Zorn's Lemma, Reverse Mathematics and Applications in Combinatorics

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Abstract

We present two hierarchies of versions of Zorn's Lemma which can be used directly in reverse mathematical analyses just as the original is used in standard mathematical arguments. We show that at the first two levels these versions are reverse mathematically equivalent (over RCA_0) to Π^1_k-CA_0 for $k = 1, 2$ and at higher levels to known choice axioms not provable in Z_2. We gives several examples of how they could be used in known proofs and a new reverse mathematical analysis of some theorems about injective choice functions (matchings) for countable families (of sets of numbers). These include a couple of unusual situations. One principle (MCSF) can be proven in Π^1_2-CA_0 using our version of Zorn's lemma at Π^1_2. It is a Π^1_4 statement and perhaps might be equivalent to Π^1_2-CA_0. Another (MRSF) is just a Π^1_3 statement and so cannot imply even Δ^1_2-CA_0 but its known proofs all use even more than Π^1_2-CA_0 (Π^1_2-CA_0^+). Both of these principles are shown to imply Π^1_1-CA_0. These results suggest several interesting reverse mathematical questions. We also briefly discuss some connections to similar work of Flood, Jura, Levin and Markkanen [2022] on matchings in general graphs.

1 Introduction

Zorn's Lemma is frequently used to prove the existence of objects with various maximality properties in many areas of mathematics. The analysis of such arguments in reverse mathematics has followed a variety of approaches. Early examples in algebra such as maximal ideals and maximal independent sets of many sorts (linear, algebraic, etc.) as, for example in Simpson [1998] were established by direct constructions in ACA_0 to which they were proven equivalent. (We refer to Simpson [1998] or [2009] for definitions of all the axiom systems of second order arithmetic used in reverse mathematics that we mention but do not define.) These results often relied on special aspects of the properties being analyzed. A common one was being of finite character (i.e. the property holds of a set if and only if it holds for all finite subsets).

Other applications of Zorn's Lemma often seemed more difficult to handle. My own first foray into reverse mathematics was Aharoni, Magidor and Shore [1992], hereafter AMS. It used a maximal set in a bipartite graph with some matching property whose existence was classically proven by Zorn's Lemma. Our substitute technique for the reverse mathematical analysis used the Kleene Basis Theorem and we have since used other basis theorems in similar applications. After all, Zorn's Lemma is famously equivalent to the axiom of choice, indeed the full version of the mathematical result analyzed in AMS easily proves the usual full set theoretic axiom of choice (Proposition 3.6) and basis theorems tell us how to choose specific simple elements from certain collections of sets. (See §4.)

Many restricted versions of the axiom of choice were introduced in reverse mathematics some of which such as Σ^1_1-AC$_0$ or the stronger Σ^1_1-DC$_0$ had previously been studied in hyperarithmetic theory along with basis theorems. (See Simpson [1998, VIII.4].). Simpson [1998, VII.6] deals with the analogous axioms Σ^1_k-AC$_0$ and Σ^1_1-DC$_0$ and introduces and studies stronger versions, Strong Σ^1_k-DC$_0$. These last versions are shown to be equivalent to reflection principles for formulas at the corresponding syntactic level. Both of these last two principles (but more often the reflection version) have been used as replacements for Zorn's Lemma in several settings. (See §2.)

Our goal in the work presented here was to find versions of Zorn's Lemma that could be stated in second order arithmetic and could be used to directly carry out the typical mathematical applications of Zorn's Lemma in the reverse mathematical analysis of standard theorems. We also wanted to determine the reverse mathematical strengths of these versions and use them in new analyses of the strength of standard theorems of combinatorics.

In §2, we provide a few of the known examples of proofs in second order arithmetic of combinatorial theorems that use various arguments to replace Zorn's Lemma as part of a reverse mathematical analysis. These arguments and others, motivated our formulations of versions Σ^1_n-ZLS$_0$ and Σ^1_n-ZLC$_0$ of Zorn's Lemma in second order arithmetic. Our principles can directly be applied to give maximal sets or collections of sets, respectively, satisfying any Σ^1_n definable property closed under increasing unions or a variation of that condition. We note they supply immediate applications for the examples previously discussed. We then analyze the reverse mathematical strength of these principles in §3. They are equivalent (over RCA$_0$) to Strong Σ^1_{n+1}-DC$_0$ and Σ^1_{n+1}-REF$_0$ (a scheme of reflection axioms that in Simpson's terminology says that for every X there is a countably coded β_{n+1}-model containing X). In particular, for $n = 0, 1$ they are equivalent to Π^1_n-CA$_0$. We describe some basis theorems in §4 for Σ^1_k collections of sets for $k = 1, 2$, along with some often unstated uniformities and note that they are provable in Π^1_k-CA$_0$. This supplies some recursion theoretic bounds that we use in our applications of Zorn's Lemma arguments in the next sections. In §5, we apply our versions of Zorn's Lemma to give a new reverse mathematical analysis of some standard theorems about representation (a.k.a. matchings, injective choice functions or marriages) in families of sets of numbers.

In particular, there are two reverse mathematically unusual results from Podewski and Steffens [1976] (hereafter, PS) which we analyze in §5. The first is the proof of MCSF, the existence or maximal critical subfamilies (mcsf) for every family. (Definitions are given in §5.) This is proven in Π^1_2-CA_0 by a straightforward application of Σ^1_2-ZLS_0. Now MCSF is a Π^1_4 statement and at least possibly equivalent to Π^1_2-CA_0 over RCA_0. If so, this would be the first example of standard theorem from the mathematical literature with this strength. The second is a related maximality result MRSF, every family has a maximal representable subfamily. The usual proofs in the literature derive this from MCSF or similar principles using a recursive construction which at each stage applies MCSF or something similar to a family that has already been constructed to extend the representation being constructed to a larger subfamily. Because of the iteration of MCSF (for which we only have a proof in Π^1_2-CA_0), the known proofs are not even in Π^1_2-CA_0. Indeed, it takes some additional effort using the Σ^1_2 basis theorem to carry out our proof even in Π^1_2-CA_0^+. (This axiom says (analogously to ACA_0^+) that for every X there is a set whose first column is X and each successive column is the complete Π^1_2 set in the previous column.) So the known proofs are reverse mathematically very complicated. On the other hand, the principle itself is a Π^1_3 statement and so by known results cannot imply even Δ^1_2-CA_0. Both principles about families, however, are shown to imply at least Π^1_1-CA_0 over RCA_0.

After we had proven these results about families and were preparing a conference talk about our work, we found Flood, Jura, Levin and Markkanen [2022] (hereafter FJLM). It analyzes variations on the results about matchings in graphs in Steffens [1976]. It has many results mathematically and reverse mathematically related to ours. In §6 we give a brief description of those of their results that are very similar in their reverse mathematical structure to ours. We also mention an application of Σ^1_2-ZLS_0 to prove a maximality result from Steffens [1976], MISG, that is not mentioned in FJLM. However, we show that it is equivalent to one, MIM, that they analyze using Σ^1_2-Reflection and additional arguments. Finally, we point out that their proof of another principle in Π^1_2-CA_0^+ by an iteration like the one we use in §5 is missing an argument like ours using the basis theorem to carry out the iteration.

We close with some open questions in §7

Before turning to our Zorn's Lemma like principles, their strength and applications we note a few perhaps not quite standard definitions and conventions

Remark 1.1. The syntactic classes of Σ^i_n, Π^i_n and Δ^i_n are defined as usual. (See, for example, Simpson [2009]. We just note that in our terminology the Σ^1_0 formulas (and the Π^1_0 ones) are just the arithmetic formulas, i.e. those that are Σ^0_n (or Π^0_n) for some n.

Trees T are downward closed subsets of $N^{<N}$ (with respect to being an initial segment). A path on T is a downward closed linearly ordered subset of T. A branch on T is an infinite path on T. $[T]$ is the set of branches of T. If we have specified a branch X on a tree T and a node $\sigma \in X$ we denote the immediate successor of σ in X by σ^+.

Every axiom system, typically marked with a 0 subscript, is assumed to include RCA_0

2 Versions of Zorn's Lemma

We begin with two simple examples of mathematical theorems which are direct application of Zorn's Lemma that have been proven in Π^1_1-CA_0. The first is from Mummert [2005, Lemma 4.1.4] .

