The Strength of Turing Determinacy within Second Order Arithmetic

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Abstract

We investigate the reverse mathematical strength of Turing determinacy up to Σ_5^0 which is itself not provable in second order arithmetic.

1 Introduction

Reverse mathematics endeavors to calibrate the complexity of mathematical theorems by determining precisely which system P of axioms are needed to prove a given theorem Θ. This is done in one direction in the usual way showing that $P \vdash \Theta$. The other direction is a "reversal" that shows that relative to some weak base theory $\Theta \vdash P$. Here one works in the setting of second order arithmetic, i.e. the usual first order language and structure $\langle M, +, \times, <, 0, 1\rangle$ supplemented by distinct variables X, Y, Z that range over

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a collection S of subsets of the domain M of the first order part and the membership relation $\in$ between elements of M and S. Most of countable or even separable classical mathematics can be developed in this setting based on very elementary axioms about the first order part of the model $\mathcal{M}$, an induction principle for sets and various set existence axioms. At the bottom one has the weak system of axioms called RCA_0 that correspond to recursive constructions. One typically then adds additional comprehension (i.e. existence) axioms to get other systems P. Many of these systems are given by Γ comprehension (Γ-CA_0) which is gotten from RCA_0 by adding on the axiom that all sets defined by formulas in some class Γ exist. So one gets ACA_0 for Γ the class or arithmetic formulas and Π^1_n-CA_0 for Γ the class of all Π^1_n formulas. (In each case the formulas may contain set parameters.) Full second order arithmetic, Z_2, is the union of all the Π^1_n-CA_0. The standard text here is Simpson [2009] to which we refer the reader for general background.

This paper is concerned with the analysis of various principles connected with axioms of determinacy. This subject has played an important role historically as an inspiration for increasingly strong axioms (as measured by consistency strength) both in reverse mathematics and set theory. We have given a brief overview of this history in §1 of Montalbán and Shore [2012] (henceforth denote by MS [2012]) and refer the reader to that paper for more historical details and other background for both reverse mathematics and determinacy. Here we give some basic definitions and cite a few results.

Definition 1.1 (Games and Determinacy)**.** Our *games* are played by two players I and II on $\{0,1\}$ [or ω]. They alternate *playing* an element of $\{0,1\}$ [or ω] with I playing first to produce a *play of the game* which is a sequence $f \in 2^\omega$ [ω^ω]. A *game* G_A *is specified* by a subset A of 2^ω [ω^ω].We say that I *wins a play* f *of the game* G_A *specified by* A if $f \in A$. Otherwise II wins that play.

Definition 1.2. A *strategy* for I (II) is a function s from strings σ in $2^{<\omega}$ [$\omega^{<\omega}$] of even (odd) length into $\{0,1\}$ [ω]. It is a *wining strategy* if any play f following it (i.e. $f(n) = s(f \restriction n)$ for every even (odd) n) is a win for for I (II). We say that the game G_A *is determined* if there is a winning strategy for I or II in this game. If Γ is a class of sets A, then we say that Γ *is determined* if G_A is determined for every $A \in \Gamma$. We denote the assertion that Γ is determined by Γ *determinacy* or Γ-DET.

The classical reverse mathematical results are (essentially Steel [1976] see also Simpson [2009 V.8]) that Σ^0_1-DET is equivalent to ATR_0 a system asserting the existence of transfinite iterations of arithmetic comprehension that lies strictly between ACA_0 and Π^1_1-CA_0; and (Tanaka [1990]) that Π^1_1-CA_0 is equivalent to determinacy for conjunctions of Π^0_1 and Σ^0_1 sets. Results on Π^0_2, Δ^0_3 and Π^0_3 determinacy (Tanaka [1991], MedSalem and Tanaka [2007] and Welch [2011]) show that they are significantly stronger with the last provable in Π^1_3-CA_0 but not Δ^1_3-CA_0. Friedman [1971] in what was really the first foray into reverse mathematics, proved that Σ^0_5-DET is not provable in full second order arithmetic and Martin [1974a], [n.d.] improved this to Σ^0_4-DET.

In MS [2012] we delineated the limits of determinacy provable in Z_2 as encompassing each level of the finite difference hierarchy on Π^0_3 sets. Indeed each level n of the this hierarchy is provable from Π^1_{n+2}-CA_0 but not at any lower level of the comprehension axiom hierarchy. (So the union of the hierarchy (which is far below Δ^0_4) is not provable in Z_2.) Then, in Montalbán and Shore [2014] (hereafter MS [2014]) we analyzed the consistency strength of all these statements, getting a much clearer picture. In this paper we analyze, to the extent we can, the reverse mathematical strength of a variation on determinacy where one is thinking of the underlying space as the Turing degrees in place of 2^ω or ω^ω.

Definition 1.3. An $A \subseteq 2^\omega[\omega^\omega]$ is *Turing invariant* or *degree closed* if $(\forall f \in 2^\omega[\omega^\omega])(\forall g \in 2^\omega[\omega^\omega])(f \equiv_T g \rightarrow (f \in A \leftrightarrow g \in A))$. We denote by Γ *Turing determinacy* or Γ-TD the assertion that every degree closed $A \in \Gamma$ is determined.

Remark 1.4. For any reasonable Γ including each of the Σ^0_n classes, it is clear that Γ-DET is equivalent (in RCA_0) to $\breve{\Gamma}$-DET where $\breve{\Gamma} = \{\bar{A} | A \in \Gamma\}$. So we can use these two assertions interchangeably and similarly for Γ-TD. We also note that while it is easy to code sets as functions recursively (and so determinacy or Turing determinacy for classes in ω^ω imply the corresponding result for 2^ω) the converse is not obvious at the very lowest level. However, for any of the arithmetic classes at or above Δ^0_3, it does not matter for determinacy or Turing determinacy if we work in 2^ω or ω^ω as we can code functions in ω^ω by sets in 2^ω as long as we include the Π^0_2 condition that the sets are infinite. So once we are at that level we work in whichever setting is more convenient.

It is a classical theorem of Martin that a degree closed set A is determined if and only if it contains a *cone*, i.e. a set of Turing degrees of the form $\{\mathbf{x} | \mathbf{x} \geq \mathbf{z}\}$ for some degree $\mathbf{z}$ called the *base of the cone* or is disjoint from a cone. (In the first case I has a winning strategy; in the second, II.) In the realm of set theory, this induces a $0-1$ valued measure on sets of degrees (with measure 1 corresponding to containing a cone). This result is the basis for many interesting set theoretic investigations. The question of the relationship between determinacy axioms and Turing determinacy axioms is an interesting one in the set theoretic setting. Perhaps the most striking early result is that for $\Gamma = \Sigma^1_1$ the two notions coincide and and are equivalent with the axiom asserting the existence of $x^\#$ for every $x \in \omega^\omega$ (Martin [1970] and Harrington [1978]). At the level of determinacy for all sets, later work by Woodin showed that for full determinacy and Turing determinacy are not only equiconsistent but are equivalent (over DC) in $L(\mathbb{R})$. (See Koellner and Woodin [2010 and other articles in the same handbook] for this and much more on the role of TD in set theory.) The main results for Turing determinacy at lower levels of the arithmetic hierarchy show some differences from full determinacy at the same levels. There are a few classical ones given in Harrington and Kechris [1975], primarily from recursion theoretic or ZFC points of view and Martin [1974] and [n. d.] from the viewpoint of working in ZFC without the power set axiom and replacement only for Σ_1 formulas.

Their results either directly, or can be refined to, give ones in reverse mathematics. In this paper we present them from the viewpoint of reverse mathematics and fill in some of the gaps. We begin with determining how much Turing determinacy is provable in weak systems. The base theory RCA_0 proves Π^0_2-TD (Theorem 2.5). The next step is, of course, Δ^0_3-TD.

The standard tool in analyzing the Δ^0_{n+1} levels is the finite level version of a classical theorem appearing in Kuratowski [1966]. It gives a representation of Δ^0_{n+1} subsets of 2^ω in terms of the transfinite difference hierarchy on Π^0_n or Σ^0_n sets. One can then use determinacy at the lower level to bootstrap up to Δ^0_{n+1}. There are various formulations and we state a couple of variants. That this theorem can be proven for $n \in \omega$ in ACA_0 with some extra recursion theoretic conclusions is due to MedSalem and Tanaka [2007]. Our notation is slightly different from theirs. It follows more closely that used by Martin [1974, 1974a, 1974b, n. d.]. We also incorporate a few normalizations of the sequences that appear in different presentations.

