# Jumps of $\Sigma^0_2$-high e-degrees and properly $\Sigma^0_2$ e-degrees

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**Abstract**

We show that the $\Sigma^0_2$ high e-degrees coincide with the high e-degrees. We also show that not every properly $\Sigma^0_2$ e-degree is high.

## 1 Introduction

Enumeration reducibility is the notion of relative enumerability of sets: a set $A$ is *enumeration reducible* (or simply *e-reducible*) to a set $B$, in symbols, $A \leq_e B$, if there is an effective procedure for enumerating $A$ given *any* enumeration of $B$. Formally, we define $A \leq_e B$ if there is some computably enumerable set $\Phi$ (called in this context an *enumeration operator* or simply an *e-operator*) such that

$$A = \{x : (\exists \text{ finite } D)[\langle x, D\rangle \in \Phi \,\&\, D \subseteq B]\}$$

(throughout the paper we identify finite sets with their canonical indices). We denote by $\equiv_e$ the equivalence relation generated by the preordering relation $\leq_e$ and $\deg_e(A)$ denotes the equivalence class (or the *e-degree*) of $A$. The partially ordered structure of the e-degrees is denoted by $\mathfrak{D}_e$; its partial ordering is denoted by $\leq$. $\mathfrak{D}_e$ is, in fact, an upper semilattice with least element $\mathbf{0}_e$.

One of the most interesting features of the e-degrees is that they extend the structure $\mathfrak{D}_T$ of the Turing degrees ([Med55] and [Rog67]). Indeed if we define $\iota : \mathfrak{D}_T \to \mathfrak{D}_e$ by $\iota(\deg_T(A)) = \deg_e(\chi_A)$ (where $\deg_T(A)$ is the Turing

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degree of the set $A$ and $\chi_A$ denotes the characteristic function of $A$) then $\iota$ is the desired embedding. (In fact, it preserves joins and least element.) The verification that $\iota$ is well defined relies on the following lemma:

**Lemma 1.1** *For every total function $f$ and $g$, we have*

$$f \leq_T g \Leftrightarrow f \leq_e g.$$

**Proof.** See e.g. [Rog67, p. 153]. □

One can define (see below) a jump operation $'$ on the e-degrees, and therefore introduce the notions of a *low* e-degree (i.e. an element of the class $\mathbf{L}_1 = \{\mathbf{a} \leq \mathbf{0}'_e : \mathbf{a}' = \mathbf{0}'_e\}$); and that of a *high* e-degree (i.e. an element of $\mathbf{H}_1 = \{\mathbf{a} \leq \mathbf{0}'_e : \mathbf{a}' = \mathbf{0}''_e\}$). Moreover, since $\iota$ preserves jump, we have that low Turing degrees are mapped to low e-degrees and high Turing degrees are mapped to high e-degrees.

A nice, useful characterization of the class $\mathbf{L}_1$ of the low e-degrees is given in [MC85]: $\mathbf{a} \in \mathbf{L}_1$ if and only if $\mathbf{a}$ contains a set $A$ such that, for every $B \leq_e A$, $B \in \Delta^0_2$ . Thus $\mathbf{a}$ and all the e-degrees below $\mathbf{a}$ consist entirely of $\Delta^0_2$ sets. In this paper, we characterize the class $\mathbf{H}_1$ of high e-degrees by a result analogous to the one characterizing the high Turing degrees as those containing a set with an approximation whose associated computation function dominates every total recursive function (Theorem 2.1). The relevant definitions of a $\Sigma^0_2$ approximation and a computation function are given below. The e-degrees of sets with such approximations are known as the $\Sigma^0_2$-high e-degrees (Definition 1.7). This characterization answers Question 7.3 of [Coo90].

Since the e-degrees below $\mathbf{0}'_e$ are exactly the e-degrees consisting of $\Sigma^0_2$ sets, in view of the above cited characterization of the low e-degrees, a natural question to ask is where an e-degree $\mathbf{a} \leq \mathbf{0}'_e$ which contains no $\Delta^0_2$ set (such an e-degree is called a *properly* $\Sigma^0_2$ e-degree) lies in the low/high hierarchy. A natural conjecture might be that the properly $\Sigma^0_2$ e-degrees are all in $\mathbf{H}_1$. Cooper and Copestake ([CC88]) show that there exist properly $\Sigma^0_2$ e-degrees that are $\Sigma^0_2$-high, and thus lie in $\mathbf{H}_1$ by Theorem 2.1. However, in Theorem 3.1 we show that the properly $\Sigma^0_2$ e-degrees are not contained in $\mathbf{H}_1$.

Our notations and terminology are mostly based on [Soa87]. The reader is referred to [Coo90] for an introduction and extensive bibliography on enumeration reducibility. We will be mostly working with $\Sigma^0_2$ sets. We recall that a $\Sigma^0_2$ *approximation* to a $\Sigma^0_2$ set $A$ is computable sequence of computable sets $\{A^s : s \in A^s\}$ such that $A = \{x : (\exists t)(\forall s \geq t)[x \in A^s]\}$. See [LS92] for an introduction to $\Sigma^0_2$ approximations, and for a proof that every $\Sigma^0_2$ set has a *good* $\Sigma^0_2$ approximation $\{A^s : s \in A^s\}$, i.e. a computable sequence of computable (in fact, finite) sets such that $\{s : A^s \subseteq A\}$ is infinite.

Let $X$ be any set of natural numbers; if $x$ is a number, then $X^{[x]} = \{z \in X : (\exists y)[z = \langle x, y\rangle]\}$, and $X \upharpoonright x = \{y \in X : y < x\}$. If $\sigma$ is a string and $x < |\sigma|$ (where $|\sigma|$ denotes the length of $\sigma$), then $\sigma \upharpoonright x$ denotes the initial segment of $\sigma$ having length $x$; likewise, if $f$ is a function, then $f \upharpoonright x$ denotes the initial segment of $f$ having length $x$.

Let $\{\varphi_i\}_{i\in\omega}$ be the standard enumeration of all partial computable functions with corresponding enumerations $\{W_i\}_{i\in\omega}$ and $\{\Phi_i\}_{i\in\omega}$ of the computably enumerable sets and the enumeration operators, respectively. Let us fix, as in [Soa87, p. 16], computable approximations $\{\varphi_{i,s}\}_{i,s\in\omega}$ to the partial computable functions. Without loss of generality, we may assume that if $\varphi_{i,s}(x)\downarrow$ then $\varphi_{i,s}(x) < s$. Correspondingly, we get computable finite approximations $\{W_{i,s}\}_{i,s\in\omega}$ and $\{\Phi_{i,s}\}_{i,s\in\omega}$ to the computably enumerable sets and the enumeration operators, respectively.