Theorem 2.1 (MF)**.** Π^1_1-CA_0 *proves MF: Every filter on a partial order extends to a maximal filter. (A filter on a partial order P is an upward closed subset F of P such that if $p, q \in F$ then the is an $r \in F$ such that $p, q \leq r$.)*

As filters are obviously closed under increasing unions, the basic mathematical fact here is an immediate application of Zorn's Lemma. Mummert's proof in Π^1_1-CA_0 uses Σ^1_1-Reflection (see Definition 3.3) in a direct construction and then proves that it produces a maximal filter as required.

The second is Theorem 8.3 of a recent paper by Fiori-Carones, Marcone, Shafer and Solda [2024], hereafter FCMSS.

Definition 2.2. A *maxless chain* in a partial order P is a linearly ordered subset C of P which has no maximal element

Theorem 2.3 (MMLC)**.** Π^1_1-CA_0 *proves MMLC: Every partial order contains a maximal (with respect to inclusion) maxless chain.*

Again the obvious mathematical proof of MMLC applies Zorn's Lemma to the collection of maxless chains in P under inclusion. FCMSS gives a direct construction of a particular maxless chain with special properties using Π^1_1-CA_0 and then prove its maximality.

We want general versions of Zorn's Lemma that can be directly cited in proofs of the existence of various maximal objects but are provable in Π^1_1-CA_0 or other appropriate systems. We begin with a hierarchy of such principles for collections of subsets of N.

Definition 2.4 (Σ^1_n-ZLS_0)**.** *Zorn's Lemma for Σ^1_n collection of sets*: For any nonempty collection of sets defined by a Σ^1_n formula $\Phi(A)$ ($n \geq 0$) which is *closed under increasing countable unions* (i.e. for every $\langle A_k \rangle$ s.t. $\forall k(A_k \subseteq A_{k+1}$ & $\Phi(A_k))$ and $A = \cup A_k$, $\Phi(A)$), there is a maximal A (with respect to set inclusion) such that $\Phi(A)$.

Note that In the setting of reverse mathematics where everything is countable and all sets are subsets of N, the standard hypothesis of being closed under increasing unions in Zorn's Lemma is equivalent to being closed under increasing unions of order type N.

We see that Σ^1_1-ZLS$_0$ is provable in Π^1_1-CA$_0$ in Theorem 3.5. As being a filter or maxless chain in a poset are arithmetic, Σ^1_1-ZLS$_0$ (or even Σ^1_0-ZLS$_0$) clearly suffices to prove Theorems 2.1 and 2.3. Most of the applications we consider are to collections of sets but one that applies to collections of classes (of sets) is a result from Barnes, Goh and Shore (BGS) [2023, 2025].

Definition 2.5. A *Graph* $G = (V, E)$ is a set V of vertices and a symmetric irreflexive binary (edge) relation E on V. A *ray* R in G is a sequence of distinct x_n, $n \in \mathbb{N}$ such that $(\forall n)E(x_n, x_{n+1})$. A class S is one of (pairwise) *disjoint rays* in G if every set in S is a ray in G and no two of them have a vertex in common.

Theorem 2.6 (MIRT)**.** *(Halin 1965): Every graph G has a (possibly finite) class S of disjoint rays which is maximal under inclusion, i.e., for any ray R in G, there is a member of the class which shares a vertex with R.*

As Halin points out, MIRT immediately follows from Zorn's Lemma. BGS prove that it is a theorem of Π^1_1-CA$_0$. Their first proof as mentioned in BGS [2023, p. 37] used a basis theorem. (See §4.) Their second proof on the same page used Strong Σ^1_1-DC$_0$ (Definition 3.1). A referee suggested using Σ^1_1-Reflection (Definition 3.2) instead (BGS [2024, Theorem 5.14]). We want a version of Zorn's Lemma (provable in Π^1_1-CA$_0$) that would obviously imply MIRT and so one that applies to collections of classes.

Notation 2.7 (Classes)**.** *We represent classes by sets in a typical way: the set C represents the class $\{C^{[n]} | n \in N\}$ where $C^{[n]} = \{x | \langle n, x\rangle \in C\}$. We say $A \in C$ if $\exists n(C^{[n]} = A)$ and define containment for classes: $C \subseteq_c D \Leftrightarrow \forall n \exists m(C^{[n]} = D^{[m]})$ and* c-equality *by $C =_c D \Leftrightarrow C \subseteq_c D$ & $D \subseteq_c C$. We let $\cup_{sc} \langle C_i\rangle = E$, the* c-union of the sequence $\langle C_i\rangle$ *of classes, be defined by $(\forall i, n)(E^{[\langle i,n\rangle]} = C_i^{[n]})$ and set $\cup_c C = \{x | \exists n(x \in C^{[n]})\}$. We think of a Σ^1_n formulas $\Phi(C)$ as determining or representing the collection of classes C such that $\Phi(C)$. We also guarantee that Φ expresses a property of the intended class and not just the representation by requiring that $(\forall C, D)(C =_c D \wedge \Phi(C) \rightarrow \Phi(D))$. We say that the collection of classes C represented by Φ is* closed under increasing countable unions *if $\forall \langle C_t\rangle [\forall t(C_t \subseteq_c C_{t+1}$ & $\Phi(C_t)) \rightarrow \Phi(\cup_{sc} \langle C_i\rangle)$*

Some care is necessary in formulating a version of Zorn's Lemma for classes as there is no set which represents the class of all sets in any model of RCA$_0$. So, for example, the trivial collection of classes defined by $\Phi(C) \Leftrightarrow C = C$ which is obviously closed under countable unions has no maximal element in any model of RCA$_0$. We suggest one version.

Definition 2.8 (Σ^1_n-ZLC$_0$)**.** *Zorn's Lemma for Σ^1_n collection of classes*: For any non-empty collection of classes represented by a Σ^1_n formula $\Phi(C)$ which is closed under increasing countable unions and such that $(\Phi(C)\ \&\ \Phi(D)\ \&\ C \subseteq_c D\ \&\ \cup_c D = \cup_c C) \to C =_c D)$, $\exists C(\Phi(C)$ and C is maximal w.r.t. $\subseteq_c$ among classes satisfying $\Phi)$.

The extra condition about $\cup_c C$ in Definition 2.8 means that to strictly increase the class represented by C by a class represented by some D, some number not in $\cup_c C$ must be put into $\cup_c D$. We see that Σ^1_k-ZLC$_0$ is provable in Π^1_k-CA$_0$ for $k = 1, 2$ in Theorem 3.5. This condition will be important in that proof. Returning to MIRT, the collection of classes of disjoint rays in the graph G is clearly Σ^1_0 and closed under increasing countable unions. It satisfies the extra condition of Definition 2.8 because the rays in each class must be pairwise disjoint. So any new ray added to C must add new elements to $\cup_c C$. Thus MIRT follows immediately from Σ^1_1-ZLC$_0$.

3 The Strength of Σ^1_n-ZLS$_0$ and Σ^1_n-ZLC$_0$

Before considering applications of our versions of Zorn's Lemma we analyze their reverse mathematical strength over RCA$_0$ by relating it to other systems such as Π^1_n-CA$_0$, Strong Σ^1_n-DC$_0$ and what we call Σ^1_n-REF$_0$. We slightly modify the notation of Simpson [2009, VI.6.1(4)] to define Strong Σ^1_n-DC$_0$.

Definition 3.1. *Strong* Σ^1_n*-DC*$_0$ is the theory containing RCA$_0$ and the scheme

$$(\exists W)(\forall n)(\forall Y)\left(\Phi\left(n, \bigoplus_{i<n} W^{[i]}, Y\right) \to \Phi\left(n, \bigoplus_{i<n} W^{[i]}, W^{[n]}\right)\right),$$

for any Σ^1_k formula $\Phi(n, X, Y)$ (in which W does not occur).

