}$ of these degrees under the induced relation $\leq_T$. Then one considers important or natural substructures of $\mathcal{D}$. The most prominent of these are the recursively enumerable degrees (those of sets with effective enumerations), $\mathcal{R}$, the degrees of sets recursive in the halting problem, $\mathcal{D}(\leq \mathbf{0}')$ and the degrees of sets definable in (first order) arithmetic, $\mathcal{A}$. These last degrees are closed under relativizations of the halting problem, i.e. the (Turing) jump sending A to A' (the halting problem for machines with oracle A). From basic facts connecting definability in arithmetic to the jump operator, it follows that $\mathcal{A}$ is the least ideal in $\mathcal{D}$ closed under the jump. ($\mathcal{D}$ is an uppersemilattice, so an ideal in $\mathcal{D}$ is a downward closed subset closed under join.) Also of interest, then, are the other ideals closed under the jump.

The complexity of the theories of all of these structures are known to be as high as possible. They are each easily seen to be (recursively) interpretable in the version of second order arithmetic with quantification over sets with degrees in the specified structures. Thus their theories are reducible (even by a one-one function) to those of the corresponding structure for arithmetic. One then proves a converse that each such structure of arithmetic is interpretable in the associated degree structure. Thus the theory of each degree structure is recursively equivalent to that of the corresponding structure of arithmetic. (These results are due to Simpson [1977] for $\mathcal{D}$, Harrington and Slaman and Slaman and Woodin (see Nies, Shore and Slaman [1998]) for $\mathcal{R}$, Shore [1981] for $\mathcal{D}(\leq \mathbf{0}')$ and Nerode and Shore [1980] for all ideals closed under jump.) By their interpretability in (some version of) arithmetic, the most one could hope for in terms of definability in these degree structures would be that all relations invariant under Turing degree and definable in the corresponding structure for arithmetic are definable in the degree structure. To date, the best general results have been that relations invariant under double jump definable in each structure of arithmetic are definable in the associated degree structure. The known results are due to Slaman and Woodin (see Slaman [1991] and [2008]) for $\mathcal{D}$, Nies, Shore and Slaman [1998] for $\mathcal{R}$ and Shore [2007] for ideals closed under jump which also contain $\mathbf{0}^{(\omega)}$, the effective join of the $\mathbf{0}^{(n)}$, the finite iterations of the jump. Possible routes to this result for $D(\leq \mathbf{0}')$ are outlined in Shore [1988] and more concretely in Nies, Shore and Slaman [1998] but a crucial ingredient is missing from these sketches (comparison maps between models as explained below). We supply this missing ingredient and so prove the result for $\mathcal{D}(\leq \mathbf{0}')$. Indeed by making use of some more recent general results on constructing 1-generics for various notions of forcing below degrees $\mathbf{z}$ in $\overline{\mathbf{GL}}_2$ ($\mathbf{z}'' > (\mathbf{z} \vee \mathbf{0}')'$) from Cai and Shore [2012] applied to the analysis of 1-genericity for two specific such types of forcing in Greenberg and Montalbán [2004], we simultaneously prove the same results for $\mathcal{D}(\leq \mathbf{z})$ for any $\mathbf{z} \leq \mathbf{0}'$ with $\mathbf{z}'' > 0''$. The corresponding question for $\mathcal{A}$ is an important open problem whose solution should lead to other interesting theorems.

Such results on definability are now generally approached through the stronger notions of biinterpretability and biinterpretability up to double jump due to Harrington for the r.e. degrees and Slaman and Woodin for other structures (see Slaman [1991] and [2008]). We define these notions in §2. There also supply statements of, and references for, the somewhat scattered results we need about coding arithmetic into degree structures and forcing in arithmetic. More details and some proofs can be found in our lecture notes Shore [2013]. Our main theorem on biinterpretability up to double jump (Theorem 3.4) is proven in §3. The major consequence is Theorem 3.5 that, for $\mathbf{z} \leq \mathbf{0}'$ with $\mathbf{z}'' > 0''$, a relation on $\mathcal{D}(\leq\mathbf{z})$ which is invariant under the double jump is definable in $\mathcal{D}(\leq\mathbf{z})$ if and only if it is definable in true first order arithmetic.

2 Background Information

We begin our path to coding arithmetic into $\mathcal{D}(\leq \mathbf{z})$ (or any degree structure $\mathcal{S}$) with a specific highly effective form of coding orderings of type ω called nice effective successor structures introduced in Shore [1981]. They have been used as well in Nies, Shore and Slaman [1998] and Shore [2007] which contains (in §3) a good presentation. For our purposes all we need to know is that the scheme provides a way of coding a sequence $\langle \mathbf{d}_n \rangle$ of independent degrees (i.e. no $\mathbf{d}_n$ is below the join of the rest of the degrees $\mathbf{d}_m$) by finitely many parameters $\bar{\mathbf{q}}$ which generate a partial lattice including the $\mathbf{d}_n$. We assume that the first element $\bar{\mathbf{q}}_0$ of $\bar{\mathbf{q}}$ is a bound on all the other elements needed to determine this partial lattice. The crucial property of this coding is the following:

Proposition 2.1. *Given a $\bar{\mathbf{q}}$ determining a nice effective successor structure, the set of indices, relative to $Q_0 \in \bar{\mathbf{q}}_0$, for the degrees in the ideal generated by the $\mathbf{d}_n$ is $\Sigma_3^{Q_0}$ and any set S such that $S = \{n | \mathbf{d}_n \leq \mathbf{g}_0, \mathbf{g}_1\}$ for any $\mathbf{g}_0, \mathbf{g}_1 \leq \mathbf{q}_0$ is also $\Sigma_3^{Q_0}$. We say that the set S is coded (with respect to the structure determined by $\bar{\mathbf{q}}$) by the degrees $\mathbf{g}_0$ and $\mathbf{g}_1$. Moreover, for every $S \in \Sigma_3^X$ with $Q_0 \leq_T X$, the set of indices relative to X, for the ideal generated by $\{\mathbf{d}_n | n \in S\}$ is Σ_3^X.*