Let

$$K^A = \{x : x \in \Phi_x^A\}.$$

**Lemma 1.2** *[McE85] Let $A, B$ be sets; then*

$$A \leq_e B \Leftrightarrow A \leq_1 K^B \Leftrightarrow K^A \leq_1 K^B.$$

Define the *jump* of a set $A$ ([McE85]) to be the set $J_e(A) = \chi_{K^A}$. (Note that we identify functions with their graphs.) Clearly $J_e(A) \equiv_e K^A \oplus \overline{K^A}$. If $\mathbf{a}$ is an e-degree, then we can define $\mathbf{a}'$ as $\deg_e(J_e(A))$ for any $A \in \mathbf{a}$ since, by the previous lemma,

$$A \equiv_e B \Rightarrow K^A \oplus \overline{K^A} \equiv_e K^B \oplus \overline{K^B}$$

this gives a well defined unary operation on the e-degrees. Moreover, $\mathbf{a} < \mathbf{a}'$ for every e-degree $\mathbf{a}$.

The following lemma records two important properties of the jump operation.

**Lemma 1.3** *[McE85] For every set $A$,*

1. $\iota((\deg_T(A))') = (\deg_e(A))'$;
2. *if $A$ is* total *(i.e. the graph of some total function), then* $J_e(A) \equiv_e \overline{K^A}$.

**Definition 1.4** A set $A$ is called *e-high* if $A \in \Sigma_2^0$ and $J_e^{(2)}(\emptyset) \leq_e J_e(A)$. An e-degree $\mathbf{a}$ is called *high*, if $\mathbf{a}$ contains an e-high set (hence $\mathbf{a}' = \mathbf{0}_e''$).

By Lemma 1.3, the embedding $\iota$ preserves highness, i.e. it maps high T-degrees to high e-degrees.

The following is a useful characterization of the e-high sets.

**Lemma 1.5** *For every set $A$,*

$$A \text{ is e-high } \Leftrightarrow Tot \leq_e J_e(A) \Leftrightarrow Tot \leq_T K^A.$$

**Proof.** First notice that, by Lemma 1.3, $A$ is e-high if and only if $\overline{K^{\overline{K}}} \leq_e J_e(A)$. Moreover, $A$ is e-high if and only if, by Lemma 1.3, $\chi_{\overline{K^{\overline{K}}}} \leq_e \chi_{K^A}$, if and only if, by Lemma 1.1, $\overline{K^{\overline{K}}} \leq_T K^A$. On the other hand, $\overline{K^{\overline{K}}} \equiv_1 Tot$. Indeed, $\overline{K^{\overline{K}}} \leq_1 Tot$ follows from the fact that $Tot$ is $\Pi^0_2$-complete. To show that $Tot \leq_1 \overline{K^{\overline{K}}}$ simply observe that $\overline{Tot} \in \Sigma^0_2$, hence $\overline{Tot} \leq_e \overline{K}$, and thus, by Lemma 1.2, $\overline{Tot} \leq_1 K^{\overline{K}}$.

It follows that $A$ is e-high if and only if $Tot \leq_e J_e(A)$ if and only if $Tot \leq_T K^A$. $\square$

Finally notice the following:

**Lemma 1.6** *For every total function $f$, $J_e(f) \equiv_T f'$.*

**Proof.** Since $K^f$ is computably enumerable in $f$, we have $K^f \leq_1 f'$, and so $\overline{K^f} \leq_T f'$. On the other hand, $f'$ is computably enumerable in $f$, hence, by the totality of $f$, $f' \leq_e f$, and thus $f' \leq_1 K^f$, by Lemma 1.2, which shows that $f' \leq_1 J_e(f)$.

Hence we conclude that $J_e(f) \equiv_T f'$. $\square$

McEvoy ([McE85]) defines the notion of a $\Sigma^0_2$-high e-degree:

**Definition 1.7** A $\Sigma^0_2$-*high* approximation $\{A^s\}_{s\in\omega}$ to a set $A$ is a $\Sigma^0_2$ approximation such that the function

$$c(x) = \mu s(s > x \;\&\; A^s \restriction x \subseteq A)$$

(called the *computation function for $A$* relative to the given approximation) is total and dominates every computable function. A set $A$ is called $\Sigma^0_2$-*high*, if it has a $\Sigma^0_2$-high approximation. Finally an e-degree is said to be $\Sigma^0_2$-*high*, if it contains a $\Sigma^0_2$-high set.

Using Lemma 1.3 (1), it is shown in [McE85] that if $\mathbf{a} \leq \mathbf{0}'_e$ is total (hence a degree of the form $\iota(\mathbf{b})$, for some Turing degree $\mathbf{b} \leq_T \mathbf{0}'_T$), and $\mathbf{a}$ is high, then $\mathbf{a}$ is $\Sigma^0_2$-high. On the other hand, the class of $\Sigma^0_2$-high e-degrees properly extends the class of all high total e-degrees (in fact, there exist quasi-minimal $\Sigma^0_2$-high e-degrees, [McE85]).

In the next section we show that the high $e$-degrees and the $\Sigma^0_2$-high e-degrees coincide. This answers Question 7.3 of [Coo90].

## 2 The $\Sigma^0_2$-high e-degrees coincide with the high e-degrees

**Theorem 2.1** *An e-degree* $\mathbf{a}$ *is* $\Sigma^0_2$*-high if and only if* $\mathbf{a}$ *is high.*

**Proof.** ($\Rightarrow$) Let $A$ be $\Sigma^0_2$-high; let $\{A^s\}_{s\in\omega}$ be a $\Sigma^0_2$-high approximation to $A$, and let $c$ be the computation function for $A$ relative to this approximation.

By Lemma 1.5 it is enough to show that $Tot \leq_e J_e(A)$. We claim that, for every $i$,

$$\begin{aligned} i \in Tot &\Leftrightarrow (\forall y)[\varphi_i(y)\downarrow] \\ &\Leftrightarrow (\exists x)(\exists s)[(\forall y < x)\varphi_{i,s}(y)\downarrow \ \& \\ &\qquad (\forall y \geq x)(\forall t > y)[A^t \restriction y \subseteq A \Rightarrow \varphi_{i,t}(y)\downarrow]]. \end{aligned}$$

Indeed, if $\varphi_i$ is total, then the function

$$\hat{\varphi}_i(x) = \mu s(\varphi_{i,s}(x)\downarrow)$$

is a total computable function, and thus dominated by $c$. Let $x$ be such that $c(y) > \hat{\varphi}_i(y)$ for every $y \geq x$; then

$$(\forall y \geq x)(\forall t > y)[A^t \restriction y \subseteq A \Rightarrow \varphi_{i,t}(y)\downarrow].$$

This establishes the left-to-right implication in the claimed equivalence. The right-to-left implication is trivial.

