# Direct and local definitions of the Turing jump

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**Abstract**

We show that there are $\Pi_5$ formulas in the language of the Turing degrees, $\mathcal{D}$, with $\leq$ ,$\vee$ and $\wedge$, that define the relations $\mathbf{x}'' \leq \mathbf{y}''$, $\mathbf{x}'' = \mathbf{y}''$ and so $\mathbf{x} \in \mathbf{L}_2(\mathbf{y}) = \{\mathbf{x} \geq \mathbf{y} | \mathbf{x}'' = y''\}$ in any jump ideal containing $\mathbf{0}^{(\omega)}$. There are also $\Sigma_6\&\Pi_6$ and $\Pi_8$ formulas that define the relations $\mathbf{w} = \mathbf{x}''$ and $\mathbf{w} = \mathbf{x}'$, respectively, in any such ideal $\mathcal{I}$. In the language with just $\leq$ the quantifier complexity of each of these definitions increases by one. For a lower bound on definability, we show that no $\Pi_2$ or $\Sigma_2$ formula in the language with just $\leq$ defines $\mathbf{L}_2$ or $\mathbf{L}_2(\mathbf{y})$. Our arguments and constructions are purely degree theoretic without any appeals to absoluteness considerations, set theoretic methods or coding of models of arithmetic. As a corollary, we see that every automorphism of $\mathcal{I}$ is fixed on every degree above $\mathbf{0}''$ and every relation on $\mathcal{I}$ which is invariant under the double jump or under join with $0''$ is definable over $\mathcal{I}$ if and only if it is definable in second order arithmetic with set quantification ranging over sets whose degrees are in $\mathcal{I}$.

## 1 Introduction

The structure of relative computability as given by Turing reductions and the corresponding structure, $\mathcal{D}$, of the Turing degrees has been the object of extensive study over the past sixty years. A central concern in this research over the past thirty years has been the issue of definability. The general question is which (interesting, apparently external) relations on $\mathcal{D}$ are actually definable in terms of relative computability alone. One important line of research has produced a sequence of results of the form that all relations on $\mathcal{D}$ which could possibly be definable, i.e. they are definable in arithmetic with quantification over both numbers and sets, are definable if restricted to "sufficiently" large degrees

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where sufficiently large has undergone a series of successive weakenings. The other major line of investigation into definability in $\mathcal{D}$ has centered on proving that specific important natural, but apparently external, degrees or relations on $\mathcal{D}$ are definable in $\mathcal{D}$.

The first major results (Jockusch and Simpson [1976]) on definability in $\mathcal{D}$ were in the structure with the Turing jump, $'$, as well as $\leq_T$. By classical results of Kleene and Post, this operator ($A \longmapsto A'$) corresponds to definability in arithmetic (extended by a predicate for membership in $A$) by formulas with only one quantifier. Its $n$th iterate $A^{(n)}$ corresponds to definability by such formulas with $n$ quantifiers. Thus, for example, $\mathcal{A} = \{\mathbf{x} | \exists n \in \omega(\mathbf{x} \leq_T \mathbf{0}^{(n)})\}$ are the degrees of the sets definable in arithmetic. This operator has played a major role in much of the work on $\mathcal{D}$ over the years and the issue of whether it is actually intrinsic to, or definable in, $\mathcal{D}$ was raised already in the fundamental paper of Kleene and Post [1954]. This question essentially asks if quantification in arithmetic can be expressed, level by level, solely in terms of relative computability. It became the overarching goal in the investigations of definability in $\mathcal{D}$.

The first approximation to a definition of the Turing jump (or of any direct definition of a nontrivial class of degrees in $\mathcal{D}$ without the jump) was the definition of the hyper-artihmetical degrees and the hyperjump (Harrington and Shore [1981]). It used codings of arithmetic and the calculation (Harrington and Kechris [1975]) that Kleene's $\mathcal{O}$ is the base of a cone of minimal covers, i.e. $\forall x \geq_T \mathcal{O} \exists y <_T x \neg \exists z (y <_T z <_T x)$. (We say that $\mathbf{x}$ is a *minimal cover of* $\mathbf{y}$ if $\mathbf{y} <_{\mathbf{T}} \mathbf{x}$ and there is no $\mathbf{z}$ with $\mathbf{y} <_T \mathbf{z} <_T \mathbf{x}$ and that $\mathbf{x}$ is a *minimal cover* if it is a minimal cover of some $\mathbf{y}$.) Harrington and Shore [1981] also showed that every automorphism of $\mathcal{D}$ is fixed on every degree above all the hyper-arithmetic ones and that every relation on such degrees (or ones invariant under joining with arbitrary hyperarithmetic degrees) that is definable in second order arithmetic is definable in $\mathcal{D}$. Jockusch and Shore [1984] then introduced and analyzed the notion of pseudojumps or iterated REA operators (e.g. $J_e(A) = A \oplus W_e^A$ and then iterations of such operators into the transfinite). This analysis lead to a proof that $\mathbf{0}^{(\omega)}$ is the base of a cone of minimal covers and it, and the $\omega$-jump ($X^{(\omega)} = \{\langle x, n \rangle | x \in X^{(n)}\}$), are definable in $\mathcal{D}$ as are all relations on degrees above the arithmetic ones (or invariant under join with these degrees) which are definable in second order arithmetic. These proofs also used codings of arithmetic but were based on one natural definition that did not: $\mathcal{A}$ is the downward closure of $\mathcal{C}_\omega = \{\mathbf{c} | \forall \mathbf{z}(\mathbf{z} \vee \mathbf{c}$ is not a minimal cover of $\mathbf{z}\}$.

Cooper [1990, 1993 and elsewhere] suggested an approach similar to that of Jockusch and Shore [1984] to the problem of defining the jump operator. It relied on two ingredients. The first was a version of a cone-avoiding join and completeness theorem like ones proven in Jockusch and Shore [1984] for certain types of 2-REA operators. The second was the existence of a specific such operator that would produce a degree with an order-theoretic property that no r.e. degree could have (even relative to any degree below it). This later claim turned out to be false as Shore and Slaman [2001] proved that no $n$-REA degree for any $n < \omega$ could have the property claimed by Cooper to hold of one 2-REA one.

The jump was then proven definable by Shore and Slaman [1999]. (Cooper later, as in [2001], made other claims for a definition along the lines of his original proposals that were either refuted or unsubstantiated. See Shore [2006] and Jockusch [2002] for more details.) Again the ingredients were a new cone-avoiding join and completeness theorem but now for all $n$-REA operators and a specific 2-REA one with the required properties. A remarkable feature of the proof was the specific operator used and the proof that it was definable in $\mathcal{D}$. The operator was the double jump and the proof of its definability followed from much earlier work of Slaman and Woodin. Although not included in the announcement of their work in Slaman [1991], their metamathematical arguments that gave many other results such as the definability of all relations on degrees above $\mathbf{0}''$ that are definable in second order arithmetic and that all such degrees are fixed under every automorphism of $\mathcal{D}$, also proved that the double jump was definable in $\mathcal{D}$. The definition requires their entire machinery to internalize their analysis of automorphisms of $\mathcal{D}$ within $\mathcal{D}$ itself. It relies on set theoretic forcing to collapse the continuum and absoluteness arguments to capture full automorphisms of $\mathcal{D}$ by countable approximations that can then be defined within the structure. The full proof appears in Slaman and Woodin [2008]. The join theorem for $n$-REA operators of Shore and Slaman [1999] then defines the Turing jump from that of the double jump: For any degree $\mathbf{x}$, $\mathbf{x}' = \max\{\mathbf{z} \geq_T \mathbf{x} | (\forall \mathbf{g} \geq_T \mathbf{x})(\mathbf{z} \vee \mathbf{g} \neq \mathbf{g}'')\}$, i.e. $\mathbf{x}'$ is the greatest degree $\mathbf{z}$ such that there is no $\mathbf{g}$ greater than or equal to $\mathbf{x}$ such that $\mathbf{z} \vee \mathbf{g}$ is equal to $\mathbf{g}''$.

Our goal in this paper is to give a direct definition of the jump operator that uses no metamathematical or set theoretic methods such as absoluteness or forcing over models of (large fragments of) ZFC. We also avoid coding models of arithmetic and using definability in them on the road to our definition. We do begin with the definition given above of $\mathcal{A}$ from Jockusch and Shore [1984] and at the end apply the definition above of the jump from the double jump of Shore and Slaman [1999]. In between, we define another class $\mathcal{C}$ (and its upward closure $\tilde{\mathcal{C}}$) that is a version of a generalization of classes from the familiar generalized high/low hierarchy: $\mathcal{C} = \{\mathbf{x} | (\forall k)(\mathbf{x}^{(3)} \not\leq (\mathbf{x} \vee \mathbf{0}^{(k)})^{(2)}\}$ (Definitions 2.5 and 2.6). This class is defined within $\mathcal{D}$ by an analysis of the finitely generated partial lattices of a specified form that can be embedded below a degree $\mathbf{x}$. (These lattices are ones whose complexity we can limit and control. They were first introduced and exploited for the analysis of the degrees below $\mathbf{0}'$ in Shore [1981].) The crucial additional ingredient from the literature is Slaman and Woodin's [1986] coding of countable sets of pairwise incomparable degrees by finitely many parameters. We also need two new technical lemmas. One, Theorem 4.1, embeds certain $\Sigma_3^X$ partial lattices below any **ANR** degree $\mathbf{x}$. (A degree $\mathbf{a}$ is **ANR** if, for any function $f \leq_{wtt} 0'$, there is a $g \leq_T \mathbf{a}$ such that there are infinitely many $n$ with $g(n) > f(n)$.) The other, Theorem 5.1, calculates the infimum of the double jumps of degrees in $\mathcal{C}$ that are above any given $\mathbf{x}$ to be $\mathbf{x}''$. Together these allow us to go from a definition of $\mathcal{C}$ (or $\tilde{\mathcal{C}}$) to one of the double jump and thence to one of the jump.

In addition to avoiding the set theoretic and metamathematical techniques of Slaman and Woodin, our approach provides definitions that define the double jump and jump inside any jump ideal of $\mathcal{D}$ that contains $\mathbf{0}^{(\omega)}$ (Theorem 6.1). (A jump ideal is a subset of $\mathcal{D}$ closed downward and under join and jump.) Because of the global nature of the arguments of Slaman and Woodin [2008], their methods give no hint as to how to define these operators in small substructures of $\mathcal{D}$. Even within all of $\mathcal{D}$, our definitions seem significantly simpler both conceptually and in terms of quantifier complexity. (We give specific quantifier complexity bounds for our definitions in Theorems 6.2 and 6.15. In the language with $\vee$ and $\wedge$, $\mathbf{L}_2(\mathbf{y}) = \{\mathbf{x} \geq \mathbf{y} | \mathbf{x}'' = \mathbf{y}''\}$ has a $\Pi_5$ definition as do $\mathbf{x}'' \leq \mathbf{y}''$ and $\mathbf{x}'' = \mathbf{y}''$ in any jump ideal containing $\mathbf{0}^{(\omega)}$. There are ones for $\mathbf{w} = \mathbf{x}''$ and $\mathbf{w} = \mathbf{x}'$ that are $\Sigma_6\&\Pi_6$ and $\Pi_8$, respectively. In the language without $\vee$ and $\wedge$ the definitions are one level higher up.) As a beginning of the investigation of lower bounds for the complexity of such definitions, we show in Proposition 7.6 that there is no definition of $\mathbf{L}_2$ or $\mathbf{L}_2(\mathbf{y})$ which is either $\Pi_2$ or $\Sigma_2$ in the language with just $\leq$.

Once we have an independent definition of the (double) jump we can also directly and simply derive the results of Slaman and Woodin [2008] on fixed points of automorphisms and definability (Theorem 2.10) and extend them to all jump ideals containing $\mathbf{0}^{(\omega)}$: If $\mathcal{I}$ is any jump ideal with $\mathbf{0}^{(\omega)} \in \mathcal{I}$ and $\varphi$ is any automorphism of $\mathcal{I}$ then $\varphi(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \geq \mathbf{0}''$. Moreover, any relation on $\mathcal{I}$ invariant under the double jump or under joining with $\mathbf{0}''$ is definable over $\mathcal{I}$ if and only if it is definable in the structure of second order arithmetic with set quantification ranging over sets with degrees in $\mathcal{I}$. Thus our approach presents the general results on fixed points and definability for sufficiently large degrees as direct consequences of a proof of the definability of natural classes and the jump operator.

In the next section, we make explicit the few properties of our coding/embedding results that are needed and present an overview of our proof that relies only on those properties. Section 3 is devoted to making these notions explicit. In §4 we provide the technical result needed about embedding partial lattices below **ANR** degrees. Section 5 contains the proof that the infimum of the double jumps of degrees $\mathbf{y} \in \mathcal{C}$ or $\tilde{\mathcal{C}}$ with $\mathbf{y} \geq \mathbf{x}$ is $\mathbf{x}''$. Section 6 further analyses all the previous constructions to see that we have definitions that work in any jump ideal containing $\mathbf{0}^{(\omega)}$ and calculates the quantifier complexity of these definitions. The final section suggests some open questions as well as indicating possible routes to partial progress on some of them.

## 2 Overview of the Proof

In this section we will give an overview of a general plan to define the jump operator from classes $\mathcal{C}$ of degrees and the properties required of $\mathcal{C}$ to be able to carry out this plan. Our penultimate goal is to define the relation $\mathbf{x}'' \leq \mathbf{y}''$ from $\mathcal{C}$. Let $\mathbf{L}_2(\mathbf{x}) = \{\mathbf{y} \geq \mathbf{x} | \mathbf{y}'' \leq \mathbf{x}''\}$. By Selman [1972], $\mathbf{x}'' = \vee \mathbf{L}_2(\mathbf{x})$ and indeed there are $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{L}_2(\mathbf{x})$ such that $\mathbf{y}_1 \vee \mathbf{y}_2 = \mathbf{x}''$. Thus a definition of the relation $\mathbf{x}'' \leq \mathbf{y}''$ and so of $\mathbf{x} \in \mathbf{L}_2(\mathbf{y})$ would suffice to define the double jump operator from $\mathcal{C}$. We then appeal to the direct definition of the jump from the double jump (Slaman and Shore [1999]): $\mathbf{x}' = \max\{\mathbf{z} \geq_T \mathbf{x} | (\forall \mathbf{g} \geq_T \mathbf{x})(\mathbf{z} \vee \mathbf{g} \neq \mathbf{g}'')\}$.

In fact, what is shown there is that if $\mathbf{w} \not\leq \mathbf{x}'$ then there is a $\mathbf{g} \geq \mathbf{x}$ such that $\mathbf{w} \vee \mathbf{z} = \mathbf{g}''$. Thus $\mathbf{x}' = \max\{\mathbf{z} \geq_T \mathbf{x} | (\forall \mathbf{g} \geq_T \mathbf{x})(\mathbf{z} \vee \mathbf{g} \not\geq \mathbf{g}'')\}$ as obviously $\mathbf{z} \vee \mathbf{g} \not\geq \mathbf{g}''$ for any $\mathbf{z} \leq \mathbf{x}'$ and $\mathbf{g} \geq \mathbf{x}$.

To begin, we note that $\mathbf{x}''$ is determined by the collection of sets $\Delta_3^X$ which is, of course, independent of the choice of $X \in \mathbf{x}$. We thus want to provide a definable (in $\mathcal{C}$) coding procedure (with free variable $\mathbf{x}$ and additional parameters) that (as the parameters vary) codes precisely the sets $\Delta_3^X$. Given such a coding procedure, we then want to have a definable comparison relation (with free variables just $\mathbf{x}$ and $\mathbf{y}$) which allows us to say that the sets coded by the specified procedure with $\mathbf{x}$ as the special parameter are also coded with $\mathbf{y}$ as the special parameter. This will then say that $\Delta_3^X \subseteq \Delta_3^Y$ and so $\mathbf{x}'' \leq \mathbf{y}''$ as desired.

There are two main tools needed to carry out this plan. The first is the coding of sets in effective successor models of $\mathbb{N}$ introduced in Shore [1981] with the use of initial segments to analyze the theory of the degrees below $0'$. Variations of this coding mechanism have since been used in a number of other settings and especially in Nies, Shore and Slaman [1998]. In each one, the crucial idea is to make the procedure used to recover the coded set from the parameters as simple as possible: positively $\Sigma_1$ in the ordering relation and join operator on the degrees involved. By positively we mean that $\leq, =$ and $\vee$ can be used but not $\not\leq$ or $\neq$ (or $\wedge$) so that, in the setting of the Turing degrees below $\mathbf{x}$, the set coded must be $\Sigma_3^X$ as $\leq_T$ restricted to the (indices of) degrees below $\mathbf{x}$ is itself $\Sigma_3^X$ and join operates recursively on the indices. The previous version of this coding procedure most closely resembling the one we use in this paper is that used in Shore [2008] to prove the rigidity of the hyperdegrees and their biinterpretability with true second order arithmetic. We spell out the precise procedure we need that defines our notion of a set $S$ being coded below a degree $\mathbf{x}$ in §3. For now, all we need to know is that any set $S$ coded (by parameters) below $\mathbf{x}$ is $\Sigma_3^X$. Thus, if both $S$ and its complement, $\bar{S}$, are coded below $\mathbf{x}$, then $S \in \Delta_3^X$, i.e. $S \leq_T \mathbf{x}''$.

The second tool that we need is the method of coding countable sets and relations on $\mathcal{D}$ by finitely many parameters uniformly definably as given by Slaman and Woodin [1986]. (The uniformity here means that there is, for each $n$, a single formula $\phi(x_1, \ldots, x_n, \vec{p})$ such that, as the parameters $\vec{p}$ vary, the formula defines all countable $n$-ary relations on $\mathcal{D}$.) Given such a way of coding arbitrary countable relations on $\mathcal{D}$ and so quantifying over them, it is clear that, in principle, we will be able to define the needed comparison relations between parameters below $\mathbf{x}$ and ones below $\mathbf{y}$ that say that they code the same set. The details are again provided in §3. In particular, as Slaman and Woodin describe in general, and we will illustrate in the detailed analysis of our specific case, one really needs only the simplest instance of their results, the coding of countable sets of pairwise incomparable degrees.

Given these two ingredients of (effective) coding and comparison, our plan is to (definably in $\mathcal{C}$) capture $\Delta_3^X$ as follows:

$\Delta_3^X = S(\mathbf{x}) \equiv \{S | S$ and $\bar{S}$ are coded below every $\mathbf{z} \in \mathcal{C}$ with $\mathbf{x} \leq_{\mathbf{T}} \mathbf{z}\}$.

In order for this description to actually capture the sets $\Delta_3^X$, we want the class $\mathcal{C}$ to have two properties:

- **Property 1**: $\mathbf{x} \in \mathcal{C} \Rightarrow$ Every $S \in \Delta_3^X$ is coded below $\mathbf{x}$.
- **Property 2**: $\forall \mathbf{x}(\wedge\{\mathbf{z}'' | \mathbf{z} \in \mathcal{C} \ \& \ \mathbf{x} \leq_{\mathbf{T}} \mathbf{z}\} = \mathbf{x}'')$.

Property 1 insures that $\Delta_3^X \subseteq S(\mathbf{x})$. In our applications we will typically show that every $S \in \Sigma_3^X$ is coded below $\mathbf{x}$. On the other hand, as any set $S$ such that $S$ and $\bar{S}$ are coded below every $\mathbf{z} \in \mathcal{C}$ with $\mathbf{x} \leq_{\mathbf{T}} \mathbf{z}$ is $\Delta_3^Z$ for every such $Z$, Property 2 guarantees that $S \leq_T \mathbf{x}''$. Thus $S(\mathbf{x}) \subseteq \Delta_3^X$ and we have that $\Delta_3^X = S(\mathbf{x})$ as required.

**Remark 2.1.** Note that Property 1 is obviously closed downward, i.e. if $\mathcal{C} \supseteq \mathcal{B}$ and $\mathcal{C}$ has Property 1 then so does $\mathcal{B}$. Similarly, Property 2 is closed upward.

We now describe some well known degree classes that have one or the other of these Properties. The first is **ANR**, the array nonrecursive degrees first introduced and studied in the setting of $\mathcal{D}$ by Downey, Jockusch and Stob [1990]: $\mathbf{a} \in \mathbf{ANR} \Leftrightarrow$ no $f \leq_{wtt} 0'$ dominates every $g \leq_T A$, i.e. if a function $f$ is wtt reducible to $0'$ then there is a function $g$ recursive in $\mathbf{a}$ such that there are infinitely many $n$ with $g(n) > f(n)$. In §4, we prove the following:

**Theorem 2.2.** **ANR** *has Property 1. In fact, every $S \in \Sigma_3^X$ is coded below* $\mathbf{x}$ *for every* $\mathbf{x} \in \mathbf{ANR}$.

As our final definable classes $\mathcal{C}$ and $\tilde{\mathcal{C}}$ will be contained in **ANR**, they too will have this Property. Similarly, any other class contained in **ANR** such as $\overline{\mathbf{GL}}_2$ has Property 1 as well. (Recall that $\mathbf{a} \in \overline{\mathbf{GL}}_2 \Leftrightarrow \mathbf{a}'' \neq (\mathbf{a} \vee \mathbf{0}')' \Leftrightarrow \mathbf{a}'' > (\mathbf{a} \vee \mathbf{0}')'$. By Downey, Jockusch and Stob [1990], $\overline{\mathbf{GL}}_2 \subseteq \mathbf{ANR}$.)

As an initial attempt at the other direction of our plan to capture the sets $\Delta_3^X$, we show that many familiar degree classes have Property 2.