We have used our definition of $W^{[n]}$ in place of Simpson's $(W)_n$ and $\bigoplus_{i<n} W^{[i]}$ for his $(W)^n$. We also note that he assumes ACA$_0$ in place of RCA$_0$ but each of the choice principles in his definition easily imply ACA$_0$ over RCA$_0$. (Σ^1_0-AC$_0$, which is obviously weaker than all the others, implies ACA$_0$: Let $\psi'(X, n)$ be the Σ^0_1 formula which says $n \in X'$ and so $\forall n \exists Y([Y = \{1\} \wedge \psi'(X, n)] \vee [Y = \{0\} \wedge \neg\psi'(X, n)]$. Applying Σ^1_0-AC$_0$ we get a set in which X' is obviously recursive.) In any case, Strong Σ^1_n-DC$_0$ is often quite cumbersome to apply and we never actually use it in this paper. It is simply the choice axiom from the literature that we show is equivalent to our Σ^1_n-ZLS$_0$ and Σ^1_n-ZLC$_0$.

We next slightly simplify and modify Simpson's [2009, VII.7.2 and V.II.7.5] definitions in RCA$_0$ of countably coded β_k models and β_k-model reflection. We still implicitly use the existence of universal Π^0_1 and Σ^1_n formulas for $n \geq 1$.

Definition 3.2. Given a set A, we say an X is a Σ^1_n*-submodel* (countably coded β_n-submodel) *containing* A if $A = X^{[0]}$ and $\mathcal{M}(X) = \langle N, \{X^{[i]} \mid i \in N\}\rangle$ is a Σ^1_n *submodel*, i.e. for any Σ^1_n formula Φ with (set) parameters from among the $X^{[i]}$, $\Phi \Leftrightarrow \mathcal{M}(X) \vDash \Phi$. Note that this is equivalent to requiring that every Σ^1_n formula with parameters in $\mathcal{M}(X)$ that has a witness has one in $\mathcal{M}(X)$.

Definition 3.3 (Σ^1_n-REF$_0$)**.** Σ^1_n-*REF*$_0$, Σ^1_n-Reflection, is the theory consisting of ACA$_0$ and the assertion that for every A there is a Σ^1_n-submodel X containing A.

We can now characterize the strength of the Σ^1_n-ZLS$_0$ and Σ^1_n-ZLC$_0$. We first introduce a notion that is useful for some our codings.

Definition 3.4. We say B *is the characteristic function of a set* if for every m exactly one of $\langle m, 0\rangle$ and $\langle m, 1\rangle$ is in B and no other numbers are in B. In this case, we naturally say that B *is the characteristic function of the set* X *defined by* $m \in X \Leftrightarrow \langle m, 1\rangle \in B$. So we can just replace instances of $y \in X$ in any Σ^1_n formula $\Psi(X, k)$ by $\langle y, 1\rangle \in B$ without increasing the complexity of Ψ.

An obvious but crucial consequence of this definition is that if A and B are the characteristic functions of sets and $A \subseteq B$, then $A = B$.

Theorem 3.5. *For each $n \geq 0$, the following implications and equivalences hold over RCA$_0$:*

1. *Σ^1_{n+1}-ZLC$_0$ $\to \Sigma^1_n$-ZLC$_0$, Σ^1_{n+1}-ZLS$_0$ $\to \Sigma^1_n$-ZLS$_0$.*
2. *Σ^1_0-ZLS$_0 \to$ ACA$_0$ and Σ^1_0-ZLC$_0 \to$ ACA$_0$.*
3. *Σ^1_n-ZLC$_0$ $\to \Sigma^1_n$-ZLS$_0$.*
4. *Σ^1_n-ZLC$_0$ $\to \Pi^1_n$-CA$_0$, Σ^1_n-ZLS$_0 \to \Pi^1_n$-CA$_0$, Σ^1_0-ZLC$_0 \to \Pi^1_1$-CA$_0$ and Σ^1_0-ZLS$_0 \to \Pi^1_1$-CA$_0$.*
5. *Σ^1_{n+1}-ZLC$_0$, Σ^1_{n+1}-ZLS$_0$, Σ^1_{n+1}-REF$_0$ and Strong Σ^1_{n+1}-DC$_0$ are all equivalent .*
6. *Σ^1_0-ZLC$_0$, Σ^1_0-ZLS$_0$, Σ^1_1-ZLC$_0$, Σ^1_1-ZLS$_0$ and Π^1_1-CA$_0$ are all equivalent as are Σ^1_2-ZLC$_0$, Σ^1_2-ZLS$_0$ and Π^1_2-CA$_0$.*

Proof.

1. These implications are all immediate from the definitions.
2. Let $\psi'(X, n)$ be the Σ^0_1 formula defining X' as above. For Σ^1_0-ZLS$_0$ let $\Phi(A) = (\forall m \in A)\psi'(X, n)$. For Σ^1_0-ZLC$_0$, let $\Phi(C)$ be a Σ^1_0 formula saying that $\forall k(C^{[k]} = \emptyset \vee \exists m((C^{[k]} = \{\langle m, 1\rangle\} \wedge \psi'(X, m)) \vee (C^{[k]} = \{\langle m, 0\rangle\} \wedge \neg\psi'(X, m)))$. It is clear that these formulas satisfy the hypotheses of Σ^1_0-ZLS$_0$ and Σ^1_0-ZLC$_0$, respectively. Let A and C be, respectively, a set and class as in their conclusion. By the maximality of A it is clear that $A = X'$. By the maximality of C, for each m either $\langle m, 1\rangle \in \cup_c C$ (in which case $m \in X'$) or $\langle m, 0\rangle \in \cup_c C$ (in which case $m \notin X'$). Thus $X' \leq_T C$ as required.
3. Let $\Phi(A)$ define a Σ^1_n collection of sets satisfying the hypotheses of Σ^1_n-ZLS$_0$. Define a Σ^1_n collection of classes by a $\Psi(C)$ equivalent to $\forall k \exists x(C^{[k]} = \{x\} \vee C^{[k]} = \emptyset)$ & $\Phi(\cup_c C)$. By our assumptions about Φ, it is clear that Ψ satisfies the hypotheses of Σ^1_n-ZLC$_0$. Thus we have a $\subseteq_c$-maximal class C satisfying Ψ. Note that $\cup_c C$ exists by (2). It is then clear that $\cup_c C$ is a maximal set satisfying Φ as required.

4. By (2) and (3), it suffices to prove that Σ^1_{n+1}-ZLS$_0 \rightarrow \Sigma^1_{n+1}$-CA$_0$ and Σ^1_0-ZLS$_0 \rightarrow \Sigma^1_1$-CA$_0$ in ACA$_0$. Consider any Σ^1_{n+1} formula $\exists X \Psi(X,k)$ where Ψ is Π^1_n. Let $\Phi(A)$ be a formula saying that $\forall k(A^{[k]}$ is $\emptyset$ or the characteristic function of a set X such that $\Psi(X,k))$. We can take Φ to be Σ^1_{n+1} and, if $n=0$, even $\Pi^1_0 = \Sigma^1_0$. Clearly Φ satisfies the hypotheses of Σ^1_{n+1}-ZLS$_0$ or even Σ^1_0-ZLS$_0$ when $n=0$. Thus we have a maximal A such that $\Phi(A)$ holds. By maximality, $\{k | \exists X \Psi(X,k)\} = \{k | \langle 0,1\rangle \in A^{[k]} \vee \langle 0,0\rangle \in A^{[k]}\}$ which is the set required by Σ^1_{n+1}-CA$_0$.

5. First note that, by Simpson [2009, VII.7.4] (and our remark above that even Σ^1_0-AC$_0$ →ACA$_0$), Σ^1_{n+1}-REF$_0$ $\Leftrightarrow$ Strong Σ^1_{n+1}-DC$_0$. Thus, by (3) it suffices to prove that Σ^1_{n+1}-ZLS$_0$ $\rightarrow \Sigma^1_{n+1}$-REF$_0 \rightarrow \Sigma^1_{n+1}$-ZLC$_0$.

   To prove the first implication, assume Σ^1_{n+1}-ZLS$_0$. Given a set Z, we want to construct a Σ^1_{n+1}-submodel containing Z. We list all the Σ^1_{n+1} formulas $\Phi_i = \exists X_0 \Theta_i(X_0, X_1, \ldots, X_{f_i})$ with Θ_i Π^1_n which have free variables $X_1, \ldots, X_{f_i}$. We then interpret every number k as a pair $\langle k_{i,} k_\sigma \rangle$ where σ is a sequence of numbers of length f_i so as include all such pairs in some standard numbering scheme. We now define a Σ^1_{n+1} collection of sets by a formula $\Phi(A)$ which says that $A^{[0]}$ is the characteristic function of Z and $\forall k > 0(A^{[k]} = \emptyset \ \vee \ [A^{[k]}$ is the characteristic function of a set and for each $l < f_i$, $A^{[\sigma(l)]}$ is the characteristic function of a set which we call Z_{l+1} and $\Theta(Z_0, Z_1, \ldots, Z_{f_i})])$. Given Definition 3.4 and the consequence noted there, it is easy to see that $\Phi(A)$ satisfies the hypotheses of Σ^1_{n+1}-ZLS$_0$ and so there is a maximal set B such that $\Phi(B)$.