Theorem 1.5 (Kuratowski; Martin; MedSalem and Tanaka for ACA_0)**.** *For any $Z \in 2^\omega$, a set $A \subseteq 2^\omega$ is Δ^Z_{n+1} if and only if there is an ordinal α recursive in Z and a sequence of uniformly Π^Z_n sets A_ξ for $\xi \leq \alpha$ which are decreasing ($A_\eta \supseteq A_\xi$ for $\eta < \xi$), continuous (for limit ordinals λ, $A_\lambda = \cap\{A_\eta | \eta < \lambda\}$) with $A_0 = 2^\omega$ and $A_\alpha = \emptyset$ such that $\forall X(X \in A \Leftrightarrow \mu\beta(X \notin A_\beta$ is odd). Dually (by taking complements) $A \in \Delta^Z_{n+1}$if and only if there is an ordinal α recursive in Z and a sequence of uniformly Σ^Z_n sets A_ξ for $\xi \leq \alpha$ which are increasing ($A_\eta \subseteq A_\xi$ for $\eta < \xi$), continuous (for limit ordinals λ, $A_\lambda = \cup\{A_\eta | \eta < \lambda\}$) with $A_0 = \emptyset$ and $A_\alpha = 2^\omega$ such that $\forall X(X \in A \Leftrightarrow \mu\beta(X \in A_\beta$ is even).*

Remark 1.6. If a Δ^0_n set A is degree invariant and $n \geq 3$ then, in the above Σ^0_n representation, we may take the A_ξ to be degree invariant as well. The first point here is that $\leq_T$ is a Σ^0_3 relation and so if A is Σ^0_n then so is its Turing closure $\hat{A} = \{f | \exists e(\Phi^f_e \in A \;\&\; \exists i(\Phi_i^{\Phi^f_e} = f)\}$. The second point is that $\hat{A}$ still gives a representation of A: If ξ is the first with X in the degree closed version $\hat{A}_\xi$ of A_ξ, then some $Y \equiv_T X$ is in A_ξ and not in any $A_\eta \subseteq \hat{A}_\eta$ and so in A.

This theorem allows us to prove Δ^0_3-TD at the expense of moving from RCA_0 to ACA_0 (Theorem 2.6). We point out that there can be no reversals from any Turing determinacy assumption to any system stronger than RCA_0. The key fact here is that the standard model of RCA_0 with just the recursive sets (or the sets recursive in any X) is obviously a model of Γ-TD for any Γ. Thus we can hope for implications from any Γ-TD only over stronger systems. In this case, we can, however, prove that Δ^0_3-TD is not provable in RCA_0 (Proposition 2.8). This supplies a natural principle that lies strictly between RCA_0 and ACA_0 but does not imply the existence of a nonrecursive set.

We next move on to Σ^0_3-TD. Combining the implication from ATR_0 to Σ^0_1-DET (Steel [1976] in RCA_0) and from Σ^0_1-DET to Σ^0_3-TD (Harrington an Kechris [1975]) we see that $\mathsf{ATR}_0 \vdash \Sigma^0_3$-$\mathsf{TD}$. In this case, we prove a reversal over ACA_0 (Theorem 3.7). This supplies an example of a natural theory strictly weaker than ATR_0 (and indeed does not even imply the existence of a nonrecursive set) but which joins ACA_0 up to it. In particular, Σ^0_1-DET is equivalent to Σ^0_3-TD over ACA_0.

Using the representation of Theorem 1.5, we can now hope to prove Δ^0_4-TD in ATR_0. We do so in Theorem 3.3 but we need an additional induction axiom.

Definition 1.7. For S a class of formulas, *S transfinite induction*, S-TI, is the scheme of axioms stating that for every well-ordering α (formally coded as a set X of ordered pairs $\langle\beta,\gamma\rangle$ prescribing its ordering relation $<_X$ on its domain which is also a subset of ω) and every formula $\varphi \in S$,
$\{(\forall\gamma)[((\forall\beta <_X \gamma)\varphi(\beta)) \ \rightarrow \ \varphi(\gamma))] \ \rightarrow \ (\forall\beta <_X \alpha)\varphi(\beta)\}$.

The version that we need to prove Δ^0_4-TD over ACA_0 in Theorem 3.3 is Π^1_1-TI_0. Over ACA_0 this axioms scheme is equivalent to the dependent choice axiom for Σ^1_1 formulas (Simpson [2009, VIII.5.12]) and so provable in Π^1_1-CA_0 but not in ATR_0.

As our last stop inside Z_2, we analyze Σ^0_4-TD and Δ^0_5-TD based on results of Harrington and Kechris [1975], Martin [1974] and Welch [2011] to show that Π^1_3-CA_0 proves both. We can have no meaningful reversal even over relatively strong systems. Even full Borel determinacy can prove neither Δ^1_2-CA_0 (even with TI for all formulas) nor Π^1_3-CA_0 even over Δ^1_3-CA_0 and TI for all formulas (MS [2012, Corollaries 6.2 and 6.6]). Still, using methods and results of MS [2012] and Montalbán and Shore [2014] working, however, with Σ^0_4-TD in place of Σ^0_3-DET, we prove that not much less than Π^1_3-CA_0 will suffice. Indeed, Π^1_1-CA_0 + Σ^0_4-TD proves the existence of a Σ_2 admissible ordinal (Lemma 4.7) and so in terms of consistency strength, Π^1_1-CA_0 + Σ^0_4-TD is much stronger than Δ^1_3-CA_0 (Corollary 4.8).

Finally, we use these methods to derive Martin's result that Σ^0_5-TD implies the existence of β_0 (the least ordinal α such that $L_\alpha \vDash Z_2$) in Π^1_1-CA_0 (Lemma 4.4). Thus Π^1_1-CA_0 + Σ^0_5-TD implies the consistency of Z_2 (and more) and so takes us well beyond the reach of full second order arithmetic (Corollary 4.6).

We close this section with some notational conventions.

Notation 1.8. *We use ω to denote the set of natural numbers. Members of 2^ω are generally called sets and symbols such as X, Y, Z are used to denote them. Members of ω^ω are often called reals and we use symbols such as f, g, h to denote them. (Of course a real may be a set when its range is contained in $\{0,1\}$.) The (Turing) degrees of these sets and functions are, as usual, denoted by the corresponding small boldface roman letter. So for example $f \in \mathbf{f}$ and $X \in \mathbf{x}$. The eth partial recursive function and r.e. set relative to f are denoted by Φ^f_e and W^f_e,respectively.*

Notation 1.9. *Subsets of both 2^ω and ω^ω are generally denoted by symbols such as A, B, C. We use symbols such as σ, τ to denote strings in either $2^{<\omega}$ or $\omega^{<\omega}$ and rely on the context to determine which is intended. The length of σ is denoted $|\sigma|$ and its initial segment of length $n \le |\sigma|$ is denoted by $\sigma \upharpoonright n$. We use standard concatenation and pairing functions, conventions and notations such as $\sigma\hat{\ }\tau, \sigma\hat{\ }f, \langle\sigma,\tau\rangle$ $\langle\sigma, f\rangle$*

$\langle \sigma, X \rangle, \langle f, g \rangle, \langle u, v, w \rangle = \langle u, \langle v, w \rangle \rangle$ in the usual way. The precise formulations do not matter as long as they are done recursively.

We assume a basic familiarity with recursive ordinals and the hyperarithmetic hierarchy and at times their formal development in ATR_0 as in Simpson [2009, VII]. Note also that we generally prove theorems in their lightface version and leave relativization to the reader unless some desired uniformity is brought out by carrying along the set parameter.

2 The trivial levels

In this section we prove Π^0_2- and Δ^0_3-Turing Determinacy. Only the first proof is carried out in RCA_0. It is helpful to introduce a weaker but more easily definable notion of closure than under $\equiv_T$.

Definition 2.1. Given $k \in \omega$ and $f \in \omega^\omega$ we define $k \times f \in \omega^\omega$ by $(k \times f)(kn) = f(n)$ and $(k \times f)(m) = 0$ for m not a multiple of k. We say that an A contained in ω^ω or 2^ω is *sufficiently closed* if $(\forall f \in A)(\forall \sigma)(\forall k)(\sigma\hat{\ }(k \times f) \in A)$. Here and elsewhere σ is in $\omega^{<\omega}$ or $2^{<\omega}$ appropriate. The smallest sufficiently closed set containing f is the *sufficient closure* of f. Let E be the set of *even strings*, i.e. those whose nonzero values occur only at even numbers such as all initial segments of $2 \times f$ for any f.

Remark 2.2. Note that, for every m, n, f, $m \times (n \times f) = mn \times f$. It is then easy to see that, for every f in ω^ω or 2^ω, $\{\sigma\hat{\ }(k \times f) | k \in \omega$ and σ a string$\}$ is the sufficient closure of $\{f\}$.