Now let

$$B = \{\langle i, x\rangle : (\exists y \geq x)(\exists t > y)[A^t \restriction y \subseteq A \ \& \ \varphi_{i,t}(y)\uparrow].$$

Clearly $B \leq_e A$. Hence, by Lemma 1.2, $B \leq_1 K^A$, via, say, the computable function $h$.

Then

$$\begin{aligned} i \in Tot &\Leftrightarrow (\exists x)(\exists s)[(\forall y < x)\varphi_{i,s}(y)\downarrow \ \& \ \langle i, x\rangle \notin B] \\ &\Leftrightarrow (\exists x)(\exists s)[(\forall y < x)\varphi_{i,s}(y)\downarrow \ \& \ h(\langle i, x\rangle) \in \overline{K^A}]. \end{aligned}$$

It follows that $Tot \leq_e \overline{K^A}$, hence $Tot \leq_e J_e(A)$, as desired.

($\Leftarrow$) Assume that $A$ is e-high. Then, by Lemma 1.5, we have that $Tot \leq_T K^A$. Let $Z = K^A$ and let $\{Z^s\}_{s\in\omega}$ be a good $\Sigma^0_2$-approximation to $K^A$. Let $\psi$ be some Turing functional such that $Tot = \psi^Z$.

We now define an enumeration operator $\Theta$ by stages, and show that $\Theta^Z$ is $\Sigma^0_2$-high. Since the $\Sigma^0_2$-high e-degrees are closed upwards in $\Sigma^0_2$-e-degrees (see [BCS97]), it follows that $\deg_e(A)$ is $\Sigma^0_2$-high as well.

**Construction:**

Stage 0: Let $\Theta_0 = \emptyset$.

Stage $s+1$: For every $i \leq s$, we distinguish the following two cases:

(a) if $\psi_s^{Z^s}(i) = 0$, then for every $x \leq s$, we enumerate the axiom

$$\langle\langle i, x\rangle, Z^s\rangle \in \Theta_{s+1};$$

(b) otherwise, do nothing.

Let $\Theta^{s+1}$ consist of $\Theta^s$ plus all the axioms enumerated at stage $s+1$ and let $\Theta = \bigcup_{s\in\omega} \Theta_s$. We now prove that $\Theta$ is the desired enumeration operator by verifying a series of claims.

**Verifications:**
**Claim 1** For every number $i$,

$i \in Tot \Rightarrow (\Theta^Z)^{[i]}$ finite;

$i \notin Tot \Rightarrow (\Theta^Z)^{[i]} = \omega^{[i]}$.

**Proof.** If $i \in Tot$ then at all sufficiently large good stages of the approximation $\{Z^s\}_{s\in\omega}$ we do nothing on behalf of $i$ (i.e. case (b) of the definition of $\Theta$ applies to $i$). To see this, assume that $i \in Tot$, and let $\sigma \subset \chi_Z$ be such that $\psi^\sigma(i) = 1$. Let $t_0$ be such that

$$(\forall x < |\sigma|)[\sigma = 1 \Rightarrow (\forall s \geq t_0)[x \in Z^s]].$$

Then

$$(\forall s \geq t_0)[s \text{ is good } \Rightarrow \psi_s^{Z^s}(i) = 1].$$

It follows that at stages $s \geq t_0$, if (a) holds then $s$ is not good, hence $\Theta_s^{Z^s} \nsubseteq \Theta^Z$. Therefore $(\Theta^Z)^{[i]}$ is finite.

If $i \notin Tot$, then at all sufficiently large good stages, case (a) of the definition of $\Theta$ applies to $i$. Indeed, if $i \notin Tot$, then $\psi^Z(i) = 0$. One thus argues as in the preceding case, but starting with a string $\sigma \subset \chi_Z$ such that $\psi^\sigma(i) = 0$.

Since $\Theta_s^{Z_s} \subseteq \Theta^Z$, for all good stages $s$, it follows in this case that $(\Theta^Z)^{[i]} = \omega^{[i]}$. □

Now let $Y = \Theta^Z$. We want to show that $Y$ has a $\Sigma_2^0$-high approximation $\{\hat{Y}^s\}_{s\in\omega}$. Let $\{Y^s\}_{s\in\omega}$ be any good $\Sigma_2^0$-approximation to $Y$. Given a partial function $\varphi$ and a number $u$, define $\varphi \upharpoonright u \downarrow$, if $\varphi(v)\downarrow$ for all $v < u$.

Define

$$\langle i, x\rangle \in \hat{Y}^s \Leftrightarrow [\langle i, x\rangle \in Y^s \vee \varphi_{i,s} \upharpoonright \langle i, x+1\rangle \uparrow].$$

**Claim 2** $\{\hat{Y}^s\}_{s\in\omega}$ is a $\Sigma_2^0$-approximation to $Y$.
**Proof.** Let $\hat{Y} = \{y : (\exists t)(\forall s \geq t)[y \in \hat{Y}^s]\}$. If $i \notin Tot$ then $Y^{[i]} = \omega^{[i]} = \hat{Y}^{[i]}$. On the other hand, assume that $i \in Tot$. If $\langle i, x\rangle \in Y$, then clearly $\langle i, x\rangle \in \hat{Y}$.

If $\langle i, x\rangle \notin Y$, then at all sufficiently large stages $\varphi_{i,s} \upharpoonright \langle i, x+1\rangle \downarrow$ and so when $\langle i, x\rangle \notin Y^s$, we have that $\langle i, x\rangle \notin \hat{Y}^s$. $\Box$

Next, let $c$ be the computation function for $Y$ relative to the $\Sigma^0_2$ approximation $\{\hat{Y}^s\}_{s\in\omega}$. The following claim completes the proof of the Theorem.

**Claim 3** The function $c$ is total and dominates all total computable functions.
**Proof.** Let us first show that $c$ is total. To this end, let $z \in \omega$ be given. Let $t > z$ be a stage such that

$$(\forall \langle i, x\rangle < z)[\varphi_i \upharpoonright \langle i, x+1\rangle \downarrow \Leftrightarrow \varphi_{i,t} \upharpoonright \langle i, x+1\rangle \downarrow].$$

Then if $s \geq t$ is a good stage of the enumeration $\{Y^s\}_{s\in\omega}$, we have that $\hat{Y}^s \upharpoonright z \subseteq Y$. Therefore $c(z)$ is defined.