**Theorem 2.3.** $\mathbf{GH}_3 = \{\mathbf{x} | \mathbf{x}^{(3)} = (\mathbf{x} \vee \mathbf{0}')^{(3)}\} = \{\mathbf{x} | \mathbf{x}^{(3)} \geq (\mathbf{x} \vee \mathbf{0}')^{(3)}\}$ *has Property 2.*

*Proof.* Consider any degree $\mathbf{x}$. By Lachlan [1966] there is a minimal pair $\mathbf{c}_0$ and $\mathbf{c}_1$ of high r.e. degrees relative to $\mathbf{x}''$, i.e. $\mathbf{c}_0 \wedge \mathbf{c}_1 = \mathbf{x}''$, $\mathbf{c}_i' = \mathbf{x}^{(4)}$ and the $\mathbf{c}_i$ are REA in (recursively enumerable in and above) $\mathbf{x}''$. Apply the Sacks jump inversion theorem (Sacks [1963]) twice to $\mathbf{c}_i$ to get $\mathbf{y}_i$ REA in $\mathbf{x}$ with $\mathbf{y}_i'' = \mathbf{c}_i$. Thus $\mathbf{y}_0'' \wedge \mathbf{y}_1'' = \mathbf{x}''$ as required for Property 2.

To see that $\mathbf{y}_i \in \mathbf{GH}_3$ note that $\mathbf{y}_i''' = \mathbf{c}_i' = \mathbf{x}^{(4)}$. As $\mathbf{y}_i$ is REA in $\mathbf{x}$, $\mathbf{y}_i \vee \mathbf{0}' \leq_T \mathbf{x}'$ and so $(\mathbf{y}_i \vee \mathbf{0}')''' \leq_T \mathbf{x}^{(4)} = \mathbf{y}_i'''$ as required. □

Thus the Turing jump is directly definable from every jump class from $\mathbf{GL}_2$ to $\mathbf{GH}_3$ as well as from **ANR**. However, it is an open question if there are natural or even simple direct definitions (not using a definition of the Turing jump) of any of these jump classes or any definition at all of **ANR**. We want a class that is directly definable and also has Properties 1 and 2. Our desired classes $\mathcal{C}$ and $\tilde{\mathcal{C}}$ were suggested by the following argument that directly defines the jump operator from a single instance.

**Proposition 2.4.** *The jump operator can be directly defined from the single degree* $\mathbf{0}'$.

*Proof.* We define $\overline{\mathbf{GL}}_3$ from $\mathbf{0}'$ and apply the previous results. We claim that $\mathbf{x} \in \overline{\mathbf{GL}}_3 \Leftrightarrow (\exists S)(S$ is coded below $\mathbf{x}$ but not both $S$ and $\bar{S}$ are coded below $\mathbf{x} \vee \mathbf{0}'\}$.

For one direction, note that if $\mathbf{x} \in \overline{\mathbf{GL}}_3$, $S = \mathbf{x}^{(3)}$ is coded below $\mathbf{x}$ by Theorem 2.2 but if $S$ and $\bar{S}$ are coded below $\mathbf{x} \vee \mathbf{0}'$ then $S = \mathbf{x}^{(3)} \leq_T (\mathbf{x} \vee \mathbf{0}')''$ for a contradiction. For the other containment, consider any $S$ coded below $\mathbf{x}$. By the effectiveness of our coding apparatus, $S \in \Sigma_3^X$. If not both $S$ and $\bar{S}$ are coded below $\mathbf{x} \vee \mathbf{0}'$ then $S \notin \Delta_3^{X \vee 0'}$ as every set $\Sigma_3^{X \vee 0'}$ is coded below $\mathbf{x} \vee \mathbf{0}'$ by Theorem 2.2 since $0' \in$ **ANR** and **ANR** is closed upward. (Both of these facts are immediate from the definition of **ANR**.) $\square$

Now we do not have a direct definition of even the single degree $\mathbf{0}'$ but we do have a natural definition of the arithmetic degrees and we can use them instead to define a more generalized jump class $\mathcal{C}$ that will still have Properties 1 and 2. The membership of a degree $\mathbf{x}$ in the standard (generalized) jump classes $\mathbf{GL}_n$ and $\mathbf{GH}_n$ is defined by the lowness or highness, respectively, of the $n^{th}$ jump relative to the appropriate jump of $\mathbf{x} \vee \mathbf{0}'$. If we replace $\mathbf{0}'$ by $\mathbf{0}^{(k)}$ we get a more generalized notion.

**Definition 2.5.** $\mathbf{GL}_{n,k} = \{\mathbf{x} | \mathbf{x}^{(n)} \leq (\mathbf{x} \vee \mathbf{0}^{(k)})^{(n-1)}\}$. $\mathbf{GH}_{n,k} = \{\mathbf{x} | \mathbf{x}^{(n)} \geq (\mathbf{x} \vee \mathbf{0}^{(k)})^{(n)}\}$.

Of course, $\mathbf{GL}_n = \mathbf{GL}_{n,1}$ and $\mathbf{GH}_{n,1} = \mathbf{GH}_n$. The first class we want is the complement of the union of the $\mathbf{GL}_{3,k}$.

**Definition 2.6.** $\mathcal{C} = \{\mathbf{x} | (\forall k)(\mathbf{x}^{(3)} \not\leq (\mathbf{x} \vee \mathbf{0}^{(k)})^{(2)}\}$. $\tilde{\mathcal{C}}$ is the upward closure of $\mathcal{C}$, i.e. $\tilde{\mathcal{C}} = \{\mathbf{x} | (\exists \mathbf{z} \in \mathcal{C})(\mathbf{x} \geq \mathbf{z})\}$.

By considering just the case that $k = 1$ in the definition of $\mathcal{C}$ it is clear that $\mathcal{C} \subseteq \overline{\mathbf{GL}}_3$ and so $\mathcal{C} \subseteq$ **ANR** and has Property 1. As **ANR** is closed upward it also contains $\tilde{\mathcal{C}}$. We prove in §5 that both $\mathcal{C}$ and $\tilde{\mathcal{C}}$ have Property 2. Thus the Turing jump is definable from $\mathcal{C}$ or $\tilde{\mathcal{C}}$. On one hand, the definition of $\mathcal{C}$ seems to be simpler than that of $\tilde{\mathcal{C}}$. On the other hand, the definition in terms of $\tilde{\mathcal{C}}$ will work in any jump ideal containing $\mathbf{0}^{(\omega)}$ while that using $\mathcal{C}$ requires closure under the $\omega$-jump.

To get our direct definition of the jump operator we need to give one of $\mathcal{C}$. We do so in terms of the class $\mathcal{C}_\omega = \{\mathbf{x} | (\forall \mathbf{z})(\mathbf{z} \vee \mathbf{x}$ is not a minimal cover of $\mathbf{z}\}$. (We say that $\mathbf{m}$ is a minimal cover of $\mathbf{z}$ if $\mathbf{m} < \mathbf{z}$ and there is no degree strictly between $\mathbf{m}$ and $\mathbf{z}$.) Jockusch and Soare [1970] show that $0^{(n)} \in \mathcal{C}_\omega$ for every $n$ while Jockusch and Shore [1984] prove that $\mathcal{C}_\omega \subseteq \mathcal{A} = \{\mathbf{x} | \exists n(\mathbf{x} \leq \mathbf{0}^{(n)})\}$. (So Jockusch and Shore [1984] provide a natural definition of $\mathcal{A}$, the degrees of the arithmetic sets, as the downward closure of $\mathcal{C}_\omega$.) We use $\mathcal{C}_\omega$ and our coding and comparison procedures to give a direct definition of $\mathcal{C}$ and so of $\tilde{\mathcal{C}}$ as well.

**Theorem 2.7.** $\mathcal{C} = \{\mathbf{x} | (\exists S)(S$ *is coded below* $\mathbf{x}$ *but not both* $S$ *and* $\bar{S}$ *are coded below* $\mathbf{x} \vee \mathbf{z}$ *for any* $\mathbf{z} \in \mathcal{C}_\omega\}$.

*Proof.* For one direction suppose $\mathbf{x} \in \mathcal{C}$. Thus in particular $\mathbf{x} \in \overline{\mathbf{GL}}_3 \subseteq \mathbf{ANR}$ and so by Theorem 2.2 the set $S = X^{(3)}$ is coded below $\mathbf{x}$. If $S$ and $\bar{S}$ were both coded below some $\mathbf{x} \vee \mathbf{z}$ with $\mathbf{z} \in \mathcal{C}_\omega$, then by taking $n$ such that $\mathbf{z} \leq \mathbf{0}^{(n)}$ we see that both are coded below $\mathbf{x} \vee \mathbf{0}^{(n)}$ and so both are $\Sigma_3^{X \vee 0^{(n)}}$, i.e. $X^{(3)} \in \Delta_3^{X \vee 0^{(n)}}$ contrary to the definition of $\mathcal{C}$.

For the other direction, suppose we have an $S$ coded below $\mathbf{x}$ such that not both $S$ and $\bar{S}$ are coded below $\mathbf{x} \vee \mathbf{z}$ for any $\mathbf{z} \in \mathcal{C}_\omega$. As $\mathbf{0}^{(n)} \in \mathcal{C}$ for every $n$, they are not both coded below $\mathbf{x} \vee \mathbf{0}^{(n)}$ for any $n$. As each of these degrees is in $\mathbf{ANR}$ (by being above $\mathbf{0}'$), the sets $T$ such that both $T$ and $\bar{T}$ are coded below them are (by Theorem 2.2 and the effectiveness of our coding) precisely the sets $\Delta_3^{X \vee 0^{(n)}}$ for some $n$. Thus $S = X^{(3)}$ is not $\Delta_3^{X \vee 0^{(n)}}$ for any $n$, i.e. $(\forall n)(\mathbf{x}^{(3)} \not\leq (\mathbf{x} \vee \mathbf{0}^{(n)})^{(2)}$ as required. □

To summarize our discussion so far, we give the crucial definitions in terms of our as yet unspecified but definable coding and comparison procedures.

**Theorem 2.8.** $\mathbf{x}'' \leq \mathbf{y}'' \Leftrightarrow$ *every set* $S$ *coded below every* $\mathbf{z}$ *such that* $\mathbf{x} \leq \mathbf{z}$ *and* $\mathbf{z} \in \mathcal{C}$ *is coded below every* $\mathbf{w}$ *such that* $\mathbf{y} \leq \mathbf{w}$ *and* $\mathbf{w} \in \mathcal{C}$. *Of course,* $\mathbf{x}'' = \mathbf{y}'' \Leftrightarrow \mathbf{x}'' \leq \mathbf{y}''$ & $\mathbf{y}'' \leq \mathbf{x}''$. *And so we have our direct definition of the jump:*
$\mathbf{x}' = \mathbf{w} \Leftrightarrow \mathbf{w} = \max\{\mathbf{z} \geq_T \mathbf{x} | (\forall \mathbf{g} \geq_T \mathbf{x})(\mathbf{z} \vee \mathbf{g} \not\geq \mathbf{g}'')\}$.

The formal versions of these definitions are given in Theorem 3.6. We will also see in §6 that a finer analysis using $\tilde{\mathcal{C}}$ in place of $\mathcal{C}$ provides local version of this theorem, i.e. a single formula $\phi(\mathbf{x}, \mathbf{w})$ such that for any jump ideal $\mathcal{I}$ (i.e. $\mathcal{I}$ is a subset of $\mathcal{D}$ closed downward and under jump and join) that contains the degree $\mathbf{0}^{(\omega)}$, $\phi$ defines the jump operator.

**Theorem 2.9.** *If* $\mathcal{I}$ *is any jump ideal with* $\mathbf{0}^{(\omega)} \in \mathcal{I}$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{I}$ *then* $\mathbf{x}'' \leq \mathbf{y}''$ *if and only if* $\mathcal{I}$ *satisfies the formula of degree theory expressing, as above, that every set* $S$ *coded below* $\mathbf{z}$ *for every* $\mathbf{z} \in \tilde{\mathcal{C}}$ *with* $\mathbf{z} \geq \mathbf{x}$ *is coded below* $\mathbf{w}$ *for every* $\mathbf{w} \in \tilde{\mathcal{C}}$ *with* $\mathbf{w} \geq \mathbf{y}$. *Thus for* $\mathbf{x}, \mathbf{w} \in \mathcal{I}$, $\mathbf{x}' = \mathbf{w} \Leftrightarrow \mathcal{I} \models \mathbf{w} = \max\{\mathbf{z} \geq_T \mathbf{x} | (\forall \mathbf{g} \geq_T \mathbf{x})(\mathbf{z} \vee \mathbf{g} \not\geq \mathbf{g}'')\}$ *where we understand the double jump relation to be defined as just specified.*

The rest of the paper is devoted to the explication and proofs of the required notions and theorems. We close this section by noting that the local definition of the jump operator allows us to prove local results about automorphisms of jump ideals containing $\mathbf{0}^{(\omega)}$ and definability in such ideals.

**Theorem 2.10.** *If $\mathcal{I}$ is any jump ideal with $\mathbf{0}^{(\omega)} \in \mathcal{I}$ and $\varphi$ is any automorphism of $\mathcal{I}$ then $\varphi(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \geq \mathbf{0}''$. Moreover, any relation on $\mathcal{I}$ invariant under the double jump or under joining with $\mathbf{0}''$ is definable over $\mathcal{I}$ if and only if it is definable in the structure of second order arithmetic with set quantification ranging over sets with degrees in $\mathcal{I}$.*

*Proof.* These consequences are pretty standard once one has the information about automorphisms being fixed on the jump or even on particular instances. The ideas go back to Jockusch and Solovay [1977] who show that all degrees above $\mathbf{0}^{(4)}$ are fixed under all automorphisms of $\mathcal{D}$ that preserve the jump operator. Transferring such fixed point theorems to definability ones have roots at least as far back as Simpson [1977]. Since these proofs, there have been many versions and improvements along with various new methods of coding. In our setting, for the first claim about fixed points, we can simply point out that if $\mathbf{x} \geq \mathbf{0}''$ then $\mathbf{x}$ is uniquely determined as the degree $\mathbf{z}$ above $\mathbf{0}''$ such that there is a $\mathbf{w} \leq \mathbf{z}$ with $\mathbf{w}'' = \mathbf{z}$ with $X, \bar{X}$ coded below $\mathbf{w}$ and such that every set $S$ with $S$ and $\bar{S}$ coded below any $\mathbf{y}$ with $\mathbf{y}'' = \mathbf{z}$ is recursive in $X$.

The first condition guarantees that $\mathbf{z} = \mathbf{w}'' \geq \mathbf{x}$ for this $\mathbf{w}$. It is satisfied by $\mathbf{x}$ because there is an **ANR** degree $\mathbf{w}$ with $\mathbf{w}'' = \mathbf{x}$. (We can easily construct such a degree directly or appeal to Downey, Jockusch and Stob [1990] who show that there is a low degree in **ANR** and relativize this to a degree with double jump $\mathbf{x}$.)

The second condition then guarantees that $\mathbf{z} \leq \mathbf{x}$ as $W''$ and $\overline{W''}$ are coded below a degree which is **ANR** and low relative to a $\mathbf{w}$ with $\mathbf{w}'' = \mathbf{z}$.

For the second claim about definability, we note that, as usual, using the coding of Slaman and Woodin [1986] we can, in $\mathcal{I}$, definably pick out standard models of arithmetic and quantify over all subsets with degrees in $\mathcal{I}$. (The point to make here is that, as Slaman and Woodin [1986] show, their coding for a set $X$ in such a model is done well within the jump ideal containing $\mathbf{x}$. In the other direction, any reasonably effective procedure for coding sets in models of arithmetic by their methods codes only sets arithmetic in the parameters used. So within $\mathcal{I}$, only sets with degrees in $\mathcal{I}$ are coded and all such are, in fact, coded.) The comparison machinery discussed above then allows us to definably move from a set $X \geq_T 0''$ coded in such a model to the degree $\mathbf{x} \geq \mathbf{0}''$ satisfying the property described in the first paragraph of this proof for the specified $X$. Given such a map between coded sets and their degrees, we can translate any property definable in second order arithmetic with set quantification over the sets with degrees in $\mathcal{I}$ which is invariant under double jump or joining with $\mathbf{0}''$ to one definable in $\mathcal{I}$. □

# 3 Coding and Comparison

Our effective coding of a set $S$ is given by an embedding of a particular partial lattice with 0 in $\mathcal{D}$. (A *partial lattice* is a partial ordering on which the operations $\vee$ and $\wedge$ may be only partial but, when defined, they obey the usual definitions in terms of the ordering.) The crucial backbone of the partial lattices we want to consider is an $\omega$-sequence of pairwise incomparable elements $d_n$ generated by five elements $d_0, e_0, e_1, f_0$ and $f_1$ satisfying the the following recursion relations for $n \geq 0$:

$(*)$ $(d_{2n} \vee e_0) \wedge f_1 = d_{2n+1}$ and

$(**)$ $(d_{2n+1} \vee e_1) \wedge f_0 = d_{2n+2}$.

These conditions clearly guarantee that we can enumerate the $d_n$ recursively in the lattice structure and write a recursive list of quantifier free formulas in this language which define each of them. Following Shore [2008], we wish to convert this procedure and these formulas into ones that are positive in the language with just $\leq$ and $\vee$ at least to the extent that we can use them to code $S$ (with the aid of other parameters $g_0$ and $g_1$). One crucial ingredient is being able to say that the (indices generated as candidates for being some) $d_n$ are strictly above 0. We do this by adding two other parameters $p$ and $q$ and require of our lattice that $p \not\geq q$ and $p \vee d_n \geq q$ for each $n$. We can now recursively generate positive existential formulas $\phi_n(x)$ using just $\leq$ , $\vee$ and parameters for the named elements such that, in any lattice $\mathcal{L}_X$ with elements $d_0, e_0, e_{1,} f_0, f_1, p$ and $q$ as described, $\phi_n(x)$ holds of $x$ if and only $0 < x \leq d_n$ and the same will be true in $\mathcal{D}$ of any degree $\mathbf{x}$ when the parameters are interpreted as their images under any (partial) lattice embedding of $\mathcal{L}$ into $\mathcal{D}$:

We begin with $x = d_0$ as $\phi_0$. Recursively, we let $\phi_{2n+1}(x)$ be $\exists z(\phi_{2n}(z)$ & $x \leq z \vee e_0, f_1$ & $q \leq x \vee p)$ and $\phi_{2n+2}(x)$ be $\exists z(\phi_{2n+1}(z)$ & $x \leq z \vee e_1, f_0$ & $q \leq x \vee p)$. Consider any $x$ such that $\phi_{2n+1}(x)$ holds. We then have a $z$ as described such that, by induction, $0 < z \leq d_{2n}$. Thus $z \vee e_0 \leq d_{2n} \vee e_0$ and so $x \leq d_{2n} \vee e_0, f_1$. As $d_{2n+1} = (d_{2n} \vee e_0) \wedge f_1$, $x \leq d_{2n+1}$ as required. Of course, $q \leq x \vee p$ guarantees that $x > 0$ as well. The argument for $\phi_{2n+2}$ is essentially the same.

We will use exact pairs for ideals to code our given set $S$ as in Shore [1981]. The idea is to have $g_0$ and $g_1$ such that $S = \{n | d_n \leq g_0, g_1\}$ as opposed to simply upper bounds on the $d_n$ with $n \in S$ and the $d_n$ with $n \notin S$ as in Shore [2008]. (Recall that $\mathbf{g}_0$ **and** $\mathbf{g}_1$ form an exact pair for the ideal generated by the degrees $\mathbf{d}_n$ for $n \in S$ if $\mathbf{x} \leq \mathbf{g}_0, \mathbf{g}_1 \Leftrightarrow \mathbf{x} \leq \mathbf{d}_{i_0} \vee \cdots \vee \mathbf{d}_{i_m}$ for some $\mathbf{d}_{i_0}, \ldots \mathbf{d}_{i_m}$ with $i_0, \ldots, i_m \in S$.) The reason for this choice is to keep the ability to make $S$, and not $\bar{S}$, positively $\Sigma_1$ in the ordering and join. The next point is, as we will only generate (indices for) elements $x \in (0, d_n)$ that are below both $g_0$ and $g_1$, we need to make sure that if we have such an $x$ it can only be associated with a single $d_n$. We thus wish to ensure that if there is a nonzero $x$ below some $d_n, g_0$ and $g_1$ then $d_n$ is itself below $g_0$ and $g_1$. This suggests that we also require of our lattice that, for $n \neq m$, $d_n \wedge d_m = 0$. Because, however, we are only determining the ideal generated by the $d_n$ with $n \in S$ by our codes $g_0$ and $g_1$, we must also guarantee that no nonzero $x \leq d_m$ with $m \notin S$ is in this ideal. The easiest way to guarantee this in general is to assure that the $d_n$ form an independent set by, for example, requiring that there be an element $\tilde{d}_n$ of our lattice that is above all $d_m$ for $m \neq n$ such that $d_n \wedge \tilde{d}_n = 0$. (This then also implies the previous requirement that the $d_n$ pairwise inf to 0 as well as the basic desideratum that $\exists x(0 < x \le d_n, g_0, g_1) \to d_n \le g_0, g_1$.)

If we have a partial lattice $\mathcal{L}$ with all these properties and an embedding of the lattice into the degrees below $\mathbf{x}$ it is not hard to see that the set $S = \{n | d_n \le g_0, g_1\}$ coded by this lattice or, as we shall say by the degrees $\mathbf{d}_0, \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1, \mathbf{p}, \mathbf{q}, \mathbf{g}_0, \mathbf{g}_1$ below $\mathbf{x}$ which are the images of the corresponding elements of $\mathcal{L}$, is $\Sigma_3^X$: $n \in S \Leftrightarrow \exists \mathbf{x}(\boldsymbol{\phi}_n(\mathbf{x})\ \&\ \mathbf{x} \le \mathbf{g}_0, \mathbf{g}_1)$. (We boldface the variables and formulas to indicate that we are interpreting them in $\mathcal{D}$ about the images of elements of $\mathcal{L}$.) As the formulas $\boldsymbol{\phi}_n$ are all positive $\Sigma_1$ formulas in $\le$ and $\vee$ which are $\Sigma_3^X$ on the indices for sets recursive in $X$, the claimed equivalent definition of $S$ is clearly $\Sigma_3^X$. As $\boldsymbol{\phi}_n(\mathbf{d}_n)$ holds, the claimed equivalent definition holds for every $n \in S$. For the other direction we rely on the prescribed properties of $\mathcal{L}$. We already know that if $\boldsymbol{\phi}_n(\mathbf{x})$ holds then $\mathbf{0} < \mathbf{x} \le \mathbf{d}_n$. Our assumptions then imply that $\mathbf{d}_n \le \mathbf{g}_0, \mathbf{g}_1$ as required. In fact, as the $\mathbf{g}_0$ and $\mathbf{g}_1$ form an exact pair for the ideal generated by the $\mathbf{d}_n$ with $n \in S$, if $\mathbf{x} \le \mathbf{d}_n, \mathbf{g}_0, \mathbf{g}_1$ and $n \notin S$, then $\mathbf{x} = \mathbf{0}$. The point here is that $\mathbf{x} \le \mathbf{d}_{i_0} \vee \cdots \vee \mathbf{d}_{i_m}$ with $i_0, \ldots, i_m \in S$ and so $\mathbf{x} \le \mathbf{d}_{i_0} \vee \cdots \vee \mathbf{d}_{i_m} \le \tilde{\mathbf{d}}_n$ while $\mathbf{d}_n \wedge \tilde{\mathbf{d}}_n = \mathbf{0}$. The crucial property needed here is that if $\exists \mathbf{x}(\mathbf{0} < \mathbf{x} \le \mathbf{d}_n, \mathbf{g}_0, \mathbf{g}_1)$ then $\mathbf{d}_n \le \mathbf{g}_0, \mathbf{g}_1$. Thus, for an exact pair $\mathbf{g}_0, \mathbf{g}_1$ for the ideal generated by $\mathbf{d}_n$ for $n \in S$, $S = \{n | \exists \mathbf{x}(\boldsymbol{\phi}_n(\mathbf{x})\ \&\ \mathbf{x} \le \mathbf{g}_0, \mathbf{g}_1)\}$.