   We now verify that if X is such that $X^{[0]} = Z = B^{[0]}$ and the $X^{[i]}$ $i > 0$ are the sets whose characteristic functions are the nonempty $B^{[j]}$ with $j > 1$, then $\mathcal{M}(X)$ is a Σ^1_{n+1}-submodel containing Z as required. Consider any Σ^1_{n+1} formula $\exists X_0 \Theta(X_0, X_1, \ldots, X_j)$ and sets E_l, $1 \le l \le j$ in $\mathcal{M}(X)$. Let k be the pair $\langle k_{i,} k_\sigma \rangle$ where k_i is the code for this Σ^1_{n+1} formula and σ is the sequence with $B^{\sigma(l)}$ the characteristic function of E_l for $1 \le l \le j$. If $B^{[k]}$ is the characteristic function of a set Z_0 then $\Theta(Z_0, E_1, \ldots, E_j)$ holds and $Z_0 \in \mathcal{M}(X)$ as required. Otherwise $B^{[k]} = \emptyset$. However, if there were a witness Z_0 such that $\Theta(Z_0, E_1, \ldots, E_j)$, we could enlarge B by making $B^{[k]}$ the characteristic function of Z_0. This larger set would then also satisfy Φ for a contradiction.

   For the second implication assume Σ^1_{n+1}-REF$_0$ and consider any Σ^1_{n+1} property of classes $\Phi(C)$ satisfying the hypotheses of Σ^1_{n+1}-ZLC$_0$. By this hypothesis we have a class C_0 such that $\Phi(C_0)$. By Σ^1_{n+1}-REF$_0$ we have an X which is a Σ^1_{n+1}-submodel with $\mathcal{M}(X)$ containing C_0 and any parameters in Φ and so $\mathcal{M}(X) \vDash \Phi(C_0)$. Using X and $\mathcal{M}(X)$ we now construct the required maximal class C for Φ by recursion. Assume by induction that we have constructed $C_k \in \mathcal{M}(X)$ such that $\mathcal{M}(X) \vDash \Phi(C_k)$. As $\mathcal{M}(X)$ is a Σ^1_{n+1} submodel there is a class $D \supseteq_c C_k$ such that $\Phi(D)$ and $k \in \cup_c D$ if and only if there is one in $\mathcal{M}(X)$ such that $\mathcal{M}(X) \vDash \Phi(D)$. The existence of such a $D \in \mathcal{M}(X)$ is arithmetic in X, indeed recursive in a fixed number of jumps depending only on the complexity of Φ. If there is one, we can find one recursively in the same number of jumps of X and let it be C_{k+1}. If there is no such D then we let $C_{k+1} = C_k$ and continue the recursion. So we have constructed the sequence $\langle C_k \rangle$ such that $\forall k(\Phi(C_k)$ and $C_k \subseteq_c C_{k+1})$. Now consider $C = \cup_{sc} \langle C_k \rangle$. By the hypotheses of Σ^1_{n+1}-ZLC$_0$, $\Phi(C)$. We claim that C is maximal for Φ as required by Σ^1_{n+1}-ZLC$_0$. If not, there would be a $D \supseteq_c C$ such that $\Phi(D)$ but $(\cup_c D) \backslash (\cup_c C) \neq \emptyset$. Say $k \in (\cup_c D) \backslash (\cup_c C)$ and consider stage $k+1$ of our construction. D would have been a witness to the question asked at that stage and so we would have extended C_k to some C_{k+1} such that $k \in \cup_c C_{k+1} \subseteq \cup_c C$ for the desired contradiction.

6. All of these equivalences follow from (2), (4), (5) and the fact that Π^1_k-CA$_0$ $\rightarrow$Strong Σ^1_k-DC$_0$ for $k = 1, 2$ (Simpson [2009,VII.6.9]).

□

We note that none of the Σ^1_{n+3}-ZLS$_0$ or Σ^1_{n+3}-ZLC$_0$ are provable even in full second order arithmetic, Z$_2$, because, as Simpson [2009, VII.6.3] points out, Feferman and Levy (see Levy [1970, Theorem 8]) have constructed a model of Z$_2$ in which even Σ^1_3-AC$_0$ fails.

In the other direction, many applications of Zorn's Lemma in the reverse mathematics literature are done in much weaker systems, often in ACA$_0$ or WKL$_0$. These all rely on special conditions for the collections defined by the $\Phi(A)$ being considered. A common one, for example, is that it be of finite character. The version of Zorn's Lemma for such collections is equivalent to ACA$_0$. This result and many others about weak versions of Zorn's Lemma and the Axiom of Choice can be found, for example, in Dzhafarov and Mummert [2012] and [2013].

4 Basis Theorems and Recursion Theoretic Bounds for ZL

In this section we provide uniform recursion theoretic bounds on the complexity of the maximal sets and classes guaranteed by Σ^1_k-ZLS$_0$ and Σ^1_k-ZLC for $k = 1, 2$.

For each of $k = 1, 2$, given a Σ^1_k $\Phi(A, Z)$ defining a collection of sets or classes as in the hypotheses of Zorn's Lemma one can find a maximal set or class as desired uniformly recursively in the complete $\Pi^1_k(Z)$ set $K^1_k(Z)$.

Theorem 4.1 (Π^1_k-CA$_0$)**.** *For $k = 1, 2$ and a $\Phi(X)$ with parameter Z which represents a Σ^1_k collection satisfying the hypotheses of Σ^1_k-ZLS$_0$ or Σ^1_k-ZLC$_0$, we can find a maximal X satisfying the conclusions of Σ^1_k-ZLS$_0$ or Σ^1_k-ZLC$_0$, respectively, such that $X \leq_T K^1_k(Z)$ with the index of the reduction given uniformly in that of Φ.*

The key ingredient in each of the required constructions for this theorem is the uniform version of the standard Σ^1_k basis theorems.

Theorem 4.2 (Π^1_k-CA$_0$)**.** *For a nonempty class of sets represented by a Σ^1_k formula Φ with parameter Z, we can find, uniformly in the index for Φ, an X such that $\Phi(X)$ with the following properties:*

1. *For $k = 1$, $K^1_1(X) \leq_T K^1_1(Z)$ (and so $X \leq_T K^1_1(Z)$) with the indices of the reductions given uniformly in that of Φ.*
2. *For $k = 2$, X is $\Delta^1_2(Z)$ (and so $X \leq_T K^1_2(Z)$) with the indices of the required formulas and reductions given uniformly in that of Φ.*

There has been some uncertainty in the literature about the status of these basis theorems and their reverse mathematical strength especially for $k = 1$. It was often cited as an exercise in Chong and Yu [2015 Exercise 2.5.6] albeit without an explicit mention of the uniformity condition. Chong told me that the intended proof was by Gandy-Harrington forcing (personal communication). Indeed the basic application of that construction easily proves the theorem with the uniformity being obvious. One explicit construction using Gandy-Harrington forcing (that also does something more) which can obviously be carried out in Π^1_1-CA$_0$ and obviously proves Theorem 4.1 for $k = 1$ can be found in Harrington, Shore and Slaman [2017 Theorem 2.1]. A recent paper proving the same result (with the uniformity) by a method more like Gandy's original proof is Calvert, Franklin and Turetsky [2022 Lemma 2.9]. They cite the theorem as folklore. It was only while preparing a talk about an earlier version of this paper that I found a much earlier proof in Π^1_1-CA$_0$ with the required simplicity property in Simpson's own book [1998 VII.2.12] albeit with other terminology and without mentioning the uniformity that can be extracted from his previous constructions that make no use of Gandy-Harrington forcing. So we now have several proofs of Theorem 4.1 for $k = 1$ in Π^1_1-CA$_0$.