Also, for any $A \subseteq 2^\omega$, the set $\hat{A} = \{X : (\forall \sigma, k)(\sigma\hat{\ }(k \times X)) \in A\}$ is sufficiently closed. The advantage of using sufficient closure instead of Turing closure is that if A is Π^0_2, then so is $\hat{A}$.

Lemma 2.3 (RCA_0)**.** *For every $Z \in \omega^\omega$, every Π^Z_2 set $A \subseteq \omega^\omega$ [2^ω] which is sufficiently closed is either empty or contains an element of every Turing degree above Z.*

Proof. Let $A \neq \emptyset$ be such a set. There is then an r.e. operator W (given by some W_e) such that $f \in A \Leftrightarrow f \in W^{Z \oplus f}$ is infinite. Let

$$X = \{\sigma : W^{Z\oplus\sigma} - W^{Z\oplus\sigma^-} \neq \emptyset\},$$

(where $W^{Z\oplus\sigma}$ only runs for $|\sigma|$ many steps and $\sigma^- = \sigma \upharpoonright |\sigma| - 1$). So, we have that $f \in A$ if and only if $f \upharpoonright n \in X$ for infinitely many n. Note that $X \leq_T Z$. Say $f \in A$, then for every σ, $\sigma\hat{\ }(2 \times f) \in A$. Thus every σ there is a $\tau \in E$ (e.g. some initial segment of $2 \times f$) such that $\sigma\hat{\ }\tau \in X$.

Now, given any infinite $Y \in 2^\omega$ with $Z \leq_T Y$, we build an $h \in A$, with $h \equiv_T Y$. We construct h as the union of finite initial segments $\emptyset = \sigma_0 \subseteq \sigma_1 \subseteq \cdots$ all of even length.

We just need to make sure that h meets X infinitely often and is of the same Turing degree as Y. Suppose we have σ_s. Let τ_s be the first $\tau \in E$ found in a standard search recursive in Z such that $\sigma_s\hat{\ }\tau \in X$. Let k_s be least such that $|\sigma_s\hat{\ }\tau_s\hat{\ }0^{k_s}| = 2\langle x, y_s\rangle + 1$ for y_s the sth element of Y and some x. Now set $\sigma_{s+1} = \sigma_s\hat{\ }\tau_s\hat{\ }0^{k_s}\hat{\ }1$. Clearly $h \leq_T Y$ (by construction as $Z \leq_T Y$), $h \in A$ and $Y \leq_T h$ (its members can be read off in order from the list of odd numbers m such that $h(m) = 1$) as required. □

As degree-invariant sets are obviously sufficiently closed we have the following corollaries.

Corollary 2.4. *For every $Z \in 2^\omega$, every nonempty degree invariant Π^Z_2 set $A \subseteq \omega^\omega$ [2^ω] contains all $f \geq_T Z$.*

Corollary 2.5. $\mathsf{RCA}_0 \vdash \mathbf{\Pi}^0_2$-$\mathsf{TD}$.

Lemma 2.6 (ACA_0). *For every $Z \in 2^\omega$, every nonempty, degree-invariant Δ^Z_3 subset A of 2^ω contains all $X \geq_T Z$.*

Proof. Let A be a Δ^Z_3 degree invariant subset of 2^ω. By Theorem 1.5, there is a decreasing, continuous sequence $\{A_\xi : \xi \leq \alpha\}$ of uniformly Π^Z_2 subsets of 2^ω with $A_\alpha = \emptyset$ such that

$$X \in A \Leftrightarrow \mu\xi(X \notin A_\xi) \text{ is odd.}$$

Now, let $\hat{A}_\xi = \{X : (\forall\sigma, k)(\sigma\hat{\ }(k \times X)) \in A_\xi\}$. The $\hat{A}_\xi$ are clearly Π^Z_2 and, by Remark 2.2, sufficiently closed and so dense. By Lemma 2.3, each $\hat{A}_\xi$ is either $\emptyset$ or contains a member Y of every degree above that of Z. As $A_\alpha = \emptyset = \hat{A}_\alpha$, there is, by ACA_0, a least ξ such that $\hat{A}_\xi = \emptyset$. (By Lemma 2.3, $\hat{A}_\xi$ being empty is equivalent to $\neg\exists X(X \equiv_T Z \;\&\; X \in A)$.) Note that as the A_ξ are continuous, so are the $\hat{A}_\xi$: Consider any limit ordinal λ. If $X \in \hat{A}_\lambda$ then its sufficient closure is contained in A_λ and so in every A_ξ for $\xi < \lambda$ and thus in $\cap\{\hat{A}_\xi | \xi < \lambda\}$. On the other hand, if $X \in \hat{A}_\xi$ for every $\xi < \lambda$, then its sufficient closure is contained in each $\hat{A}_\xi \subseteq A_\xi$ and so in A_λ and in $\hat{A}_\lambda$. Thus ξ cannot be a limit ordinal by the Baire category theorem. (The $\hat{A}_\xi$ are dense Π^Z_2 and so themselves intersections of open (and hence dense open) sets.)

We now claim that A is either $\emptyset$ or contains every $Y \geq_T Z$ depending on the parity of ξ (or if it is α). To see this consider any degree $\mathbf{y} \geq \mathbf{z}$. By the leastness of ξ and Lemma 2.3, there is a $Y \in \hat{A}_{\xi-1}$ of degree $\mathbf{y}$. Of course, $Y \notin \hat{A}_\xi = \emptyset$ and so, by definition, there are σ and k such that $\sigma\hat{\ }(k \times Y) \notin A_\xi$. On the other hand, since $\hat{A}_{\xi-1}$ is sufficiently closed, $\sigma\hat{\ }(k \times Y) \in \hat{A}_{\xi-1}$. Thus the membership of $\sigma\hat{\ }(k \times Y)$ (and so of $Y \equiv_T \sigma\hat{\ }(k \times Y)$) in A is determined by the parity of ξ as required. □

Corollary 2.7. $\mathsf{ACA}_0 \vdash \mathbf{\Delta}^0_3$-$\mathsf{TD}$.

Note that by Remark 1.4 the corollary holds in ω^ω as well as 2^ω. From now on we will be concerned with Turing determinacy at levels above Δ^0_3 and so work in whichever setting is more convenient.

We conclude this section by showing that $\mathbf{\Delta}^0_3$-TD is not provable in RCA_0. If we are looking for a standard model of RCA_0 in which $\mathbf{\Delta}^0_3$-TD fails we have serious restrictions on the method of attack. Suppose the formulas (with parameter Z) defining a Δ^0_3 set A of reals determining a game in a standard model $\mathcal{M}$ actually define a Δ^0_3 set of reals in the universe or even in any extension of the sets of $\mathcal{M}$ to a model of ACA_0. In this case, Theorem 1.5 provides a representation of A in the extension and Lemma 2.6 applies. Its conclusions, however, are clearly absolute downwards to $\mathcal{M}$ and so the given game is determined in $\mathcal{M}$. Thus the only hope of finding a standard model counter-example is to consider formulas which define a Δ^0_3 set in $\mathcal{M}$ but not in any extension to a model of ACA_0.

Proposition 2.8. $\mathsf{RCA}_0 \nvdash \mathbf{\Delta}^0_3$-$\mathsf{TD}$.

Proof. Consider an initial segment of the degrees below $0'$ of order type ω given by representatives X_n which are uniformly Δ^0_2 (Lerman [1983, XII.5.1], Epstein [1983]). Our model $\mathcal{M}$ of RCA_0 consists of all sets recursive in some X_n. Our Δ^0_3 degree invariant class is given by two Σ^0_3 formulas φ and ψ. The first says that there is an n such that $X \equiv_T X_{2n}$ and the second that there is an n such that $X \equiv_T X_{2n+1}$. These sets are clearly complementary in $\mathcal{M}$. To see that they are Σ^0_3 write out the definitions, for example, $\varphi(X) \Leftrightarrow \exists n\{\exists e(\Phi^X_e$ is total & $\forall m(\Phi^X_e(m) = X_{2n}(m))$ & $\exists i(\Phi_i^{\Phi^X_e} = X)\}$ and remember that the X_n are uniformly Δ^0_2. Thus φ and ψ define a Δ^0_3 set of reals in $\mathcal{M}$ while both sets are clearly unbounded in the Turing degrees of $\mathcal{M}$. Thus $\mathcal{M} \nvDash \mathbf{\Delta}^0_3$-$\mathsf{TD}$ as required. ☐

3 $\mathbf{\Sigma}^0_3$ and $\mathbf{\Delta}^0_4$ sets

In this section we show that Σ^0_3-TD is equivalent to ATR_0 over ACA_0 and that Δ^0_4-TD is provable from ATR_0 + Π^1_1-TI_0. As mentioned in §1, Π^1_1-TI_0 is equivalent to Σ^1_1-DC_0 over ACA_0 and ATR_0 + Π^1_1-TI_0 the lies strictly between ATR_0 and Π^1_1-CA_0. On the other hand, we show that Δ^0_4-TD is not provable from ATR_0. The situation here is similar to, but much more subtle than that for Δ^0_3-TD in Proposition 2.8.