We next want to extend the sequence of parameters $\bar{\mathbf{q}}$ to one $\bar{\mathbf{p}}$ such that there are formulas with parameters $\bar{\mathbf{p}}$ providing an interpretation of arithmetic on the domain consisting of the $\mathbf{d}_n$ given by $\bar{\mathbf{q}}$ that identifies $\mathbf{d}_n$ with the nth element of this model, $\mathcal{M}(\bar{\mathbf{p}})$. (See Hodges [1993] for a general explanation of interpretations of one structure in another.) For our purposes, we just need formulas $\varphi_D(x)$, $\varphi_+(x,y,z)$, $\varphi_\times(x,y,z)$, $\varphi_<(x,y)$ all with parameters $\bar{\mathbf{p}}$ and an additional one $\varphi_c(\bar{\mathbf{p}})$ called a correctness condition that asserts at least that, for any $\bar{\mathbf{p}}$, the structure $\mathcal{M}(\bar{\mathbf{p}})$ with domain $D(\bar{p}) = \{x \in \mathcal{S} | \mathcal{S} \vDash \varphi_D(x)\}$ and relations $+$, $\times$ and $<$ defined by $\varphi_+(x,y,z)$, $\varphi_\times(x,y,z)$, $\varphi_<(x,y)$, respectively, is a model of some standard finite axiomatization of arithmetic and that the $\mathbf{d}_n$ form an initial segment of its domain. Providing the translation of the axioms of arithmetic is a general fact about interpretations as is saying that $\mathbf{d}_0$ is the 0 of the structure. Saying that the $\mathbf{d}_n$ form an initial segment is phrased by using the definition of the way $\mathbf{d}_{n+1}$ is generated (in terms of $\vee$ and $\wedge$) from the degrees in $\bar{\mathbf{q}}$. We also want to add a condition to φ_c that, in the degree structures we study, will guarantee that the models $\mathcal{M}(\bar{\mathbf{p}})$ for which $\bar{\mathbf{p}}$ satisfies it are all standard. As we code sets S in such models by exact pairs $\mathbf{g}_0$, $\mathbf{g}_1$ for the ideal generated by $\{\mathbf{d}_n | n \in S\}$, we can translate the sentence $x \in S$ into arithmetic in $\mathcal{M}(\bar{\mathbf{p}})$ by $\varphi_D(x)$ & $x \leq \mathbf{g}_0, \mathbf{g}_1$. (Degrees $\mathbf{g}_0$ and $\mathbf{g}_1$ are an exact pair for an ideal $\mathcal{I}$ if $\mathcal{I} = \{\mathbf{x}|\mathbf{x} \leq \mathbf{g}_0, \mathbf{g}_1\}$. Note also that by the independence of the $\mathbf{d}_n$ the ideal generated by $\{\mathbf{d}_n|n \in S\}$ contains no $\mathbf{d}_m$ for $m \notin S$.) We write this as $\varphi_S(x, \bar{\mathbf{g}}, \bar{\mathbf{p}})$. To say that $\mathcal{M}(\bar{\mathbf{p}})$ is standard, it suffices to say that every nonempty bounded subset of its domain as coded by a pair $\mathbf{g}_0$, $\mathbf{g}_1$ has a greatest element as long as the set $\{\mathbf{d}_n|n \in \mathbb{N}\}$ is so coded. To this end, we want a theorem that guarantees the existence of exact pairs for the ideals generated by $\Sigma_3^{Q_0}$ sets of indices. We give one which follows from Shore [1981] where it is proved for $\mathbf{a}$ r.e. in $\mathbf{b}$ and Ambos-Spies et al. [2009] or Cai and Shore [2012] where it is proven that, even for $\mathbf{a} \in \overline{\mathbf{GL}}_2(\mathbf{b})$, there is an $\hat{\mathbf{a}} \in [\mathbf{b}, \mathbf{a})$ with $\mathbf{a}$ r.e. in $\hat{\mathbf{a}}$.

Theorem 2.2. *If* $\mathbf{b} <_T \mathbf{a}$ *and* $\mathbf{a} \in \bar{\mathbf{L}}_2(\mathbf{b})$ *and* $\mathcal{I}$ *is a* Σ_3^B *ideal in* $\mathcal{D}(\leq \mathbf{b})$ *then there is an exact pair for* $\mathcal{I}$ *below* $\mathbf{a}$.

To see that this suffices for our added correctness condition to define a class of standard models requires a further condition that restricts $\bar{\mathbf{q}}_0$ to $\mathbf{L}_2$. The point then is that if $\bar{\mathbf{q}}_0 < \mathbf{z}$ and $\bar{\mathbf{q}}_0 \in \mathbf{L}_2$ then $\mathbf{z} \in \bar{\mathbf{L}}_2(\bar{\mathbf{q}}_0)$ and so we have enough exact pairs below $\mathbf{z}$ to define the standard part of any $\mathcal{M}(\bar{\mathbf{p}})$ with $\bar{\mathbf{p}} \in \mathcal{D}(\leq \mathbf{z})$. Once we argue that this is possible (Theorem 3.1), Theorem 2.2 and our calculation that the ideal generated by the $\mathbf{d}_n$ is $\Sigma_3^{Q_0}$ shows that our correctness condition picks out only standard models as long as $\bar{\mathbf{q}}_0 \in \mathbf{L}_2$. In the other direction, it also shows that (with these expanded correctness conditions) the sets coded in $\mathcal{M}(\bar{\mathbf{p}})$ by pairs below any $\mathbf{x} \geq \bar{\mathbf{q}}_0$ not in $\mathbf{L}_2$ are precisely those Σ_3^X. This suggests a route to the characterization of $\mathbf{x}''$ via the following standard fact.