We note that any sequence of degrees $\mathbf{d}_0, \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1, \mathbf{p}, \mathbf{q}, \mathbf{g}_0, \mathbf{g}_1 \le \mathbf{x}$ can be viewed as coding the set $S = \{n | \exists x(\boldsymbol{\phi}_n(\mathbf{x})\ \&\ \mathbf{x} \le \mathbf{g}_0, \mathbf{g}_1)\}$ which is always $\Sigma_3^X$. When we want to claim that (for certain degrees $\mathbf{x}$) every $\Sigma_3^X$ set is coded by such a sequence, we have to guarantee all these properties of the required (embedding of) $\mathcal{L}$. We do this in Theorem 4.1 for $\mathbf{x} \in \mathbf{ANR}$. What remains for us to do now is to explain how we can assert, in a first order way in $\mathcal{D}$, that two such sequences code the same set or complementary sets. For this we need to guarantee some of the structural properties of $\mathcal{L}$ already discussed as well as a comparison procedure. The crucial ingredient is the coding of Slaman and Woodin [1986]:

**Theorem 3.1.** (Slaman and Woodin [1986])*: For any set* $\{\mathbf{c}_n\}$ *of pairwise incomparable degrees uniformly recursive in* $\mathbf{u}$ *there are degrees* $\mathbf{h}_0$ *and* $\mathbf{h}_1$ *below* $\mathbf{u}''$ *such that* $\mathbf{y} \in \{\mathbf{c}_n\} \Leftrightarrow \mathbf{y} \le \mathbf{u}\ \&\ \exists \mathbf{w}(\mathbf{w} \le \mathbf{h}_0 \vee \mathbf{y}, \mathbf{h}_1 \vee \mathbf{y}\ \&\ \mathbf{w} \not\le \mathbf{y})\ \&\ (\forall \mathbf{u} < \mathbf{y}) \neg \exists \mathbf{w}(\mathbf{w} \le \mathbf{h}_0 \vee \mathbf{u}, \mathbf{h}_1 \vee \mathbf{u}\ \&\ \mathbf{w} \not\le \mathbf{u})$. *We will denote this relation by* $\mathbf{y} \in Kd(\mathbf{u}, \mathbf{h}_0, \mathbf{h}_1)$ *and the set of such* $\mathbf{y}$ *by* $Kd(\mathbf{u}, \mathbf{h}_0, \mathbf{h}_1)$.

**Notation 3.2.** *For notational convenience we will use* $\mathbf{a}$ *to stand for the sequence* $\mathbf{d}_0, \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1, \mathbf{p}, \mathbf{q}, \mathbf{g}_0, \mathbf{g}_1$ *(to code one set) and decorated variants such as* $\hat{\mathbf{a}}$ *for the correspondingly decorated sequence. We thus write, for example,* $\mathbf{a} \le \mathbf{x}$ *to mean* $\mathbf{d}_0, \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1, \mathbf{p}, \mathbf{q}, \mathbf{g}_0, \mathbf{g}_1 \le \mathbf{x}$. *We use* $\mathbf{b}$ *and its decorated variants to stand for an additional pair of degrees* $\mathbf{g}_2, \mathbf{g}_3$ *which we add on to* $\mathbf{a}$ *to code a second set in the obvious way. We use* $\mathbf{c}$ *and its variants to stand for sequences of the form* $\mathbf{u}, \mathbf{h}_0, \mathbf{h}_1$ *as used in Theorem 3.1 and so, for example, write* $\mathbf{y} \in Kd(\mathbf{c})$ *for* $\mathbf{y} \in Kd(\mathbf{u}, \mathbf{h}_0, \mathbf{h}_1)$.

We now give the formal definition in $\mathcal{D}$ of various degrees coding a set below $\mathbf{x}$. For ease of reading (to the extent possible) we expand our language by adding on the definable (partial) operations $\vee$ and $\wedge$ as well as a constant symbol for $\mathbf{0}$. Our intention is that whenever a term of the form $\mathbf{x} \wedge \mathbf{y}$ occurs we intend to assert the existence of an infimum for $\mathbf{x}$ and $\mathbf{y}$. The formal version for the sentences that we use will be made precise near the end of §6 where we explain how to eliminate these symbols.

**Definition 3.3.** Two sequences of degrees $\mathbf{a}$ and $\mathbf{c}$ *code a set below* $\mathbf{x}$ if the following conditions hold:

1. $\mathbf{a} \leq \mathbf{x}$.
2. $\mathbf{q} \not\leq \mathbf{p}$.
3. $\mathbf{d}_0 \in Kd(\mathbf{c})$.
4. $\forall \mathbf{d}(\mathbf{d} \in Kd(\mathbf{c}) \rightarrow (\mathbf{d} \vee \mathbf{e}_0) \wedge \mathbf{f}_1 \in Kd(\mathbf{c})\ \&\ (\mathbf{d} \vee \mathbf{e}_1) \wedge \mathbf{f}_0 \in Kd(\mathbf{c}))$.
5. $\forall \mathbf{d}, \hat{\mathbf{d}}(\mathbf{d} \neq \hat{\mathbf{d}}\ \&\ \mathbf{d}, \hat{\mathbf{d}} \in Kd(\mathbf{c}) \ \rightarrow\ \mathbf{d} \wedge \hat{\mathbf{d}} = \mathbf{0}\ \&\ \mathbf{q} \leq \mathbf{d} \vee \mathbf{p}\ \&\ \exists \mathbf{x}(\mathbf{0} < \mathbf{x} \leq \mathbf{d}, \mathbf{g}_0, \mathbf{g}_1) \rightarrow \mathbf{d} \leq \mathbf{g}_0, \mathbf{g}_1)$.

We denote the conjunction of these properties by $Cd(\mathbf{x}, \mathbf{a}, \mathbf{c})$ and, if we omit (1) and so $\mathbf{x}$, by $Cd(\mathbf{a}, \mathbf{c})$ . We then say that the set $S$ coded by the degrees $\mathbf{a}$ and $\mathbf{c}$ is $\{n | \mathbf{d}_n \leq \mathbf{g}_0, \mathbf{g}_1\}$ where the $\mathbf{d}_n$ are defined by the recursions $(*)$ and $(**)$ at the beginning of this section. Similarly, we use $Cd(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c})$ to say that, in addition to $Cd(\mathbf{x}, \mathbf{a}, \mathbf{c})$ we have coded a second set $S' = \{n | \mathbf{d}_n \leq \mathbf{g}_2, \mathbf{g}_3\}$ with the additional parameters $\mathbf{g}_2, \mathbf{g}_3$. As written, this definition only guarantees that each $\mathbf{d}_n \in Kd(\mathbf{c})$. We can require that the set $Kd(\mathbf{c})$ consists precisely of the $\mathbf{d}_n$ which we designate as $PrCd$ by adding on the condition

6 $Kd(\mathbf{c})$ is the smallest set satisfying (3) and (4), i.e. $\forall \hat{\mathbf{c}}([\mathbf{d}_0 \in Kd(\hat{\mathbf{c}})\ \&\ \forall \mathbf{d}(\mathbf{d} \in Kd(\hat{\mathbf{c}}) \rightarrow (\mathbf{d} \vee \mathbf{e}_0) \wedge \mathbf{f}_1 \in Kd(\hat{\mathbf{c}})\ \&\ (\mathbf{d} \vee \mathbf{e}_1) \wedge \mathbf{f}_0 \in Kd(\hat{\mathbf{c}}))] \rightarrow \forall \mathbf{d}(\mathbf{d} \in Kd(\mathbf{c}) \rightarrow \mathbf{d} \in Kd(\hat{\mathbf{c}})))$.

It may be easier to see that our definitions have the desired properties with the precise coding clause but it will not be really necessary to include it. Even without it, we have specified enough properties of the partial lattice involving $\mathbf{a}$ to guarantee that $S = \{n | \exists \mathbf{x}(\phi_n(\mathbf{x})\ \&\ \mathbf{x} \leq \mathbf{g}_0, \mathbf{g}_1)\}$ and so $S$ is $\Sigma_3^X$ (and so also for $S'$). We can then guarantee that $\bar{S} = S'$ by saying, in addition to $Cd(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c})$, that $(\forall \mathbf{d} \in Kd(\mathbf{c}))(\mathbf{d} \leq \mathbf{g}_0, \mathbf{g}_1 \leftrightarrow \mathbf{d} \not\leq \mathbf{g}_2, \mathbf{g}_3)$. We denote this relation by $Compl(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c})$. To compare the sets $S$ and $\tilde{S}$ coded by $\mathbf{a}, \mathbf{c}$ and $\tilde{\mathbf{a}}, \tilde{\mathbf{c}}$, respectively, we need to know a bit more about Slaman-Woodin coding.

Given any two sets such as $\{\mathbf{d}_n\}$ and $\{\tilde{\mathbf{d}}_n\}$ each consisting of pairwise incomparable degrees with $Kd(\mathbf{c}) = \{\mathbf{d}_n\}$ and $Kd(\tilde{\mathbf{c}}) = \{\tilde{\mathbf{d}}_n\}$ there are $\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}$ and $\hat{\mathbf{c}}$ such that, for any $\mathbf{z} \in Kd(\mathbf{c})$ and $\tilde{\mathbf{z}} \in Kd(\tilde{\mathbf{c}})$, $(\exists \mathbf{w}, \tilde{\mathbf{w}} \in Kd(\check{\mathbf{c}}))(\mathbf{z} \vee \mathbf{w} \in Kd(\dot{\mathbf{c}})\ \&\ \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}} \in Kd(\ddot{\mathbf{c}})\ \&$ $(\mathbf{z} \vee \mathbf{w} \vee \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}}) \in Kd(\hat{\mathbf{c}})) \leftrightarrow (\exists n)(\mathbf{z} = \mathbf{d}_n\ \&\ \tilde{\mathbf{z}} = \tilde{\mathbf{d}}_n)$. As pointed out in Slaman and

Woodin [1986], one can see this directly from Theorem 3.1 by taking a set $\mathbf{h}_i$ of mutually 1-generic degrees relative to any degree above all the $\mathbf{d}_n$ and $\tilde{\mathbf{d}}_n$ and let $\check{\mathbf{c}}$ code $\{\mathbf{h}_i\}$. Next let $\dot{\mathbf{c}}$ code the set $\{\mathbf{d}_n \vee \mathbf{h}_{2n}\}$ of pairwise incomparable degrees and $\ddot{\mathbf{c}}$ code $\{\tilde{\mathbf{d}}_n \vee \mathbf{h}_{2n+1}\}$. Finally let $\hat{\mathbf{c}}$ code $\{\mathbf{d}_n \vee \mathbf{h}_{2n} \vee \tilde{\mathbf{d}}_n \vee \mathbf{h}_{2n+1}\}$ which is also pairwise incomparable. We abbreviate $(\exists \mathbf{w}, \tilde{\mathbf{w}} \in Kd(\check{\mathbf{c}}))(\mathbf{z} \vee \mathbf{w} \in Kd(\dot{\mathbf{c}})\ \&\ \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}} \in Kd(\ddot{\mathbf{c}})\ \&\ (\mathbf{z} \vee \mathbf{w} \vee \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}}) \in Kd(\hat{\mathbf{c}}))$ as $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{z}, \tilde{\mathbf{z}})$. We can now definably say that the sets $S$ and $\tilde{S}$ coded by $\mathbf{a}, \mathbf{c}$ and $\tilde{\mathbf{a}}, \tilde{\mathbf{c}}$, respectively, are the same.

**Proposition 3.4.** *If* $\mathbf{a}, \mathbf{c}$ *and* $\tilde{\mathbf{a}}, \tilde{\mathbf{c}}$ *code the sets* $S$ *and* $\tilde{S}$, *respectively, then* $S = \tilde{S}$ *if and only if* $(\exists \check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}})[(\forall \mathbf{x}, \mathbf{y}, \mathbf{z})(Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{x}, \mathbf{y})\ \& Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{x}, \mathbf{z}) \rightarrow \mathbf{y} = \mathbf{z})\ \&$ $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{d}_0, \tilde{\mathbf{d}}_0)\ \&\ (\forall \mathbf{d} \in Kd(\mathbf{c}))(\forall \tilde{\mathbf{d}} \in Kd(\tilde{\mathbf{c}}))(Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{d}, \tilde{\mathbf{d}}) \rightarrow Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, (\mathbf{d} \vee \mathbf{e}_0) \wedge \mathbf{f}_1, (\tilde{\mathbf{d}} \vee \tilde{\mathbf{e}}_0) \wedge \tilde{\mathbf{f}}_1)\ \&\ Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, (\mathbf{d} \vee \mathbf{e}_1) \wedge \mathbf{f}_0, (\tilde{\mathbf{d}} \vee \tilde{\mathbf{e}}_1) \wedge \tilde{\mathbf{f}}_0)\ \&\ (\mathbf{d} \leq \mathbf{g}_0, \mathbf{g}_1 \leftrightarrow \tilde{\mathbf{d}} \leq \tilde{\mathbf{g}}_0, \tilde{\mathbf{g}}_1)]$. *We denote this relation by* $Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})$.

*Proof.* The formula given says that $Mp$ defines a one-one relation that takes $\mathbf{d}_0$ to $\tilde{\mathbf{d}}_0$ and, by induction, $\mathbf{d}_n$ to $\tilde{\mathbf{d}}_n$. It then guarantees that $\mathbf{d}_n \leq \mathbf{g}_0, \mathbf{g}_1$, i.e. $n \in S$, if and only if $\tilde{\mathbf{d}}_n \leq \tilde{\mathbf{g}}_0, \tilde{\mathbf{g}}_1$, i.e. $n \in \tilde{S}$. (Note that for $\mathbf{d} = \mathbf{d}_n$ one of $(\mathbf{d} \vee \mathbf{e}_0) \wedge \mathbf{f}_1$ and $(\mathbf{d} \vee \mathbf{e}_1) \wedge \mathbf{f}_0$ is $\mathbf{d}_{n+1}$ and the other is $\mathbf{d}_n$.) We also point out that even without the added condition for precisely coding sets, this relation has the correct meaning since if $S = \tilde{S}$ we can choose the parameters to define $Mp$ only on the degrees $\mathbf{d}_n$. □

We now have our formal version of the definition of $\mathcal{C}$ in $\mathcal{D}$ given in Theorem 2.7. In our current style of abbreviations we use $\mathbf{z} \in \mathcal{C}_\omega$ to abbreviate the formula $\forall \mathbf{z}(\mathbf{z} \leq \mathbf{x}$ or $\exists \mathbf{w}(\mathbf{x} < \mathbf{w} < \mathbf{x} \vee \mathbf{z})$.

**Theorem 3.5.** $\mathcal{C} = \{\mathbf{x} | \exists \mathbf{a}, \mathbf{c}(Cd(\mathbf{x}, \mathbf{a}, \mathbf{c})\ \&\ (\forall \tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}, \mathbf{z})(\mathbf{z} \in \mathcal{C}_\omega\ \&\ Compl(\mathbf{x} \vee \mathbf{z}, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}) \rightarrow \neg Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}}))\}$.

Using the notation $\mathbf{x} \in \mathcal{C}$ in the obvious way, we now have our formal analog of the definition of $\mathbf{x}'' \leq \mathbf{y}''$ given in Theorem 2.8 and so of jump operation.

**Theorem 3.6.** *1.* $\mathbf{x}'' \leq \mathbf{y}'' \Leftrightarrow (\forall \mathbf{a}, \mathbf{c})[Cd(\mathbf{a}, \mathbf{c})\ \&\ (\forall \mathbf{z})(\mathbf{z} \in \mathcal{C}\ \&\ \mathbf{x} \leq \mathbf{z} \rightarrow$
$(\exists \tilde{\mathbf{a}}, \tilde{\mathbf{c}})(Cd(\mathbf{z}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})\ \&\ Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})) \rightarrow (\forall \mathbf{w})(\mathbf{w} \in \mathcal{C}\ \&\ \mathbf{y} \leq \mathbf{w} \rightarrow$
$(\exists \hat{\mathbf{a}}, \hat{\mathbf{c}})(Cd(\mathbf{w}, \hat{\mathbf{a}}, \hat{\mathbf{c}})\ \&\ Eq(\mathbf{a}, \mathbf{c}, \hat{\mathbf{a}}, \hat{\mathbf{c}}))]$.

*2.* $\mathbf{x}' = \mathbf{w} \Leftrightarrow \mathbf{w} = \max\{\mathbf{z} \geq_{\mathbf{T}} \mathbf{x} | (\forall \mathbf{g} \geq_{\mathbf{T}} \mathbf{x})(\mathbf{z} \vee \mathbf{g} \not\geq \mathbf{g}'') \Leftrightarrow \forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{w} \vee \mathbf{g}) \neq \mathbf{g}'')\ \&$ $\forall \mathbf{v}(\forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{v} \vee \mathbf{g}) \neq \mathbf{g}'' \rightarrow \mathbf{v} \leq \mathbf{w})$.

To see that these definitions have the intended meaning, we must prove Theorems 2.2 and 5.1. After we prove these theorems we will be able to see in §6 (Theorem 6.1) that the same definitions using $\tilde{\mathcal{C}}$ in place of $\mathcal{C}$ have their intended meaning in every jump ideal containing $0^{(\omega)}$:

**Theorem 3.7.** *The equivalences for* $\mathbf{x}'' \leq \mathbf{y}''$ *and* $\mathbf{x}' = \mathbf{w}$ *given in Theorem 3.6 are valid in every jump ideal containing* $\mathbf{0}^{(\omega)}$ *if we replace* $\mathcal{C}$ *by* $\tilde{\mathcal{C}}$.

# 4 Coding Below an ANR Degree

In this section we prove that **ANR** and so $\mathcal{C}$ and $\tilde{\mathcal{C}}$ have Property 1. There is clearly a recursive (partial) lattice $\mathcal{L}$ with the properties described in §3: There are elements $d_0, e_0, e_1, f_0$ and $f_1$ that generate a $\omega$-sequence of pairwise incomparable elements $d_n$ by the recursion formulas $(*)$ and $(**)$ of §3. There are elements $p \not\geq q$ such that $p \vee d_n \geq q$ for each $n$. For every $d_n$ there is an element $\tilde{d}_n$ such that $\tilde{d}_n \geq d_m$ for $m \neq n$ and $d_n \wedge \tilde{d}_n = 0$. Let us fix such a (partial) lattice $\mathcal{L}$. Given any $\mathbf{x} \in \mathbf{ANR}$ and any $S \in \Sigma_3^X$ it clearly suffices to embed $\mathcal{L}$ in the degrees below $\mathbf{x}$ and to simultaneously construct degrees $\mathbf{g}_0, \mathbf{g}_1 \leq_T \mathbf{x}$ such that $\mathbf{g}_0$ and $\mathbf{g}_1$ are a minimal pair for the ideal generated by $\{\mathbf{d}_n | n \in S\}$. We prove a more general result.

**Theorem 4.1.** *If $\mathcal{L}$ is a recursive (partial) lattice with a recursive list $d_i$ of elements forming an independent set (no join of a finite subset is above any one not in the given finite set), $X \in \mathbf{x} \in \mathbf{ANR}$ and $S$ is $\Sigma_3^X$, then there is an embedding of $\mathcal{L}$ into the degrees below $\mathbf{x}$ (taking $d_i$ to $\mathbf{d}_i$) and degrees $\mathbf{g}_0, \mathbf{g}_1 \leq_T \mathbf{x}$ such that $\mathbf{g}_0$ and $\mathbf{g}_1$ are a minimal pair for the ideal generated by $\{\mathbf{d}_n | n \in S\}$. Indeed, given another $\tilde{S} \in \Sigma_3^X$ we can also simultaneously get $\mathbf{g}_2, \mathbf{g}_2 \leq_T \mathbf{x}$ that form a minimal pair for the ideal generated by $\{\mathbf{d}_n | n \in \tilde{S}\}$.*

The basis for our construction is the method of embedding (partial) lattices in $\mathcal{D}$ by Cohen type forcing (i.e. finite conditions) introduced in Shore [1982]. It begins with standard lattice representation results (originally from Jonsson [1953] but translated into the language of Lerman [1971] or [1983]:

**Theorem 4.2.** *Let $\mathcal{L}$ be a recursively presentable partial lattice with $0$ its least element. There is a uniformly recursive array $\Theta$ of functions $\alpha_n : \mathcal{L} \to \omega$ such that for all $p, q, r \in \mathcal{L}$ and $n, m \in \omega$:*

*0.* $\alpha_n(0) = 0$,

*1.* $p \leq q \Rightarrow \alpha_n(q) = \alpha_m(q) \to \alpha_n(p) = \alpha_m(p)$ *and*
$p \not\leq q \Rightarrow \exists k, l(\alpha_k(q) = \alpha_l(q)\ \&\ \alpha_k(p) \neq \alpha_l(p))$,

*2.* $p \vee q = r \Rightarrow [\alpha_n(p) = \alpha_m(p)\ \&\ \alpha_n(q) = \alpha_m(q) \ \to\ \alpha_n(r) = \alpha_m(r)]$,

*3.* $p \wedge q = r\ \&\ \alpha_n(r) = \alpha_m(r) \ \Rightarrow\ \exists n_1, n_2, n_3[\alpha_n(p) = \alpha_{n_1}(p)\ \&\ \alpha_{n_1}(q) = \alpha_{n_2}(q)$
$\&\ \alpha_{n_2}(p) = \alpha_{n_3}(p)\ \&\ \alpha_{n_3}(q) = \alpha_m(q)]$. *The $\alpha_{n_i}$ here are called* interpolants for (or between) $\alpha_n$ *and* $\alpha_m$.