Theorem 3.1 (essentially Harrington and Kechris [1975]). *$ATR_0 \vdash \mathbf{\Sigma}^0_3$-TD.*

Proof. We follow the proof of Harrington and Kechris [1975, §2] but make explicit a property of their construction that we will need in the proof of Theorem 3.3. Let a given game be specified by a Σ^0_3 degree invariant subset of ω^ω, $B = \{f | \exists i \forall j \exists k R(i,j,\bar{f}(k))\}$ where R is a recursive predicate and $\bar{f}(k)$ is the sequence $\langle f(0), \ldots, f(k-1)\rangle$. We define a Π^0_1 set A which has members of the same degrees as B: $A = \{\langle i,f,g\rangle \, | \forall j(g(j) = \mu k R(i,j,\bar{f}(k))\}$. Clearly, if $\langle i,f,g\rangle \in A$ then $g \leq_T f$ and so $\langle i,f,g\rangle \equiv_T f \in B$. Conversely, if $f \in B$ then there is an i such $\forall j \exists k R(i,j,\bar{f}(k))$ and so a $g \leq_T f$ such that $\langle i,f,g\rangle \in A$. Thus A and B have elements of exactly the same degrees.

We next consider another Π^0_1 set $C = \{\langle\langle i, f, g\rangle, h\rangle \,|g \in A$ & $\forall n(\Phi^g_i(n)$ converges in exactly $f(n)$ many steps) & h is II's play when he follows the strategy given by Φ^g_i against I playing $\langle i, f, g\rangle\}$. Note that if $\langle\langle i, f, g\rangle, h\rangle \in C$ then $\langle\langle i, f, g\rangle, h\rangle \equiv_T g$ and $g \in A$.

Now apply Π^0_1-determinacy (which follows from ATR_0 as in Simpson [2009, V.8.2]), to the game specified by C. If I has a strategy s then, we claim that every degree $\mathbf{t} \geq \mathbf{s}$ has a representative in A: As usual, let I play s against any real $t \in \mathbf{t}$. The resulting play $\langle s(t), t\rangle$ has degree $\mathbf{t}$ and is in C and so of the form $\langle\langle i, f, g\rangle, h\rangle$ with $g \in A$ and $\langle\langle i, f, g\rangle, h\rangle \equiv_T g$ as required. Thus, in this case, as B is degree invariant, it contains a cone with base the strategy for I in the game specified by C. On the other hand, if II has a strategy s for this game, we claim that B is disjoint from the cone above $\mathbf{s}$. If not then there is a $\hat{g} \in B$ and hence one $g \in A$ which computes s. Suppose $\Phi^g_i = s$. Let $f(n) \leq_T g$ be the number of steps it takes $\Phi^g_i(n)$ to converge and h be II's play following his supposedly winning strategy given by $\Phi^g_i = s$ against I playing $\langle i, f, g\rangle$. It is clear from the definitions that the play of this game is $\langle\langle i, f, g\rangle, h\rangle$ and it is in C for the desired contradiction. $\square$

We now calculate the complexity of the property of a Σ^0_3 degree invariant subset of ω^ω containing a cone. We use this calculation in the proof of Theorem 3.3.

Proposition 3.2 (ATR_0)**.** *The predicate that (the formula defining) a $\boldsymbol{\Sigma}^0_3$ degree invariant set of reals contains a cone of degrees is Σ^1_1.*

Proof. Let B be a degree invariant Σ^0_3 set of reals. Define Π^0_1 sets A and C as in the proof of Theorem 3.1. It is easy to see that the existence of a strategy s for the closed game given by C is a Σ^1_1 property: for every σ the result of playing s against σ satisfies the Σ^0_1 predicate of not being in C. If this condition holds then the proof of Theorem 3.1 shows that A intersects every degree above that of a strategy and hence B contains a cone. On the other hand, If there is no such strategy, then by ATR_0 there is one for II in this game and so again as in the proof of Theorem 3.1, B is disjoint from the cone above II's strategy. $\square$

We now give a proof of Δ^0_4-TD in ATR_0 + Π^1_1-TI_0 which, as pointed after Definition 1.7, is lies strictly between ATR_0 and Π^1_1-CA_0. Thus Δ^0_4-TD is strictly weaker than Π^1_1-CA_0 even over ATR_0.

Theorem 3.3. ATR_0 + Π^1_1-$\mathsf{TI}_0 \vdash \Delta^0_4$-*$\mathsf{TD}$.*

Proof. Represent a given Δ^0_4 degree invariant set $B \subseteq 2^\omega$ using the difference hierarchy on Σ^0_3 sets as in Theorem 1.5. By Remark 1.6, we may assume that each B_ξ, $\xi \leq \alpha$ is itself Turing invariant and so (by Theorem 3.1) either disjoint from a cone or contains one. As $B_\alpha = 2^\omega$ and the B_ξ are increasing, there is, by Proposition 3.2 and Π^1_1-TI_0, a least γ such that B_γ contains a cone. If γ is a successor ordinal, then we have a cone disjoint from $B_{\gamma-1}$ and contained in B_γ. Depending on the parity of γ, this cone is either disjoint from, or contained in, B as required.

To finish the proof we show that γ cannot be a limit. For each $\xi < \gamma$, let A_ξ be a Π^0_1 set reals with members of the same Turing degrees as B_ξ and C_ξ the associated Π^0_1 set as defined in Theorem 3.1. Consider the Π^0_1 game specified by $C = \{\langle\langle\langle\xi, i\rangle, f, g\rangle, h\rangle \,|\, \langle\langle i, f, g\rangle, h\rangle \in C_\xi\}$, i.e. I first chooses a ξ and then plays the game determined by C_ξ. If I has a winning strategy in this game, say his first move is to play $\langle\xi, i\rangle$. The rest of his strategy then gives him a wining strategy in C_ξ which (by the proof of Theorem 3.1) would be the base of cone in B_ξ contrary to the assumption that it is disjoint from a cone. Thus (by Π^0_1-DET), II has a strategy s for the game specified by C. Restricting I to play a given $\xi < \gamma$ as the first part of his first move gives a strategy s_ξ for II in C_ξ uniformly recursive in s. As, by the proof of Theorem 3.1, each s_ξ is the base of a cone disjoint from B_ξ, s is the base of a cone disjoint from all the B_ξ for $\xi < \gamma$ and so disjoint from $B_\gamma = \cup_{\xi<\gamma} B_\xi$ for the desired contradiction. □

We now prove that one cannot get Δ^0_4-TD from ATR_0 alone. A crucial ingredient is H. Friedman's [1967, II] ω-incompleteness theorem (see Simpson [2009, VIII.5.6]). Note that a *countable coded ω model* specified by a set $\mathcal{M}$ is a structure for second order arithmetic in which the numbers are the numbers (in the ambient universe) and its sets are the columns $(\mathcal{M})_n = \{x | \langle x, n\rangle \in \mathcal{M}\}$.

Theorem 3.4 (H. Friedman)**.** *Let S be a recursive set of sentences of second order arithmetic which includes ACA_0. If there exists a countable coded ω-model of S, then there exists a countable coded ω-model of $S \cup \{\neg\exists$countable coded ω-model of $S\}$.*

Theorem 3.5. *$\mathsf{ATR}_0 \nvdash \Delta^0_4$-$\mathsf{TD}$.*

Proof. For convenience we work in the real world although certainly Π^1_1-CA_0 suffices. All models $\mathcal{M}$ or $\mathcal{M}_n$ in our proof, beginning with $\mathcal{M}_0 \vDash T_0$, will be countable coded ω-models of $T_0 = \mathsf{ATR}_0$. By Theorem 3.4, there is an $\mathcal{M}_1 \vDash S_1$ where $S_1 = T_0 \;\&\; \neg\exists\mathcal{M} \vDash T_0$. As $\mathcal{M}_1$ is a coded ω-model there is an $\hat{\mathcal{M}}_1$ containing it such that $\hat{\mathcal{M}}_1 \vDash T_1$ where $T_1 = T_0 \;\&\; \exists\mathcal{M} \vDash S_1$. Applying Theorem 3.4 again, we get an $\mathcal{M}_2 \vDash S_2$ where $S_2 = T_1 \;\&\; \neg\exists\mathcal{M} \vDash T_1$. We now set $T_2 = T_0 \;\&\; \exists\mathcal{M} \vDash S_2$ and continue similarly to get $\hat{\mathcal{M}}_2 \vDash T_2$ and $\mathcal{M}_3 \vDash S_3$ with $S_3 = T_2 \;\&\; \neg\exists\mathcal{M} \vDash T_2$ and $\hat{\mathcal{M}}_3 \vDash T_0 \;\&\; \exists\mathcal{M} \vDash S_3$. Then we proceed similarly by induction to get $\mathcal{M}_{n+1} \vDash S_{n+1}$ with $S_{n+1} = T_n \;\&\; \neg\exists\mathcal{M} \vDash T_n$ and $\hat{\mathcal{M}}_{n+1} \vDash T_{n+1}$ with $T_{n+1} = T_0 \;\&\; \exists\mathcal{M} \vDash S_{n+1}$.