Proposition 2.3. *For any sets* A *and* B *and* $n \in \mathbb{N}$, $A^{(n)} \equiv_T B^{(n)}$ *if and only if* $\Sigma_{n+1}^A = \Sigma_{n+1}^B$.

Proof. By the hierarchy theorem $\Sigma_{n+1}^X = \Sigma_1^{X^{(n)}}$, so if $A^{(n)} \equiv_T B^{(n)}$, then $\Sigma_n^A = \Sigma_n^B$. On the other hand, for any Z and W, $\Sigma_1^Z = \Sigma_1^W$ iff $Z \equiv_T W$, since the equality implies that both Z and $\bar{Z}$ (W and $\bar{W}$) are Σ_1, i.e. r.e., in W (Z) and so each is recursive in the other. Thus if $\Sigma_{n+1}^A = \Sigma_{n+1}^B$ then $\Sigma_1^{A^{(n)}} = \Sigma_1^{B^{(n)}}$ and so $A^{(n)} \equiv_T B^{(n)}$ as required. $\square$

We now turn to the notions of invariance under, and biinterpretability up to, the double jump.

Definition 2.4. A relation $R(x_1, \ldots x_n)$ on degrees is *invariant under the double jump* if, for all degrees $\mathbf{x}_1, \ldots \mathbf{x}_n$ and $\mathbf{y}_1, \ldots \mathbf{y}_n$ such that $\mathbf{x}_i'' = \mathbf{y}_i''$ for all $i \leq n$, $R(\mathbf{x}_1, \ldots \mathbf{x}_n) \Leftrightarrow R(\mathbf{y}_1, \ldots, \mathbf{y}_n)$.

Definition 2.5. A degree structure $\mathcal{S}$ is *biinterpretable with arithmetic* if it is interpretable in arithmetic and we have formulas in parameters $\bar{\mathbf{p}}$ (including a correctness condition) and a formula $\varphi_S(x, \bar{y})$ which defines sets (coded) in the model given by $\bar{\mathbf{p}}$ as described above which provide an interpretation of true arithmetic in $\mathcal{S}$ (i.e. the models $\mathcal{M}(\bar{\mathbf{p}})$ satisfying the correctness condition are all standard). Moreover, there is an additional formula $\varphi_R(x,\bar{y},\bar{\mathbf{p}})$ such that $\mathcal{S} \vDash \forall x \exists \bar{y} \varphi_R(x,\bar{y},\bar{\mathbf{p}})$ and for every $\mathbf{a}, \bar{\mathbf{g}} \in \mathcal{S}$, $\mathcal{S} \vDash \varphi_R(\mathbf{a},\bar{\mathbf{g}},\bar{\mathbf{p}})$ if and only if the set $\{n | \varphi_S(\mathbf{d}_n,\bar{\mathbf{g}},\bar{\mathbf{p}})\}$ (where $\mathbf{d}_n$ is the nth element of the model $\mathcal{M}(\bar{\mathbf{p}})$ coded by the parameters $\bar{\mathbf{p}}$) is of degree $\mathbf{a}$. These last conditions then say that the set coded in $\mathcal{M}(\bar{\mathbf{p}})$ by $\bar{\mathbf{g}}$ is of degree $\mathbf{a}$ and that all degrees $\mathbf{a}$ in $\mathcal{S}$ have codes $\bar{\mathbf{g}}$ for a set of degree $\mathbf{a}$. If we weaken the second condition on φ_R so that for every $\mathbf{a}, \bar{\mathbf{b}} \in S$, $\mathcal{S} \vDash \varphi_R(\mathbf{a},\bar{\mathbf{b}},\bar{\mathbf{p}})$ if and only if the set $\{n | \varphi_S(\mathbf{d}_n,\bar{\mathbf{b}},\bar{\mathbf{p}})\}$ has the same double jump as $\mathbf{a}$, we say that $\mathcal{S}$ is *biinterpretable with arithmetic up to double jump*

It is not hard to see that, if a degree structure $\mathcal{S}$ is biinterpretable with arithmetic up to double jump, then we know all there is to know about the definability in $\mathcal{S}$ of relations invariant under the double jump.

Theorem 2.6. *If a degree structure $\mathcal{S}$ is biinterpretable with arithmetic up to double jump, then a relation on $\mathcal{S}$ which is invariant under double jump is definable in $\mathcal{S}$ if and only if it is definable in arithmetic.*

Proof. Consider any definable relation $Q(\bar{x})$ on $\mathcal{S}$. By the assumption that $\mathcal{S}$ is interpretable in arithmetic, we know that Q is definable in arithmetic. For the other direction, suppose Q is definable by a formula $\Theta(\bar{X})$ of arithmetic with additional set variables $\bar{X}$ and relation $\in$ between numbers and elements of $\bar{X}$, i.e. Θ defines the property that the sequence of the degrees of $\bar{X}$ satisfies Q. Q is then defined in $\mathcal{S}$ by the formula $\Psi(\bar{z}) \equiv \exists \bar{p}, \bar{g}_0 \ldots \exists \bar{g}_{n-1}(\varphi_c(\bar{p})\ \&\ \bigwedge\limits_{i<n} \varphi_R(z_i,\bar{g}_i,\bar{p}) \to \Theta^T(\bar{g}_i,\bar{p}))$ where T is the translation of formulas of second order arithmetic given after Proposition 2.1. Here our correctness condition φ_c guarantees that the model $\mathcal{M}(\bar{p})$ is standard and we also assume that the requirements of the definition of biinterpretability are satisfied. So the translation of Θ asserts (because of the properties of φ_R) that a sequence of sets of degree $\mathbf{z}_i$ satisfy Θ (in $\mathbb{N}$), i.e. Q holds of $\bar{\mathbf{z}}$. $\square$