The basic classical proof of the Σ^1_2-Basis theorem (as in Moschovakis. [1980, 4E.5] is a simple elementary application of the Novikov-Kondo-Addison Π^1_1-Uniformization Theorem. Simpson [1998, VII.6.7] proves it in Π^1_1-CA$_0$ as the Σ^1_2-Uniformization Theorem based on either his earlier proofs of the Kondo Π^1_1-Uniformization Theorem or of the Shoenfield. Absoluteness Theorem in Π^1_1-CA$_0$. The required uniformity follows perhaps more directly from the second proof.

Now we can prove our recursion theoretically bounded versions of Σ^1_k-ZLS$_0$ and Σ^1_k-ZLC for $k = 1, 2$.

Proof of Theorem 4.1. Fix Φ as in the theorem. We begin with an X_0 as given by the Σ^1_k basis theorem applied to Φ and construct an increasing sequence X_m recursively in $K^1_k(Z)$ with the properties (and indices) specified in the basis theorems such that $\forall m \Phi(X_m)$. At step $m+1$ ask K^1_k if there is a $Y \supseteq X_m$ such that $m \in Y$ (for Σ^1_k-ZLS$_0$) or a $Y \supseteq_c X_n$ with $m \in \cup_c Y$ (for Σ^1_k-ZLC$_0$) and $\Phi(Y)$. If so choose the Y given by the basis theorem with X_m as an additional parameter. Then set $X_{m+1} = Y$. (Note that if X_m is simple relative to Z

in the sense of the basis theorem then so is Y and the indices are all computed recursively in K_k^1 by uniformity: For $k = 1$ and X_m as given by an application of Theorem 4.2(1) the question is if there is a Y satisfying a $\Sigma_1^1(X_m)$ property. As $K_1^1(X_m) =_T K_1^1(Z)$ by induction, the answer is recursive in $K_1^1(Z)$. If the answer is yes then that basis theorem applied (relative to X_m) gives a Y with $K_1^1(Y) =_T K_1^1(X_m) =_T K_1^1(Z)$ uniformly. For $k = 2$ and an X_m as given by an application of Theorem 4.2(2), the question is if there is a Y satisfying a property which is a conjunction of a formula arithmetic in X_m and one that is Σ_2^1. Now formulas arithmetic in $\Delta_2^1(Z)$ sets are $\Delta_2^1(Z)$ by the usual quantifier manipulation rules since Σ_2^1-AC_0 follows from Π_2^1-CA_0. Thus the whole property is $\Sigma_2^1(Z)$ and whether there is such a Y is recursive in $K_2^1(Z)$. If there is one, Theorem 4.2(2), supplies one which is itself $\Delta_2^1(Z)$ uniformly.) Otherwise, let $X_{m+1} = X_m$. Let $X = \cup X_m$ for Σ_k^1-ZLS_0 and $X = \cup_{sc} \langle X_m \rangle$ for Σ_k^1-ZLC_0. This X is clearly recursive in $K_k^1(Z)$. (For $k = 1$, we have decided if $m \in X$ by step $m + 1$ of the construction. For $k = 2$, we have decided all of $X^{[\langle m,j \rangle]}$ for every j by stage $m + 1$ of the construction.)

By the assumptions of Σ_k^1-ZLS_0 and Σ_k^1-ZLC_0, $\Phi(X)$. For any m, if $m \notin X_{m+1}$ ($m \notin \cup_c X_{m+1}$) then no $Y \supseteq X_m$ ($Y \supseteq_c X_m$) (and so none properly extending X) can satisfy Φ. Thus X is maximal as required. □

As we mentioned above, Theorem 2.6 was first proved in Π_1^1-CA_0 by the use of the Σ_1^1 basis theorem. That proof required the uniform version. Another maximality result in matching theory proved in Π_1^1-CA_0 using a simpler version of this basis theorem is in Aharoni, Magidor and Shore [1992], hereafter AMS, in the "Proof of Lemma 3.2 in Π_1^1-CA_0" on p. 276. We will see an application that uses the uniformity in the Σ_2^1 case in the proof of Theorem 5.7 and in Remark 6.7 where we point to one for the Σ_1^1 case as well.

5 Representable and Critical Families

We now turn to a new reverse mathematical analysis of other combinatorial applications of Zorn's Lemma with a couple of unusual features. We essentially follow Podewski and Steffens [1976] which we denote by PS.

Definition 5.1. A *family* is a function F with domain some $I \subseteq N$ such that $\forall i \in I(\emptyset \neq F(i) \subseteq \mathbb{N})$. A *subfamily* of F is the family $F \upharpoonright S$ for some $S \subseteq I$. We may abuse notation and denote $F \upharpoonright S$ by S when the intended F is clear from context. A family F is *representable* if there is an injective choice function, i.e. a one-one $f : I \to \mathbb{N}$ such that $\forall i \in I(f(i) \in F(i))$. Any such f is a *representation of* F. Note that as being a representation is arithmetic, $S \subseteq I$ being a representable subfamily *(rsf)* is Σ_1^1 (in F). Hereafter in this section, F will always denote a family.

Theorem 5.2 (MRSF)**.** *(PS, Corollary 9): Every countable family has a maximal representable subfamily* (mrsf)*.*

AMS proves (Theorem 4.26) that Theorem 5.2 implies Π^1_1-CA_0 (by a much more circuitous route than we do in Theorem 5.18 below). It then should look like a candidate for an application of the Kleene-Gandy basis theorem (or now Σ^1_1-ZLS_0) to prove the theorem in Π^1_1-CA_0 and complete the equivalence. After all, being representable is a Σ^1_1 property. The problem is that representability is not closed under unions of chains (see the comments after Theorem 5.7). So how does PS prove MRSF? Well, they do prove it by Zorn's Lemma but by applying it to another property of subfamilies.

Definition 5.3. A family F is *critical* if it is representable and for every representation f, $rg(f) = \cup rg(F)$. Note that F being critical is a Σ^1_2 (indeed a $\Sigma^1_1 \wedge \Pi^1_1$) property and $G = \emptyset$ is always a csf.

Theorem 5.4 (MCSF)**.** *(PS) Every family has a maximal critical subfamily* (mcsf)*. Indeed, every critical subfamily* (csf) *can be extended to a mcsf.*

PS (Lemma 1) proves that critical subfamilies are closed under unions and so the theorem follows by Zorn's Lemma. It is clear that being a critical family is a Σ^1_2 property and so we can apply Σ^1_2-ZLS_0 to get a mcsf as long as we can prove closure under countable unions in Π^1_2-CA_0. Their proof of closure is by transfinite recursion. As we only need the countable case we can simplify their proof and see that it works in Π^1_2-CA_0.

Theorem 5.5 (Π^1_2-CA_0)**.** *Every family has a mcsf. Indeed, every csf can be extended to a mcsf.*

Proof. Given a csf G, we show that being a csf extending G is closed under increasing countable unions. Let $\langle A_i \rangle$ be an increasing sequence of subsets of N such that each is a csf of a given family F with $A_0 = G$. By Σ^1_1-AC_0 we have $\langle f_i \rangle$ with each f_i a representation of A_i. We now construct a representation f of $A = \cup A_i$ Let $f(k) = f_i(k)$ where i is least such that $k \in A_i$. We claim $f \restriction A_i$ is a representation of A_i for every i and so for A. If not, let $k+1$ be the least counterexample, i.e. for some $l \in A_{k+1} \backslash A_k$, $f_{k+1}(l) = f_k(m)$ for some $m \in A_k$. Now $f_{k+1} \restriction A_k$ is a representation of A_k and so by criticality its range is $\cup\{F(i) | i \in A_k\}$ which contains $f_k(m) = f_{k+1}(l)$ contradicting the fact that f_{k+1} is a representation of A_{k+1}. Now argue that A is critical. If f represents A then $f \restriction A_i$ represents A_i. By the criticality of A_i, $rg(f \restriction A_i) = \cup rg(F \restriction A_i)$ and so, as the A_i are nested, $rg(f) = \cup rg(F \restriction A)$. We are now done by Σ^1_2-ZLS_0. □

It might be tempting to think that one could weaken the theory needed in Theorem 5.5 by exploiting the fact that the definition of critical family is only $\Sigma^1_1 \wedge \Pi^1_1$ rather than Σ^1_2. No such improvement is possible in general, however, as it is easy to see that ZLS_0 for even just Π^1_1 formulas Φ already implies Σ^1_2-CA_0 and so Σ^1_2-ZLS_0.