(A simple proof without the requirement for 0 is in Shore [1982]. Adding the requirement that the value of each function in the representation is 0 at 0 at the beginning presents no difficulties. A proof in this case (even with the additional assumption that the lattice has a greatest element as well as a least) can be found in Greenberg and Montalbán [2004].)

Standard arguments as in Shore [1982] show that we can define an embedding of $\mathcal{P}$ into $\mathcal{D}$ from any sufficiently generic function $h : \mathbb{N} \to \Theta$ by letting the image of $p \in \mathcal{L}$ be the degree of the function $h_p$ defined by $h_p(n) = h(n)(p)$. In our setting, we proceed more cautiously so as to be able to do the embedding and simultaneously build the desired $\mathbf{g}_i$ all below $\mathbf{x}$ when $\mathbf{x} \in \mathbf{ANR}$. We consider the case with only one set $S$ as a second set simply adds extra notation with no change in the types of requirements or strategies.

Let $S$ be defined by $n \in S \Leftrightarrow \exists u \forall v \exists w R(n, u, v, w, X)$ where $R$ is recursive (in $X$). Let $\Theta = \{\alpha_i | i \in \mathbb{N}\}$ be a recursive (partial) lattice table representation for $\mathcal{L}$ in the sense of Theorem 4.2. We will build a function $h : \mathbb{N} \to \Theta$ recursively in $X$ by a sequence of finite approximations $h_s$ which will be initial segments of $h$. Our embedding of $\mathcal{L}$ into the degrees below $\mathbf{x}$ will then be given as usual by $p \mapsto \deg(h_p) = \mathbf{h}_p$ where $h_p(n) = (h(n))(p)$ for $p \in \mathcal{L}$. For notational convenience we let $D_n = h_{d_n} \in \mathbf{d}_n$. By the definition of a lattice table, $p \leq_{\mathcal{L}} q \to h_p \leq_T h_q$ and $p \vee q = r \to h_p \oplus h_q \equiv_T h_r$. In our construction we must act to satisfy the requirements for nonorder and infimum:

$N_{e,p,q}$ for $p \nleq q$: $\{e\}^{h_p} \neq h_q$.

$P_{e,p,q,r}$ for $p \wedge q = r$: If $\{e\}^{h_p} = \{e\}^{h_q} = C$ then $C \leq_T h_r$.

In addition, we have the requirements for the sets $G_i$ that we are constructing simultaneously via initial segment approximations $g_{i,s}$ to represent the desired degrees $\mathbf{g}_i$:

$Q_e$: If $\{e\}^{G_0} = D = \{e\}^{G_1}$, then $D \leq_T \oplus\{D_i | i \leq s \ \& \ i \in S\}$ for some $s$.

$R_n$: If $n \in S$ then $D_n \leq_T G_0, G_1$.

Our mechanism to satisfy the requirements $R_n$ will proceed by coding $D_n$ into $G_i^{[\langle n,u \rangle]} = \{x | \langle n, u, x \rangle \in G_i\}$ if $\forall v \exists w R(n, u, v, w, X)$. The overriding rule that will be observed for every $\langle n, u \rangle$ is that for each $t$,$m$ and $i \in \{0, 1\}$, there will be at most one $\langle n, u, t, m, k \rangle \in G_i$. We divide the requirement $R_n$ into subrequirements $R_{n,u,m}$.

$R_{n,u,m}$: At stage $s$ these requirements will ask that $\langle n, u, t, m, D_{n,s}(m) \rangle \in G_i$ for some $t$ (actually the least $t$ allowed) if $\forall v \leq m \exists w \leq s R(n, u, v, w, X)$. Thus if $n \in S$ and $u$ is its witness then for each $m$ there will be a stage $s(m)$ after which we will always try to code $D_n(m)$ into $G_i^{[\langle n,u \rangle]}$ in this way. Our decoding procedure is to calculate $D_n(m)$ from $G_i^{[\langle n,u \rangle]}$ by searching for a $\langle t, m, k \rangle$ such that $\langle t, m, k \rangle \in G_i^{[\langle n,u \rangle]}$ and declaring that $D_n(m) = k$. Our overriding rule will guarantee that we find at most one answer and we must argue that for almost all $m$ we eventually insert the correct answer into $G_i$.

We put all the requirements $N_{e,p,q}$, $P_{e,p,q,r}$, $Q_e$ and $R_{n,u,m}$ into a single priority list $S_e$.

The requirements $N_{e,p,q}$ and $P_{e,p,q,r}$ are handled in a fairly standard way if they do not appear satisfied (as will be defined in the construction).

$N_{e,p,q}$ at stage $s$: We look for an $x$ and an extension $\tilde{h}$ of $h_s$ such that $\{e\}^{\tilde{h}_p}(x) \downarrow \neq \tilde{h}_q(x)$. Making such an extension would, of course, satisfy the requirement. We will argue that if we never get to make such an extension, then $\{e\}^{h_p}$ is not total.

$P_{e,p,q,r}$ at stage $s$: Here we have a multistep procedure. We first look for an $x$ and extensions $h^0, h^4$ of $h_s$ of the same length such that $\{e\}^{h^0_p}(x)\downarrow \neq \{e\}^{h^4_q}(x)\downarrow$ and $h^0(m)(r) = h^4(m)(r)$ for $m \notin \operatorname{dom} h_s$. If we find such, we choose interpolants $\rho_{1,m}, \rho_{2,m}$ and $\rho_{3,m}$ between $h^0(m)$ and $h^4(m)$ for $m \in \operatorname{dom} h^0 - \operatorname{dom} h_s$ as in Theorem 4.2(3). Let $h^j$ $(j = 1, 2, 3)$ extend $h_s$ by making $h^j(m) = \rho_{j,m}$ for $m \in \operatorname{dom} h^0 - \operatorname{dom} h_s$. Now look in turn (for $j = 1, 2, 3$) for extensions $\hat{h}^j$ of $h^j$ such that $\{e\}^{\hat{h}^j_p}(x)\downarrow$, $\{e\}^{\hat{h}^j_q}(x)\downarrow$ and $\hat{h}^j(m) = \hat{h}^{j'}(m)$ for $m \notin \operatorname{dom} h^j$ for $j, j' \leq 4$, i.e. first try to extend $h^1$ to get the convergences at $x$ and then, if successful, add the new values on to $h^2$ and try to extend that to get the convergences. If successful add the new values onto $h^3$ and try to extend that to get the convergences. If successful, extend $h^0, h^4$ and the $\hat{h}^j$ $(1 \leq j \leq 3)$ to $\tilde{h}^j$ $(j \leq 4)$ so that they have the same length and the same values on all numbers at which they are not yet defined. If we have been successful all the way to the end, then by the properties of the interpolants and the fact that all later extensions of the $h^j$ were the same for all new values, we see that there is a $j \leq 4$ such that $\{e\}^{\tilde{h}^j_p}(x)\downarrow \neq \{e\}^{\tilde{h}^j_q}(x)$. If we find and choose such an $\tilde{h}^j$ as $h_{s+1}$ we, of course, satisfy requirement $P_{e,p,q,r}$. We will argue that if we never choose such an extension and $\{e\}^{h_p} = \{e\}^{h_q} = C$ then $C \leq_T h_r$ as required.

$Q_e$ at stage $s$: If $Q_e$ appears unsatisfied, we will search for an $x$ and $\tilde{h}$ extending $h_s$ and $\tilde{g}_i$ extending $g_{i,s}$ such that $\{e\}^{\tilde{g}_0}(x)\downarrow \neq \{e\}^{\tilde{g}_1}(x)\downarrow$, the $\tilde{g}_i$ obey the overriding rule of coding, add no new elements $\langle n, u, t, m, y\rangle$ to $G_i$ if $\langle n, u\rangle < e$ unless $\forall v \leq e \exists w \leq s R(n, u, v, w, X)$. If this last condition holds, then such elements may be added to $G_i$ by $\tilde{g}_i$ but only if they are of the right form, i.e. $\tilde{h}_{d_n}(m) = y$. Again, if we find and make such an extension, we satisfy $Q_e$. As in the other cases, we must argue that if we never make such an extension and $\{e\}^{G_0} = D = \{e\}^{G_1}$ for some $n$ then $D \leq_T \oplus\{D_i | i \leq s \ \& \ i \in S\}$ for some $s$.

$R_{n,u,m}$ at stage $s$: If $R_{n,u,m}$ is not yet satisfied and $\forall v \leq m \exists w \leq s R(n, u, v, w, X)$ then we look for extensions of $g_{i,s}$ to include $\langle n, u, t, m, D_{n,s}(m)\rangle$ for some $t$ if this does not violate the overriding rule of coding.

As usual for constructions below an ANR set $X$, we must define an appropriate function $\hat{f} \leq_{wtt} 0'$, choose one $f \leq_T X$ not dominated by $\hat{f}$ and restrict our searches for witnesses and extensions at stage $s$ to ones with codes less than $f(s)$. We now define $\hat{f}$ by specifying, recursively in advance for each $s$, a finite list of questions to be asked of $0'$ and recursive search procedures that will terminate if $0'$ answers the associated question positively. The questions and searches corresponding to the desired actions for each requirement are as follows:

$N_{e,p,q}$: For each finite function $\hat{h} : \mathbb{N} \to \Theta$ with code less than $s$ and requirement $N_{e,p,q} = S_t$ for $t < s$ ask if there is an $x$ and an extension $\tilde{h}$ of $\hat{h}$ such that $\{e\}^{\tilde{h}_p}(x)\downarrow \neq \tilde{h}_q(x)$. If the answer is yes, we include a search for one such extension.

$P_{e,p,q,r}$: For each finite function $\hat{h} : \mathbb{N} \to \Theta$ with code less than $s$ and requirement $P_{e,p,q,r} = S_t$ for $t < s$ ask if there is an $x$ and extensions $h^0, h^4$ of $\hat{h}$ of the same length such that $\{e\}^{h^0_p}(x)\downarrow \neq \{e\}^{h^4_q}(x)\downarrow$ and $h^0(m)(r) = h^4(m)(r)$ for $m \notin \operatorname{dom} h_s$. If the answer to this question is yes, we include a search for the first such extensions in a standard search procedure. Also ask if there is an extension $\hat{h}^1$ of the $h^0$ of the first pair satisfying the previous search condition extended by the first interpolants between it and $h^4$ (the second element of the first pair found) as described above such that $\{e\}^{\hat{h}^1_p}(x)\downarrow$ and $\{e\}^{\hat{h}^1_q}(x)\downarrow$. If the answer from $0'$ is yes, include a search for (the first such) $\hat{h}^1$. We also ask if there is an extension $\hat{h}^2$ of the finite function that would be produced as $h^2$ by the first search and extending first by the associated interpolants and then also by the new values determined by the witness $\hat{h}^1$ just described. If the answer from $0'$ is yes, include a search for (the first such) $\hat{h}^2$. Similarly, we ask if all the searches needed to define $\hat{h}^3$ terminate and if so include that search as well.

$Q_e$: For each requirement $Q_e = S_t$ for $t < s$ and finite functions $\hat{h}, \hat{g}_0, \hat{g}_1$ with codes less than $s$ and finite set $F$ of numbers less than $e$, we ask if there is an $x$ and extensions $\tilde{h}, \tilde{g}_0, \tilde{g}_1$ of $\hat{h}, \hat{g}_0$ and $\hat{g}_1$, respectively, such that $\{e\}^{\tilde{g}_0}(x)\downarrow\neq\{e\}^{\tilde{g}_1}(x)\downarrow$, the $\tilde{g}_i$ obey the overriding rule of coding, add no new elements $\langle n,u,t,m,y\rangle$ to $G_i$ if $\langle n,u\rangle < e$ unless $\langle n,u\rangle \in F$ and, if $\langle n,u\rangle \in F$, they add such elements only if $\tilde{h}_{d_n}(m) = y$. If the answer is yes, we include a search for such extensions.

$R_{n,u,m}$: For each requirement $R_{n,u,m} = S_t$ for $t < s$ and finite functions $\hat{h}, \hat{g}_0, \hat{g}_1$ with codes less than $s$ find (recursively including answering the question if they exist) the least $t$ such that $\langle n,u,t,m,\hat{h}_{d_n}(m)\rangle$ can be added to $G_i$ without violating the overriding coding rule and the codes for the corresponding extensions of $\hat{g}_i$.

We let $\hat{f}(s)$ be the max of $s$ and the codes of all finite functions and witnesses found by all of the search procedures that $0'$ says will terminate. Note that $\hat{f}$ is nondecreasing and *wtt* below $0'$. We let $f \leq_{wtt} X$ be such that $f$ is nondecreasing and not dominated by $\hat{f}$, i.e. $\exists^\infty s(\hat{f}(s) < f(s))$.

**Construction:** We begin at stage 0 with $h_0, g_{i,0} = \emptyset$. At stage $s+1$ we have $h_s$ and $g_{i,s}$ with codes less than $s+1$ and act to define $h_{s+1}, g_{i,s+1}$ with codes less than $s+2$. For each requirement $S_e$ with $e < s$ that does not now appear to be satisfied we search for extensions (and witness $x$ if needed) with codes less than $f(s)$ as requested in the initial descriptions above of the requirements. If we find any such, we act for the one found of highest priority by choosing as $h_{s+1}$ and $g_{i,s+1}$ the longest possible extensions of $h_s$ and $g_{i,s}$, respectively, contained in the first found extensions which have codes less than $s+2$ (these are the targets for this requirement at stage $s$). If there are no such extensions for any requirement less than $s$ then $h_{s+1} = h_s$ and $g_{i,s+1} = g_{i,s}$.

This description is unambiguous for requirements of the form $N_{e,p,q}$, $Q_e$ and $R_{n,u,m}$. If, by our action in these cases, we reach the desired target extensions for the requirement of highest priority for which we found desired extensions, we declare this requirement satisfied and it will remain satisfied forever. The procedure for requirements $P_{e,p.q,r}$ needs further elaboration.

For each currently unsatisfied $P_{e,p.q,r} = S_k$ for $k < s$, we mimic the search procedure as described in the initial account of this requirement but bounding our searches by $f(s)$. If we reach the end of the procedure with an $\tilde{h}^j$ as described then that is our target for this requirement. If we reach this target at stage $s$, $P_{e,p.q,r}$ is declared satisfied and remains so forever. If there is no such $\tilde{h}^j$ as a target, we see where the search procedure for $P_{e,p.q,r}$ failed.

If the search failed at the first step, i.e. we found no extensions $h^0, h^4$ of $h_s$ of the same length such that $\{e\}^{h^0_p}(x) \downarrow \neq \{e\}^{h^4_q}(x) \downarrow$ and $h^0(m)(r) = h^4(m)(r)$ for $m \notin \operatorname{dom} h_s$, we let $h_s$ and $g_{i,s}$ be the targets for $P_{e,p.q,r}$. If we act for no requirement of higher priority, we act for $P_{e,p.q,r}$ by setting $h_{s+1} = h_s$, $g_{i,s+1} = g_{i,s}$ and declare $P_{e,p.q,r}$ to be satisfied. If at the beginning of any later stage $t$ we see that there are extensions $h^0, h^4$ of $h_s$ with codes below $f(t)$ as were desired at stage $s$, then $P_{e,p.q,r}$ becomes unsatisfied.

If we found extensions $h^0, h^4$ of $h_s$ as desired below $f(s)$, we take the first ones found in our standard search and define $h^j$ for $j = 1, 2, 3$ as above. We now search (below $f(s)$) in turn for $j = 1, 2, 3$ for $\hat{h}^j$ as described. If we find them all then we would have an $\tilde{h}^j$ as required and so by our case assumption, one of the searches fails. Say the first to fail is for $\hat{h}^j$. It failed because we had found an extension $\bar{h}^j$ of $h^j$ as described so far but no extension $\tilde{h}$ of $\bar{h}^j$ such that $\{e\}^{\tilde{h}_p}(x) \downarrow$ and $\{e\}^{\tilde{h}_q}(x) \downarrow$. In this case, we set our target for $P_{e,p.q,r}$ to be this $\bar{h}^j$ (and no changes for $g_{i,s}$). If we reach this target at stage $s$ we declare $P_{e,p.q,r}$ to be satisfied. However, if at the beginning of any later stage $t$ we see (by a search below $f(t)$) that there is an extension $\tilde{h}$ of $\bar{h}^j$ such that $\{e\}^{\tilde{h}_p}(x) \downarrow$ and $\{e\}^{\tilde{h}_q}(x) \downarrow$, $P_{e,p.q,r}$ becomes unsatisfied.

**Verifications:** We wish to show that we stop acting for each requirement and that each of their goals is met.

**Lemma 4.3.** *For each requirement $S_t$ there is a stage $s(t)$ after which $S_t$ is never the requirement supplying the target chosen at stage $s$.*

*Proof.* We proceed by induction on $t$ and by cases dictated by the type of requirement.

$S_t = N_{e,p,q}$, $Q_e$ or $R_{n,u,m}$: If there is any stage $s > s(t-1)$ at which $S_t$ supplies a target then, by the rules of the construction and the assumption that $s > s(t-1)$, that target remains the one of highest priority available until we reach it. At that point $S_t$ is satisfied and remains so forever. It therefore never supplies a target again.

$S_t = P_{e,p.q,r}$: Choose an $s > s(t-1)$ such that $f(s) > \hat{f}(s)$. If $P_{e,p.q,r}$ entered this stage apparently satisfied and is not declared unsatisfied by our check below $f(s)$ at the beginning of this stage then there will be no later stage at which it becomes unsatisfied. The point is that any extension that we are looking for has code less than $\hat{f}(s)$ if one exists at all by definition and so less than $f(s)$ by our choice of $s$. Thus we may assume that $P_{e,p.q,r}$ appears unsatisfied. By construction it always has a target and as $s > s(t-1)$ it is the one of highest priority. Thus we head toward this target, eventually reach it and declare $P_{e,p.q,r}$ satisfied. Again, as this target was chosen when $f(s) > \hat{f}(s)$, we will never discover that there was some desired extension not found at stage $s$. Thus $P_{e,p.q,r}$ will remain satisfied forever and never supply a target again. $\square$

**Lemma 4.4.** *All the requirements are satisfied.*

*Proof.* $S_t = N_{e,p,q}$: If we ever declare $N_{e,p,q}$ satisfied at $t$ then $\{e\}^{h_{p,t}}(x) \downarrow \neq h_{q,t}(x)$ and so $\{e\}^{h_p} \neq h_q$ as required. If not, consider an $s > s(t)$ such that $f(s) > \hat{f}(s)$. As there are no targets found for $N_{e,p,q}$ at $s$ then as $f(s) > \hat{f}(s)$ there is no $x$ with an extension $\tilde{h}$ of $h_s$ such that $\{e\}^{\tilde{h}_p}(x) \downarrow \neq \tilde{h}_q(x)$. If, however, $\{e\}^{h_p}(x) \downarrow$ for some $x \notin \operatorname{dom} h_{q,s}$ then there would be an $\hat{s} > s$ such that $\{e\}^{h_p}(x) \downarrow = \{e\}^{h_{p,\hat{s}}}(x) \downarrow$. By the properties of the lattice table there would then be an $\tilde{h}$ such that $\tilde{h}_p = h_{p,\hat{s}}$ and $\tilde{h}_q(x) \neq h_{q,\hat{s}}(x)$. Clearly one of $\tilde{h}$ and $h_{\hat{s}}$ would be an extension of $h_s$ satisfying the desired property for a contradiction to our case assumption. Thus, in this case, $\{e\}^{h_p}$ is not total.

$S_t = P_{e,p,q,r}$: By the proof of the previous Lemma there is a stage $s$ at which we declare $P_{e,p,q,r}$ satisfied and it never becomes unsatisfied again. If the declaration was based on making an extension such that $\{e\}^{\tilde{h}^j_p}(x) \downarrow \neq \{e\}^{\tilde{h}^j_q}(x) \downarrow$ then $\{e\}^{h_p}(x) \downarrow \neq \{e\}^{h_q}(x) \downarrow$ and we satisfy $P_{e,p,q,r}$. Otherwise, our search procedure terminated at an intermediate step because of our failure to find certain extensions with specified properties. As we never declare $P_{e,p,q,r}$ to be unsatisfied a later stage, there are, in fact, no such extensions.

If the failure occurred at the first step, there are no extensions $h^0$ and $h^4$ of $h_s$ such that $\{e\}^{h^0_p}(x) \downarrow \neq \{e\}^{h^4_q}(x) \downarrow$ and $h^0(m)(r) = h^4(m)(r)$ for $m \notin \operatorname{dom} h_s$. In this case, if $\{e\}^{h_p} = C = \{e\}^{h_q}$ we claim that $C \leq_T h_r$. To compute $C(x)$ find any $\hat{h}$ extending $h_s$ such that $\hat{h}(m)(r) = h(m)(r)$ for $m \in \operatorname{dom} h$ with $\{e\}^{\hat{h}_p}(x) \downarrow$. Such an $\hat{h}$ exists since $h \upharpoonright u$ is one where $u$ is sufficiently large so that $\{e\}^{(h\upharpoonright u)_p}(x) \downarrow$ and $\{e\}^{(h\upharpoonright u)_q}(x) \downarrow$. If $\{e\}^{\hat{h}_p}(x) \downarrow \neq C(x)$ then the pair $\hat{h}$ and $h \upharpoonright u$ would supply a pair as desired and not found at stage $s$ (by possibly extending one of them to make them of the same length) contradicting our case assumption. Thus, in this case, we satisfy $P_{e,p,q,r}$ by computing $C$ from $h_r$ if $\{e\}^{h_p} = C = \{e\}^{h_q}$.