We also need some facts about forcing in arithmetic. The setting is standard. We consider only recursive notions of forcing, i.e. recursive partial orders $\mathcal{P}$ for which forcing sentences of the form $\Phi_e^{G \oplus A}(x) \downarrow = y$ is recursive in A for any set A. Hence forcing existential sentences (in A) is Σ_1 (Σ_1^A) and for universal ones it is Π_1 (Π_1^A). For a class $\mathcal{C}$ of dense sets, $\mathcal{C}$-generic filters $\mathcal{G}$ and $\mathcal{C}$-generic sequences $\langle p_k \rangle$ are defined as usual: they meet all the dense sets in $\mathcal{C}$. If $\mathcal{C}$ is the collection of all sets $D_e = \{p | p \Vdash \Phi_e^G(e) \downarrow$ or $(\forall q \leq_{\mathcal{P}} p)(q \nVdash \Phi_e^{G)}(e) \downarrow\}$ ($D_e = \{p | p \Vdash \Phi_e^{A \oplus G}(e) \downarrow$ or $(\forall q \leq_{\mathcal{P}} p)(q \nVdash \Phi_e^{A \oplus G}(e) \downarrow\}$) then the associated generic filters, sequences and objects G are called 1-generic (over A). We use the following results (and various relativizations):

Theorem 2.7 (Cai and Shore [2012]). *If $A \notin GL_2$ then for any recursive notion of forcing $\mathcal{P}$ and any collection $\mathcal{C}$ of dense sets uniformly recursive in $A \oplus 0'$ (such as those for 1-genericity) there is $\mathcal{C}$-generic sequence $\langle p_s \rangle \leq_T A$ and so the associated $\mathcal{C}$-generic G is also recursive in A.*

The next two theorems reduce constructions of Shore [1982] and Slaman and Woodin [1986], respectively, to constructing 1-generics for recursive notions of forcing. The last Proposition is a standard argument.

Theorem 2.8 (Greenberg and Montalbán [2004]). *Given a recursive partial lattice $\mathcal{L}$, there is a recursive notion of forcing for which any 1-generic G computes an embedding of $\mathcal{L}$ into the degrees below G which is uniformly recursive in G. So, in particular, one for which any 1-generic G computes degrees $\bar{\mathbf{q}}$ determining a nice effective successor structure in which the $\mathbf{d}_n$, and indeed all the elements of the embedded lattice, are uniformly recursive in G.*

Theorem 2.9 (Greenberg and Montalbán [2004]). *Given any $\mathbf{c}$ which uniformly bounds sets C_i and relations R_j on $\{\deg(C_i)\}$ there is a recursive notion of forcing for which any 1-generic over C computes degrees $\bar{\mathbf{p}}_j$ which code the relations R_j on the degrees of the C_i in the sense that there are fixed formulas φ_n independent of C, C_i and R_j) such that, if R_j is of parity n, $R_j(\bar{\mathbf{x}}) \Leftrightarrow \varphi_n(\bar{\mathbf{x}}, \bar{\mathbf{p}}_j)$ and, moreover, $\varphi_n(\bar{\mathbf{x}}, \bar{\mathbf{p}}_j)$ holds if and only if it holds in any (equivalently all) ideals containing the degrees in $\bar{\mathbf{p}}_j$.*

Proposition 2.10. *For any recursive notion of forcing $\mathcal{P}$, set A and 1-generic over A sequence $\langle p_k \rangle$ with an associated set G, $(A \oplus G)' \leq_T A' \oplus \langle p_k \rangle$.*

Proof. To decide if $e \in (A \oplus G)'$, find a k such that $p_k \in D_e$ (as in the definition of 1-genericity over A). This can clearly be done recursively in $A' \oplus \langle p_k \rangle$. Then A' can decide which clause of the definition of D_e holds and so if $e \in (A \oplus G)'$. $\square$

3 Biinterpretability up to double jump

We fix a degree $\mathbf{z} \leq \mathbf{0}'$ which is in $\overline{\mathbf{L}}_2$, work in $\mathcal{D}(\leq \mathbf{z})$ and plan to show that $\mathcal{D}(\leq \mathbf{z})$ is biinterpretable with arithmetic up to double jump. We want to show that there are degrees $\bar{\mathbf{p}} < \mathbf{z}$ which define a standard model of arithmetic and a correctness condition as described in §2 that guarantees that $\mathcal{M}(\bar{\mathbf{p}})$ is standard for any $\bar{\mathbf{p}}$ satisfying it.

We begin by defining the set $\{\mathbf{x} \leq \mathbf{z} | \mathbf{x} \in \mathbf{L}_2\}$. Of course, that it is definable would be an immediate corollary of biinterpretability up to double jump by Theorem 2.6. Now the proof of this specific result uses and illustrates several of the ingredients needed for the full result. They would suffice to define all the double jump classes in $\mathcal{D}(\leq \mathbf{z})$ and, with some additional effort, all the subsets of $\mathcal{D}(\leq \mathbf{z})$ invariant under double jump. More is needed for the full result but, most importantly, we actually need this special case to prove the full result.

The crucial point for both this special case and the full result is that the sets we can code below a $\overline{\mathbf{GL}}_2$ degree $\mathbf{x}$ are precisely the ones Σ_3^X and the class of sets Σ_3^X uniquely determines $\mathbf{x}''$ by Proposition 2.3.