Now consider any total $\varphi_i$. Let $x$ be such that $\langle i, y\rangle \notin Y$, for every $y \geq x$. Let $z \geq \langle i, x\rangle$, and let $y$ be the least number such that $\langle i, y\rangle \leq z < \langle i, y+1\rangle$. Let $s$ be the least stage such that $\varphi_{i,s}(z) \downarrow$, hence $\varphi_{i,t} \upharpoonright \langle i, y+1\rangle \uparrow$ for every $t < s$. Then $\langle i, y\rangle \in \hat{Y}^t$, for every $t < s$. Therefore $\varphi_i(z) < s \leq c(z)$. $\Box$

# 3 Jumps of properly $\Sigma^0_2$ e-degrees

A $\Sigma^0_2$ e-degree $\mathbf{a}$ is called *properly* $\Sigma^0_2$ ([CC88]) if $\mathbf{a}$ contains no $\Delta^0_2$ set. Copestake and Cooper, [CC88, Theorem 1], show that there exist e-degrees that are properly $\Sigma^0_2$ and $\Sigma^0_2$-high. Since every high computably enumerable Turing degree corresponds, under the embedding $\iota$, to a high e-degree, it follows that not every $\Sigma^0_2$-high e-degree is properly $\Sigma^0_2$-high. (A trivial counterexample is $\mathbf{0}'_e = \deg_e(\overline{K})$). It is shown in [MC85] that $\deg_e(A)$ is low if and only if $B \in \Delta^0_2$, for every $B \leq_e A$. This characterization of the low e-degrees seems to suggest the possibility that the properly $\Sigma^0_2$ e-degrees are all high. We show in this section that this is not the case.

**Theorem 3.1** *Let $C$ be such that $C$ is computably enumerable in $\emptyset'$, $\emptyset' \leq_T C <_T \emptyset''$ and $C' \equiv_T \emptyset'''$. Then there exists a set $A$ of properly $\Sigma^0_2$ e-degree, such that $J_e(A) \leq_e \chi_C$.*

**Corollary 3.2** *There exist properly $\Sigma^0_2$ e-degrees that are not high.*

**Proof.** Let $C$ and $A$ be as in the previous theorem. If $A$ were e-high, then $J^{(2)}_e(\emptyset) \leq_e \chi_C$, from which, by totality, $J^{(2)}_e(\emptyset) \leq_T \chi_C$; but $J^{(2)}_e(\emptyset) \equiv_T \emptyset''$, by Lemma 1.6. Hence $\emptyset'' \leq_T C$, contradiction. $\Box$

## 3.1 Proof of Theorem 3.1

Let $C$ satisfy the hypotheses of the theorem; let $C = W^K$, for some computably enumerable set $W$. For every $t$, let $\kappa_t = \chi_{K^t} \upharpoonright k(t)$, where $k$ is some $1-1$ computable function such that $K = range(k)$ and $K^t = \{k(s)|s \leq t\}$. Define a $\Sigma^0_2$ approximation $\{C^t\}_{t\in\omega}$ to $C$ by letting

$$C^t = W_t^{\kappa_t},$$

As $C' \equiv_T K''$, there is an $f \leq_T C$ that dominates all $\Delta^0_2$ total functions (see e.g. [Ler83, p. 85]). Let $f = \Psi^C$, for some Turing functional $\Psi$, be such a function.

We need the following lemma:

**Lemma 3.3** *There exists a computable sequence $\{B_i^s\}_{i,s\in\omega}$ of finite sets such that, if*

$$B_i = \{x : (\exists t)(\forall s \geq t)[x \in B_i^s]\}$$

*then*

1. *for every $B \in \Delta^0_2$, there is an $i$ such that $B =^* B_i$ and, for almost all $x$, $\lim_s B_i^s(x)$ exists;*
2. *the relation $x \in B_i$ (as one of $x$ and $i$) is computable in $C$.*

**Proof.** Given $u$ and $X$, with $X = K$, or $X = K^v$ for some $v \geq u$, we say that $u$ is $X$-*true* if $\kappa_u \subseteq \chi_X$. We will use the fact that for every $B \in \Delta^0_2$ there exists some $i$ such that $\chi_B = \varphi_i^K$. Roughly speaking, we will have $x \in B_i^s$ if there exists some $K^s$-true stage $t < w$, with $\varphi_{i,t}^{\kappa_t}(x) = 1$, where $w$ is the least $K^s$-true stage such that $\Psi_w^\sigma(x) \downarrow$, for some $\sigma \subset \chi_C$. Then we use the fact that $\Psi^C$ dominates all $\Delta^0_2$ functions to verify that, for all but finitely many $x$, there exists a $K$-true stage $t$ such that $\varphi_{i,t}^{\kappa_t}(x) = 1$ and $t < \Psi^C(x) < w$. The main difficulty here is that one can not find, in a computable way, the right $w$ at $s$. For every $i, x, s$, we will therefore define the values of a finite set $B_i^s \subseteq \omega$, a finite set $L(x,s) \subseteq \omega \times 2^{<\omega}$ and a linear ordering $<_{x,s}$ on $L(x,s)$. We "assign preconditions" to elements of $\omega \times 2^{<\omega}$ subject to the following rules: $L(x,s)$ may contain only pairs $\langle r, \rho\rangle$ with preconditions which have been satisfied at some stage $u \leq s$. At stage $s$, we will choose the $<_{x,s}$-first element $\langle r, \rho\rangle$ of $L(x,s)$. We will argue that infinitely many times we choose the correct $\langle w, \sigma\rangle$ and we eventually choose only pairs $\langle r, \rho\rangle$ with $r \geq w$.

Let $i, x$ be given. The formal definitions are given by induction on $s$.

Stage 0: Define $B_i^0 = \emptyset$ and $L(x, 0) =<_{x,0}= \emptyset$. No $\langle r, \rho\rangle$ has a precondition at 0.

Stage $s+1$: If $x \geq s+1$ then $x \notin B_i^{s+1}$; otherwise, we distinguish two cases:

- if $L(x,s) = \emptyset$, then

$$x \in B_i^{s+1} \Leftrightarrow x \in B_i^s;$$

- otherwise, let $\langle w, \sigma \rangle$ be the $<_{x,s}$-least element of $L(x,s)$. Then,

  (a) if there is no $t < w$ such that $t$ is $K^{s+1}$-true, then $x \notin B_i^{s+1}$;

  (b) otherwise, for the least such $t$, $x \in B_i^{s+1}$ if and only if $\varphi_{i,t}^{K_t}(x) = 1$.

In the latter case, i.e. when $L(x,s) \neq \emptyset$, we extract $\langle w, \sigma \rangle$ from $L(x, s+1)$ and cancel the related precondition. Hence $\langle w, \sigma \rangle$ has no precondition at any stage $v \geq s+1$ prior to the smallest stage $v' > s+1$ (if any) at which we again assign a precondition to $\langle w, \sigma \rangle$.