If the failure occurred later, we had an $\bar{h}^j$ with no extension $\hat{h}$ of $\bar{h}^j$ such that $\{e\}^{\hat{h}_p}(x) \downarrow$ and $\{e\}^{\hat{h}_q}(x) \downarrow$. In this case we set this $\bar{h}^j$ to be our target and eventually realized it as an initial segment of our final $h$. Thus not both $\{e\}^{h_p}(x)$ and $\{e\}^{h_q}(x)$ are convergent and we satisfy $P_{e,p,q,r}$ in this way.

$S_t = Q_e$: If we satisfy $Q_e$ at some stage $s$ then $\{e\}^{g_{0,s}}(x) \downarrow \neq \{e\}^{g_{1,s}}(x) \downarrow$ and so $\{e\}^{g_0} \neq \{e\}^{g_1}$ as required. Otherwise, consider an $s > s(t-1)$ such that $f(s) > \hat{f}(s)$ and, for every $\langle n, u \rangle < e$, $\forall v \leq e \exists w R(n,u,v,w,X) \rightarrow \forall v \leq e \exists w \leq s R(n,u,v,w,X)$. We also require that if $\langle n, u \rangle < e$ and $\exists v \forall w \neg R(n,u,v,w,X)$ with $v(n,u)$ being the least such $v$ then $R_{n,u,m}$ never supplies a target after stage $s$ for $m \leq v(n,u)$. If there are no targets found for $Q_e$ at $s$ then, as $f(s) > \hat{f}(s)$, there is no $x$ with extensions $\hat{g}_i$ of $g_{i,s}$ such that $\{e\}^{\hat{g}_0}(x) \downarrow \neq \{e\}^{\hat{g}_1}(x) \downarrow$ also satisfying the other conditions required for them. We claim that for every $v \geq s$, the $h_v$ and $g_{i,v}$ satisfy these extra conditions. As we always obey the overriding rule of coding, it is satisfied by every $g_{i,v}$. If it is not the case that $\forall v \leq e \exists w \leq s R(n,u,v,w,X)$ for some $\langle n, u \rangle < e$ then this situation remains true at every $v > s$ by our choice of $s$. Thus no $Q_i$ with $i > e$ can add an element $\langle n,u,t,m,y \rangle$ with such $\langle n, u \rangle < e$ to $G_i$. The only other requirements that can add elements $\langle n,u,t,m,y \rangle$ with $\langle n, u \rangle < e$ to $G_i$ are the $R_{n,u,m}$. Our final condition on the size of $s$, however, guarantees that none of these for which $\exists v \forall w \leq s \neg R(n,u,v,w,X)$ will ever act after stage $s$. The other $R_{n,u,m'}$ will put numbers into $G_i$ at a stage $v > s$ only if $y = h_{d_n,v}(m')$. Thus in every case we maintain the fact that $h_v, g_{i,v}$ satisfy the extra conditions required for the extensions desired at stage $s$. Now suppose $\{e\}^{g_0}(x) \downarrow = \{e\}^{g_1}(x) \downarrow = z$. We can calculate $z$ by finding any $\hat{h}$, $\hat{g}_i$ extending $h_s$ and $g_{i,s}$, respectively and obeying the extra conditions on $\hat{g}_i$ such that $\{e\}^{\hat{g}_0}(x) \downarrow = \{e\}^{\hat{g}_1}(x) \downarrow$ and such that $\hat{h}_{d_n}(m') = h_{d_n}(m')$ for every $m' \in \operatorname{dom} \hat{h}_{d_n}$, $n < e$ and $n \in S$. Such exist by our assumption. If one had $\{e\}^{\hat{g}_0}(x) \downarrow \neq z$ then $h_v, \hat{g}_0$ and $g_{1,v}$ would be extensions of $h_s$ and $g_{i,s}$ as desired in our original search at stage $s$ contradicting our case assumption. Clearly, we can find such $\hat{h}$, $\hat{g}_i$ and so $z$ recursively in $\oplus\{D_n | n < e \ \&\ n \in S\}$. Thus if $\{e\}^{g_0}(x) = D = \{e\}^{g_1}(x)$, then $D \leq_T \oplus\{D_n | n < e \ \&\ n \in S\}$ as required.

$R_n$: Suppose that $n \in S$ and $u$ is a witness, i.e. $\forall v \exists w R(n, u, v, w, X)$. By our overriding coding condition, there is, for each $m$, at most one $k$ such that $\langle n, u, t, m, k\rangle \in G_i$ for each $i \in \{0, 1\}$. We wish to show that, for each $i \in \{0, 1\}$ and almost every $m$, $\langle n, u, t, m, D_n(m)\rangle \in G_i$ for some $t$. This will suffice to show that $D_n \leq_T G_i$ as we can then correctly calculate $D_n(m)$ at almost every $m$ by searching for a $\langle n, u, t, m, k\rangle \in G_i$ and, upon finding one, declaring that $k = D_n(m)$. Let $s_0$ be such that no $Q_e$ with $e \leq \langle n, u\rangle$ ever supplies a target after stage $s_0$. The only actions that put a number $\langle n, u, t, m, k\rangle$ into $G_i$ at $s > s_0$ are ones (by other $Q_{e'}$ and $R_{n,u.m}$) that do so only if $k = h_{d_n,s}(m) = D_n(m)$. Thus there at most finitely many incorrect values coded into $G_i$. Consider any $m$ larger than all of these values, the requirement $R_{n,u,m}$ and any stage $s$ after $s_0$ and $t(j)$ where $S_j = R_{n,u,m}$ such that $\forall v \leq m \exists w \leq s R(n, u, v, w, X)$ and $h_{d_n,s}(m) \downarrow$. If no $\langle n, u, t, m, k\rangle \in g_{i,s}$, then an extension putting $\langle n, u, t, m, h_{d_n,s}(m)\rangle$ in is declared a target and it is the one of highest priority not yet satisfied. Thus we will eventually put such an element into $g_i$ as required. $\square$

This concludes the proof of Theorem 4.1.

**Corollary 4.5.** **ANR** *and hence* $\mathcal{C}$ *and* $\tilde{\mathcal{C}}$ *have Property 1.*

# 5 Forcing argument

In this section prove that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ have Property 2.

**Theorem 5.1.** *For every degree* $\mathbf{x}$*, there are* $\mathbf{a}_0, \mathbf{a}_1 \leq \mathbf{x}^{(\omega)}$ *(indeed recursive in* $\oplus\{(\mathbf{x} \vee \mathbf{0}^{(n)})''' | n \in \mathbb{N}\}$*) such that* $\mathbf{a}_0, \mathbf{a}_1 \geq \mathbf{x}$, $\mathbf{a}_0'' \wedge \mathbf{a}_1'' = \mathbf{x}''$ *and* $(\forall i \in \{0, 1\})(\forall n)(\mathbf{a}_i''' \not\leq (\mathbf{a}_i \vee \mathbf{0}^{(n)})''$*, i.e.* $\mathbf{a}_i \in \mathcal{C}$*. Moreover, there are* $\mathbf{b}_0, \mathbf{b}_1 \leq (\mathbf{x} \vee \mathbf{0}^{(\omega)})'''$ *such that* $(\mathbf{x} \vee \mathbf{b}_0)'' \wedge (\mathbf{x} \vee \mathbf{b}_1)'' = \mathbf{x}''$ *and* $(\forall i \in \{0, 1\})(\forall n)(\mathbf{b}_i''' \not\leq (\mathbf{b}_i \vee \mathbf{0}^{(n)})''$ *and so* $(\mathbf{x} \vee \mathbf{b}_i) \in \tilde{\mathcal{C}}$*.*

Working towards our proof, we first describe a forcing language and a specific notion of forcing. After analyzing the forcing relation for two quantifier sentences, we construct a sequence of forcing conditions that decide all two quantifier sentences in our language.

We also act along the way to satisfy requirements corresponding to the conditions on the degrees in the theorem.

Our *forcing language* is that of first order arithmetic with a unary predicate $G$ for the generic set as usual plus additional unary predicates for a fixed $X \in \mathbf{x}$ and the sets $0^{(n)}$. Our *notion of forcing* consists of triples $\langle \sigma, F, I \rangle$. Here $\sigma \in 2^{<\omega}$ is thought of as a finite initial segment of the characteristic function for the generic $G$, so if we have a sequence $p_s = \langle \sigma_{p_s}, F_{p_s}, I_{p_s} \rangle$ of conditions the corresponding generic set is $G = \cup \sigma_s$. $F$ and $I$ are disjoint finite subsets of $\omega$. We say that $p' = \langle \sigma', F', I' \rangle$ *extends* $p = \langle \sigma, F, I \rangle$, $p' \leq p$, if

- $\sigma' \supseteq \sigma$, $F' \supseteq F$, $I' \supseteq I$ and
- $(\forall j \in F)(\forall x \in \operatorname{dom} \sigma' - \operatorname{dom} \sigma)(x \in \omega^{[j]} \to \sigma'(x) = 0)$.

The intuition here is that once $j \in F_p$ no more numbers in column $j$ can be put into $G$ and so $G^{[j]} = \{x | \langle j, x \rangle \in G\}$ will be finite. On the other hand, once $j \in I_p$, $j$ can never be put into $F_q$ for any $q \leq p$ and so if $G$ is even slightly generic, $G^{[j]}$ will be infinite.

We say that $p \Vdash \theta(\bar{x}, G, X, 0^{(n)})$ for $\theta$ with only bounded quantifiers if $\mathbb{N} \models \theta(\bar{x}, \sigma_p, X, 0^{(n)})$, in the usual sense of $\sigma_p$ having enough information to verify the statement which depends only on an initial segment of the predicate $G$. Thus $p \Vdash \theta(\bar{x}, G, X, 0^{(n)}) \Leftrightarrow (\sigma_p, \emptyset, \emptyset) \Vdash \theta(\bar{x}, G, X, 0^{(n)})$. Note that this relation is uniformly recursive in $X \oplus 0^{(n)}$ (or in $X(0^{(n)})$ if $0^{(n)}(X)$ does not appear in $\theta$. The forcing relation on more complicated sentences is then defined in the usual inductive fashion.

**Lemma 5.2.** *The relations $p \Vdash \exists \bar{x} \forall \bar{y} \theta(\bar{x}, \bar{y}, G, X, 0^{(n)})$ and $p \Vdash \forall \bar{x} \exists \bar{y} \theta(\bar{x}, \bar{y}, G, X, 0^{(n)})$ for $\theta$ with only bounded quantifiers are uniformly $\Sigma_2$ and $\Pi_2$ in $X \oplus 0^{(n)}$, respectively, or in $X(0^{(n)})$ alone if $0^{(n)}(X)$ does not appear in $\theta$. Moreover, given any condition $p$ and sentence $\exists \bar{x} \forall \bar{y} \theta(\bar{x}, \bar{y}, G, X, 0^{(n)})$ either there is a $q \leq p$ and $\bar{x}$ such that $q \Vdash \forall \bar{y} \theta(\bar{x}, \bar{y}, G, X, 0^{(n)})$ and $I_q = I_p$ or $p \Vdash \forall \bar{x} \exists \bar{y} \neg \theta(\bar{x}, \bar{y}, G, X, 0^{(n)})$. In either, case we see that there is a $q \leq p$ deciding $\exists \bar{x} \forall \bar{y} \theta(\bar{x}, \bar{y}, G, X, 0^{(n)})$ with $I_q = I_p$.*

*Proof.* To fix our notation we carry along both $X$ and $0^{(n)}$ in $\theta$ and the full analysis. Omitting either one is purely a notational change. As usual, $p \Vdash \exists \bar{x} \theta(\bar{x}, G, X, 0^{(n)}) \Leftrightarrow \exists \bar{x} p \Vdash \theta(\bar{x}, G, X, 0^{(n)})$ for $\theta$ with only bounded quantifiers and so $p \Vdash \exists \bar{x} \theta(\bar{x}, G, X, 0^{(n)}) \Leftrightarrow (\sigma_p, \emptyset, \emptyset) \Vdash \exists \bar{x} \theta(\bar{x}, G, X, 0^{(n)})$. Thus if, for any given $p$, there is a $q \leq p$ such that $q \Vdash \exists \bar{x} \theta(\bar{x}, G, X, 0^{(n)})$ then there is one $q'$ with $F_{q'} = F_p$ and $I_{q'} = I_p$. (Just let $q' = (\sigma_q, F_p, I_p)$.) For $\Pi_1$ sentences we have by definition $p \Vdash \forall \bar{x} \theta(\bar{x}, G, X, 0^{(n)}) \Leftrightarrow \forall q \leq p (q \nVdash \exists \bar{x} \neg \theta(\bar{x}, G, X, 0^{(n)})$. Thus these relations are uniformly $\Sigma_1$ and $\Pi_1$ in $X \oplus 0^{(n)}$. We also see that $p \Vdash \forall \bar{x} \theta(\bar{x}, G, X, 0^{(n)}) \Leftrightarrow \forall q \leq p (F_q = F_p \ \& \ I_q = I_p \to q \nVdash \exists \bar{x} \neg \theta(\bar{x}, G, X, 0^{(n)}))$. So $p \Vdash \forall \bar{x} \theta(\bar{x}, G, X, 0^{(n)}) \Leftrightarrow (\sigma_p, F_p, \emptyset) \Vdash \forall \bar{x} \theta(\bar{x}, G, X, 0^{(n)})$. (For the right to left direction suppose, for the sake of a contradiction that $(\sigma_p, F_p, \emptyset) \nVdash \forall \bar{x} \theta(\bar{x}, G, X, 0^{(n)})$. So we have a $q \leq (\sigma_p, F_p, \emptyset)$ such that $q \Vdash \exists \bar{x} \neg \theta(\bar{x}, G, X, 0^{(n)})$ and so by the analysis of forcing for existential sentences, $(\sigma_q, \emptyset, \emptyset) \Vdash \exists \bar{x} \neg \theta(\bar{x}, G, X, 0^{(n)})$ and so $(\sigma_q, F_p, I_p) \exists \bar{x} \neg \theta(\bar{x}, G, X, 0^{(n)})$ and extends $p$ for the required contradiction.)

At the two quantifier level, $p \Vdash \exists\bar{x}\forall\bar{y}\theta(\bar{x},\bar{y},G,X,0^{(n)}) \Leftrightarrow \exists\bar{x}p \Vdash \forall\bar{y}\theta(\bar{x},\bar{y},G,X,0^{(n)})$ and $p \Vdash \forall\bar{x}\exists\bar{y}\theta(\bar{x},\bar{y},G,X,0^{(n)}) \Leftrightarrow \forall\bar{x}\forall q \leq p\exists\bar{y}\exists r \leq q(r \Vdash \theta(\bar{x},\bar{y},G,X,0^{(n)})$ and so these relations are uniformly $\Sigma_2$ and $\Pi_2$ in $X \oplus 0^{(n)}$, respectively. Moreover, given any $p$ if there is any $x$ and $q \leq p$ such that $q \Vdash \forall\bar{y}\theta(\bar{x},\bar{y},G,X,0^{(n)})$ then, by the fact above about forcing $\Pi_1$ sentences, $(\sigma_q, F_q, \emptyset) \Vdash \forall\bar{y}\theta(\bar{x},\bar{y},G,X,0^{(n)})$ and so does $(\sigma_q, F_q, I_p)$ which is an extension $q'$ of $p$ with $I_p = I_{q'}$. On the other hand, if there is no $x$ and no $q \leq p$ with $I_p = I_q$ that forces $\forall\bar{y}\theta(\bar{x},\bar{y},G,X,0^{(n)})$, then $p \Vdash \forall\bar{x}\exists\bar{y}\neg\theta(\bar{x},\bar{y},G,X,0^{(n)})$. □

We now show how to construct the degrees $\mathbf{a}_i$ required in the Theorem. Minor modifications will then suffice to construct the desired $\mathbf{b}_i$. Our plan is to construct sequences $p^i_s$ of forcing conditions for $i \in \{0,1\}$ with $p^i_{s+1} \leq p^i_s$. We let $G_i = \cup\sigma_{p^i_s}$ and $A_i$ be $X \oplus G_i$ which for notational convenience we take to be $\{0\} \times X \cup \{1\} \times G_i$. We make a list $\theta_{j,n}(\bar{x},\bar{y},G,X,0^{(n)})$ of the formulas with only bounded quantifiers and so of the $\Sigma_2$ sentences of our language: $\varphi_{j,n} = \exists\bar{x}\forall\bar{y}\theta_{j,n}(\bar{x},\bar{y},G,X,0^{(n)})$. Deciding these sentences in our construction will be requirements $S_{j,n}$. We also deal with the requirements $N_e$ : $\{e\}^{A''_0} = C = \{e\}^{A''_1} \Rightarrow C \leq_T X''$ and $D_{i,e,n} : A'''_i \neq \{e\}^{(A_i \oplus 0^{(n)})''}$. We let $R_s$ list all these requirements. To handle the diagonalization requirements, we choose a recursive function $f$ such that $(\forall A, s, t)(f(s,t) \notin A''' \Leftrightarrow (\forall w \geq t)(A^{[\langle 1,s,t\rangle]}$ is finite).

**Construction:** We begin with $p^i_0 = (\emptyset, \emptyset, \emptyset)$. At the beginning of stage $s+1$ we will have forcing conditions $p^i_s$. We act at stage $s+1$ according to the following cases:

$R_s = S_{j,n}$: For each $i \in \{0,1\}$, ask if $\exists\bar{x}\exists q^i \leq p^i_s$ such that $q^i \Vdash \forall\bar{y}\theta_{j,n}(\bar{x},\bar{y},G,X,0^{(n)})$. If so, we can choose such $q^i$ with $I_{q^i} = I_{p^i_s}$ by Lemma 5.2. We then set $p^i_{s+1} = q^i$. If there is no such $q^i$, we set $p^i_{s+1} = p^i_s$. Note that in either case $p^i_{s+1}$ decides $\varphi_{j,n}$ and $I_{p^i_{s+1}} = I_{p^i_s}$. By Lemma 5.2, this procedure is uniformly recursive in $(\mathbf{x} \vee \mathbf{0}^{(n)})''$.

$R_s = D_{i,e,n}$: Let $t = \max I_{p^i_s} + 1$ and ask if there is a $\tau$ such that $\{e\}^\tau(f(s,t)) = 0$ and a $q \leq p^i_s$ that forces the $\Sigma_2$ and $\Pi_2$ facts about $G, X, 0^{(n)}$ needed to make $\tau$ an initial segment of $(X \oplus G_i \oplus 0^{(n)})''$ such that there is no $\langle k, v\rangle \in I_q - I_{p^i_s}$ with $k \leq s$. If so, let $w = \max F_q + t + 1$ and $p^i_{s+1} = (\sigma_q, F_q, I_q \cup \{\langle s, w\rangle\})$. (The condition $p^i_{s+1}$ guarantees that at the end of the construction $\{e\}^{(A_i \oplus 0^{(n)})''}(f(s,t)) = 0$. On the other hand, it also guarantees that $A_i^{[\langle 1,s,w\rangle]}$ is infinite and so that $f(s,t) \in A'''_i$ for the desired diagonalization.) If there is no such $q$ then let $p^i_{s+1} = p^i_s$. (In this case we will argue that every $A_i^{[\langle 1,s,t'\rangle]}$ is finite for $t' \geq t$ but $\{e\}^{(A_i \oplus 0^{(n)})''}(f(s,t)) \neq 0$ and so we also diagonalized.) Again by Lemma 5.2 this procedure is recursive in $(\mathbf{x} \vee \mathbf{0}^{(n)})'''$.

$R_s = N_e$: Ask if there are $\tau^i$ and $x$ such that $\{e\}^{\tau^0}(x) \downarrow \neq \{e\}^{\tau^1}(x) \downarrow$ and $q^i \leq p^i_s$ such that there is no $\langle k, v\rangle \in I_q - I_{p^i_s}$ with $k \leq s$ and $q^i$ forces all the $\Sigma_2$ and $\Pi_2$ facts needed to make $\tau^i$ an initial segment of $(X \oplus G_i)''$. If so, let $p^i_{s+1} = q^i$. (Clearly in this case we have guaranteed that $\{e\}^{A''_0} \neq \{e\}^{A''_1}$.) Otherwise, we let $p^i_{s+1} = p^i_s$. (In this case, we will argue that we also satisfy the requirement $N_e$.) Again by Lemma 5.2 this procedure is recursive in $\mathbf{x}'''$.

**Verification**: We argue that at stage $s+1$ we satisfy requirement $R_s$. For $R_s = S_{j,n}$ this is immediate and so, as usual, every $\Sigma_2(\Pi_2)$ sentence about $G_i, X$ and any $0^{(n)}$ (and so also the appropriate "translations" of such sentences about $A_i$ and $0^{(n)}$) is true if and only if it is forced. For the other requirements, we first note that, by construction, no $\langle k, v\rangle$ with $k \le s$ is ever added to $I_{p^i_{t+1}}$ for $t > s$.