We now let T be the theory containing T_0 with new constants $\mathcal{M}_n$ and assertions saying that for all n, the $\mathcal{M}_n$ are countable coded ω-models of S_n and that $\mathcal{M}_n$ is a member of $\mathcal{M}_{n+1}$ (in the sense that as a set it is coded in $\mathcal{M}_{n+1}$ by being one of the columns of $\mathcal{M}_{n+1}$). Any finite subset of T is satisfied by one of the $\mathcal{M}_n$ just constructed. (Unravelling the definitions of T_n and S_n shows that any model $\mathcal{M}_{n+1}$ of S_{n+1} contains an $\mathcal{M}_n \vDash S_n$ and so by induction a sequence of $\mathcal{M}_i$ for $i < n$ as required in T.) Thus there is a model $\hat{\mathcal{N}}$ of T. (Note that this model is only given by a compactness argument so is expected to be nonstandard.)

We now consider the ω-submodel $\mathcal{N}$ of $\hat{\mathcal{N}}$ specified by taking as its second order part all sets coded in any of the $\mathcal{M}_n$ in $\hat{\mathcal{N}}$. First note that $\mathcal{N} \vDash \mathsf{ATR}_0$: If there is a well-ordering α in $\mathcal{N}$ then it is a member of some $\mathcal{M}_n \subset \mathcal{N}$ and so also well-ordered in $\mathcal{M}_n$. If we have any arithmetic predicate S for which we want a hierarchy to witness ATR_0 in $\mathcal{N}$, consider the same formula interpreted in $\mathcal{M}_n$ (which we may assume contains the set parameters in S as well as α). As $\mathcal{M}_n \vDash \mathsf{ATR}_0$, the desired hierarchy of sets exists in M_n. As the properties required of it are arithmetic they hold in $\mathcal{N}$ as well.

We now define, in $\mathcal{N}$, degree invariant classes $A, B \subset \mathcal{N}$: $A = \{X|$ the least n, such that X does not compute both an $\mathcal{M} \vDash S_n$ and its satisfaction predicate, is even$\}$ and $B = \{X|$ the least n, such that X does not compute both an $\mathcal{M} \vDash S_n$ and its satisfaction predicate, is odd$\}$. Clearly A and B are disjoint. We claim that $A \cup B = \mathcal{N}$. Consider any $X \in \mathcal{N}$ so $X \in \mathcal{M}_i$ for some i. We see, by the definition of the $\mathcal{M}_n$, that no member of $\mathcal{M}_i$ can be an $\mathcal{M} \vDash S_i$ and so no such is computable from X. (If $\mathcal{M} \in \mathcal{M}_i$ and $\mathcal{M} \vDash S_i$ then, by the definition of S_i, $\mathcal{M} \vDash T_{i-1}$ but , again by the definition of S_i, $\mathcal{M}_i \vDash \neg\exists\mathcal{M} \vDash T_{i-1}$ for the desired contradiction.) Thus there is some $n \in \omega$ and so a least one such that no $\mathcal{M}$ computable from X can be a model of S_n. (Notice that if $X \in \mathcal{M}_i$ computes a model $\mathcal{M}$ then, as $\mathcal{M}_i$ is a model of ATR_0, the satisfaction predicate for $\mathcal{M}$ is also in $\mathcal{M}_i$.) Thus $X \in A \cup B$ as required. Next, we claim that both A and B are unbounded in the Turing degrees of $\mathcal{N}$. The point here is that $\mathcal{M}_n \vDash S_n$ but it and every model $\mathcal{M}$ computable from it together with its satisfaction predicate is in $\mathcal{M}_{n+1}$ and so $\mathcal{M} \nvDash S_{n+1}$. Thus $\mathcal{M}_n \oplus Sat(\mathcal{M}_n) \in A$ for n odd and $\mathcal{M}_n \oplus Sat(\mathcal{M}_n) \in B$ for n even where $Sat(\mathcal{M})$ is the full satisfaction predicate (elementary diagram) for $\mathcal{M}$. Of course, the degrees of the $\mathcal{M}_n$ are cofinal in those of $\mathcal{N}$ for both the even and odd n.

All that remains to see that A is a counterexample to Δ_4-TD in $\mathcal{N}$ is to show that it (and analogously B) is Σ^0_4. To this end we write out the definition of A: $X \in A \Leftrightarrow \exists n(\exists m(n = 2m)$ & $(\forall Y, W \leq_T X(Y$ is not a countable coded ω-model of S_n with W its satisfaction predicate) & $(\exists Z, V \leq_T X)(Z$ is a countable coded ω-model of S_{n-1} with satisfaction predicate V). As usual, we represent a set $Z \leq_T X$ by an index of a characteristic function Φ^X_e computable from X. Thus to say $(\exists Z, V \leq_T X)\Theta(Z, V)$ is to say $\exists e, i(\Phi^X_e$ and Φ^X_i are total characteristic functions & $\Theta(\Phi^X_e, \Phi^X_i))$. Now being total is a Π^X_2 property. Once we have guaranteed totality for Φ^X_e and Φ^X_i, the substitution of Φ^X_e and Φ^X_i for Z and V can be done at no additional quantifier costs since quantifier free formulas in Z, V and X now have Δ^X_1 equivalents. Thus if Θ is $\Sigma^{X,Z,V}_3$, $(\exists Z, V \leq_T X)\Theta(Z, V)$ is equivalent to a Σ^X_3 formula. Similarly, $(\forall Y, W \leq_T X)\Psi(Y, W)$ is equivalent to a Π^X_3 formula if Ψ is $\Pi^{X,Z,V}_3$. Thus we are left with analyzing the rest of the relations in the formula.

Any set Z can be effectively viewed as a sequence of its columns $\langle (Z)_n \rangle$ and the associated structure for second order arithmetic is given by specifying the $(Z)_n = \{m| \langle n, m \rangle \in X\}$ as its second order part. The first order part remains the same as in the ambient universe. So each Z is, in this way, recursively interpreted as an ω-model. That V is the satisfaction predicate for the model coded in this way by Z is then a Π^0_2 relation. (See Simpson [2009, V.2] for these definitions.) Once we have the satisfaction set V for Z, to say that a formula is true in Z is then, of course, a Δ^0_1 relation. Thus the whole formula is of the form $\exists(\exists \ \& \ \Pi_3 \ \& \ \Sigma_3)$ and so Σ^0_4 as required. $\square$

Next we prove a reversal of Theorem 3.1 over ACA_0. We begin by pointing out that a standard fact on iterations of the Turing jump holds in ACA_0.

Lemma 3.6 (ACA_0)**.** *Let α be a well-ordering. If $0^\eta \leq_T X$ for every $\eta < \alpha$, i.e. there is an e such that Φ^X_e is a total characteristic function for a set satisfying the Π^0_2 formula determining 0^η, then 0^α exists and indeed $0^\alpha \leq_T X''$.*

Proof. If α is a successor ordinal, the result follows immediately from ACA_0. Otherwise, say α is a limit ordinal. The function f taking $\eta < \alpha$ to the least e satisfying the conditions of the Lemma is total by hypothesis and exists by ACA_0. Indeed, $f \leq_T X''$. The set $\{\langle n, \eta\rangle \,|\Phi^X_{f(e)}(n) = 1\}$ then also exists, satisfies the definition of 0^α and is recursive in X''. $\square$

Theorem 3.7. ACA_0 + $\mathbf{\Sigma}^0_3$-TD $\vdash \mathit{ATR}_0$.

Proof. Let α be a well-ordering. We want to prove that 0^α exists. Let W be a low nonrecursive REA operator, i.e. $\forall X(X <_T W^X \ \& \ X' \equiv_T (W^X)')$ and the indices for the required Turing reductions are the same for all X. (The standard construction for such an operator clearly works in ACA_0.)