We *assign a precondition* to each pair $\langle r, \rho \rangle$ such that

1. $r$ is $K^{s+1}$-true;
2. $\Psi_r^\rho(x) \downarrow$;
3. $\rho \subseteq \chi_{C^r}$;
4. $\rho$ of minimal length, satisfying 2. and 3. (i.e. if $\Psi_r^{\rho'}(x) \downarrow$ and $\rho' \subseteq \chi_{C^r}$ then $\rho \subseteq \rho'$; notice that $|\rho'| < r$, for each such $\rho'$, by the definition of the use function as in [Soa87, p. 49]);
5. $\langle r, \rho \rangle$ does not have a precondition at $s+1$.

At any $v > s+1$, we say that this precondition becomes *satisfied at* $v$ if

$$(\forall i < |\rho|)[\rho(i) = 0 \Rightarrow (\exists t)[s+1 \leq t \leq v \,\&\, i \notin C^t]].$$

Let

$$L(x, s+1) = (L(x,s) - \{\langle w, \sigma \rangle\}) \cup \\ \{\langle r, \rho \rangle : \langle r, \rho \rangle \text{ has a precondition that becomes satisfied at } s+1\}.$$

Finally, we order $L(x, s+1)$ as follows: if $\langle r, \rho \rangle, \langle r', \rho' \rangle \in L(x, s+1)$, then let $\langle r, \rho \rangle <_{x,s+1} \langle r', \rho' \rangle$ if either

1. $\langle r, \rho \rangle, \langle r', \rho' \rangle \in L(x,s)$ and $\langle r, \rho \rangle <_{x,s} \langle r', \rho' \rangle$, or
2. $\langle r, \rho \rangle \in L(x,s)$ and $\langle r', \rho' \rangle \notin L(x,s)$, or
3. $\langle r, \rho \rangle, \langle r', \rho' \rangle \notin L(x,s)$ and $r < r'$.

We now check that the sequence $B_i^s$ has the desired properties.

**Claim** Let $x$ be given, let $\sigma$ be the least string such that $\sigma \subset \chi_C$ and $\Psi^\sigma(x)\downarrow$. Let $w$ be the least $K$-true stage such that $\Psi_w^\sigma(x)\downarrow$. Then

1. at infinitely many stages $s$ we extract $\langle w, \sigma\rangle$ from $L(x,s)$;
2. there exists a stage $t_0$ such that we do not extract any pair $\langle r, \rho\rangle$ with $r < w$ from $L(x,s)$ at any stage $s \geq t_0$.

**Proof.** Since $\sigma \subset \chi_C$, it is clear that there are infinitely many stages at which the requirements (1-4) for assigning a precondition to $\langle w, \sigma\rangle$ are fulfilled. Moreover, once assigned at a stage $s_0$, there exists a stage $s_1 > s_0$ such that the precondition becomes satisfied at $s_1$ and so is then in $L(x,s)$ until extracted. As there are only finitely many elements of $L(x,s_1)$ before $\langle w,\sigma\rangle$ in the ordering and no new ones can later be inserted before it $\langle w,\sigma\rangle$ is eventually extracted. Hence, there exist infinitely many stages $s$ such that $\langle w,\sigma\rangle \in L(x,s)$ and we extract $\langle w,\sigma\rangle$ from $L(x,s)$ at infinitely many stages.

Let $t < w$, and assume for a contradiction that at infinitely many stages $s$, we extract $\langle t, \rho_s\rangle$, for some string $\rho_s$. Thus $|\rho_s| < t$ by the definition of the use function, since $\Psi_t^{\rho_s}(x)\downarrow$. Then there exist a $\rho$, with $|\rho| < t$, and infinitely many stages $u_s$ at which we assign a precondition to $\langle t,\rho\rangle$ which becomes satisfied at some stage $v_s \leq s$ and $\langle t,\rho\rangle \in L(x,v)$ for every $v$ such that $v_s \leq v \leq s$. Then $t$ is $K$-true. Let $t_0$ be a stage such that

$$(\forall s \geq t_0)(\forall i < t)[i \in C \Rightarrow i \in C^s].$$

By the minimality of $\sigma$ and $w$, and since $C^t \subseteq C$ and $t$ is $K$-true, it follows that there exists some $i < |\rho| < t$ such that $i \in C$ and $\rho(i) = 0$. But no pair $\langle r,\rho\rangle$ with $\rho(i) = 0$, for some $i < |\rho|$ such that $\chi_C(i) = 1$, can have a precondition assigned to $\langle r,\rho\rangle$ at some stage $u \geq t_0$ which becomes later satisfied. □

We now conclude the proof of the lemma. Let $B \in \Delta_2^0$, and let $i$ be such that $\chi_B = \varphi_i^K$. Let

$$t(x) = \min\{t : t \text{ is } K\text{-true and } \varphi_{i,t}^{K_t}(x)\downarrow\}.$$

Then $t$ is total and so a $\Delta_2^0$ function. It follows that there exists some number $x_0$ such that $f(x) > t(x)$, for all $x \geq x_0$.

Given $x \geq x_0$, let $w$ and $\sigma$ be as in the previous claim (for $x$). Then $t(x) < f(x) < w$ (since $f(x) = \Psi_w^\sigma(x)$). Moreover, if $t_0$ is as in the proof of the previous claim, then, for every pair $\langle r,\rho\rangle$ such that $\langle r,\rho\rangle$ is extracted from $L(x,s)$ at any stage $s \geq t_0$, we have $t(x) \leq r$. Hence for all $x \geq x_0$, $\chi_{B_i}(x) = \varphi_i^K(x)$.

Since $f(x), \psi(x)$, and $w$ can be computed by $C$, we easily conclude that the relation $x \in B_i$ is computable in $C$. □

**Remark 3.4** Note that if $t(x) \geq w$, then $\lim_s B_i^s(x)$ need not exist, but, in any case, $x \notin B_i$, since at every large enough stage at which we extract $\langle w, \sigma \rangle$ from $L(x,s)$ we have $x \notin B_i^s$.

We now go back to the proof of the theorem. We will build a $\Sigma_2^0$ set $A$ such that, for every $\Delta_2^0$ set $B$, $A \not\equiv_e B$, and $K^A \leq_T C$. This implies that $\deg_e(A)$ is properly $\Sigma_2^0$ and, by Lemma 1.1, $J_e(A) \leq_e \chi_C$.

## 3.2 The strategies

**The properly $\Sigma_2^0$-strategy.** Let $\{\Phi_e, \Psi_e\}_{e \in \omega}$ be some effective listing of all pairs of e-operators. To make $A$ of properly $\Sigma_2^0$ e-degree, it is enough to satisfy the following requirements, for every $e, i \in \omega$:

$$\mathcal{P}_{e,i}: \qquad A = \Phi_e^{B_i} \;\&\; B_i = \Psi_e^A \Rightarrow (\exists^\infty x)[\lim_s B_i^s(x) \uparrow]$$

where $\{B_i\}_{i \in \omega}$ and $\{B_i^s\}_{i,s \in \omega}$ are as given in Lemma 3.3.