$R_s = D_{i,e,n}$: Clearly if we extended $p^i_s$ to a $q$ as desired in the construction, we have $\{e\}^{(A_i \oplus 0^{(n)})''}(f(s,t)) = 0$. We also put some $\langle s, w\rangle$ into $I_{p^i_{s+1}}$ and so it never enters $F_{p^i_r}$ for any $r$. Thus when we reach any stage $r > s$ with $R_r$ devoted to a sentence that says that $\exists x > v(\langle s, x\rangle \in G_i)$ then we choose an extension that forces it to be true and so $A_i^{[\langle 1,s,t\rangle]}$ is infinite and $f(s,t) \in A_i'''$ for the desired diagonalization. On the other hand, if there is no $q \le p^i_s$ as desired in the construction, then no number $\langle s, v\rangle$ with $v \ge t$ is ever put into $I_{p^i_r}$ for any $r$. Thus when we reach any stage $r$ with $R_r$ devoted to a sentence that says that $\exists w \forall u > w(u \notin G_i^{[\langle s,v\rangle]})$ for any $v \ge t$, we will choose an extension which forces it to be true by putting $\langle s, v\rangle$ into $F_{p^i_r}$. Thus $G_i^{[\langle s,v\rangle]}$ and so $A_i^{[\langle 1,s,v\rangle]}$ is finite for every $v \ge t$ and $f(s,t) \notin A'''$. On the other hand, if $\{e\}^{(A_i \oplus 0^{(n)})''}(f(s,t)) \downarrow = 0$ then there is some oracle information $\tau$ about $(X \oplus G_i \oplus 0^{(n)})''$ that gives the correct computation. Each $\Sigma_2$ or $\Pi_2$ fact about $X \oplus G_i \oplus 0^{(n)}$ reflected in $\tau$ is decided at some stage of the construction and must be decided in accordance with the information in $\tau$. So some $p^i_y$ forces all the facts needed by $\tau$. By construction, no numbers of the form $\langle s, v\rangle$ are added to $I_{p^i_r}$ after stage $s$ and so $p^i_y$ is an extension of $p^i_s$ that could have been chosen at stage $s$ contradicting our assumption. So again we have that, if convergent, $\{e\}^{(A_i \oplus 0^{(n)})''}(f(s,t))$ does not equal $A'''(f(s,t))$ as required.

$R_s = N_e$: Again, if at stage $s$ we found $q^i$ as described and extended $p^i_s$ accordingly, $\{e\}^{A_0''} \ne \{e\}^{A_1''}$. If not, but $\{e\}^{A_0''} = C = \{e\}^{A_1''}$, then we claim that $C \le_T X''$. To compute $C(x)$, find any $\tau$ such that $\{e\}^{\tau}(x) \downarrow$ and any extension $q$ of $p^0_s$ such that $q$ forces all the $\Sigma_2$ and $\Pi_2$ facts about $X \oplus G_0$ needed to make $\tau$ an initial segment of $A_0''$ and there is no $\langle k, v\rangle \in I_q - I_{p^0_s}$ with $k \le s$. As some $p^0_y$ forces all the true facts about $A_0$ needed to get the correct computation of $\{e\}^{A_0''}(x)$ and, by construction, there is no $\langle k, v\rangle \in I_{p^0_y} - I_{p^0_s}$ with $k \le s$, this search terminates (with some $\tau$ and $q$). As the forcing relation for $\Sigma_2(G, X)$and $\Pi_2(G, X)$ sentences are $\Sigma_2(X)$ and $\Pi_2(X)$ respectively, the search is recursive in $X''$. Finally, we claim that the search terminates with a $\tau$ such that $\{e\}^{\tau}(x) = \{e\}^{A_1}(x) = C(x)$ as desired. The point here is that there is some $y$ such that $p^1_y$ forces all the all the true facts $\tau^1$ about $A_1''$ needed to get the correct computation of $\{e\}^{A_1''}(x)$ and, of course, $p^1_y \le p^1_s$ and there is no $\langle k, v\rangle \in I_{p^1_y} - I_{p^1_s}$ with $k \le s$. Thus the pairs $\tau, \tau^1$ and $q, p^1_y$ would be as desired at stage $s$ of the construction, contrary to our assumption. This concludes the construction of the sets $A_0$ and $A_1$ and so the desired degrees $\mathbf{a}_0$ and $\mathbf{a}_1$.

To construct the degrees $\mathbf{b}_i$ required in the theorem repeat the above argument with $B_i = G_i$ in place of $A_i$ omitting $X$ from the requirements $D_{i,e,n}$ making that step of the construction recursive in $0^{(n+3)}$. We also adjust the list of formulas $\theta_{j,n}$ in $S_{j,n}$ so that they omit either $X$ or $0^{(n)}$ but still contain all instances of such formulas with at most one of these two parameters. This adjustment makes the corresponding steps of the construction recursive in either $\mathbf{x}'''$ or $0^{(n+3)}$. Thus the $B_i$ so constructed will have all the desired properties.

# 6 Localization and Counting Quantifiers

In this section we will examine our proof of the definability of the jump more carefully to see that it is a correct definition in every sufficiently large jump ideal. The definition with $\mathcal{C}$ works in all ideals closed under the $\omega$-jump but that with $\tilde{\mathcal{C}}$ works in all jump ideals simply containing the single degree $\mathbf{0}^{(\omega)}$. We will also calculate bounds on the quantifier complexity of these definitions. We begin with the localization issue. Let $\mathcal{I}$ be a jump ideal.

We start with the definition $\mathcal{C}_\omega = \{\mathbf{x}|(\forall\mathbf{z})(\mathbf{z}\vee\mathbf{x}$ is not a minimal cover of $\mathbf{z}\}$. We need to verify that, for each $n$, $\mathcal{I}\models\mathbf{0}^{(n)}\in\mathcal{C}_\omega$ and, for every $\mathbf{x}\in\mathcal{I}$, there is an $n\in\mathbb{N}$ such that $\mathcal{I}\models(\forall\mathbf{z})(\mathbf{z}\vee\mathbf{x}$ is not a minimal cover of $\mathbf{z})\rightarrow\mathbf{x}\leq\mathbf{0}^{(n)}$. Of course, fact that $\mathbf{0}^{(n)}\vee\mathbf{z}$ is not a minimal cover of $\mathbf{z}$ in $\mathcal{D}$ means that it is not one in any ideal containing $\mathbf{0}^{(n)}$ and $\mathbf{z}$. For the second fact, note that Jockusch and Shore [1984, Corollary 3.3] actually prove that if $\mathbf{x}\not\leq\mathbf{0}^{(n)}$ for every $n$ then there is a $\mathbf{z}\leq\mathbf{x}\vee\mathbf{0}^{(\omega)}$ such that $\mathbf{z}\vee\mathbf{x}$ is a minimal cover of $\mathbf{z}$. Thus the facts we need about $\mathcal{C}_\omega$ are true in any jump ideal containing $\mathbf{0}^{(\omega)}$ as required.

Next, we note that all the procedures connected with Slaman-Woodin coding are arithmetic by the results of Slaman and Woodin [1986]. In particular, any countable set of pairwise incomparable degrees uniformly recursive in $\mathbf{x}$ are coded by a $\mathbf{c}\leq\mathbf{x}''$. Moreover, by definition, the set $Kd(\mathbf{c})$ coded by $\mathbf{c}$ is uniformly recursive in just a few jumps of $\mathbf{c}$. Thus it, and all the notions developed in §3 about coding and comparing sets, work in (i.e. are absolute for) $\mathcal{I}$ for any sets with degrees in $\mathcal{I}$ and any codes inside $\mathcal{I}$ for any jump ideal $\mathcal{I}$. The relations absolute to any jump ideal $\mathcal{I}$ thus include $\mathbf{y}\in Kd(\mathbf{c})$, $Mp(\check{\mathbf{c}},\dot{\mathbf{c}},\ddot{\mathbf{c}},\hat{\mathbf{c}},\mathbf{d},\tilde{\mathbf{d}})$, $Compl(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{c})$ and $Eq(\mathbf{a},\mathbf{c},\tilde{\mathbf{a}},\tilde{\mathbf{c}})$.

This brings us to the definitions of $\mathcal{C}$ and $\tilde{\mathcal{C}}$ in Theorem 3.5. We must verify that the classes so defined when interpreted in $\mathcal{I}$ still have Properties 1 and 2 in $\mathcal{I}$. The lattice coding needed for Property 1 is done in Theorem 2.2 by degrees below the given $\mathbf{x}$ and so is available in any ideal containing $\mathbf{x}$. The Slaman-Woodin coding needed in the formal definition of sets being coded below $\mathbf{x}$ work, as we just remarked, in every jump ideal containing $\mathbf{x}$ as they do in $\mathcal{D}$. In particular, in the definition of $Cd(\mathbf{x},\mathbf{a},\mathbf{c})$, a $\mathbf{c}$ needed for any $\mathbf{a}<\mathbf{x}$ exists in the degrees arithmetic in $\mathbf{x}$ as the set $\{\mathbf{d}_n\}$ being coded is uniformly recursive in $\mathbf{x}^{(5)}$ (to calculate the infimum and other operations on degrees below $\mathbf{x}$). As this set is the minimal one coded anywhere satisfying condition (3) and (4) of Definition 3.3, it is also the minimal one coded in $\mathcal{I}$ satisfying these conditions. Thus, for any $\mathbf{a}\leq\mathbf{x}$ coding a set, the existence of a $\mathbf{c}$ such that $Cd(\mathbf{x},\mathbf{a},\mathbf{c})$ is absolute to any jump ideal so all the desired sets are coded below $\mathbf{x}$ by degrees in $\mathcal{I}$. Similarly, the existence of codes needed for any instances of the relations $M$, $Compl$ and $Eq$ for parameters in

$\mathcal{I}$ is absolute for any jump ideal. All that remains to check for the absoluteness of our definition of $\mathbf{x}''\leq \mathbf{y}''$ is Theorem 5.1 that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ have Property 2.

From the statement of Theorem 5.1, we see that the degrees $\mathbf{a}_0, \mathbf{a}_1$ required for $\mathcal{C}$ exists below $\mathbf{x}^{(\omega)}$. As the properties of these degrees required in the definition are all specified in terms of lattice coding of sets and Slaman-Woodin comparisons, the degrees constructed have the desired properties inside any jump ideal containing them. Thus the definition of $\mathbf{x}''\leq \mathbf{y}''$ from $\mathcal{C}$ works in any ideal closed under the $\omega$-jump as desired. For the definition in terms of $\tilde{\mathcal{C}}$, we need the degrees $\mathbf{b}_0, \mathbf{b}_1$ of Theorem 5.1. Again the statement of the theorem assures us that they are computable in $(\mathbf{x} \vee \mathbf{0}^{(\omega)})'''$ and so exist in any jump ideal containing $\mathbf{0}^{(\omega)}$.

All that remains now is to verify the absoluteness of the definition of the jump from the relation $\mathbf{x}'' \leq \mathbf{y}''$. Here we note that Shore and Slaman [1999, Theorem 2.3 $n = 2$ relativized to $\mathbf{x}$] prove that if $\mathbf{w} \not\leq \mathbf{x}'$ then there is a $\mathbf{g} \leq \mathbf{w} \vee \mathbf{x}''$ such that $\mathbf{w} \vee \mathbf{g} = \mathbf{g}''$. As argued there, this suffices to show that $\mathbf{x}'$ is the maximal $\mathbf{w}$ such that $\forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{w} \vee \mathbf{g}) \neq \mathbf{g}''$ as required in the definition of $\mathbf{x}'$ from the relation $\mathbf{x}'' \leq \mathbf{y}''$: Clearly no degree less than or equal to $\mathbf{x}'$ can join any $\mathbf{g} \geq \mathbf{x}$ to $\mathbf{g}''$. For the other direction, the cited theorem provides, for every $\mathbf{w} \not\leq \mathbf{x}'$, a $\mathbf{g} \leq \mathbf{w} \vee \mathbf{x}''$, and so one in any jump ideal containing $\mathbf{x}$ and $\mathbf{w}$, such that $\mathbf{w} \vee \mathbf{g} = \mathbf{g}''$.

We have thus proven Theorem 3.7 and a version for the definition with $\tilde{\mathcal{C}}$ as well.

**Theorem 6.1.** *The definitions of* $\mathbf{x}'' \leq \mathbf{y}''$ *and* $\mathbf{x}' = \mathbf{w}$ *in terms of* $\mathcal{C}$ *are absolute to every ideal* $\mathcal{I}$ *closed under the* $\omega$*-jump, i.e. for* $\mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathcal{I}$*,*

1. $\mathbf{x}'' \leq \mathbf{y}'' \Leftrightarrow \mathcal{I} \models (\forall \mathbf{a}, \mathbf{c})[Cd(\mathbf{a}, \mathbf{c})\ \&\ (\forall \mathbf{z})(\mathbf{z} \in \mathcal{C}\ \&\ \mathbf{x} \leq \mathbf{z} \rightarrow (\exists \tilde{\mathbf{a}}, \tilde{\mathbf{c}})(Cd(\mathbf{z}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})\ \&\ Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})) \rightarrow (\forall \mathbf{w})(\mathbf{w} \in \mathcal{C}\ \&\ \mathbf{y} \leq \mathbf{w} \rightarrow (\exists \hat{\mathbf{a}}, \hat{\mathbf{c}})(Cd(\mathbf{w}, \hat{\mathbf{a}}, \hat{\mathbf{c}})\ \&\ Eq(\mathbf{a}, \mathbf{c}, \hat{\mathbf{a}}, \hat{\mathbf{c}}))]$.
2. $\mathbf{x}' = \mathbf{w} \Leftrightarrow \mathcal{I} \models \forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{w} \vee \mathbf{g}) \neq \mathbf{g}'')\ \&\ \forall \mathbf{v}(\forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{v} \vee \mathbf{g}) \neq \mathbf{g}'' \rightarrow \mathbf{v} \leq \mathbf{w})$.

*When given in terms of* $\tilde{\mathcal{C}}$ *they are absolute for every jump ideal* $\mathcal{I}$ *containing* $\mathbf{0}^{(\omega)}$*, i.e. for* $\mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathcal{I}$*,*

1. $\mathbf{x}'' \leq \mathbf{y}'' \Leftrightarrow \mathcal{I} \models (\forall \mathbf{a}, \mathbf{c})[Cd(\mathbf{a}, \mathbf{c})\ \&\ (\forall \mathbf{z})(\mathbf{z} \in \tilde{\mathcal{C}}\ \&\ \mathbf{x} \leq \mathbf{z} \rightarrow (\exists \tilde{\mathbf{a}}, \tilde{\mathbf{c}})(Cd(\mathbf{z}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})\ \&\ Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})) \rightarrow (\forall \mathbf{w})(\mathbf{w} \in \tilde{\mathcal{C}}\ \&\ \mathbf{y} \leq \mathbf{w} \rightarrow (\exists \hat{\mathbf{a}}, \hat{\mathbf{c}})(Cd(\mathbf{w}, \hat{\mathbf{a}}, \hat{\mathbf{c}})\ \&\ Eq(\mathbf{a}, \mathbf{c}, \hat{\mathbf{a}}, \hat{\mathbf{c}}))]$.
2. $\mathbf{x}' = \mathbf{w} \Leftrightarrow \mathcal{I} \models \forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{w} \vee \mathbf{g}) \neq \mathbf{g}'')\ \&\ \forall \mathbf{v}(\forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{v} \vee \mathbf{g}) \neq \mathbf{g}'' \rightarrow \mathbf{v} \leq \mathbf{w})$.

We now provide an analysis of these definitions to calculate their quantifier complexity. We follow the process of definition in §3 and work first in the language with $\vee$, $\wedge$ and 0.

**Theorem 6.2.** *There are* $\Pi_5$ *formulas of* $\mathcal{D}$ *in the language with* $\leq$, $\vee$ *and* $\wedge$ *that define the relations* $\mathbf{x}''\leq \mathbf{y}''$, $\mathbf{x} \in \mathbf{L}_2(\mathbf{y})$ *and* $\mathbf{x}'' = \mathbf{y}''$ *in any jump ideal containing* $\mathbf{0}^{(\omega)}$*, a* $\Sigma_6\&\Pi_6$ *one that defines* $\mathbf{w} = \mathbf{x}''$ *and a* $\Pi_8$ *one that defines the relation* $\mathbf{w} = \mathbf{x}'$ *in any such ideal.*

**Lemma 6.3.** *The relation* $\mathbf{y} \in Kd(\mathbf{c})$ *is* $\exists\&\forall$*.*

*Proof.* By definition (Theorem 3.1) $\mathbf{y} \in Kd(\mathbf{c}) \Leftrightarrow \mathbf{y} \leq \mathbf{u}\ \&\ \exists \mathbf{w}(\mathbf{w} \leq \mathbf{h}_0 \vee \mathbf{y}, \mathbf{h}_1 \vee \mathbf{y}\ \&\ \mathbf{w} \not\leq \mathbf{y})$ $\&\ (\forall \mathbf{u} < \mathbf{y})\neg\exists \mathbf{w}(\mathbf{w} \leq \mathbf{h}_0 \vee \mathbf{u}, \mathbf{h}_1 \vee \mathbf{u}\ \&\ \mathbf{w} \not\leq \mathbf{u})$. □

**Lemma 6.4.** *The relations $Cd(\mathbf{x}, \mathbf{a}, \mathbf{c})$, $Cd(\mathbf{a}, \mathbf{c})$ and $Cd(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c})$ are $\forall\exists$. The precise versions are $\forall\exists\forall$.*

*Proof.* Consider the clauses of Definition 3.3. Clauses (1) and (2) are quantifier free. Clause (3) is $\exists\&\forall$ by the previous Lemma. Similarly, clause (4) is $\forall(\exists\&\forall \to \exists\&\forall\&\exists\&\forall)$ and so $\forall\exists$. Clause (5) is $\forall(\exists\&\forall \to \forall)$. Using this analysis of (4), clause (6) is $\forall(\forall\exists \to \forall(\exists\&\forall \to \exists\&\forall))$ and so $\forall\exists\forall$. □

**Lemma 6.5.** *The relation $Compl(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c})$ is $\forall\exists$.*

*Proof.* $Compl(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c}) \Leftrightarrow Cd(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c})\ \&\ (\forall \mathbf{d} \in Kd(\mathbf{c}))(\mathbf{d} \leq \mathbf{g}_0, \mathbf{g}_1 \leftrightarrow \mathbf{d} \not\leq \mathbf{g}_2, \mathbf{g}_3)$. By the previous Lemmas this relation is $\forall\exists$. □

**Lemma 6.6.** *The relation $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{z}, \tilde{\mathbf{z}})$ is $\exists\forall$.*

*Proof.* $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{z}, \tilde{\mathbf{z}}) \Leftrightarrow (\exists \mathbf{w}, \tilde{\mathbf{w}} \in Kd(\check{\mathbf{c}}))(\mathbf{z} \vee \mathbf{w} \in Kd(\dot{\mathbf{c}})\ \&\ \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}} \in Kd(\ddot{\mathbf{c}})\ \&$ $(\mathbf{z} \vee \mathbf{w} \vee \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}}) \in Kd(\hat{\mathbf{c}}))$. By Lemma 6.3 this is $\exists\forall$. □

**Lemma 6.7.** *The relation $Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})$ is $\exists\forall\exists\forall$, i.e. $\Sigma_4$.*

*Proof.* $Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}}) \Leftrightarrow Cd(\mathbf{a}, \mathbf{c})\&Cd(\tilde{\mathbf{a}}, \tilde{\mathbf{c}})\&(\exists \check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}})\{(\forall \mathbf{x}, \mathbf{y}, \mathbf{z})(Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{x}, \mathbf{y})\ \&$ $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{x}, \mathbf{z}) \to \mathbf{y} = \mathbf{z})\ \&\ Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{d}_0, \tilde{\mathbf{d}}_0)\ \&\ (\forall \mathbf{d})(\forall \tilde{\mathbf{d}})$ $(Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{d}, \tilde{\mathbf{d}}) \to Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, (\mathbf{d} \vee \mathbf{e}_0) \wedge \mathbf{f}_1, (\tilde{\mathbf{d}} \vee \tilde{\mathbf{e}}_0) \wedge \tilde{\mathbf{f}}_1)\ \&\ Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, (\mathbf{d} \vee \mathbf{e}_1) \wedge$ $\mathbf{f}_0, (\tilde{\mathbf{d}} \vee \tilde{\mathbf{e}}_1) \wedge \tilde{\mathbf{f}}_0)\ \&\ (\mathbf{d} \leq \mathbf{g}_0, \mathbf{g}_1 \leftrightarrow \tilde{\mathbf{d}} \leq \tilde{\mathbf{g}}_0, \tilde{\mathbf{g}}_1)\}$. Applying the previous Lemmas we have that the relation is $\forall\exists\&\forall\exists\&\exists\{\forall(\exists\forall\&\exists\forall \to)\ \&\ \exists\forall\&\forall[(\exists\forall \to \exists\forall\&\exists\forall)]\}$. Some simplification yields $\forall\exists\ \&\ \exists\{\forall(\exists\forall \to)\ \&\ \forall[(\exists\forall \to \exists\forall]\}$. One more round gives $\exists\{\forall\exists\ \&\ \forall\exists\forall\}$ and so $\exists\forall\exists\forall$. □

We can actually improve this calculation by being more explicit in our definitions of $Mp$ and $Eq$. Recall that we defined $Mp$ between the sets $\{\mathbf{d}_n\}$ and $\{\tilde{\mathbf{d}}_n\}$ corresponding to two codes $\mathbf{a}, \mathbf{c}$ and $\tilde{\mathbf{a}}, \tilde{\mathbf{c}}$, respectively, by choosing $\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}$ and $\hat{\mathbf{c}}$ such that, for any $\mathbf{z} \in Kd(\mathbf{c})$ and $\tilde{\mathbf{z}} \in Kd(\tilde{\mathbf{c}})$, $(\exists \mathbf{w}, \tilde{\mathbf{w}} \in Kd(\check{\mathbf{c}}))(\mathbf{z} \vee \mathbf{w} \in Kd(\dot{\mathbf{c}})\ \&\ \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}} \in Kd(\ddot{\mathbf{c}})\ \&$ $(\mathbf{z} \vee \mathbf{w} \vee \tilde{\mathbf{z}} \vee \tilde{\mathbf{w}}) \in Kd(\hat{\mathbf{c}})) \leftrightarrow (\exists n)(\mathbf{z} = \mathbf{d}_n\ \&\ \tilde{\mathbf{z}} = \tilde{\mathbf{d}}_n)$. The intention here was that we would have a set $\mathbf{h}_i$ of mutually 1-generic degrees relative to any degree above all the $\mathbf{d}_n$ and $\tilde{\mathbf{d}}_n$ and let $\check{\mathbf{c}}$ code $\{\mathbf{h}_i\}$. Next we let $\dot{\mathbf{c}}$ code the set $\{\mathbf{d}_n \vee \mathbf{h}_{2n}\}$ of pairwise incomparable degrees and $\ddot{\mathbf{c}}$ code $\{\tilde{\mathbf{d}}_n \vee \mathbf{h}_{2n+1}\}$. Finally we let $\hat{\mathbf{c}}$ code $\{\mathbf{d}_n \vee \mathbf{h}_{2n} \vee \tilde{\mathbf{d}}_n \vee \mathbf{h}_{2n+1}\}$ which is also pairwise incomparable. We can now add on a new code $\grave{\mathbf{c}}$ for the set $\{\mathbf{h}_n \vee \mathbf{h}_{n+2} | n \in \omega\}$ and so guarantee that (for these codes) if $\mathbf{w}, \tilde{\mathbf{w}}$ are the witnesses that $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{d}_n, \tilde{\mathbf{d}}_n)$ for $\mathbf{n} > 1$ and $\mathbf{w} \vee \mathbf{h}, \tilde{\mathbf{w}} \vee \tilde{\mathbf{h}}, \mathbf{w} \vee \mathbf{j}, \tilde{\mathbf{w}} \vee \tilde{\mathbf{j}} \in Kd(\grave{\mathbf{c}})$ with $\mathbf{h} \neq \mathbf{j}$ and $\tilde{\mathbf{h}} \neq \tilde{\mathbf{j}}$ then some choice of one each of $\{\mathbf{h}, \mathbf{j}\}$ and $\{\tilde{\mathbf{h}}, \tilde{\mathbf{j}}\}$ provides the witnesses that $Mp(\check{\mathbf{c}}, \dot{\mathbf{c}}, \ddot{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{d}_{n+1}, \tilde{\mathbf{d}}_{n+1})$. (The reason we need two candidates here is that for a given $\mathbf{h}_n$, $n > 1$, both $\mathbf{h}_n \vee \mathbf{h}_{n+2}$ and $\mathbf{h}_n \vee \mathbf{h}_{n-2}$ are in $Kd(\grave{\mathbf{c}})$.) We can now use the code $\grave{\mathbf{c}}$ to provide a $\Sigma_3$ equivalent for $Eq$.