Consider the set $P = \{X|(\exists\beta < \alpha)(0^\beta \oplus X \equiv_T W^X)\}$. To see that this set is Σ^0_3 rewrite its defining condition by saying that there is a $\beta < \alpha$ and an e such that $\Phi^{W^X}_e$ is a total characteristic function for a set that satisfies the Π^0_2 defining condition for 0^β and $W^X \equiv_T \Phi^{W^X}_e \oplus X$. As $(W^X)'$ is uniformly recursive in X', totality of $\Phi^{W^X}_e$ is Π^X_2 as is every $\Pi^{W^X}_2$ predicate (uniformly). Thus the condition defining P is $\mathbf{\Sigma}^0_3$. Let $\hat{P}$ be the Turing closure of P, i.e. $\hat{P} = \{X|\exists Y(Y \in P \ \& \ X \equiv_T Y)$. Similarly, $\hat{P} \in \mathbf{\Sigma}^0_3$.

By $\mathbf{\Sigma}^0_3$-TD there is a cone of degrees in $\hat{P}$ or its complement. Let $\hat{X}$ be a set in the base of such a cone. If $\hat{X} \in \hat{P}$ let $X \equiv_T \hat{X}$ be in P. If not, let $X = \hat{X}$. By Lemma 3.6, it suffices to prove that $0^\eta \leq_T X$ for every $\eta < \alpha$ to get that 0^α exists. If not, then, by ACA_0, there is a least $\gamma < \alpha$ such that $0^\gamma \nleq_T X$. Note that, again by Lemma 3.6, 0^γ exists. We now work toward a contradiction.

If $X \in P$, let $\beta < \alpha$ be as required in the definition of P and so by the leastness of γ, $\gamma \leq \beta$ (and $0^\gamma \leq_T 0^\beta$) as $0^\beta \oplus X \equiv_T W^X >_T X$. Now we have $X <_T X \oplus 0^\gamma \leq_T X \oplus 0^\beta \equiv_T W^X <_T (W^X)' \equiv_T X'$. By Posner and Robinson [1981, Theorem 3 relativized to X], which can easily be proven in ACA_0, there is a $\hat{Y}$ such that $X <_T \hat{Y}$ and $X' \equiv_T \hat{Y}' \equiv_T \hat{Y} \oplus X \oplus 0^\gamma$. By our choice of W, $\hat{Y} <_T W^{\hat{Y}} <_T \hat{Y}'$. On the other hand, our assumptions guarantee that $\hat{Y} \in \hat{P}$ and so that there is a $Y \in P$ with $Y \equiv_T \hat{Y}$. Let δ be the witness for Y being in P, i.e. $Y \oplus 0^\delta \equiv_T W^Y$. If $\delta < \gamma$, $0^\delta \leq_T X \leq_T Y$ which contradicts $Y <_T W^Y$. On the other hand, if $\delta \geq \gamma$, $0^\gamma \leq_T 0^\delta$ and so $0^\delta \oplus Y \geq_T Y' >_T W^Y$ for another contradiction.

Finally, suppose $X \notin P$. As $0^\gamma \not\leq_T X$, we have, again by Posner and Robinson, a $Y >_T X$ with $Y' \equiv_T Y \oplus 0^\gamma \equiv_T Y \oplus 0^\gamma \oplus X'$. By pseudojump inversion for REA operators (Jockusch and Shore [1984]), which can also easily be proven in ACA_0, there is an Z with $Z >_T Y$ such that $W^Z \equiv_T Y'$. Now, as $Z \oplus 0^\gamma \equiv_T Y \oplus 0^\gamma \equiv_T Y' \equiv_T W^Z$, γ is a witness that $Z \in P \subseteq \hat{P}$. This is the desired final contradiction to $\hat{X}$ being the base of a cone outside of $\hat{P}$ and so to the existence of γ as required. □

4 Σ^0_4, Δ^0_5 and Σ^0_5 sets

We now prove generalizations to all levels of the arithmetic hierarchy of weaker versions of Theorems 3.1 and 3.3 due to Harrington and Kechris [1975] and Martin [1974], respectively. We prove the first in RCA_0 and the second in Π^1_1-CA_0.

Lemma 4.1 (essentially Harrington and Kechris). *$\mathsf{RCA}_0 \vdash \Sigma^0_n$-Determinacy $\rightarrow \Sigma^0_{n+1}$-$\mathsf{TD}$.*

Proof. We follow the proof of Theorem 3.1. Given a Σ^0_{n+1} degree invariant set $B = \{f|\exists n Q(n,f)\}$ with $Q \in \Pi^0_n$, set $A = \{\langle n,f\rangle\,|Q(n,f)\}$. Clearly, A is Π^0_n and has elements of exactly the same degrees as B. Now as in Theorem 3.1 let $C = \{\langle\langle i,f,g\rangle, h\rangle\,|g \in A \;\&\; \forall n(\Phi^g_i(n)$ converges in exactly $f(n)$ many steps) & h is II's play when he follows the strategy given by Φ^g_i against I playing $\langle i,f,g\rangle\}$. Note that $C \in \Pi^0_n$ and if $\langle\langle i,f,g\rangle, h\rangle \in C$ then $\langle\langle i,f,g\rangle, h\rangle \equiv_T g$ and $g \in A$. By Σ^0_n-Determinacy, C is determined. The analysis to show that B contains or is disjoint from a cone is now exactly as in Theorem 3.1. □

Lemma 4.2 (essentially Martin). *Π^1_1-$\mathsf{CA}_0 \vdash \Sigma^0_n$-$\mathsf{TD} \leftrightarrow \Delta^0_{n+1}$-$\mathsf{TD}$.*

Proof. As Δ^0_{n+1}-TD is a Π^1_3 sentence, we can use Δ^1_2-CA_0 and its equivalent Σ^1_2-AC_0 to prove it as Δ^1_2-CA is Π^1_3-conservative over Π^1_1-CA_0. (See Simpson [2009, VII.6.9.1 and IX.4.9].) Let a game be specified by a degree invariant $\mathbf{\Delta}^0_{n+1}$ set $A \subseteq 2^\omega$. Apply the Kuratowski analysis (Theorem 1.5 and Remark 1.6) to represent A by a sequence A_ξ of degree invariant uniformly Σ^0_n sets. By Σ^0_n-TD each of these sets either contains or is disjoint from a cone. By Δ^1_2-CA_0 we have the sequence telling us which is the case. We may then take the least γ such that A_γ contains a cone ($A_\alpha = 2^\omega$ if no other). Now by Σ^1_2-AC_0 we have a sequence s_η of bases of cones disjoint from A_η for $\eta < \gamma$. The degree of this sequence is then the base of a cone disjoint from all the A_η for $\eta < \gamma$. Its join with the base of a cone contained in A_γ is then the base of a cone contained in or disjoint from A depending on the parity of γ. □

As Π^1_3-CA_0 proves Σ^0_3-Determinacy (Welch [2011]), we now have a bound on what is needed to prove Σ^0_4 and so Δ^0_5 Turing determinacy.

Corollary 4.3. *Π^1_3-$\mathsf{CA}_0 \vdash \Sigma^0_4$-$\mathsf{TD}$ & Δ^0_5-TD.*

4.1 A lower bound for Σ_5^0-TD

As mentioned in §1, there can be no reversals here. While we have seen that Δ_5^0-TD is provable already in Π_3^1-CA_0 (Corollary 4.3), this is the end of provable Turing determinacy in full second order arithmetic, Z_2. Martin ([1974] and [1974a]; see [n.d.]) has shown that Σ_5^0-TD implies the existence of β_0, the least ordinal γ such that L_γ is a model of Z_2. None of these results have been published so we indicate how to modify arguments of Martin's and ours from MS [2014] to give a slightly different proof of this result in Π_1^1-CA_0.

Lemma 4.4 (Martin)**.** Π_1^1-CA_0 + Σ_5^0-$\mathsf{TD} \vdash \beta_0$ *exists.*

Proof. Here we work in Π_1^1-CA_0 + Σ_5^0-TD but assume β_0 does not exist and consider the same theory as in MS [2014]:

$$T = KP + \text{“}V = L\text{”} + \forall\gamma(L_\gamma \text{ is countable inside } L_{\gamma+1})$$

which implies that β_0 does not exist.

We first note that as in MS [2014, Lemma 2.1] the set

$$A = \{\alpha \mid L_\alpha \vDash T \text{ and every member of } L_\alpha \text{ is definable in } L_\alpha\}$$

is unbounded in the ordinals: If not, let $\delta = \sup A$, and let α be the least admissible ordinal greater than δ. (Note that Π_1^1-CA_0 implies that for every X the least ordinal admissible in X exists.) Let $\mathcal{M}$ be the elementary submodel of L_α consisting of all its definable elements. Then $\delta \in \mathcal{M}$. Since β_0 does not exist, every ordinal is countable, and hence there is a bijection between ω and δ and the $<_L$-least such bijection belongs to $\mathcal{M}$. Thus $\delta \subseteq \mathcal{M}$, indeed $\delta + 1 \subseteq \mathcal{M}$. Since the Mostowski collapse of $\mathcal{M}$ is admissible and contains $\delta + 1$, it must be L_α. It follows that every member of L_α is definable in L_α and hence that $\alpha \in A$ for the desired contradiction.