Indeed, if we satisfy these requirements for every $e, i$, then $\deg_e(A)$ is properly $\Sigma_2^0$. Suppose, for the sake of a contradiction, that $A \equiv_e B$ and $B \in \Delta_2^0$. Then, by the previous lemma, $A = \Phi_e^{B_i}$ and $B_i = \Psi_e^A$ for some $e, i$, with $B =^* B_i$ and so $\lim_s B_i^s(x)$ does not exist for infinitely many $x$ for the desired contradiction, since $\lim_s B_i^s(x)$ exists for almost all $x$.

The strategy to meet $\mathcal{P}_{e,i}$ is a slight modification of the canonical properly $\Sigma_2^0$ strategy as given in [CC88], and described as follows:

($a$) appoint a witness $x$ and let $x \in A$;

($w_1$) wait for finite sets $D, E$ such that $x \in \Phi_e^D$ and $D \subseteq \Psi_e^E$;

($b$) fix $E - \{x\} \subseteq A$;

($w_2$) wait for $D \subseteq B_i$;

($w_3$) let $x \notin A$, wait for $D \not\subseteq B_i$;

($\ell$) let $x \in A$; go back to ($w_2$).

A triple $x, D, E$ as above is called a *follower* of $\mathcal{P}_{e,i}$.

As described in [CC88, Theorem 1], for a given follower $x, D, E$ this strategy may have the following outcomes: ($w_1$) yields $x \in A - \Phi_e^{B_i}$ or $y \in B_i - \Psi_e^A$ for some $y$; ($w_2$) corresponds to the case $D \subseteq \Psi^A$, $D \not\subseteq B_i$; ($w_3$) corresponds to the case $x \in \Phi^{B_i} - A$; finally, the infinitary outcome $\ell$ entails that $\lim_s B_i^s(y)$ does not exist for some $y \in D$.

**The subrequirements $\mathcal{P}_{e,i,j}$.** It follows by the analysis of the outcomes of the previous strategy that if $B_i^s(x)$ does have limit on every $x \in D$, then $A \neq \Phi_e^{B_i}$ or $B_i \neq \Psi_e^A$. The only complication here (see Remark 3.4) is that there might exist finitely many numbers $x$ such that $\lim_s B_i^s(x)$ does not exist, thus, for some $y \in D$, $\lim_s B_i^s(y)$ need not exist. We cope with this difficulty by attacking $\mathcal{P}_{e,i}$ through infinitely many subrequirements $\mathcal{P}_{e,i,j}$, with $j \in \omega$. The strategy for $\mathcal{P}_{e,i,j}$ consists in looking for a follower $x, D, E$ such that $D \upharpoonright j = B_i \upharpoonright j$ : thus, for almost all $j$, if we appoint a follower $x, D, E$ as before, we are bound to conclude that $B_i^s(y)$ exists on every $y \in D$. Thus $\mathcal{P}_{e,i}$ is satisfied through some subrequirement $\mathcal{P}_{e,i,j}$ (in fact cofinitely many such subrequirements). Before acting, the subrequirement $\mathcal{P}_{e,i,j}$ must therefore be provided with some knowledge of what numbers $x < j$ are in fact in $B_i$. This information is coded in the first component, $h(\sigma, s)$, of the outcome of the node corresponding to $\mathcal{P}_{e,i,j}$ in the tree of outcomes.

**The strategy for $K^A \leq_T C$.** For every $i$, we will look for a finite set $D$ such that $i \in \Phi_i^D$. If such a $D$ exists then we let $D \subseteq A$. Notice that we can determine computably in $\emptyset'$ and, thus, in $C$, whether or not such a finite set exists.

## 3.3 The tree of outcomes

For notation and terminology for strings and trees, the reader is referred to [Soa87]. The tree of outcomes is the smallest set $T$ of strings $\sigma$ such that

1. if $|\sigma|$ is even then $\sigma\widehat{\ }(h, r) \in T$, for every $h \in \omega$ and $r \in \{0, 1\}$;
2. if $|\sigma|$ is odd then $\sigma\widehat{\ }r \in T$, for every $r \in \{0, 1\}$.

The strings of even length are assigned to the (sub)requirements $\mathcal{P}_{e,i,j}$, according to some fixed priority listing. The first component, $h(\sigma, s)$, of the outcome of $\sigma$ at stage $s$ will be an assessment as to which numbers $x < j$ are in fact in $B_i$: at stage $s + 1$, $h(\sigma, s)$ will be chosen to be the first element of a list $\mathcal{L}(\sigma, s)$ of numbers. Each element $h$ of the list is the canonical index of a finite subset of $\{x : x < j\}$. Its position in the list measures how well the set $\{x : x < j\} - B_i$ is approximated by the finite set $D_h$. Having decided on the first component, $h$, of the outcome at $\sigma$, the strategy for $\mathcal{P}_{e,i,j}$ is ready to act at $\sigma^+ = \sigma\widehat{\ }h$. The outcome 1 at $\sigma^+$ corresponds to $(w_1)$ or $(w_2)$; the outcome 0 corresponds to $(w_3)$ or $(\ell)$.

The strings of odd length are devoted to guaranteeing that $K^A \leq_T C$: if $|\sigma| = 2i + 1$ then we have outcome 0 if there exists (modulo higher priority constraints) some finite set $D$ such that $i \in \Phi_i^D$; otherwise we have outcome 1.

Let $\hat{T} = T \cup \{\sigma^\frown h : |\sigma| \text{ even } \& h \in \omega\}$. For $\sigma \in \hat{T}$, the parameter $\alpha(\sigma, s)$ is intended to record some finite set which we want to keep in $A$ for the sake of our actions at $\sigma$; the parameter $\epsilon(\sigma, s)$ is meant to record some finite set of elements which we want to keep out of $A$.

The ordering $\preceq$ of $T$ is determined in the usual way by the ordering of the outcomes given that we define $(h, r) < (h', r')$ if

$$h > h' \text{ or } [h = h' \,\&\, r < r'].$$

We extend $\preceq$ to $\hat{T}$ in the obvious way.

Finally, let $\{\xi_\sigma\}_{\sigma \in \hat{T}}$ be a computable partition of $\omega$ into infinite computable sets.

## 3.4 The construction

The construction proceeds by stages. At stage $s$ we define a finite set $A^s$, a string $\delta_s$, and the values of several parameters. Unless otherwise specified, at each stage each parameter retains the same value as at the preceding stage.