**Lemma 6.8.** *There is a $\Sigma_3$ equivalent of $Eq$.*

*Proof.* We use $\mathbf{d}_1$ and $\tilde{\mathbf{d}}_1$ as abbreviations for $(\mathbf{d}\vee\mathbf{e}_0)\wedge\mathbf{f}_1$ and $((\tilde{\mathbf{d}}\vee\tilde{\mathbf{e}}_\mathbf{0})\wedge\tilde{\mathbf{f}}_\mathbf{1})$, respectively.

$Eq(\mathbf{a},\mathbf{c},\tilde{\mathbf{a}},\tilde{\mathbf{c}}) \Leftrightarrow Cd(\mathbf{a},\mathbf{c})\&Cd(\tilde{\mathbf{a}},\tilde{\mathbf{c}})\&(\exists\check{\mathbf{c}},\dot{\mathbf{c}},\ddot{\mathbf{c}},\hat{\mathbf{c}},\grave{\mathbf{c}})\{$
$(\forall\mathbf{x},\mathbf{y},\mathbf{z})(Mp(\check{\mathbf{c}},\dot{\mathbf{c}},\ddot{\mathbf{c}},\hat{\mathbf{c}},\mathbf{x},\mathbf{y})\ \&\ Mp(\check{\mathbf{c}},\dot{\mathbf{c}},\ddot{\mathbf{c}},\hat{\mathbf{c}},\mathbf{x},\mathbf{z})\rightarrow\mathbf{y}=\mathbf{z})\ \&\ Mp(\check{\mathbf{c}},\dot{\mathbf{c}},\ddot{\mathbf{c}},\hat{\mathbf{c}},\mathbf{d}_0,\tilde{\mathbf{d}}_0)$
$\&\ Mp(\check{\mathbf{c}},\dot{\mathbf{c}},\ddot{\mathbf{c}},\hat{\mathbf{c}},\mathbf{d}_1,\tilde{\mathbf{d}}_1)\ \&\ (\forall\mathbf{d}\in Kd(\mathbf{c}))(\forall\tilde{\mathbf{d}}\in Kd(\tilde{\mathbf{c}}))(\forall\mathbf{h},\tilde{\mathbf{h}},\mathbf{j},\tilde{\mathbf{j}}\in Kd(\check{\mathbf{c}})[\mathbf{h}\neq\mathbf{j}\&\tilde{\mathbf{h}}\neq\tilde{\mathbf{j}}$
$\&\ (\exists\mathbf{w},\tilde{\mathbf{w}}\in Kd(\check{\mathbf{c}}))(\mathbf{h}\vee\mathbf{w}\in Kd(\grave{\mathbf{c}})\ \&\ \tilde{\mathbf{h}}\vee\tilde{\mathbf{w}}\in Kd(\grave{\mathbf{c}})\ \&\ \mathbf{j}\vee\mathbf{w}\in Kd(\grave{\mathbf{c}})\ \&\ \tilde{\mathbf{j}}\vee\tilde{\mathbf{w}}\in Kd(\grave{\mathbf{c}})\&$
$\mathbf{d}\vee\mathbf{w}\in Kd(\dot{\mathbf{c}})\ \&\ \tilde{\mathbf{d}}\vee\tilde{\mathbf{w}}\in Kd(\ddot{\mathbf{c}})\ \&\ (\mathbf{d}\vee\mathbf{w}\vee\tilde{\mathbf{d}}\vee\tilde{\mathbf{w}})\in Kd(\hat{\mathbf{c}}))\rightarrow\bigvee_{\mathbf{k}\in\{\mathbf{h},\mathbf{j}\},\tilde{\mathbf{k}}\in\{\tilde{\mathbf{h}},\tilde{\mathbf{j}}\}}\{$
$[((\mathbf{d}\vee\mathbf{e}_1)\wedge\mathbf{f}_0=\mathbf{d}\ \rightarrow((\mathbf{d}\vee\mathbf{e}_0)\wedge\mathbf{f}_1)\vee\mathbf{k}\in Kd(\dot{\mathbf{c}})\ \&\ ((\tilde{\mathbf{d}}\vee\tilde{\mathbf{e}}_\mathbf{0})\wedge\tilde{\mathbf{f}}_\mathbf{1})\vee\tilde{\mathbf{k}}\in Kd(\ddot{\mathbf{c}})\ \&$
$(((\mathbf{d}\vee\mathbf{e}_0)\wedge\mathbf{f}_1)\vee\mathbf{k}\vee((\tilde{\mathbf{d}}\vee\tilde{\mathbf{e}}_\mathbf{0})\wedge\tilde{\mathbf{f}}_\mathbf{1})\vee\tilde{\mathbf{k}})\in Kd(\hat{\mathbf{c}}))]\ \&$
$[((\mathbf{d}\vee\mathbf{e}_0)\wedge\mathbf{f}_1=\mathbf{d}\ \rightarrow((\mathbf{d}\vee\mathbf{e}_1)\wedge\mathbf{f}_0)\vee\mathbf{k}\in Kd(\dot{\mathbf{c}})\ \&\ ((\tilde{\mathbf{d}}\vee\tilde{\mathbf{e}}_\mathbf{1})\wedge\tilde{\mathbf{f}}_\mathbf{0})\vee\tilde{\mathbf{k}}\in Kd(\ddot{\mathbf{c}})\ \&$
$(((\mathbf{d}\vee\mathbf{e}_1)\wedge\mathbf{f}_0)\vee\mathbf{k}\vee((\tilde{\mathbf{d}}\vee\tilde{\mathbf{e}}_\mathbf{1})\wedge\tilde{\mathbf{f}}_\mathbf{0})\vee\tilde{\mathbf{k}})\in Kd(\hat{\mathbf{c}})]\}]\}.$

This formula is of the form $\forall\exists\&\forall\exists\&\exists\{\forall(\exists\forall\&\exists\forall\rightarrow)\ \&\ \exists\forall\ \&\ \exists\forall\ \&\ \forall\forall[\exists\&\forall\ \&\ \exists\&\forall...\ \&$ $\exists(\exists\&\forall\ \&\ \exists\&\forall...)\rightarrow\bigvee\{[\rightarrow\exists\&\forall\ \&\ \exists\&\forall\ \&\ \exists\&\forall]\ \&\ [\rightarrow\exists\&\forall\ \&\ \exists\&\forall\ \&\ \exists\&\forall]\}]\}$. Simplifying we get $\exists\{\forall\exists\&\exists\forall\ \&\ \forall[\exists\&\forall\ \&\ \exists(\exists\&\forall)\rightarrow\bigvee\{\exists\&\forall\}]\}$ and so $\exists\{\forall[\exists\forall\rightarrow\exists\&\forall]\}$, i.e. $\exists\forall\exists$. □

**Lemma 6.9.** *The relation $\mathbf{x}\in\mathcal{C}_\omega$ is $\forall\exists$.*

*Proof.* $\mathbf{x}\in\mathcal{C}_\omega\Leftrightarrow(\forall\mathbf{z})(\mathbf{z}\vee\mathbf{x}$ is not a minimal cover of $\mathbf{z}\}\Leftrightarrow$
$(\forall\mathbf{z})(\mathbf{z}\vee\mathbf{x}\leq\mathbf{x}$ or $\exists\mathbf{w}(\mathbf{x}<\mathbf{w}<\mathbf{z}\vee\mathbf{x}))$. □

**Lemma 6.10.** *The relation $\mathbf{x}\in\mathcal{C}$ is $\Sigma_4$.*

*Proof.* By Theorem 3.5, $\mathbf{x}\in\mathcal{C}\Leftrightarrow\exists\mathbf{a},\mathbf{c}(Cd(\mathbf{x},\mathbf{a},\mathbf{c})\ \&$ $(\forall\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}},\mathbf{z})(\mathbf{z}\in\mathcal{C}_\omega\ \&\ Compl(\mathbf{x}\vee\mathbf{z},\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}})\rightarrow\neg Eq(\mathbf{a},\mathbf{c},\tilde{\mathbf{a}},\tilde{\mathbf{c}}))$. By the previous Lemmas this is $\exists(\forall\exists\ \&\ \forall(\forall\exists\ \&\ \forall\exists\rightarrow\Pi_3))$ and so $\Sigma_4$. □

**Lemma 6.11.** *The relation $\mathbf{x}\in\tilde{\mathcal{C}}$ is also $\Sigma_4$.*

*Proof.* $\mathbf{x}\in\tilde{\mathcal{C}}\Leftrightarrow\exists\mathbf{y}(\mathbf{y}\in\mathcal{C}\ \&\ \mathbf{y}\leq\mathbf{x})$. □

**Lemma 6.12.** *The relation $\mathbf{x}''\leq\mathbf{y}''$ using $\mathcal{C}$ or $\tilde{\mathcal{C}}$ is $\Pi_5$.*

*Proof.* By Theorem 3.6, $\mathbf{x}''\leq\mathbf{y}''\Leftrightarrow(\forall\mathbf{a},\mathbf{c})[Cd(\mathbf{a},\mathbf{c})\ \&\ (\forall\mathbf{z})(\mathbf{z}\in\mathcal{C}\ \&\ \mathbf{x}\leq\mathbf{z}\rightarrow(\exists\tilde{\mathbf{a}},\tilde{\mathbf{c}})$ $(Cd(\mathbf{z},\tilde{\mathbf{a}},\tilde{\mathbf{c}})\ \&\ Eq(\mathbf{a},\mathbf{c},\tilde{\mathbf{a}},\tilde{\mathbf{c}}))\rightarrow(\forall\mathbf{w})(\mathbf{w}\in\mathcal{C}\ \&\ \mathbf{y}\leq\mathbf{w}\rightarrow(\exists\hat{\mathbf{a}},\hat{\mathbf{c}})(Cd(\mathbf{w},\hat{\mathbf{a}},\hat{\mathbf{c}})\ \&\ Eq(\mathbf{a},\mathbf{c},\hat{\mathbf{a}},\hat{\mathbf{c}}))]$. So this relation is $\forall[\Pi_2\ \&\ \forall(\Sigma_4\rightarrow\exists(\Pi_2\ \&\ \Sigma_3))\rightarrow\forall(\Sigma_4\rightarrow\exists(\Pi_2\ \&\ \Sigma_3))]$. Simplifying gives $\forall[\Pi_4\rightarrow\Pi_4]$, i.e. $\Pi_5$. If we use $\tilde{\mathcal{C}}$ in place of $\mathcal{C}$, nothing changes as both are $\Sigma_4$. □

**Lemma 6.13.** *The relation $\mathbf{u}\geq\mathbf{x}''$ is $\Pi_6$, $\mathbf{u}\leq\mathbf{x}''$ is $\Sigma_6$ and so $\mathbf{u}=\mathbf{x}''$ is $\Sigma_6\&\Pi_6$.*

*Proof.* As pointed out at the beginning of §2, $\mathbf{x}'' = \vee\{\mathbf{y} \geq \mathbf{x} | \mathbf{y}'' \leq \mathbf{x}''\}$ and so $\mathbf{u} \geq \mathbf{x}'' \Leftrightarrow (\forall \mathbf{y} \geq \mathbf{x})(\mathbf{y}'' \leq \mathbf{x}'' \rightarrow \& \ \mathbf{u} \geq \mathbf{y})$ and by the previous Lemma this relation is $\Pi_6$. As we also noted there, we have $\mathbf{y}_1, \mathbf{y}_2 \geq \mathbf{x}$ with $\mathbf{y}_1'', \mathbf{y}_2'' = \mathbf{x}''$ such that $\mathbf{y}_1'' \vee \mathbf{y}_2'' = \mathbf{x}''$ so that $\mathbf{u} \leq \mathbf{x}'' \Leftrightarrow (\exists \mathbf{y}_1, \mathbf{y}_2 \geq \mathbf{x})(\mathbf{y}_1'', \mathbf{y}_2'' \leq \mathbf{x}'' \ \& \ \mathbf{u} \leq \mathbf{y}_1 \vee \mathbf{y}_2)$. This relation is then $\Sigma_6$. □

**Lemma 6.14.** *The relation* $\mathbf{x}' = \mathbf{w}$ *is* $\Pi_8$.

*Proof.* By Theorem 3.6, $\mathbf{x}' = \mathbf{w} \Leftrightarrow \mathbf{w} = \max\{\mathbf{z} \geq_{\mathbf{T}} \mathbf{x} | (\forall \mathbf{g} \geq_{\mathbf{T}} \mathbf{x})(\mathbf{z} \vee \mathbf{g} \ngeq \mathbf{g}'') \Leftrightarrow \forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{w} \vee \mathbf{g}) \ngeq \mathbf{g}'') \ \& \ \forall \mathbf{v}(\forall \mathbf{g}(\mathbf{g} \geq \mathbf{x} \rightarrow (\mathbf{v} \vee \mathbf{g}) \ngeq \mathbf{g}'') \rightarrow \mathbf{v} \leq \mathbf{w})$. By the previous Lemmas, this relation is $\forall(\Sigma_6) \ \& \ \forall(\forall \Sigma_6 \rightarrow)$, i.e. $\Pi_7 \ \& \ \Pi_8$. □

We turn now to the issue of eliminating the defined symbols 0, $\vee$ and $\wedge$. The only use of 0 is in clause (5) of Definition 3.3. We can eliminate it and the uses of $\vee$ and $\wedge$ there as well without increasing its quantifier complexity by rewriting the clause as

$5'$ $\forall \mathbf{d}, \hat{\mathbf{d}}(\mathbf{d} \neq \hat{\mathbf{d}} \ \& \ \mathbf{d}, \hat{\mathbf{d}} \in Kd(\mathbf{c}) \ \rightarrow \ \forall \mathbf{u}, \mathbf{v}(\mathbf{u} \leq \mathbf{d}, \hat{\mathbf{d}} \rightarrow \mathbf{u} \leq \mathbf{v}) \ \& \ \forall \mathbf{u}(\mathbf{u} \geq \mathbf{d}, \mathbf{p} \rightarrow \mathbf{q} \leq \mathbf{u})$
$\& \ \exists \mathbf{x}, \mathbf{y}(\mathbf{y} < \mathbf{x} \leq \mathbf{d}, \mathbf{g}_0, \mathbf{g}_1) \rightarrow \mathbf{d} \leq \mathbf{g}_0, \mathbf{g}_1)$.

Now, in general, we can eliminate $\vee$ at the expense of increasing the quantifier complexity by at most one. Given a formula $Q\tilde{\mathbf{x}}\boldsymbol{\varphi}$ where $\boldsymbol{\varphi}$ is quantifier free and includes the symbol $\vee$, we can find an equivalent without $\vee$ that has at most one additional alternation of quantifiers as follows: for each term of the form $\mathbf{x} \vee \mathbf{y}$ in $\boldsymbol{\varphi}$ we can add a new variable $\mathbf{u}$ and replace $\boldsymbol{\varphi}$ by $\exists \mathbf{u}(\mathbf{x}, \mathbf{y} \leq \mathbf{u} \ \& \ \forall \mathbf{v}(\mathbf{x}, \mathbf{y} \leq \mathbf{v} \rightarrow \mathbf{u} \leq \mathbf{v}) \ \& \ \boldsymbol{\varphi}(\mathbf{u}/(\mathbf{x} \vee \mathbf{y})))$. Similarly, $\boldsymbol{\varphi}$ can be replaced by $\forall \mathbf{u}(\mathbf{x}, \mathbf{y} \leq \mathbf{u} \ \& \ \forall \mathbf{v}(\mathbf{x}, \mathbf{y} \leq \mathbf{v} \rightarrow \mathbf{u} \leq \mathbf{v}) \ \rightarrow \ \boldsymbol{\varphi}(\mathbf{u}/(\mathbf{x} \vee \mathbf{y})))$. Of course, one of these replacements has its initial quantifier the same as the final one of $Q\tilde{\mathbf{x}}$ and so increases the quantifier complexity by exactly one. Clearly, we can iterate and even chain this procedure to eliminate all occurrences of $\vee$ at the same cost. We could perform the dual procedure for $\wedge$ except for the fact that $\wedge$ is not always defined in $\mathcal{D}$. Checking all our formulas we see that the only remaining occurrences of $\wedge$ are in terms of the form $(\mathbf{d} \vee \mathbf{e}_0) \wedge \mathbf{f}_1$ or $(\mathbf{d} \vee \mathbf{e}_1) \wedge \mathbf{f}_0$ with $\mathbf{d} \in Kd(\mathbf{a}, \mathbf{c})$ and the other elements $\mathbf{e}_i$ and $\mathbf{f}_i$ are the standard ones in $\mathbf{a}$. In each such instance, we have also asserted in the formula that $Cd(\mathbf{a}, \mathbf{c})$ or we are within the definition of $Cd$ itself. Thus we wish to add a condition that guarantees that all infima of this form exist. We do this by adding the following to clause (3) of Definition 3.3:

$(\forall \mathbf{d})(\mathbf{d} \in \widehat{Kd}(\mathbf{c})) \rightarrow (\exists \mathbf{u}_0, \mathbf{u}_1, \mathbf{v}_0, \mathbf{v}_1)(\mathbf{d}, \mathbf{e}_0 \leq \mathbf{u}_0 \ \& \ \forall \mathbf{w}(\mathbf{d}, \mathbf{e}_0 \leq \mathbf{w} \rightarrow \mathbf{u}_0 \leq \mathbf{w}) \ \& \ \mathbf{v}_0 \leq \mathbf{u}_0, \mathbf{f}_1 \ \& \ \forall \mathbf{w}(\mathbf{w} \leq \mathbf{u}_0, \mathbf{f}_1 \rightarrow \mathbf{w} \leq \mathbf{v}_0) \ \& \ \mathbf{d}, \mathbf{e}_1 \leq \mathbf{u}_1 \ \& \ \forall \mathbf{w}(\mathbf{d}, \mathbf{e}_1 \leq \mathbf{w} \rightarrow \mathbf{u}_1 \leq \mathbf{w}) \ \& \ \mathbf{v}_1 \leq \mathbf{u}_1, \mathbf{f}_0 \ \& \ \forall \mathbf{w}(\mathbf{w} \leq \mathbf{u}_1, \mathbf{f}_0 \rightarrow \mathbf{w} \leq \mathbf{v}_1))$

where $\widehat{Kd}(\mathbf{c})$ is the version of $Kd$ with $\vee$ eliminated and so on general grounds of complexity at most $\exists\forall\&\forall\exists$. This new clause is then $\Pi_3$ in the language with just $\leq$. The whole translation of $Cd$ into the language with just $\leq$ is then $\Pi_3$ as we can now apply the general elimination procedure to clause (4). All later uses of $\wedge$ take place within contexts in which we have, by this addition to clause (3) of the definition of $Cd$, guaranteed that the required infimum exists. We can thus apply the general elimination rules for $\wedge$ as well as $\vee$ to all the remaining formulas at the cost of one level in quantifier complexity.

**Theorem 6.15.** *There are $\Pi_6$ formulas of $\mathcal{D}$ in the language with just $\leq$ that define the relations $\mathbf{x}'' \leq \mathbf{y}''$, $\mathbf{x} \in \mathbf{L}_2(\mathbf{y})$ and $\mathbf{x}'' = \mathbf{y}''$ in any jump ideal containing $\mathbf{0}^{(\omega)}$, a $\Sigma_7\&\Pi_7$ one that defines $\mathbf{w} = \mathbf{x}''$ and a $\Pi_9$ one that defines the relation $\mathbf{w} = \mathbf{x}'$ in any such ideal.*

# 7 Questions

Before turning to the major topic of this paper, the definability of the jump operator, we would like to point out an interesting class of problems raised by the introduction of our new hierarchies of generalized high and low classes in Definition 2.5. There are clearly many natural questions that these notions suggest. In general terms, one would want to know what properties of the usual generalized high and low hierarchies carry over to these new ones. These properties would include classifications by growth rates and structural properties of the degrees in the various classes.

Returning to our concern with definitions, perhaps someone so well steeped in the ways of the Turing degrees that the lattice and Slaman-Woodin coding procedures are second nature might been inclined to view our definitions as natural. At least in terms of invariance under automorphisms, one can dispense with the Slaman-Woodin coding apparatus. In this case, they simply say that one can determine various classes of degrees by the order types embeddable below them. (More precisely, they are determined by the finitely generated copies of independent degrees of order type $\omega$ along with an additional pair of degrees above some subset of these degrees.) In any case, there still seems room for a definition that the casual observer would see as natural. As we are perhaps already close to the border of the natural, it is even more difficult to make a precise claim as to what form of definition would fit the bill. There are, however, a couple of ways in which our results can be improved that do have precise measures.