The route from this theorem to the proof of the one for representable subfamilies (Theorem 5.2) is, however, nontrivial. PS uses several other notions and at the end provides a construction by recursion applying MCSF at each step. This argument uses an iteration of applications of MCSF and so of Π^1_2-CA_0 in a way that goes even beyond

Π^1_2-CA$_0$. Note that our proof even in this stronger theory uses the uniform complexity bounded version of Σ^1_2-ZLS$_0$ of Theorem 4.1.

Definition 5.6 (Π^1_n-CA$_0^+$)**.** For $n \geq 1$, Π^1_n-CA$_0^+$ is the theory containing RCA$_0$ and the axioms asserting that for every Z there is a set X such that $X^{[0]} = Z$ and for every m, $X^{[m+1]}$ is $K^1_n(X^{[m]})$, the complete Π^1_n set in $X^{[m]}$.

Theorem 5.7 (Π^1_2-CA$_0^+$)**.** *Every family has a maximal representable subfamily.*

As in PS, it suffices to prove the next two Lemmas. The first is elementary and provable in RCA$_0$. The second uses Π^1_2-CA$_0^+$ and Theorem 4.1. PS (p. 44, Remark) shows that not every representable subfamily can be extended to a mrsf by providing an example in a countable family. That example also shows that representable subfamilies are not closed under increasing unions of countable chains.

Notation 5.8. *If F is a family, $j \in dom(F)$, $a \in F(j)$ and for no $k \neq j$ does $F(k) = \{a\}$, then $F(j,a)$ is the family with domain $dom(F)\backslash\{j\}$ defined by $F(j,a)(i) = F(i)\backslash\{a\}$. For G a subfamily of F, F_G is the family with domain $\{i \in dom(F) | F(i) \subsetneq \cup rg(G)\}$ defined by $F_G(i) = F(i)\backslash \cup rg(G)\}$.*

Lemma 5.9. *If G is a mcsf of F and F_G is representable, then $G \cup dom(F_G)$ is a mrsf of F.*

Proof. Let g and h represent G and F_G, respectively. First note that, by the definition of F_G, no $h(i) \in \cup rgG$ and, by the criticality of G, $dom(G) \cap dom(F_G) = \emptyset$. Thus $g \cup h$ represents $G \cup F_G$. If there were an $i \in dom(F)\backslash(dom(G) \cup dom(F_G))$ and an h representing $dom(G) \cup dom(F_G) \cup \{i\}$ then $h(i) \notin \cup rg(G)$ as, by the criticality of G, $h[dom(G)] = \cup rg(G)$. Then, by the definition of F_G, $i \in dom(F_G)$ for a contradiction. $\square$

Lemma 5.10. *If G is a mcsf of F then F_G is representable.*

Proof. We construct a sequence of families F^l with domains contained in that of $F_G = F^0$ and compatible representations for subfamilies by a recursion up to some $k \leq \omega$ so that their domains eventually cover all of $dom(F_G)$. Each step of the recursion uses the operations of Notation 5.8 to get the next family. The first operation produces a family H^l uniformly recursive in F^l. The second operation is applied to H^l and a mcsf of H^l to get a family uniformly recursive in $\Pi^1_2(F^l)$ by Theorem 4.1. Choosing the representations being constructed will be simpler (recursive or recursive in the complete Π^1_1 set of what we have already constructed by Theorem 4.2(1)). Thus the whole construction will be recursive in the set supplied by Π^1_2-CA$_0^+$ applied to $F \oplus F_G$.

Given F^l, $l \geq 0$, if $F^l = \emptyset$ we terminate the recursion and set $k = l$. Otherwise, we let $i_l = \mu n(n \in dom(F^l))$, choose any $a_l \in F^l(i_l)$, set $H^l = F^l(i_l, a_l)$ and choose G_{l+1} as a mcsf of H^l with a representation g_{l+1}. We now let $F^{l+1} = H^l_{G_{l+1}}$. If we never reach an l with $F^l = \emptyset$, we set $k = \omega$. In any case we want to show that $\cup\{g_l | l < k\} \cup \{\langle i_l, a_l\rangle \,| l < k\}$

is the desired representation of F_G. We want to verify by induction that for, $l < k$, H^l is a family as defined above; $dom(H^l) = dom(F^l)\setminus\{i_l\}$; $dom(F^{l+1}) = dom(H^l)\setminus dom(G_{l+1})$ and that making the choices (a_l and g_{l+1}) as specified for i_l and $dom(G_{l+1})$ maintains $\cup\{g_l | l < k\} \cup \{\langle i_l, a_l\rangle \, | l < k\}$ as an injective function whose domain is that of F_G at the end. These verifications follow from a series of lemmas in PS that we now describe to finish this proof. $\square$

We have to eliminate the explicit use of some named classes of functions or sets (e.g. $IA(F)$ the collection of representations of F and $\mathcal{G}_F$ the collection of critical subfamilies of F) that may not exist in all models of Π^1_2-CA$_0$. This amounts only in notational variations such as $IA(F) \neq \emptyset$ means that F is representable and noticing that other sets that we do define provably exist in Π^1_2-CA$_0$ (e.g. $\ker(F)$ is Σ^1_2). We state the relevant Lemmas from PS in a suitable terminology and point out why the proofs there work in (mostly much less than) Π^1_2-CA$_0$.

Definition 5.11. The *kernel of F*, $\ker(F) = \{i | \exists G(G$ is a csf of F and $i \in dom(G)\}$. (In PS $\ker(F)$ is defined as $\cup\mathcal{G}_F$.)

Lemma 5.12 (PS Lemma 2)**.** *If F is representable then $\{n | \forall f'(f'$ represents $F \rightarrow \exists i(f'(i) = n))\} = \{n | \exists G(G$ is a csf of F and $\exists i(i \in dom(G)$ and $n \in F(i)\}$.*

The sets in the statement of the Lemma exist by Π^1_2-CA$_0$. The proof of equality in PS which is really all that is needed only uses a recursion with arithmetical steps and more elementary procedures. They next give the property of families that allows us to continue the induction for Lemma 5.10 without loosing any elements of $dom(F)$ as we define the F^l and g_l

Definition 5.13. F is a *K-family* if there is no $i \in dom(F)$ and G a csf of F such that $i \notin dom(G)$ but $F(i) \subseteq \cup rg(G)$.

PS then states a few Lemmas without proof about K-families that can easily be verified from the definitions.

Lemma 5.14 (PS Lemma 3)**.** *If $\ker(F) = \emptyset$ then F is a K-family.*

Lemma 5.15 (PS Lemma 4)**.** *If G is a mcsf of F, then $\ker(F_G) = \emptyset$ and so F is a K-family.*

Lemma 5.16 (PS Lemma 5)**.** *If G is a mcsf of a K-family F, then $dom(F_G) = dom(F)\setminus dom(G)$.*

This last Lemma lets us verify that we don't miss elements of $dom(F)$ in the construction for Lemma 5.10 when we define F^{l+1}. The final lemma from PS needed to carry out the verifications for Lemma 5.10 when we define H^l is the following one.

Lemma 5.17 (PS Lemma 6)**.** *If* $\ker(F) = \emptyset$*,* $j \in dom(F)$ *and* $a \in F(j)$*, then* $F(j, a)$ *is a* K*-family.*

The proof of this Lemma in PS relies Lemma 5.12 (Lemma 2 of PS) and is otherwise elementary. This completes the proof of Lemma 5.10 and so Theorem 5.7 in Π^1_2-CA_0^+.

If we are looking for reversals for Theorems 5.7 or 5.5, MRSF cannot imply even Δ^1_2-CA_0 even over Π^1_1-CA_0 as its quantifier complexity (Π^1_3) is too low (Montalbán and Shore [2012, §6 and especially 6.2]). On the other hand, MCSF is a Π^1_4 sentence so could, in theory, imply and hence be equivalent to Π^1_2-CA_0 even over RCA_0.