Theorem 3.1. *The set $\mathbf{L}_2 = \{\mathbf{x} \leq \mathbf{z} | \mathbf{x}'' = \mathbf{0}''\}$ is definable in $\mathcal{D}(\leq \mathbf{z})$.*

Proof. Our analysis of coding in models of arithmetic in §2 shows that we have a way to, definably in $\mathcal{D}(\leq \mathbf{z})$, pick out, via correctness conditions, parameters $\bar{\mathbf{p}}$ that define structures $\mathcal{M}(\bar{\mathbf{p}})$ which are models of our finite axiomatization of arithmetic. We have a correctness condition in §2 which would pick out $\bar{\mathbf{p}}$ such that $\mathcal{M}(\bar{\mathbf{p}})$ is standard if we also knew that $\bar{\mathbf{p}}_0 \in \mathbf{L}_2$. Now consider then the formula $\psi(x)$ which says of an $\mathbf{x} < \mathbf{z}$ that for any $\bar{\mathbf{p}}$ satisfying this correctness condition with $\bar{\mathbf{p}}_0 < \mathbf{x}$ any set S coded in $\mathcal{M}(\bar{\mathbf{p}})$ by a pair $\mathbf{g}_0, \mathbf{g}_1 < \mathbf{x}$ satisfies the (translation of) the formula of arithmetic saying that $S \in \Sigma_3$. We claim that $\mathbf{x} \in \mathbf{L}_2$ if and only this formula holds of $\mathbf{x}$.

First suppose that $\mathbf{x} \in \mathbf{L}_2$ and consider any $\bar{\mathbf{p}}$ as described by $\psi(x)$. As $\bar{\mathbf{p}}_0 < \mathbf{x}$, there is a pair $\mathbf{g}_0, \mathbf{g}_1 < \mathbf{z}$ (not necessarily below $\mathbf{x}$) defining the standard part of the model by Proposition 2.1 and Theorem 2.2 as $\mathbf{z} \in \bar{\mathbf{L}}_2(\bar{\mathbf{p}}_0)$ and so the model must be standard. In this case, only $\Sigma_3^{P_0} = \Sigma_3$ sets can be defined in the model by codes below $\mathbf{x}$ by Proposition 2.1 and so $\mathbf{x}$ satisfies ψ

Next, if $\mathbf{x} \notin \mathbf{L}_2$, then by Theorems 2.8 and 2.7 and Proposition 2.10, there are parameters $\bar{\mathbf{q}}$ defining a nice effective successor model with $\bar{\mathbf{q}}_0 < \mathbf{x}$ and $\bar{\mathbf{q}}_0' = \mathbf{0}'$. By Theorems 2.9 and 2.7, we can extend these parameters to ones $\bar{\mathbf{p}}$ defining a standard model of arithmetic which, of course, satisfies the correctness condition for being standard and, as $\bar{\mathbf{p}}_0' = \bar{\mathbf{q}}_0' = \mathbf{0}'$, $\psi(\bar{\mathbf{p}}_0)$ holds as well. Thus $\mathbf{x} \in \bar{\mathbf{L}}_2(\bar{\mathbf{p}}_0)$ and so every set $S \in \Sigma_3^X$ is coded in $\mathcal{M}(\bar{\mathbf{p}})$ by some $\mathbf{g}_0, \mathbf{g}_1 \leq_T \mathbf{x}$ by Proposition 2.1 and Theorem 2.2. Since $\mathbf{x}'' > \mathbf{0}''$ there is an $S \in \Sigma_3^X - \Sigma_3$ by Proposition 2.3 and so a code for such an S below $\mathbf{x}$ as required to show that $\mathbf{x}$ does not satisfy $\psi(x)$. □

Corollary 3.2. *The theory of $\mathcal{D}(\leq \mathbf{z})$ is recursively equivalent to that of true arithmetic, $Th(\mathbb{N})$.*

Proof. By the arguments in the proof of the theorem we can pick out the standard models by our correctness conditions. Thus we can interpret arithmetic in $\mathcal{D}(\leq \mathbf{z})$ and so $Th(\mathbb{N}) \leq_{1-1} Th(\mathcal{D}(\leq \mathbf{z}))$. The other directions follows as we have pointed out from the interpretability of $\mathcal{D}(\leq \mathbf{z})$ in arithmetic. □

To discuss even the binary relation of two degrees having the same double jump given by biinterpretability up to double jump, we must have a way to analyze, for any $\mathbf{a}$ and $\mathbf{b}$ the sets coded below them in a single common model. This is the crux of biinterpretability to which we now turn.

Our plan for being able to talk about the sets coded in one model $\mathcal{M}(\bar{\mathbf{p}})$ in another $\mathcal{M}(\bar{\mathbf{p}}')$ is similar in outline to that used in Nies, Shore and Slaman [1998] for $\mathcal{R}$. We provide a scheme defining isomorphisms between two arbitrary standard models satisfying the standard correctness condition plus the requirement that $\bar{\mathbf{p}}_0$ (and $\bar{\mathbf{p}}_0'$) are in $\mathbf{L}_2$ which is definable in $\mathcal{D}(\leq \mathbf{z})$ by Theorem 3.1. We call this the strong correctness condition and assume from now on that any parameters $\bar{\mathbf{p}}$ used to define a model $\mathcal{M}(\bar{\mathbf{p}})$ satisfy it.