We now define a Σ_5^0 set and so a game Q using the same r.e. operator W as in the proof of Theorem 3.7 as well as some notions from MS [2014]. As there, we consider complete extensions of T defined from the play of the game whose term models are ω-models (albeit in ways more complicated than simply being the plays of the two players). (The term model of such an extension is the structure whose members are (equivalence classes) of formulas $\varphi(x)$ which, in the appropriate theory, define unique elements. It is an ω-model if its natural numbers are the terms $x = 1 + 1 + \ldots + 1$.)

The idea of the following definition is that Q is the set of all X such that there is a completion of T with degree W^X which is “better” than all completions of degree X. Here, the “better” of two completions is the one whose term model is either well-founded or has a larger well-founded part than the other.

$Q = \{X \mid \exists\hat{T}[\hat{T} \equiv_T W^X \;\&\; \hat{T}$ is a complete extension of T

whose term model $\mathcal{M}_{\mathrm{I}}$ is an ω-model &

$$\forall \tilde{T}(\tilde{T} \equiv_T X \ \& \ \tilde{T} \text{ is a complete extension of } T \text{ whose term model } \mathcal{M}_{\mathrm{II}} \text{ is an } \omega\text{-model} \rightarrow On^{\mathcal{M}_{\mathrm{I}}} \backslash On^{\mathcal{A}_{\mathrm{I}}} \text{ is either empty or has a least element})]\}.$$

We need some terminology from MS [2014] to explain the notation in this definition. Here $\mathcal{A}_{\mathrm{I}}$ is the image inside $\mathcal{M}_{\mathrm{I}}$ of the "intersection" of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$, i.e. the union of all the L_β in $\mathcal{M}_{\mathrm{I}}$ which can be coded by reals that belong to both $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. (Recall that since every set in these models is countable, every such L_β can be coded by a real in $\mathcal{M}_{\mathrm{I}}$.) Note that by MS [2014, Claim 2.6] $\mathcal{A}_{\mathrm{I}}$ is Σ^0_2. We use $On^{\mathcal{M}_{\mathrm{I}}}$ to denote the set of ordinals in $\mathcal{M}_{\mathrm{I}}$.

To see that Q is Σ^0_5, we rewrite the definition in terms of indices of reductions (from W^X and X) as in Theorem 3.7 and use the quantifier counting from MS [2014]. To say that some Z is a complete (consistent) extension of T is Π^0_1 and that its term model is an ω-model is Π^0_2 (MS [2014, Claim 2.4]). The term model of such a theory is obviously recursive in the theory as is its satisfaction relation. Since $\mathcal{A}_{\mathrm{I}}$ is Σ^0_2, saying that $On^{\mathcal{M}_{\mathrm{I}}} \backslash On^{\mathcal{A}_{\mathrm{I}}}$ is empty is Π^0_3 and that it has no least element is Π^0_4. Using these calculations the defintiion of Q has the form $\exists[\Sigma_3 \ \& \ \Pi_1 \ \& \ \Pi_2 \ \& \ \forall(\Sigma_3 \ \& \ \Pi_1 \ \& \ \Pi_2 \ \rightarrow \ \Pi_3 \ \vee \ \Pi_4)]$. The set Q is thus Σ^0_5, and so is its closure $\hat{Q}$ under Turing degree.

By Σ^0_5-TD, $\hat{Q}$ contains or is disjoint, from a cone. By Shoenfield's absoluteness theorem (which is provable in Π^1_1-CA_0 (Simpson [2009, VII.4.14]), the base $\mathbf{z}$ of the cone can be taken to be in L. Let α be an admissible ordinal such that $L_\alpha \models T$ and every element of L_α is definable in L_α and such that $Z \in L_\alpha$. (Such an ordinal exists by the unboundedness result at the beginning of this proof.) Let Th_α be the theory of L_α. So, in particular $Z, Z' \leq_T Th_\alpha$.

We first claim that $Th_\alpha \notin \hat{Q}$. Take $Y \equiv_T Th_\alpha$; we will show that $Y \notin Q$. To see this, consider any $\hat{T} \equiv_T W^Y$ with term model $\mathcal{M}_{\mathrm{I}}$ as in the definition of Q. Let $\tilde{T} = Th_\alpha$ with term model $\mathcal{M}_{\mathrm{II}} = L_\alpha$. So, we have that $\mathcal{M}_{\mathrm{I}} \neq L_\alpha$ because their theories have different Turing degrees, and we have that $(Th(\mathcal{M}_{\mathrm{I}}))' \equiv_T (Th_\alpha)'$ because W^Y is low over Y.

Claim 4.5. *If $\mathcal{M}_{\mathrm{I}} \neq L_\alpha$, $\mathcal{M}_{\mathrm{I}} \models T$ and $(Th(\mathcal{M}_{\mathrm{I}}))' \equiv_T (Th_\alpha)'$, then $\mathcal{M}_{\mathrm{I}}$ is ill-founded and its well-founded part is at most L_α.*

Let L_β be the well-founded part of $\mathcal{M}_{\mathrm{I}}$. We cannot have $\beta > \alpha$ because we would then have $(Th_\alpha)' \leq_T (Th(\mathcal{M}_{\mathrm{I}}))$. If $\beta = \alpha$, then $\mathcal{M}_{\mathrm{I}}$ must be ill-founded because $\mathcal{M}_{\mathrm{I}} \neq L_\alpha$. If $\beta < \alpha$, then $\mathcal{M}_{\mathrm{I}}$ must be ill-founded because otherwise $(Th(\mathcal{M}_{\mathrm{I}}))' = (Th(L_\beta))' \leq_T Th_\alpha$ contradicting our assumption that $(Th(\mathcal{M}_{\mathrm{I}}))' \equiv_T (Th_\alpha)'$. This proves the claim.

It follows that $\mathcal{A}_{\mathrm{I}} = L_\beta$ and that $On^{\mathcal{M}_{\mathrm{I}}} \backslash On^{\mathcal{A}_{\mathrm{I}}}$ is nonempty and has no least element, showing that $Y \notin Q$.

Second, we find another degree $X \geq_T Z$ which is in Q and hence in $\hat{Q}$. As $Z' \leq_T Th_\alpha$, there is (by pseudojump inversion) an $X >_T Z$ such $W^X \equiv_T Th_\alpha$. Let $\hat{T} = Th_\alpha$ and its term model $\mathcal{M}_{\mathrm{I}} = L_\alpha$. Since $\mathcal{M}_{\mathrm{I}}$ is well-founded, whatever $\mathcal{A}_{\mathrm{I}}$ is, $On^{\mathcal{M}_{\mathrm{I}}} \backslash On^{\mathcal{A}_{\mathrm{I}}}$ is always either empty or has a least element.

Thus, we have $Th_\alpha \notin \hat{Q}$ and $X \in \hat{Q}$ with both above $\mathbf{z}$, the supposed base of a cone inside or disjoint from $\hat{Q}$, for the final contradiction. $\square$

Corollary 4.6 (Martin)**.** *Z_2 does not prove Σ^0_5-TD. Indeed, Π^1_1-CA_0 + Σ^0_5-TD proves that for every set Y there is a β-model of Z_2 containing Y and hence much more than the consistency of Z_2.*

Proof. Recall that $L_{\beta_0} \cap \mathbb{R}$ is a model of Z_2 and indeed a β-model. Thus Π^1_1-CA_0 + Σ^0_5-TD proves the consistency of Z_2. As the Lemma relativizes to any Y, Π^1_1-CA_0 + Σ^0_5-TD proves that, for every set Y, there is a β-model of Z_2 containing Y. $\square$

4.2 A lower bound for Σ^0_4-TD

As mentioned before we cannot find reversals from Σ^0_4-TD. Relying on several notions and results of MS [2012] and [2014], we do, however, show that we cannot get by with much less than Corollary 4.3. The following proof is somewhat complicated and builds on the proof of Lemma 4.4. Recall that α_2 is the least 2-admissible ordinal, and equivalently, the least ordinal such that $L_{\alpha_2} \cap \mathbb{R} \models \Delta^1_3$-$\mathsf{CA}_0$.