We know of very few theorems that follow from, and seem to need, Π^1_2-CA_0. These include some about determinacy and almost none that also imply it. One older equivalence involving topological spaces is given by Mummert and Simpson [2005] but their proof of Π^1_2-CA_0 is only over Π^1_1-CA_0 which is needed to even make sense of the notions involved. A recent one about minimal bad arrays in bqo theory by Freund, Phakhamov and Solda [2024] is done over ATR_0 and is known to be weak over ACA_0 (Freund, Marcone, Pakhomov and Solda [ta]). So we know of no equivalences to Π^1_2-CA_0 over RCA_0. MCSF seems like a good candidate.

We have two partial results in terms of reversals. Both principles imply Π^1_1-CA_0 over RCA_0.

Theorem 5.18. *Each of MRSF and MCSF imply* Π^1_1-CA_0 *over* RCA_0*.*

As an ingredient for the proof of Theorem 5.18, we prove ACA_0

Lemma 5.19. *Each of MRSF and MCSF imply* ACA_0 *over* RCA_0*.*

Proof. Define a family F with domain $\{\langle n, i\rangle \,| n \in N \wedge i \in \{0, 1\}\}$ by $F(\langle n, 0\rangle) = \{\langle n\rangle\}$ and $F(\langle n, 1\rangle) = \{\langle n\rangle\} \cup \{\langle n, s\rangle \,| n$ is enumerated in a one-one enumeration of $0'$ at stage $s\}$. Let S be a mrsf or a mcsf for F with a representing function f. Clearly, for $n \notin 0'$, not both of $\langle n, 0\rangle$ and $\langle n, 1\rangle$ can be in S (as f is injective). If $n \in 0'$ and not both $\langle n, 0\rangle$ and $\langle n, 1\rangle$ are in S then we could properly extend S to S' by making sure both are in S'. We could then replace f by g where $g(\langle n, 0\rangle) = \langle n\rangle$ and $g(\langle n, 1\rangle) = \langle n, s\rangle$ for the required s and other wise $g(x) = f(x)$. This would show that S' is representable and also critical if S were. Thus, if S is maximal, $n \in 0' \Leftrightarrow \langle n, 0\rangle$ and $\langle n, 1\rangle$ are in S. So $0' \leq_T S$ as required. $\square$

We now define a kind of coding of trees as families and some Lemmas about it.

Notation 5.20. *Given a tree* T *define a family* F_T *with domain* T*:* $F_T(\emptyset) = \{\sigma \in T \,||\sigma| = 1\}$*; for* $\emptyset \neq \sigma \in T$*,* $F(\sigma) = \{\sigma\} \cup \{\tau \in T | \exists n(\tau = \sigma\hat{\ }n)\}$*.*

Lemma 5.21. *For any tree* T*,* $[T] \neq \emptyset \Leftrightarrow F_T$ *is representable.*

Proof. If f represents F_T then $\{f^n(\emptyset)|n \in N\} \in [T]$. If $X \in [T]$ and $\sigma \in X$ we let σ^+ be *the immediate successor of* σ *in* X. We define a representing function f by letting $f(\sigma)$ be σ^+ for $\sigma \in X$ and $f(\sigma) = \sigma$ otherwise. □

Lemma 5.22. *If* $[T] = \emptyset$ *the mcsfs of* F_T *are precisely the* $T - \{\sigma\}$ *for* $\sigma \in T$.

Proof. First consider any $\sigma \in T$. We can define a representing function f for $T - \{\sigma\}$ by $f(\rho) = \sigma \restriction |\rho| + 1$ for $\rho \subsetneqq \sigma$ and $f(\rho) = \rho$ for $\rho \nsubseteq \sigma$. This function is clearly injective and total. We next claim that $T - \{\sigma\}$ is critical. So suppose that f represents it. If $f(\rho) \neq \rho$ for some $\rho \nsubseteq \sigma$, then as $f \restriction T^\rho$ where $T^\rho = \{\tau \in T|\tau \supseteq \rho\}$ (properly relabelled so as to identify ρ with $\emptyset$) is a representing function for T^ρ. Lemma 5.21 applied to T^ρ provides a branch in T^ρ and so one in T contradicting our hypothesis. If $\sigma = \emptyset$ we are done as for every $\rho \neq \emptyset$ on T, $f(\rho) = \rho$ and these are precisely the elements of $T - \{\emptyset\} = \cup rg(F_T \restriction (T - \{\emptyset\})$. If $\sigma \neq \emptyset$, consider $\{f^n(\emptyset)|n \in N\}$. As in the proof of Lemma 5.21, this is a path in T starting at $\emptyset$ as long as it is defined. As $[T] = \emptyset$ by hypothesis, this path must reach σ. So we have determined the values of $f(\rho)$ for $\rho \subsetneqq \sigma$ in addition to the previously determined $f(\rho) = \rho$ for $\rho \nsubseteq \sigma$. Together they include all of $T - \{\sigma\}$ which is then critical. As by Lemma 5.21 F_T is not representable, the $T - \{\sigma\}$ are mcsfs and so the $T - \{\sigma\}$ for $\sigma \in T$ are precisely its mcsfs. □

Lemma 5.23. *If* $[T] \neq \emptyset$ *no* $T - \{\sigma\}$ *for* $\sigma \in T$ *is critical.*

Proof. Suppose we have an $X \in [T]$ and $\sigma \in T$. We will define a representation f of $T - \{\sigma\}$ omitting a node from $T - \{\emptyset\} = \cup rg(F \restriction (T - \{\sigma\}))$ from its range for a contradiction. If $\sigma \in X$ we define f as follows: For $\tau \in X - \{\sigma\}$, $f(\tau) = \tau^+$. For $\tau \notin X$ (so $\tau \neq \emptyset$), $f(\tau) = \tau$. Now $\sigma^+ \notin rg(f)$ as desired. If $\sigma \notin X$ (so $\sigma \neq \emptyset$), we define f as follows: For $\tau \subsetneqq \sigma$, $f(\tau) = \tau^+$ ($\subseteq \sigma$). For $\sigma \subsetneqq \tau$, $f(\tau) = \tau$ (which is not in X). For $\tau \in X$ and $\tau \nsubseteq \sigma$, $f(\tau) = \tau^+$. Otherwise, $f(\tau) = \tau$. This f represents $T - \{\sigma\}$ but omits the least $\tau \in X$ with $\tau \nsubseteq \sigma$. □

Proof of Theorem 5.18. By Lemma 5.19 we may assume ACA_0. So we have a list T_e of all the recursive trees. Let F be the disjoint union of the F_{T_e}. Suppose S is a mrsf of F. By Lemma 5.21, $[T_e] = \emptyset$ if and only if S contains the copy of F_{T_e} in F, an arithmetic property of S. As the question of which $[T_e] = \emptyset$ is complete Π^1_1, we have Π^1_1-CA_0. If S is a mcsf of F, then Lemmas 5.22 and 5.23 show that $[T_e] = \emptyset$ if and only if the part of S in the copy of F_{T_e} in F is one of the (copies of) $T_e - \{\sigma\}$ for $\sigma \in T_e$. As this is also an arithmetic question, we again derive Π^1_1-CA_0. □

6 Matchings

An important topic in combinatorics related to families and representations with interesting reverse mathematical problems similar to those in the last section is matchings in graphs. When restricted to bipartite graphs as studied in AMS, this topic is especially close to §5. As we were preparing a conference talk about our work in this paper, we found Flood, Jura, Levin and Markkanen [2022] (hereafter FJLM), that has many reverse mathematical related results about matchings in graphs. Their work includes several principles about matchings in arbitrary graphs with reverse mathematical properties similar the ones we have discussed above. We omit most of the definitions but attempt to give a view of the approaches and results there that are parallel to ours in §5. In addition they have an extensive reverse mathematical analysis of restrictions of the results of Steffens [1976] to locally finite or bounded graphs. We expect that a similar analysis can be done for the work on families considered in §5.

Steffens [1976] deals with matchings in arbitrary graphs. Its final result (Corollary 8) is what FJLM calls MM.

Definition 6.1. A matching in a graph $G = (V, E)$ is a set M of edges in G such that no two edges in M have a vertex in common. A *perfect matching* of $V' \subseteq V$ is a set M' of edges of G such that every vertex in V' is in exactly one edge in M'.