Such isomorphisms would allow us to definably transfer assertions about (codes for) sets in different models to ones expressing the same facts about the same sets in a single model and so define the required relation φ_R. More precisely, our procedure provides a formula $\theta(x, y, \bar{z}, \bar{z}')$ such that for any $\bar{\mathbf{p}}$ and $\bar{\mathbf{p}}'$ satisfying the strong correctness condition $\theta(\mathbf{n}, \mathbf{m}, \bar{\mathbf{p}}, \bar{\mathbf{p}}')$ holds if and only if $\mathbf{n}$ and $\mathbf{m}$ represent the same natural number in $\mathcal{M}(\bar{\mathbf{p}})$ and $\mathcal{M}(\bar{\mathbf{p}}')$, respectively. Using this formula, we can talk about a set S coded in $\mathcal{M}(\bar{\mathbf{p}}')$ by parameters $\bar{\mathbf{g}}$ inside $\mathcal{M}(\bar{\mathbf{p}})$ by, in our translation of arithmetic, replacing $\varphi_S(x, \bar{\mathbf{g}}, \bar{\mathbf{p}}')$ (which says the number represented by x in $\mathcal{M}(\bar{\mathbf{p}}')$ is in S) by $\exists y[\varphi_S(y, \bar{\mathbf{g}}, \bar{\mathbf{p}}') \mathbin{\&} \theta(x, y, \bar{\mathbf{p}}, \bar{\mathbf{p}}')]$ which then says that the number represented in $\mathcal{M}(\bar{\mathbf{p}})$ by x is in S. Thus we can talk about the set S defined in $\mathcal{M}(\bar{\mathbf{p}}')$ in $\mathcal{M}(\bar{\mathbf{p}})$ as desired.

Arguments similar to those in the proof of 3.1 then allow us to identify degrees up to double jump with the sets coded below them in appropriate models. The transfer procedure we have just suggested then allows us to make the identification in any model and so define the formula φ_R required for biinterpretability up to double jump.

We begin with a lemma used to build such isomorphisms by interpolating a sequence of additional models between the two given ones and isomorphisms between each successive pair of such models.

Lemma 3.3. *If $\mathbf{d} \leq \mathbf{0}'$, $\mathbf{d} \in \overline{\mathbf{L}}_2$, $\mathbf{a}_0, \mathbf{a}_1 \in \mathbf{L}_1$ and $\mathcal{P}$ is a recursive notion of forcing, then there is a $G \leq_T D$ which is 1-generic for $\mathcal{P}$ and such that $(A_0 \oplus G)' \equiv_T 0' \equiv_T (A_1 \oplus G)'$.*

Proof. Let $\mathcal{C}$ consist of the dense sets for 1-genericity over A_0 and over A_1 (each separately). As $A_0' \equiv_T A_1' \equiv_T 0'$, these sets are uniformly recursive in $0'$ and so there is a $\mathcal{C}$-generic sequence recursive in $\mathbf{d}$ by Theorem 2.7. Proposition 2.10 then says that $(A_0 \oplus G)' \equiv_T 0' \equiv_T (A_1 \oplus G)'$. □

We now turn to our main result.

Theorem 3.4. *$\mathcal{D}(\leq \mathbf{z})$ is biinterpretable with arithmetic up to double jump.*

Proof. Given two models $\mathcal{M}(\bar{\mathbf{p}}_0)$ and $\mathcal{M}(\bar{\mathbf{p}}_4)$ we want to show that there are additional models $\mathcal{M}(\bar{\mathbf{p}}_k)$ for $k \in \{1, 2, 3\}$ and uniformly definable isomorphisms between the domain of these models taking $\mathbf{d}_{i,n}$ to $\mathbf{d}_{i+1,n}$ for $i < 4$. (Given parameters $\bar{\mathbf{p}}_k$ defining a model $\mathcal{M}(\bar{\mathbf{p}}_k)$ we write $\mathbf{d}_{k,n}$ for the degree representing the nth element of this model. Similarly, we write $\bar{\mathbf{p}}_{k,0}$ for the first element of $\bar{\mathbf{p}}_k$ and $\bar{\mathbf{q}}_k$ for the parameters in $\bar{\mathbf{p}}_k$ determining the effective successor structure which provides the domain of $\mathcal{M}(\bar{\mathbf{p}}_k)$.) Thus (as we explain below) we produce a single formula $\theta(x, y, \bar{z}, \bar{z}')$ which uniformly defines isomorphisms between any two of our models $\mathcal{M}(\bar{\mathbf{p}}_0)$ and $\mathcal{M}(\bar{\mathbf{p}}_4)$ as described above (with $\bar{z}$ and $\bar{z}'$ replaced by $\bar{\mathbf{p}}_0$ and $\bar{\mathbf{p}}_4$).

We begin by choosing $\bar{\mathbf{q}}_1 < \mathbf{z}$ as given by a 1-generic over $\mathbf{p}_{0,0}$ sequence for the recursive notion of forcing of Theorem 2.8. As $\mathbf{p}_{0,0} \in \mathbf{L}_2$, $\mathbf{z}$ is $\overline{\mathbf{L}}_2(\mathbf{p}_{0,0})$ and so such $\bar{\mathbf{q}}_1$ exists by Theorem 2.7 (relativized to $\mathbf{p}_{0,0}$). Note that $\bar{\mathbf{q}}_1$ (and so $\bar{\mathbf{q}}_{1,0}$) is in $\mathbf{L}_1$ by Proposition 2.10 as it is associated with a 1-generic sequence recursive in $\mathbf{z}$. We may now extend $\bar{\mathbf{q}}_1$ to $\bar{\mathbf{p}}_1$ defining a standard model $\mathcal{M}(\bar{\mathbf{p}}_1)$ by Theorem 2.9 and Theorem 2.7 as $\mathbf{z}$ is $\overline{\mathbf{GL}}_2(\bar{\mathbf{q}}_1)$. Similarly, we see that there are $\bar{\mathbf{q}}_3$ and $\bar{\mathbf{p}}_3$ bearing the same relation to $\mathcal{M}(\mathbf{p}_4)$ as $\bar{\mathbf{q}}_1$ and $\bar{\mathbf{p}}_1$ do to $\mathcal{M}(\mathbf{p}_0)$. Now as $\bar{\mathbf{q}}_{1,0}$ and $\bar{\mathbf{q}}_{3,0}$ are both low we may apply Lemma 3.3 to the forcing of Theorem 2.8 to get $\bar{\mathbf{q}}_2 < \mathbf{z}$ (again as $\mathbf{z} \in \bar{\mathbf{L}}_2(\bar{\mathbf{q}}_{1,0}), \bar{\mathbf{L}}_2(\bar{\mathbf{q}}_{3,0})$) such that both $\bar{\mathbf{q}}_{1,0} \oplus \bar{\mathbf{q}}_{2,0}$ and $\bar{\mathbf{q}}_{2,0} \oplus \bar{\mathbf{q}}_{3,0}$ are in $\mathbf{L}_1$ and then extend $\bar{\mathbf{q}}_2$ to $\bar{\mathbf{p}}_2$ defining $\mathcal{M}(\bar{\mathbf{p}}_2)$ as we did for $\bar{\mathbf{q}}_1$.