The first is obviously the quantifier complexity of the definitions. Simpler is better and so we ask the following:

**Question 7.1.** Are there definitions of $\mathbf{L}_{2,}$ the double jump and the jump which are at lower levels of the alternating quantifier hierarchy than those established here?

The second way is the extent to which the definitions are local. Of course, a definition of the (double) jump can only make sense in jump ideals. (Individual instances such as a definition of $0'$ can make sense in arbitrary ideals.) Our results require just a bit more: the presence of the single degree $\mathbf{0}^{(\omega)}$. Thus we ask for the best possible results:

**Question 7.2.** Is there a formula that defines the relations $\mathbf{x}' = \mathbf{w}$ in every jump ideal? Is there a formula which defines the degree $\mathbf{0}'$ in every ideal containing it?

It seems reasonably likely that a definition that supplies positive answers to both questions will also be viewed by all as natural.

We have one suggestion for an approach for a minor improvement along the lines of the first of these questions. It is based on an alternate approach to a lattice coding of sets $\Sigma_3^X$ supplied by recursive enumerability.

**Theorem 7.3.** *If* $\mathbf{c} > \mathbf{b}$ *and* $\mathbf{c}$ *is r.e. in* $\mathbf{b}$ *then every* $\Sigma_3^C$ *set is coded below* $\mathbf{a}$.

*Proof.* Choose $\mathbf{a} \in (\mathbf{b}, \mathbf{c})$ r.e. in $\mathbf{b}$ with $\mathbf{a}' = \mathbf{b}'$. As described in Shore [1982, p. 262], the lattice $\mathcal{L}$ of Theorem 2.2 can be embedded in $[\mathbf{b}, \mathbf{a})$ with the images of $d_n$ becoming uniformly recursive in $\mathbf{a}$. By Shore [1981, Lemma 4.2], given any $X \in \Sigma_3^C$ there is an exact pair for the ideal generated by $\{\mathbf{d}_n | n \in X\}$ below $\mathbf{c}$. □

Thus we could hope to use a class of degrees every member of which bounds an r.e. degree as a stepping stone to a definition of the (double) jump. A natural definable candidate for such a class is $\mathcal{C}_\omega$ as it is known to contain all the n-REA degrees (Shore and Slaman [2001] but not known to contain any other degrees. (The 1-REA degrees are the r.e. degrees and the (n+1)-REA degrees are those that are REA in an n-REA degree. So all of them bound an r.e. degree.)

**Conjecture 7.4.** $\mathcal{C}_\omega$ *is the union over* $n \in \mathbb{N}$ *of the n-REA degrees and indeed for any* $\mathbf{a}$ $\mathcal{C}_\omega^{\mathbf{a}} = \{\{\mathbf{x} \geq \mathbf{a} | (\forall \mathbf{z} \geq \mathbf{a})(\mathbf{z} \vee \mathbf{x}$ *is not a minimal cover of* $\mathbf{z}\}$ *is the union of the degrees n-REA in* $\mathbf{a}$.

**Proposition 7.5.** *If this conjecture holds then we can define the double jump of* $\mathbf{h}$ *as* $\vee \mathbf{T}^{\mathbf{h}}$ *where* $\mathbf{T}^{\mathbf{h}} = \{\mathbf{x} \geq \mathbf{h} | (\exists \mathbf{z} \in \mathcal{C}_\omega^{\mathbf{h}})[(\forall \mathbf{u} < \mathbf{v} \leq \mathbf{z})(\mathbf{u} \geq \mathbf{h} \rightarrow \mathbf{u} \vee \mathbf{x} < \mathbf{v} \vee \mathbf{x})$ & $\forall \mathbf{a}, \mathbf{c}(Cd(\mathbf{x} \vee \mathbf{z}, \mathbf{a}, \mathbf{c}) \rightarrow (\forall \mathbf{y} \in \mathcal{C}_\omega^{\mathbf{h}})(\exists \tilde{\mathbf{a}}, \tilde{\mathbf{c}})(Cd(\mathbf{y}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})$ & $Eq(\mathbf{a}, \mathbf{c}, \tilde{\mathbf{a}}, \tilde{\mathbf{c}})]\}$. *Moreover, the relation* $\mathbf{x} \in \mathbf{T}^{\mathbf{h}}$ *is* $\Sigma_5$ *and so the resulting definition of the relation* $\mathbf{h}'' = \mathbf{w}$ *is* $\Pi_6$. *This then would give a* $\Pi_7$ *definition of* $\mathbf{x}' = \mathbf{w}$ *in the language with* $\leq$, $\vee$ *and* $\wedge$.

*Proof.* First, if $\mathbf{x} \in \mathbf{T}^{\mathbf{h}}$ let $\mathbf{z}$ be its witness and let $\mathbf{h} \leq \mathbf{u} < \mathbf{v} \leq \mathbf{z}$ be such that $\mathbf{v}$ is r.e. in $\mathbf{u}$. Thus $\mathbf{u} \vee \mathbf{x} < \mathbf{v} \vee \mathbf{x}$ and $\mathbf{v} \vee \mathbf{x}$ is r.e. in $\mathbf{u} \vee \mathbf{x}$. By the Theorem 7.3, every $\Sigma_3^{V \oplus X}$ set $S$ is coded below $\mathbf{v} \vee \mathbf{x}$. By the definition of $\mathbf{T}^{\mathbf{h}}$, $S$ is coded below every $\mathbf{y} \in \mathcal{C}_\omega^{\mathbf{h}}$. Now choose $\mathbf{y}$ to be REA in $\mathbf{h}$ with $\mathbf{h}' = \mathbf{y}'$. Thus $S \in \Sigma_3^Y = \Sigma_3^H$ and so $(\mathbf{v} \vee \mathbf{x})'' \leq \mathbf{h}''$ and so *a fortiori* $\mathbf{x}'' \leq_T \mathbf{h}''$.

On the other hand, consider any $\mathbf{z} \in \mathcal{C}_\omega^{\mathbf{h}}$ with $\mathbf{z}$ REA in $\mathbf{h}$ and $\mathbf{h}' = \mathbf{z}'$. Let $\mathbf{x}$ be any set 2-generic with respect to $\mathbf{z}$ so that $(\forall \mathbf{u} < \mathbf{v} \leq \mathbf{z})(\mathbf{u} \geq \mathbf{h} \rightarrow \mathbf{u} \vee \mathbf{x} \vee \mathbf{h} < \mathbf{v} \vee \mathbf{x} \vee \mathbf{h})$ and also $(\mathbf{x} \vee \mathbf{h} \vee \mathbf{z})'' = (\mathbf{x} \vee \mathbf{h})'' = \mathbf{z}''$. Thus every $S$ coded below $\mathbf{x} \vee \mathbf{h} \vee \mathbf{z}$ is $\Sigma_3^A$ and coded below every $\mathbf{y} \in \mathcal{C}_\omega^{\mathbf{h}}$ and so $\mathbf{x} \in \mathbf{T}$. As $\mathbf{z}'' = \mathbf{h}''$ is the join of two degrees $\mathbf{x}_1$ and $\mathbf{x}_2$ each 2-generic with respect to $\mathbf{z}$, both $\mathbf{x}_1 \vee \mathbf{h}$ and $\mathbf{x}_2 \vee \mathbf{h}$ are in $\mathbf{T}^{\mathbf{h}}$ and so $\mathbf{h}'' = \vee \mathbf{T}^{\mathbf{h}}$ as desired.

We leave the quantifier counting as an exercise given the information already in §6. □

A very interesting problem is to attack this issue from the other end and put lower bounds on the complexity of such definitions. Some beginnings to this approach are in Lerman and Shore [1988] who show, for example, that no nonzero degree has a $\Sigma_2$ definition in $\mathcal{D}$. We offer one more such example in the style of that paper that shows that, for at least one of our definitions, the gap between what we have and what is possible is not too large. By Lemma 6.12 $\mathbf{L}_2 = \{\mathbf{x} | \mathbf{x}'' = \mathbf{0}''\} = \{\mathbf{x} | \forall \mathbf{v}(\mathbf{x}'' \leq \mathbf{v}'')\}$ is $\Pi_6$ in the language with just $\leq$. We show that it is neither $\Pi_2$ nor $\Sigma_2$.

**Proposition 7.6.** *There is no $\Pi_2$ or $\Sigma_2$ definition in $\mathcal{D}$ (with just $\leq$) of $\mathbf{L}_2$ (or of $\mathbf{L}_2(\mathbf{y})$ by relativization).*

*Proof.* Suppose, for the sake of a contradiction to the assumed existence of a $\Pi_2$ definition of $\mathbf{L}_2$, that $\mathbf{x} \in \mathbf{L}_2 \Leftrightarrow \forall \tilde{\mathbf{v}} \exists \tilde{\mathbf{u}} \theta(\mathbf{x}, \tilde{\mathbf{v}}, \tilde{\mathbf{u}})$. Choose a minimal degree $\mathbf{x}_1 \in \mathbf{L}_2$ such that there are extensions of $[\mathbf{0}, \mathbf{x}]$ to initial segments realizing all possible uppersemilattices of size at most $2^{|\tilde{\mathbf{v}}|+\mathbf{2}}$ with $\mathbf{x}$ as a minimal element. (Clearly there is a finite lattice containing copies of each such extension of a single minimal element as an initial segment. Choose a realization of this lattice inside $\mathbf{L}_2$ by Lerman [1983] and take the degree corresponding to the single distinguished minimal element as $\mathbf{x}$.) Now choose an $\mathbf{x}_2 \notin \mathbf{L}_2$ and witnesses $\tilde{\mathbf{v}}_2$ such that $\neg \forall \tilde{\mathbf{v}} \exists \tilde{\mathbf{u}} \theta(\mathbf{x}_2, \tilde{\mathbf{v}}, \tilde{\mathbf{u}})$. Consider the uppersemilattice $\mathcal{U}$ generated in $\mathcal{D}$ by $\mathbf{x}_2$ and $\tilde{\mathbf{v}}_2$. Now take a realization of $\mathcal{U}$ as an initial segment extending $\mathbf{x}_1$ and let its elements corresponding to $\tilde{\mathbf{v}}_2$ be $\tilde{\mathbf{v}}_1$. As $\mathbf{x}_1 \in \mathbf{L}_2$, there are degrees $\tilde{\mathbf{u}}_1$ such that $\theta(\mathbf{x}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{u}}_1)$. As $\tilde{\mathbf{v}}_1$ generates an usl initial segment, the degrees in $\tilde{\mathbf{u}}_1$ consist of members of this initial segment (i.e. joins of some of the $\tilde{\mathbf{v}}_1$) plus an end extension of it. For those $i$ such that $\mathbf{u}_{1,i}$ is a join of elements of $\tilde{\mathbf{v}}_1$ take as $\mathbf{u}_{2,i}$ the corresponding join of elements of $\tilde{\mathbf{v}}_2$. All the others form a partial order end extension of the initial segment realization of $\mathcal{U}$. By the usual Kleene-Post construction, there is an isomorphic end extension in $\mathcal{D}$ of the usl generated by $\mathbf{x}_2$ and $\tilde{\mathbf{v}}_2$. Use such an extension to define the remaining $\mathbf{u}_{2,i}$ and note that we now have constructed witnesses $\tilde{\mathbf{u}}_2$ such that $\theta(\mathbf{x}_2, \tilde{\mathbf{v}}_2, \tilde{\mathbf{u}}_2)$ for the desired contradiction.

To consider a proposed $\Sigma_2$ definition of $\mathbf{L}_2$ and so a $\Pi_2$ one $\forall \tilde{\mathbf{v}} \exists \tilde{\mathbf{u}} \theta(\mathbf{x}, \tilde{\mathbf{v}}, \tilde{\mathbf{u}})$ for $\bar{\mathbf{L}}_2$, just interchange the roles of $\mathbf{L}_2$ and $\bar{\mathbf{L}}_2$ by choosing $\mathbf{x}_1$ and its extensions as initial segments of $\mathcal{D}$ not in $\mathbf{L}_2$ (e.g. not below $\mathbf{0}''$). Now choose $\mathbf{x}_2 \in \mathbf{L}_2$ and appropriate witnesses $\tilde{\mathbf{v}}_2$ such that $\neg \forall \tilde{\mathbf{v}} \exists \tilde{\mathbf{u}} \theta(\mathbf{x}_2, \tilde{\mathbf{v}}, \tilde{\mathbf{u}})$ and continue on as above to get witnesses $\tilde{\mathbf{u}}_2$ such that $\theta(\mathbf{x}_2, \tilde{\mathbf{v}}_2, \tilde{\mathbf{u}}_2)$ for the desired contradiction. □

Finally, the major global fact about $\mathcal{D}$ not addressed by our results is that it has at most countably many automorphisms (Slaman and Woodin [2008]). We ask for a local version:

**Question 7.7.** Does every jump ideal of $\mathcal{D}$ have at most countable many automorphisms?

We note that a positive answer would have Slaman and Woodin's result as a corollary. First a lemma.

**Lemma 7.8.** *The following conditions are equivalent:*

1. There are at most countably many automorphisms of $\mathcal{D}$.

2. There is a countable automorphism base for $\mathcal{D}$, i.e. a countable set $A$ of degrees such that any two automorphisms of $\mathcal{D}$ that agree on $A$ are identical.

3. There is a finite automorphism base for $\mathcal{D}$.

*Proof.* (1) $\Rightarrow$ (2): Suppose $\{\varphi_i | i \in \mathbb{N}\}$ lists all the automorphisms of $\mathcal{D}$. Choose $\mathbf{d}_{i,j}$ for $i \neq j$ such that $\varphi_i(\mathbf{d}_{i,j}) \neq \varphi_j(\mathbf{d}_{i,j})$. Clearly $\{\mathbf{d}_{i,j}\}$ is an automorphism base since any two distinct automorphisms must be $\varphi_i$ and $\varphi_j$ for some $i \neq j$ and so differ on $\mathbf{d}_{i,j}$.

(2) $\Rightarrow$ (3): As pointed out in Odifreddi and Shore [1991], Slaman-Woodin coding shows that the restriction of any automorphism $\varphi$ to the degrees below any $\mathbf{x}$ is determined by the action of $\varphi$ on finitely many parameters which are arithmetic in $\mathbf{x}$.

(3) $\Rightarrow$ (1): Any automorphism takes the members of the finite base to degrees of the same arithmetic degree by Jockusch and Shore [1984] (indeed to ones below their join with $\mathbf{0}''$ as all degrees above $\mathbf{0}''$ are fixed under all automorphisms). Thus there are only countably many possible images for this base and so only countably many automorphisms. $\square$

**Proposition 7.9.** *If every countable jump ideal which is sufficiently large (i.e. contains some fixed degree* $\mathbf{h}$*) or sufficiently closed (i.e. closed under the* $\omega$*-jump or any other fixed function* $F$ *on* $\mathcal{D}$*) has at most countably many automorphisms then so does* $\mathcal{D}$.

*Proof.* As we know, every automorphism of $\mathcal{D}$ restricts to an automorphism of each jump ideal $\mathcal{I}$. Now note that if a countable jump ideal $\mathcal{I}$ (containing $\mathbf{h}$ or closed under $F$) is not an automorphism base and $\varphi$ is an automorphism of $\mathcal{D}$ then there is another automorphism $\psi$ of $\mathcal{D}$ that agrees with $\varphi$ on $\mathcal{I}$: If $\rho$ and $\tau$ are any distinct automorphisms that agree on $\mathcal{I}$, then $\rho^{-1}\tau\varphi$ is the desired $\psi$. Next we can choose a degree $\mathbf{d}$ on which $\varphi$ and $\psi$ differ. With this procedure as the basic step and the assumption that no countable ideal is an automorphism base, we can start with $\mathcal{I}_{-1} = \mathcal{I}$ and any automorphisms $\varphi = \varphi_{\emptyset}$ and build a binary tree with a degree $\mathbf{d}_\sigma$ at each node and a nested sequence of jump ideals $\mathcal{I}_n$ (closed under $F$) for each level $n$ of the tree such that, for every string $\sigma$, $\mathbf{d}_\sigma \in \mathcal{I}_{|\sigma|}$ and there are automorphisms $\varphi_\sigma$ of $\mathcal{D}$ such that $\varphi_{\sigma\hat{}0}$ and $\varphi_{\sigma\hat{}1}$ differ at $\mathbf{d}_\sigma$ but agree on $\mathcal{I}_{|\sigma|-1}$. Now consider the jump ideal $\mathcal{I}_\omega = \cup\mathcal{I}_n$ which contains $\mathbf{h}$ (and is closed under $F$). For every $P \in 2^\omega$ the function $\cup\{\varphi_\sigma \restriction \mathcal{I}_{|\sigma|-1} | \sigma \subset P\}$ is an automorphisms of $\mathcal{I}_\omega$ and they are all distinct for the desired contradiction to our assumption that no such countable jump ideal is an automorphism base. Thus, by the Lemma, there are at most countably many automorphisms of $\mathcal{D}$. $\square$

# 8 Bibliography


Cooper, S. B. [1990], The jump is definable in the structure of the degrees of unsolvability (research announcement), *Bull. Am. Math. Soc.* **23**, 151-158.

Cooper, S. B. [1993], On a conjecture of Kleene and Post, *University of Leeds, Department of Pure Mathematics Preprint Series* No. 7.

Cooper, S. B. [2001], On a conjecture of Kleene and Post. *Math. Log. Q.* **47**, 3–33.

Downey, R. G., Jockusch, C. G. Jr. and Stob, M. [1990], Array nonrecursive sets and multiple permitting arguments, in *Recursion Theory Week*, K. Ambos-Spies, G. H. Müller and G. E. Sacks eds., Springer-Verlag, Berlin, 141-174.

Greenberg, N. and Montalbán, A. [2004], Embedding and coding below a 1-generic degree, *Notre Dame J. Formal Logic* **44**, 200-216.

Harrington, L. and Kechris, A. S. [1975], A basis result for $\Sigma^0_3$ sets of reals with an application to minimal covers, *Proc. Am. Math. Soc.* **53**, 445-448

Harrington, L. and Shore, R. A. [1981], Definable degrees and automorphisms of $\mathcal{D}$, *Bul. Am. Math. Soc. (NS)* **4**, 97-100.

Jockusch, C. G. Jr. [2002], review of Cooper [2001], Math. Rev. MR1808943 (2002m:03061).

Jockusch, C. G. Jr. and Shore, R. A. [1984], Pseudo-jump operators II: transfinite iterations, hierarchies and minimal covers, *J. Symb. Logic* **49**, 1205-1236.

Jockusch, C. G. Jr. and Simpson, S. G. [1976], A degree theoretic definition of the ramified analytic hierarchy, *Ann. Math. Logic* **10**, 1-32.

Jockusch, C. G. Jr. and Soare, R. I. [1970], Minimal covers and arithmetical sets, *Proc. Amer. Math. Soc.* **25**, 856-859.

Jockusch, C. G. Jr. and Solovay, R. M., [1977], Fixed points of jump preserving automorphisms of degrees, *Israel J. Math.* **26**, 91-94.

Jónsson, B. [1953] On the representations of lattices, *Math. Scand.* 1, 193-206.

Kleene, S. C. and Post, E. L. [1954] The upper semi-lattice of degrees of recursive unsolvability, *Ann. of Math.* (2) **59**, 379–407.

Lachlan, A. [1966], Lower bounds for pairs of recursively enumerable degrees, *Proc. Lon. Math. Soc.* **16**(3), 53-569.

Lerman, M. [1971], Initial segments of the degrees of unsolvability, *Ann. Math.* **93**, 365-89.

Lerman, M. [1983], *Degrees of Unsolvability*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin.

Lerman, M. and Shore, R. A. [1988], Decidability and invariant classes for degree structures, *Trans. Am. Math. Soc.* **310**, 669-692.

Nies, A., Shore, R. A. and Slaman, T. [1998], Interpretability and definability in the recursively enumerable degrees, *Proceedings of the London Mathematical Society* (3) **77**, 241-291.

Odifreddi, P. and Shore, R. A. [1991], Global properties of local degree structures, *Bul. U. Mat. Ital.* **7**, 97-120

Sacks, G. E. [1963], Recursive enumerability and the jump operator, *Trans. Am. Math. Soc.* **108**, 223-239.

Selman, A. L. [1972], Applications of forcing to the degree theory of the arithmetic hierarchy, *Proc. Lon. Math. Soc.* **25**,586-602.

Shore, R. A. [1981], The theory of the degrees below $\mathbf{0}'$, *J. London Math. Soc.* (3) **24** (1981), 1-14.

Shore, R. A. [1982], Finitely generated codings and the degrees r.e. in a degree **d**, *Proc. Am. Math. Soc.* **84**, 256-263.

Shore, R. A. [2006], Degree Structures: Local and Global Investigations, *Bulletin of Symbolic Logic*, **12**, 369-389

Shore, R. A. [2008], Rigidity and biinterpretability in the hyperdegrees, in *Computational Prospects of Infinity, Part II: Presented Talks*, C. T. Chong, F. Qi and Y. Yang eds., Lecture Note Series, Institute for Mathematical Sciences, National University of Singapore **15,** World Scientific Publishing Co., Singapore.

Shore, R. A. and Slaman, T. A. [1999], Defining the Turing jump, *Math. Research Letters* **6** (1999), 711-722.

Shore, R. A. and Slaman, T. A. [2001], A splitting theorem for $n$-REA degrees, *Proc. Am. Math. Soc.* **129**, 3721-3728.

Simpson, S. G. [1977], First order theory of the degrees of recursive unsolvability, *Ann. Math. (2)*, **105**, 121-139.

Slaman, T. A. [1991], Degree structures, in *Proc. Int. Cong. Math., Kyoto 1990*, Springer-Verlag, Tokyo, 303-316.

Slaman, T. A. and Woodin, H. W. [1986], Definability in the Turing degrees, *Illinois J. Math.* **30**, 320-334.

Slaman, T. A. and Woodin, H. W. [2008], Definability in degree structures, *J. Math. Logic,* to appear.