Notation 6.2. MM*, the* Maximal Matching Theorem *is the assertion that for each countable graph $G = (V, E)$ there is a maximal $V' \subseteq V$ that has a perfect matching.*

One of Steffens [1997] basic results (Lemma 3) which is crucial in his proof of MM as well as a characterization result about graphs with perfect matchings, is proved by a use of Zorn's Lemma. It can be proved directly in Π^1_2-CA_0 using Σ^1_2-ZLS_0:

Theorem 6.3 (Π^1_2-CA_0)**.** *MISG: Every graph has a maximal independent subgraph.*

Proof. Being an independent subgraph is a Σ^1_2 property so just follow Steffens's proof of closure under unions (for the countable case) and then apply Σ^1_2- ZLS_0. □

FJLM does not state or prove MISG but derives MM (Theorem 4.12) by a different argument. It first proves in Π^1_2-CA_0 (Theorem 4.7) a maximality principle for independent matchings (MIM). That proof involves both a very clever use of Σ^1_2 Reflection and a couple of applications of absoluteness. It then proves (Theorem 8) PM a kind of characterization theorem for the existence of perfect matchings. That proof uses something called Π^1_2-CA_0^+. (This principle is not precisely defined but Definition 5.6 is what was intended (personal communication).) It includes an iterated use of MIM similar to the proof of MRSF from MCSF in §5. It also shows that MM implies both MIM and PM (Propositions 1.5 and 4.14). Then it shows that MIM implies Π^1_1-CA_0 (over RCA_0) but that MM and so neither MIM nor PM implies Π^1_2-CA_0 because MM is a Π^1_3 statement.

We point out that MISG (proved above in Π^1_2-CA_0 directly by Σ^1_2-ZLS) is essentially equivalent to MIM and that MM even restricted to bipartite graphs easily implies MSRF (over RCA_0) which is known to imply Π^1_1-CA_0 by AMS (Theorem 4.26) and is also reproved more simply in Theorem 5.18 above. Moreover, as MM is Π^1_3 it cannot imply even Δ^1_2-CA_0 by using Montalbán and Shore [2012] as we did for MSRF after Lemma

5.17. Indeed, all the principles studied in FJLM are consequences of MM and so none of them can imply Δ^1_2-CA$_0$.

Lemma 6.4. *A subgraph G' of G is independent if and only if there is an independent perfect matching of G'.*

This is Lemma 2 of Steffens [1977] where the proof is left to the reader. FJLM (Lemma 4.6) notes that is provable in RCA$_0$.

Proposition 6.5 (RCA$_0$)**.** *MISG$\leftrightarrow$MIM. In fact, every independent perfect matching of a misg is a mim and the subgraph specified by the vertices of any mim is an misg.*

Proof. Given a graph G first consider any misg G' with a perfect matching M' independent in G. We claim that M' is a mim. If not, there is an independent matching M'' with $V(M'') \supset V(M')$. Then by Lemma 6.4, the subgraph G'' with vertices $V(M'')$ is independent in G, contradicting the maximality of G'. For the other direction, suppose that M' is a mim in G. Again by Lemma 6.4, the subgraph G' with vertices those of M' is independent. If there were an independent subgraph $G'' \supset G'$ then it has an independent perfect matching which would contradict the maximality of M. □

Proposition 6.6 (RCA$_0$)**.** *MM for bipartite graphs$\rightarrow$MSRF.*

Proof. Given a family F for which we can assume wlog that $\cup rgF \cap I = \emptyset$, consider the associated bipartite graph G with vertices divided into the sides I and $\cup rgF$ and edges $\{(i,x)|x \in F(i)\}$. Let V' be as in MM and M a perfect matching of V'. Let $S = V' \cap I$. Clearly, the matching M gives a representation f of S. Assume, for the sake of a contradiction, that S is not a mrsf. Then, there is an $i_0 \in I \backslash S$ and a representation g of $S \cup \{i_0\}$. We now show that V' is not maximal for a contradiction. If $g(i_0) \notin rg(f)$, then we can add the edge $(i_0, g(i_0))$ to M to get a perfect matching of $V' \cup \{i_0, g(i_0)\} = V'_0$ as desired. If not, $g(i_0) = f(i_1)$ for some $i_1 \in S$. If $g(i_1) \notin rg(f)$, we can get a perfect matching of $V'_1 = V'_0 \cup \{g(i_1)\}$ by matching i_0 with $f(i_1)$ and i_1 with $g(i_1)$. Otherwise, we continue the recursion to get $f(i_2) = g(i_1)$ and check to see if $g(i_2) \notin rg(f)$. If so, we can again define a perfect matching of $V'_0 \cup \{g(i_2)\}$ matching i_2 with $g(i_2)$. If not, we have i_3 with $g(i_2) = f(i_3)$. We continue this recursive procedure until we reach an i_k such that $g(i_k) \notin rg(f)$ and so the desired extension of V' or we define i_j for every j. In this last case, we can define the desired perfect matching of $V'_0 \supset V'$ by matching i_j with $f(i_{j+1})$ for $j \geq 0$. □

Remark 6.7. We also want to point out a lacuna in the proof in FJLM of PM from Π^1_2-CA$_0^+$ (Theorems 4.8) that can be remedied by using Theorem 4.1 as we did in the proof of Lemma 5.10. On its own Π^1_2-CA$_0^+$is not enough to do arbitrary recursions finding and using some set whose existence is guaranteed by Π^1_2-CA$_0$ to produce new sets at each step. It needs an argument that shows that each set constructed is uniformly recursive in some (truly finite) iteration of taking the complete Π^1_2 set relative to the set constructed at the previous stage of the recursion. We do such a recursion in Π^1_2-CA^+_0 using a basis theorem to get a bound on complexity of a witness in the proof of Theorem 5.7. One can see that something is need by looking at the naive "proof" of Σ^1_n-ZLS using a recursion where each step is guaranteed to exist even in RCA_0 (and induction). Start with any C_0 satisfying Φ by assumption. At step $i+1$ consider the set A_i which is $\{1\}$ if there is a $C \supseteq C_i$ such that $i \in C$ and $\{0\}$ otherwise. This set exists and then let C_{i+1} be such a set if $A_i = \{1\}$ and C_i if $A_i = \{0\}$. This sequence and so its union would then be "proven" to exist but this cannot be proven even in Z_2 for $n > 2$ as mentioned above after the proof of Theorem 3.5. Of course, it is easy to see that, for every n, $Z_2 \vdash \Pi^1_n$-CA^+_0.

It seems that a similar lacuna and correction using the Σ^1_1-basis theorem apply to the proof of Finite Path PM from Π^1_1-CA^+_0 suggested for Corollary 6.2 of FJLM. Alternatively, for both results, one can go through the relevant proofs of Theorems VII.6.9 and VII.7.4 of Simpson [2009] used in FJLM to verify that the needed Turing reductions and uniformities hold.

7 Questions

The most appealing question about MCSF and MRSF is the possible reversal to Π^1_2-CA_0:

Question 7.1. Does MCSF$\rightarrow$ Π^1_2-CA_0? (Over RCA_0 but by Theorem 5.18 we can assume Π^1_1-CA_0.)

As we mentioned above, MRSF$\nrightarrow$ Δ^1_2-CA_0. So if one could prove the weaker result that MCSF$\rightarrow$ Δ^1_2-CA_0, that would suffice to show that MRSF$\nrightarrow$MCSF. Thus, it is worth considering the weaker result.

Question 7.2. Does MCSF$\rightarrow$ Δ^1_2-CA_0? (Over RCA_0 but by Theorem 5.18 we can assume Π^1_1-CA_0)?

As for the status of MRSF itself the main question is what is sufficient to prove it?

Question 7.3. Does Π^1_1-$CA_0 \rightarrow$ MRSF (and so is equivalent to it)?

If not, MRSF would be strictly above Π^1_1-CA_0 but, as above, it cannot imply even Δ^1_2-CA_0. Thus such a result would provide an example of an an interesting phenomena.

Clearly more can be asked about matchings in general graphs and the relationships among various principles in FJLM and their extensive work on the reverse mathematical properties of those principles. Many interesting questions are explicitly raised in FJLM. In addition, we can now add questions about the relationships between the principles studied there and here in §5.

Question 7.4. What reverse mathematical relations (other than Proposition 6.6) hold between MRSF, MCSF and each of MIM (MSIG), MM and PM?

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