We now apply Theorem 2.9 and Theorem 2.7 again (many times) to get the desired schemes defining our desired isomorphisms: Given any $n \in \mathbb{N}$ and $i < 4$, consider the finite sequences of degrees $\langle \mathbf{d}_{i,0}, \ldots, \mathbf{d}_{i,n} \rangle$ and $\langle \mathbf{d}_{i+1,0}, \ldots, \mathbf{d}_{i+1,n} \rangle$. We want to show that there are parameters $\bar{\mathbf{r}}_i < \mathbf{z}$ such that the formula $\varphi_2(x, y, \bar{\mathbf{r}}_i)$ (where $\varphi_2(x, y, \bar{z})$ ranges over binary relations as $\bar{z}$ varies as in Theorem 2.9) defines an isomorphism taking $\mathbf{d}_{i,k}$ to $\mathbf{d}_{i+1,k}$ for each $k \leq n$. By the results just cited it suffices to show that the $\bigoplus_{k<n} \mathbf{d}_{i,k} \oplus \bigoplus_{k<n} \mathbf{d}_{i+1,k}$ are in $\mathbf{L}_2$ for each $i < 4$. For $i = 0$, note that $\bar{\mathbf{q}}_1$ is associated with a 1-generic/$\mathbf{p}_{0.0}$ sequence which is recursive in $\mathbf{z}$. Thus by Proposition 2.10 (suitably relativized) $(\bar{\mathbf{q}}_1 \oplus \mathbf{p}_{0,0})' = \mathbf{p}'_{0,0}$ and so $(\bar{\mathbf{q}}_{1,0} \oplus \bar{\mathbf{p}}_{0,0})' = \mathbf{p}'_{0,0}$. As $\mathbf{p}_{0,0} \in \mathbf{L}_2$, $\mathbf{0}'' = \bar{\mathbf{p}}''_{0,0} = (\bar{\mathbf{q}}_{1,0} \oplus \bar{\mathbf{p}}_{0,0})''$ as required. The argument for $i = 3$ is similar. For the other pairs, we have already guaranteed that $\bar{\mathbf{q}}_{1,0} \oplus \bar{\mathbf{q}}_{2,0}$ and $\bar{\mathbf{q}}_{2,0} \oplus \bar{\mathbf{q}}_{3,0}$ are in $\mathbf{L}_1$. (Note that we could not directly define the isomorphisms between $\mathcal{M}(\bar{\mathbf{p}}_1)$ and $\mathcal{M}(\bar{\mathbf{p}}_3)$ as the sequences of degrees required in the argument above would be recursive only in $\bar{\mathbf{p}}_{1,0} \oplus \bar{\mathbf{p}}_{3,0}$ which could be $\mathbf{z}$ and so not in $\mathbf{L}_2$. It is by the use of Proposition 2.10 and the model $\mathcal{M}(\bar{\mathbf{p}}_3)$ that we can make all the required joins be in $\mathbf{L}_2$ as required to apply Theorem 2.7.)

We can now define the desired isomorphism $\theta(\mathbf{n}, \mathbf{m}, \bar{\mathbf{p}}_0, \bar{\mathbf{p}}_4)$ between $\mathcal{M}(\bar{\mathbf{p}}_0)$ and $\mathcal{M}(\bar{\mathbf{p}}_4)$. We say that an $\mathbf{n}$ in the domain of $\mathcal{M}(\bar{\mathbf{p}}_0)$ (i.e. $\varphi_D(\mathbf{n}, \bar{\mathbf{p}}_0)$) is taken to $\mathbf{m}$ in the domain of $\mathcal{M}(\bar{\mathbf{p}}_4)$ (with $\bar{\mathbf{p}}_{0,0}, \bar{\mathbf{p}}_{4,0} \in \mathbf{L}_2$) if and only if there are degrees $\bar{\mathbf{p}}_k$ for $k \in \{1, 2, 3\}$ defining models of arithmetic $\mathcal{M}(\bar{\mathbf{p}}_k)$ and ones $\bar{\mathbf{r}}_i$ for $i < 4$ as above such that each $\varphi_2(x, y, \bar{\mathbf{r}}_i)$ defines an isomorphism between initial segments of (the domains of) $\mathcal{M}(\bar{\mathbf{p}}_i)$ and $\mathcal{M}(\bar{\mathbf{p}}_{i+1})$ where the initial segment in $\mathcal{M}(\bar{\mathbf{p}}_0)$ is the one with largest element $\mathbf{n}$ and that in $\mathcal{M}(\bar{\mathbf{p}}_4)$ has largest element $\mathbf{m}$. Clearly this can all be expressed using the formulas $\varphi_D(x, \bar{\mathbf{p}}_k)$ and $\varphi_<(x, y, \bar{\mathbf{p}}_k)$ defining the domains of $\mathcal{M}(\bar{\mathbf{p}}_k)$ and the orderings on them. Note that the definition of this isomorphism is uniform in $\bar{\mathbf{p}}_0$ and $\bar{\mathbf{p}}_4$ and that we have shown that, for any $\bar{\mathbf{p}}_0$ and $\bar{\mathbf{p}}_4$ defining standard models as guaranteed by our strong correctness condition, there are parameters below $\mathbf{z}$ defining all these isomorphisms. In other words, we have described the desired formula $\theta(x, \bar{y}, \bar{z}, \bar{z}')$.