Lemma 4.7. *Π^1_1-CA_0 + Σ^0_4-TD $\vdash \alpha_2$ exists.*

Proof. We assume, for the sake of a contradiction, that α_2 does not exist. We extend the theory T of MS [2014, §2] by setting

$$T = KP + \text{“}V = L\text{”} + \forall\gamma(L_\gamma \text{ is countable inside } L_{\gamma+1}) + \text{ no ordinal is } \Sigma_2\text{-admissible.}$$

By the same proof as in the second paragraph of the proof of Lemma 2.1 of MS [2014] (or at the beginning of the proof of Lemma 4.4 above), if α_2 does not exist then

$$A = \{\alpha \mid L_\alpha \vDash T \text{ and every member of } L_\alpha \text{ is definable in } L_\alpha\}$$

is unbounded in the ordinals.

We now define a set P which plays the role of Q in the previous proof. Again, P is the set of all X such that there is a model of T of degree W^X which is better than any of degree X, but this time we need P to be Σ^0_4.

$$\begin{aligned} P = \{X \mid \exists\hat{T}[&\hat{T} \equiv_T W^X \;\&\; \hat{T} \text{ is a complete extension of } T \\ &\qquad \text{whose term model } \mathcal{M}_{\mathrm{I}} \text{ is an } \omega\text{-model } \& \\ &\forall\tilde{T}(\tilde{T} \equiv_T X \;\&\; \tilde{T} \text{ is a complete extension of } T \\ &\qquad \text{whose term model } \mathcal{M}_{\mathrm{II}} \text{ is an } \omega\text{-model} \to \\ &\quad \text{conditions } R_I new \text{ or } R_I 3 \text{ hold})]\}. \end{aligned}$$

The conditions $R_I new$ and $R_I 3$ are defined in Section 2 of MS [2014]. Instead of repeating the whole background developed there, we just use a few lemmas from that section to prove below the few properties we need. Before doing that, let us notice that since every element of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ is definable by a real (because T says that every set is countable), we can compare their elements by looking at the reals coding them. Thus, when we say $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$, we mean that every element of $\mathcal{M}_{\mathrm{I}}$ is coded by a real in $\mathcal{M}_{\mathrm{I}}$ which also belongs to $\mathcal{M}_{\mathrm{II}}$. (As both models are standard, we can confidently talk about reals, i.e. subsets of ω, being in one or both of them.) The main properties about $R_I new$, $R_I 3$, and $R_{II} 3$ are the following:

1. If one of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ is well-founded, then $R_I new$ holds if and only if $\mathcal{M}_{\mathrm{I}}$ is isomorphic to the well-founded part of $\mathcal{M}_{\mathrm{II}}$.

2. If $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable, then either $R_I 3$ or $R_{II} 3$ holds.

3. If $R_I 3$ holds, then $\mathcal{M}_{\mathrm{II}}$ is ill-founded, and if If $R_{II} 3$ holds then $\mathcal{M}_{\mathrm{I}}$ is ill-founded.

4. The conditions $R_I new$ and $R_I 3$ are Π^0_3.

The first property is proved in Lemma 2.9 of MS [2014] with the fact that the definition of $R_I new$ implies that $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$. For the second, we have, by MS [2014, Lemma 2.17], that if neither of $R_I 3$ and $R_{II} 3$ hold, then there are ordinals β_1 and β_2 such that $\star_1(\beta_1, \beta_2)$ holds, which by MS [2014, Lemma 2.18(b)] implies that α is 2-admissible, where α is such that $\mathcal{A} = L_\alpha$. But since α_2 doesn't exists, there are no 2-admissible ordinals, and hence this is a contradiction. The third property follows from the definition of $R_I 3$, MS [2014, Definition 2.16] which asserts that a subset of the ordinals in $\mathcal{M}_{\mathrm{II}}$ has no least element. Finally, the fourth property follows from MS [2014, Claim 2.7] for $R_I new$ and from MS [2014, Definition 2.16 and Claim 2.11] for $R_I 3$.

The rest of the proof is similar to that of the previous lemma. To see that P is Σ^0_4, we again rewrite the definition in terms of indices of reductions (from W^X and X). The conditions $R_I new$ and $R_I 3$ are Π^0_3. As remarked above, $\Pi^{0\,W^X}_2$ relations are uniformly Π^X_2 and, of course, the relation $Z \leq_T Y$ is Σ^0_3. It is then routine to calculate that P is Σ^0_4.

The closure $\hat{P}$ of P under $\equiv_T$ is then also a Σ^0_4 set. By Σ^0_4-TD, $\hat{P}$ contains or is disjoint, from a cone. By Shoenfield's absoluteness theorem, the base $\mathbf{z}$ of the cone can be taken to be in L. Let α be an admissible ordinal such that $L_\alpha \models T$ and every element of L_α is definable in L_α and such that $Z \in L_\alpha$. (Such an ordinal exists by the unboundedness result at the beginning of this proof.) Let Th_α be the theory of L_α. So, in particular $Z, Z' \leq_T Th_\alpha$.

We first claim that $Th_\alpha \notin \hat{P}$. Take $Y \equiv_T Th_\alpha$; we will show that $Y \notin P$. To see this, consider any $\hat{T} \equiv_T W^Y$ with term model $\mathcal{M}_{\mathrm{I}}$ as in the definition of P. Let $\tilde{T} = Th_\alpha$ with term model $\mathcal{M}_{\mathrm{II}} = L_\alpha$. So, we have that $\mathcal{M}_{\mathrm{I}} \neq L_\alpha$ because their theories have different Turing degrees. Thus, $\mathcal{M}_{\mathrm{I}}$ cannot be the well-founded part of $\mathcal{M}_{\mathrm{II}}$, and hence $R_I new$ cannot hold. Since $\mathcal{M}_{\mathrm{II}}$ is well-founded, $R_I 3$ cannot hold either. So $Y \notin P$.

Second, we find a degree $X \geq_T Z$ which is in P, and hence in $\hat{P}$. As $Z' \leq_T Th_\alpha$, there is (by pseudojump inversion) an $X >_T Z$ such $W^X \equiv_T Th_\alpha$. We claim that $X \in P$.

Let $\hat{T} = Th_\alpha$ with term model $\mathcal{M}_{\mathrm{I}} = L_\alpha$. Consider any $\tilde{T} \equiv_T X$ with term model $\mathcal{M}_{\mathrm{II}}$ as in the definition of P. So, we have that $\mathcal{M}_{\mathrm{II}} \neq L_\alpha$ because their theories have different Turing degrees, and we have that $(Th(\mathcal{M}_{\mathrm{II}}))' \equiv_T (Th_\alpha)'$ because W^X is low over X. By Claim 4.5, $\mathcal{M}_{\mathrm{II}}$ is ill-founded and its well-founded part is at most L_α. If $\mathcal{A}_{\mathrm{II}} = L_\alpha$, then $\mathcal{M}_{\mathrm{I}}$ is isomorphic to the well-founded part of $\mathcal{M}_{\mathrm{II}}$, and hence $R_I new$ holds. Otherwise, $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable. Since $R_{II}3$ does not hold (because $\mathcal{M}_{\mathrm{II}}$ is well-founded) $R_I 3$ must hold, proving that $X \in P$.

As $Th_\alpha, X \geq_T Z$, we see that $\mathbf{z}$ is not the base of a cone for $\hat{P}$ for the final contradiction and so α_2 exists as required. □

Corollary 4.8. *Δ^1_3-CA$_0$ does not prove Σ^0_4-TD. Indeed, Π^1_1-CA$_0$ + Σ^0_4-TD proves that for every set Z there is a β-model of Δ^1_3-CA$_0$ containing Z and hence much more than the consistency of Δ^1_3-CA$_0$.*

Proof. By Simpson [2009, VII.5.17 and the notes thereafter], $L_{\alpha_2} \cap \mathbb{R}$ is a model of Δ^1_3-CA$_0$ and indeed a β-model. Thus Π^1_1-CA$_0$ + Σ^0_4-TD proves the consistency of Δ^1_3-CA$_0$. As the Lemma relativizes to any Z, Π^1_1-CA$_0$ + Σ^0_4-TD proves that, for every set Z, there is a β-model of Δ^1_3-CA$_0$ containing Z. □

5 Questions

There are several natural questions left open here. We first point to two for which we expect that answers should require some new interesting models of fragments of Z_2.

Question 5.1. Does WKL$_0$ or some other known principle strictly between RCA$_0$ and ACA$_0$ prove Δ^0_3-TD?

Question 5.2. Does Δ^0_3-TD (or some stronger version) prove ACA$_0$ over WKL$_0$?

Question 5.3. Clarify the status of Δ^0_4-TD over ACA$_0$. In particular does ATR$_0$ + Σ^1_1-TI$_0$ (or equivalently Σ^1_1-IND) or ATR$_0$ with full induction prove Δ^0_4-TD? If not, does ACA$_0$ + Δ^0_4-TD prove Π^1_1-TI$_0$?

Question 5.4. Does Δ^0_4-TD (or some stronger version) prove Π^1_1-CA$_0$ over ATR$_0$?

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