Stage 0: Define $\delta_0 = \emptyset$. For every $\sigma \in \hat{T}$, let

$$\alpha(\sigma, 0) = \epsilon(\sigma, 0) = \mathcal{L}(\sigma, 0) = \emptyset.$$

Let $x(\sigma, 0)$ and $p(\sigma, h, 0)$ be undefined for every $x, h \in \omega$. Finally, let $A^0 = \emptyset$.

Stage $s + 1$: Suppose that we have defined $\delta_{s+1} \restriction n$, where $n < s + 1$: let $\sigma = \delta_{s+1} \restriction n$. Our aim is to define a string $\sigma^{++}$ which we will be $\delta_{s+1} \restriction n + 1$.

$|\sigma|$ **even.** Let $\mathcal{P}_{e,i,j}$ be the requirement assigned to $\sigma$. For simplicity, drop subscripts, and let $\Phi_e = \Phi$, $\Psi_e = \Psi$ and $B_i = B$.

Our first task is to define the first component, $h(\sigma, s+1)$, of the outcome. We define $h(\sigma, s+1)$ to be the least element of $\mathcal{L}(\sigma, s)$ if $\mathcal{L}(\sigma, s) \neq \emptyset$, otherwise $h(\sigma, s + 1) = 0$. Then we cancel the precondition for $h(\sigma, s + 1)$ by letting $p(\sigma, h(\sigma, s + 1), s + 1) \uparrow$.

To every $h$ such that $\max D_h < j$ and $h$ does not have a precondition, we assign the precondition $p(\sigma, h, s + 1)$ which becomes satisfied at some later stage $v > s + 1$ if, for every $x < j$ and $x \in D_h$, there exists $u$ such that $s + 1 \leq u \leq v$ and $B^u(x) = 0$.

Define

$$\mathcal{L}(\sigma, s + 1) = (\mathcal{L}(\sigma, s) - \{h(\sigma, s + 1)\}) \cup \{h : h \text{ has a precondition that is satisfied at } s + 1\}$$

and order $\mathcal{L}(\sigma, s + 1)$ in the usual way: for every $h, h' \in \mathcal{L}(\sigma, s + 1)$, define $h <_{\sigma,s+1} h'$ if either

1. $h, h' \in \mathcal{L}(\sigma, s)$ and $h <_{\sigma,s} h'$, or
2. $h \in \mathcal{L}(\sigma, s)$ and $h' \notin \mathcal{L}(\sigma, s)$, or
3. $h, h' \notin \mathcal{L}(\sigma, s)$ and $h < h'$.

Let $\sigma^+ = \sigma\hat{\ }h(\sigma, s+1)$.

Now we are ready to activate the strategy for $\mathcal{P}$.

Let $x = x(\sigma^+, s+1)$ be the least number in $\xi_{\sigma^+}$ such that $x \notin \alpha(\rho, s+1)$, for every $\rho \prec \sigma^+$.

Case 1).

$$(\exists D)(\exists E)[x \in \Phi_s^D \,\&\, D \cap D_{h(\sigma,s+1)} = \emptyset \,\&\, D \subseteq \Psi_s^E \\ \&\, E \cap \bigcup\{\epsilon(\rho, s+1) : \rho \preceq \sigma\} = \emptyset].$$

Choose the least such pair $D, E$.

In this case, let $\alpha(\sigma^+, s+1) = E - \{x\}$:

1. if $D \subseteq B^s$, then let $\sigma^{++} = \sigma^+\hat{\ }0$ and $\epsilon(\sigma^{++}, s+1) = \{x\}$;
2. otherwise, let $\sigma^{++} = \sigma^+\hat{\ }1$ and $\alpha(\sigma^{++}, s+1) = \{x\}$.

Case 2). Otherwise, let $\sigma^{++} = \sigma^+\hat{\ }1$ and $\alpha(\sigma^{++}, s+1) = \{x\}$.

$|\sigma|$ **odd.** Let $|\sigma| = 2i + 1$. We distinguish two cases.

Case 1). $(\exists D)[i \in \Phi_i^D \,\&\, D \cap \bigcup\{\epsilon(\rho, s+1) : \rho \preceq \sigma\} = \emptyset]$.

In this case, let $\sigma^{++} = \sigma\hat{\ }0$, and let $\alpha(\sigma, s+1) = D$ for the least such $D$.

Case 2). Otherwise, let $\sigma^{++} = \sigma\hat{\ }1$.

**Definition of $A^{s+1}$.** At the end of stage $s+1$, let

$$A^{s+1} = (A^s \cup \bigcup\{\alpha(\rho, s+1) : \rho \preceq \delta_{s+1}\}) - \bigcup\{\epsilon(\rho, s+1) : \rho \preceq \delta_{s+1}\}.$$

## 3.5 Verification

The verification is based upon the following lemmas.

**Lemma 3.5** *For every $n$, $\sigma_n = \liminf_s \delta_s \restriction n$ exists.*

**Proof.** Assume by induction that the claim is true of $n$. The only nontrivial case is when $|\sigma_n|$ is even, where, say, the requirement $\mathcal{P}_{e,i,j}$ is assigned to $\sigma_n$.

Let $h$ be the canonical index of $\overline{B_i} \restriction j$. It is clear that, whenever we assign a precondition to $h$, then this precondition becomes satisfied at some later stage. Hence, at infinitely many stages $s$, $h \in \mathcal{L}(\sigma_n, s)$, and at infinitely many stages $t$, $h = h(\sigma_n, t)$. On the other hand, it is also clear that for almost all stages $s$, if $h' \in \mathcal{L}(\sigma_n, s)$, then $D_{h'} \subseteq D_h$, hence $h' \leq h$ by the usual coding of canonical sets. Therefore it follows that $\sigma_{n+1} = \sigma_n{}^\frown(h, r)$, for some $r \in \{0, 1\}$. $\square$

Let $f = \bigcup_{n\in\omega} \sigma_n$.

**Lemma 3.6** *For every $\tau \in \hat{T}$, if $\tau \subset f$, then $\alpha(\tau) = \lim_s \alpha(\tau, s)$, $\epsilon(\tau) = \lim_s \epsilon(\tau, s)$ and $x(\tau) = \lim_s x(\tau, s)$ exist. Moreover, if $\tau = \sigma_n$ for some $n$, then the requirement assigned to $\sigma_n$ is satisfied.*

**Proof.** By induction on $n$, we show that if $\tau = \sigma_n$ or $\tau = \sigma_n^+$, where $\sigma_{n+1} = \sigma_n^{+\frown} r$ for some $r \in \{0, 1\}$ (of course $\sigma_n^+ = \sigma_n$ if $n$ is odd), then $\lim_s \alpha(\tau, s)$, $\lim_s \epsilon(\tau, s)$ and $\lim_s x(\tau, s)$ exist, and the requirement assigned to $\sigma_n$ is satisfied. The case $n = 0$ is trivial as are the existence of the required limits for all nodes to the left of the true path.