We now wish to define the formula $\varphi_R(x, \bar{y}, \bar{\mathbf{p}}_0)$ required in the definition of biinterpretability up to double jump. (We have replaced $\bar{\mathbf{p}}$ in Definition 2.5 by $\bar{\mathbf{p}}_0$ to match our current notation.) First, φ_R says that, if $x \in \mathbf{L}_2$ (as defined by Theorem 3.1), then $\bar{y}$ codes (via our standard φ_S) the empty set in $\mathcal{M}(\bar{\mathbf{p}}_0)$. In addition, φ_R says that, if $x \notin \mathbf{L}_2$ and S is the set coded in $\mathcal{M}(\bar{\mathbf{p}}_0)$ by $\bar{y}$, then for every set $\hat{S} \in \Sigma_3^S$ (as given by a definition in arithmetic in $\mathcal{M}(\bar{\mathbf{p}}_0)$ using the definition of S via $\varphi_S(x, \bar{y}, \mathbf{p}_0)$), there are $\bar{g}{<}x$ and $\bar{p}_4$ with $\bar{p}_{4,0} < x$ such that $\bar{g}$ codes a set $\hat{S}_4$ in $\mathcal{M}(\bar{p}_4)$ and $\theta(n,m, \bar{\mathbf{p}}_0, \bar{\mathbf{p}}_4)$; then, n satisfies the definition of $\hat{S}$ in $\mathcal{M}(\bar{\mathbf{p}}_0)$ if and only if $\varphi_S(m,\bar{g}, \bar{\mathbf{p}}_4)$, i.e. $\hat{S} = \hat{S}_4$. By all that we have done already, this guarantees that every $\hat{S} \in \Sigma_3^S$ is Σ_3^X. For the other direction, φ_R also says that if $\bar{g}{<}x$ and $\bar{p}_4$ with $\bar{p}_{4,0} < x$ are such that $\bar{g}$ codes a set $\hat{S}_4$ in $\mathcal{M}(\bar{p}_4)$

then there is a set $\hat{S}$ (coded in $\mathcal{M}(\bar{\mathbf{p}}_0)$ by some $\bar{h}$) which is Σ_3^S (as defined in arithmetic in $\mathcal{M}(\bar{\mathbf{p}}_0)$) such that $\hat{S} = \hat{S}_4$ as expressed as above using θ. So again by what we have already done, this guarantees that every $\hat{S}_4 \in \Sigma_3^X$ is Σ_3^S. Thus, by Proposition 2.3, S has the same double jump as X as required. $\square$

Theorem 2.6 now gives us our desired result on definability in $\mathcal{D}(\leq \mathbf{z})$.

Theorem 3.5. *A relation on $\mathcal{D}(\leq \mathbf{z})$ which is invariant under the double jump is definable in $\mathcal{D}(\leq \mathbf{z})$ if and only if it is definable in true first order arithmetic. In particular the jump classes $\mathbf{L}_n$ ($\mathbf{x}^{(n)} = 0^{(n)}$) and $\mathbf{H}_n$ ($\mathbf{x}^{(n)} = 0^{(n+1)}$) are definable in $\mathcal{D}(\leq \mathbf{z})$ for $n \geq 2$.*

We can draw one more conclusion about definability of one of the two jump classes not invariant under double jump ($\mathbf{H}_1$ and $\mathbf{L}_1$).

Corollary 3.6. *$\mathbf{H}_1$ is definable in $\mathcal{D}(\leq \mathbf{z})$.*

Proof. This follows from the fact that $\mathbf{x} < \mathbf{0}'$ is in $\mathbf{H}_1$ if and only if $\forall w \leq \mathbf{0}' \exists y \leq \mathbf{x}(w'' = y'')$ (Nies, Shore and Slaman [1998, Theorem 2.21 and the remarks there on p. 257]). We can now capture this characterization in $\mathcal{D}(\leq \mathbf{z})$ using our results. An $\mathbf{x} \leq \mathbf{z}$ is in $\mathbf{H}_1$ if and only if there is a $\bar{\mathbf{p}}$ such that in $\mathcal{M}(\bar{\mathbf{p}})$ for every set $W \leq_T 0'$ (with W coded by any $\bar{\mathbf{h}}$ and its being recursive in $0'$ expressed in arithmetic in $\mathcal{M}(\bar{\mathbf{p}})$) there is a $\mathbf{y} \leq \mathbf{x}$ such that the sets coded by $\bar{\mathbf{g}} < \mathbf{y}$ in models $\mathcal{M}(\bar{\mathbf{p}}')$ with $\bar{\mathbf{p}}'$ satisfying the strong correctness condition and with $\bar{\mathbf{p}}'_0 < \mathbf{y}$ are precisely the sets Σ_3^W (and so $\mathbf{y}'' = \mathbf{w}''$). Our isomorphisms between any two models $\mathcal{M}(\bar{\mathbf{p}})$ and $\mathcal{M}(\bar{\mathbf{p}}')$ allows us to express this formula in $\mathcal{D}(\leq \mathbf{z})$ and our results on what sets can be coded below any $\mathbf{y}$ show that we get precisely those which are Σ_3^Y and so the formula says that $\mathbf{y}'' = \mathbf{w}''$ as required. $\square$

The definability of $\mathbf{L}_1$ in $\mathcal{D}(\leq \mathbf{z})$ (or even $\mathcal{D}(\leq \mathbf{0}')$) and, more generally, full biinterpretability, even up to single jump, are major open questions.

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