Assume that the claim is true of $n$. For every $\tau \in \hat{T}$ such that $\tau \preceq \sigma_n$, let $\alpha(\tau) = \lim_s \alpha(\tau, s)$, $\epsilon(\tau) = \lim_s \epsilon(\tau, s)$ and $x(\tau) = \lim_s x(\tau, s)$; and let $t$ be a stage such that, for every $s \geq t$ and $\tau \preceq \sigma_n$, $\alpha(\tau) = \alpha(\tau, s)$, $\epsilon(\tau) = \epsilon(\tau, s)$ and $x(\tau) = x(\tau, s)$.

Suppose first that $|\sigma_n|$ is even, and let $\mathcal{P}_{e,i,j}$ be the requirement assigned to $\sigma_n$. Let $\sigma_{n+1} = \sigma_n{}^\frown(h, i)$, and let $\sigma_n^+ = \sigma_n{}^\frown h$. Then

$$x(\sigma_n^+) = \min x \in (\xi_{\sigma_n^+} - \bigcup_{\tau\preceq\sigma_n} \alpha(\tau)).$$

- If there are no finite sets $D, E$ such that $D \cap D_h = \emptyset$, $x(\sigma_n^+) \in \Phi_e^D$, $E \cap \bigcup_{\tau\preceq\sigma_n} \epsilon(\tau) = \emptyset$ and $D \subseteq \Psi_e^E$, then $\sigma_{n+1} = \sigma_n^{+\frown}1$,

  $$\lim_s \epsilon(\sigma_n^+, s) = \lim_s \epsilon(\sigma_{n+1}, s) = \emptyset,$$

  $\lim_s \alpha(\sigma_n^+, s) = \emptyset$, $\lim_s \alpha(\sigma_{n+1}, s) = \{x(\sigma_n^+)\}$ and $x(\sigma_n^+) \in A$. Moreover, either $x(\sigma_n^+) \in A - \Phi_e^{B_i}$, or $x(\sigma_n^+) \in \Phi_e^D$, for some $D \subseteq B_i$, but $D \not\subseteq \Psi_e^A$.

- If $D, E$ exist, then we eventually choose the least such pair $D, E$, hence $\alpha(\sigma_n^+) = E - \{x(\sigma_n^+)\}$, $\epsilon(\sigma_n^+) = \emptyset$, and either (a) or (b) holds:

  (a) $\sigma_{n+1} = \sigma_n^+{}^\frown 0$ and $\alpha(\sigma_{n+1}) = \emptyset$, $\epsilon(\sigma_{n+1}) = \{x(\sigma_n^+)\}$;

  (b) $\sigma_{n+1} = \sigma_n^+{}^\frown 1$ and $\alpha(\sigma_{n+1}) = \{x(\sigma_n^+)\}$, $\epsilon(\sigma_{n+1}) = \emptyset$.

  In case (a) either

  ($a_1$) there exist infinitely many stages $s$ such that $\sigma_n^+{}^\frown 1 \subseteq \delta_s$, in which case, there exists some $y \in D$ such that $\lim_s B_i^s(y)$ does not exist; or

  ($a_2$) $D \subseteq B_i$ but $x(\sigma_n^+) \notin A$, giving $x(\sigma_n^+) \in \Phi_e^{B_i} - A$.

  In (b) we have $D \nsubseteq B_i$, but $E \subseteq A$, hence $D \subseteq \Psi_e^A$.

**Remark 3.7** Notice that if $j$ is such that $\lim_s B_i^s(y)$ exists for every $y \geq j$, then ($a_1$) does not occur, by Lemma 3.3.

If $|\sigma_n| = 2i + 1$ is odd and $i \in \Phi_i^D$, for some finite set $D$ such that $D \cap \bigcup_{\tau \preceq \sigma_n} \epsilon(\tau) = \emptyset$, then $\sigma_{n+1} = \sigma_n{}^\frown 0$ and $\alpha(\sigma_{n+1}) = D$, for some such $D$; otherwise $\sigma_{n+1} = \sigma_n{}^\frown 1$ and $\alpha(\sigma_{n+1}) = \emptyset$. In either case $\epsilon(\sigma_{n+1}) = \emptyset$.

The proof of the lemma is now complete. □

**Lemma 3.8** $K^A \leq_T C$.

**Proof.** We will show that, for every $n$, one can compute $\sigma_n$ recursively in $C$.

Now, $\sigma_0 = \emptyset$. Assume by induction that we can compute $\sigma_n$ and a stage $s_n$ such that $\tau \nsubseteq \delta_s$, for every $s \geq s_n$ and $\tau \prec_L \sigma_n$ and each parameter at any $\tau \preceq \sigma_n$ has reached its limit by stage $s_n$. Assume first that $|\sigma_n|$ is even, and let $\mathcal{P}_{e,i,j}$ be the requirement assigned to $\sigma_n$. Since $C$ can compute $B_i \upharpoonright j$, it follows that $C$ can compute the first component, $h = \lim_s h(\sigma_n, s)$, of the outcome of $\sigma_n$. Moreover, since $\emptyset' \leq_T C$, $C$ can compute the least stage $s_n^+ \geq s_n$ such that $\tau \nsubseteq \delta_s$, for every $\tau \prec_L \sigma_n{}^\frown h$ (since for every $x \in B_i \upharpoonright j$ and every $t$, one can compute in $\emptyset'$ whether there exists some $s \geq t$ such that $x \notin B_i^s$). Again, using $\emptyset'$ as an oracle, one can compute whether Case 1 or Case 2 of the construction holds, thus computing $\sigma_{n+1}$ and the corresponding $s_{n+1}$.

A similar argument applies in the case $|\sigma_n| = 2i + 1$, for some $i$, since the oracle $\emptyset'$ can compute whether or not there exists some stage $s \geq s_n$ and some finite $D$ such that $i \in \Phi_i^D$, and $D \cap \bigcup_{\tau \preceq \sigma_n} \epsilon(\tau, s_n) = \emptyset$.

It follows that $i \in K^A$ if and only if $\sigma_{2i+2} = \sigma_{2i+1}{}^\frown 0$, thus $K^A \leq_T C$. □

**Remark 3.9** We expect that, by combining the above construction of $A$ with a variant of the coding procedure and the associated guessing at outcomes used in the tree proof of the Sacks' jump inversion theorem, one can actually guarantee that $J_e(A) \equiv_e \chi_C$.

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