INTERVALS WITHOUT CRITICAL TRIPLES

PETER CHOLAK, ROD DOWNEY AND RICHARD SHORE

Abstract. This paper is concerned with the construction of intervals of computably enumerable degrees in which the lattice M_5 (see Figure 1) cannot be embedded. Actually, we construct intervals $\mathcal{I}$ of computably enumerable degrees without any weak critical triples (this implies that M_5 cannot be embedded in $\mathcal{I}$, see Section 2). Our strongest result is that there is a low_2 computably enumerable degree $\mathbf{e}$ such that there are no weak critical triples in either of the intervals $[\mathbf{0}, \mathbf{e}]$ or $[\mathbf{e}, \mathbf{0}']$.

1. Introduction

A set of natural numbers is computably (or recursively) enumerable if it is the range of a function computed by a Turing machine. We say one set of natural numbers, A, is Turing computable from another, B, if there is a Turing machine, which using an oracle for B, computes A. Equivalence classes under this reduction are called Turing degrees or just degrees. In this paper we will restrict our attention to those degrees which contain a computably enumerable set; the computably enumerable degrees.

The computably enumerable degrees form an upper semilattice, $\mathcal{R}$. Despite the fact that $\mathcal{R}$ is not a lattice, there has been a long series of results each demonstrating larger and larger classes of lattices that could be embedded into $\mathcal{R}$. Examples include the results of Lachlan [1972] and Ambos-Spies and Lerman [1989]. On the other hand, in Lachlan and Soare [1980] it is shown that not every lattice can be embedded into $\mathcal{R}$. Naturally, understanding precisely which lattices can be embedded into $\mathcal{R}$ is central to the question of the decidability

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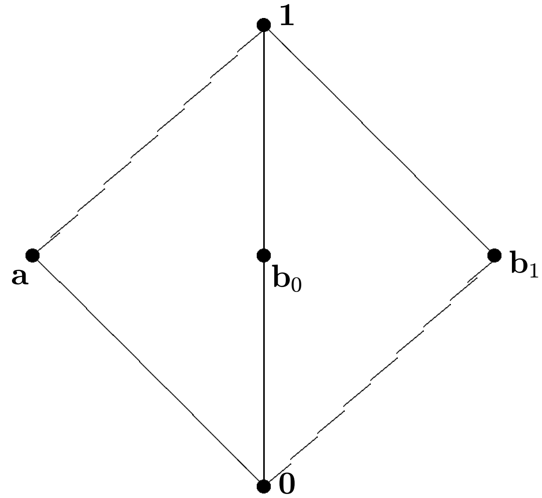

FIGURE 1. The lattice M_5

of the $\exists\forall$ theory of $\mathcal{R}$. For instance, in the language of $\{\leq\}$, one needs to demonstrate an algorithm that will decide which lattices can be embedded into $\mathcal{R}$ (or the nonexistence of such an algorithm).

In the present paper, our concern is the lattice M_5 (see Figure 1), the modular 5 element lattice and its distribution in $\mathcal{R}$. In Lachlan [1972] it is shown that M_5 is embeddable into $\mathcal{R}$. However, Lachlan's proof exhibited certain technical features such as "continuous tracing" which had not been necessary in previous lattice embedding theorems. Lachlan and Soare's proof that S_8 (see Figure 2) is not embeddable in some sense demonstrates that to embed M_5 requires such features and furthermore these features can be incompatible with other lattice properties such as simultaneously controlling the infima of a pair (or more) of degrees as with $\mathbf{a_1} \cap \mathbf{a_2} = \mathbf{a_3}$ in Figure 2).

These observations led Lerman to conjecture that essentially the "M_5 phenomenon" interacting with the infima was the only blockage to embeddability. The central role of the "M_5 phenomenon" was also demonstrated in Downey [1990] and Weinstein [1988] where it was proved that there are initial segments of $\mathcal{R}$ into which M_5 cannot be embedded.

Actually, the Downey and Weinstein results were stated in terms of (weak) critical triples (see Section 2 for a definition). Critical triples is a lattice theoretic condition which reflects the need for "continuous tracing" in the construction of such a triple. These results as well as Cholak and Downey [1993] demonstrate that embedding of critical triples and the structure of $\mathcal{R}$ interact in very interesting ways.

Indeed, embeddings of nondistributive lattices is closely tied to the structure of $\mathcal{R}$. For instance, following the work in Downey and Lempp

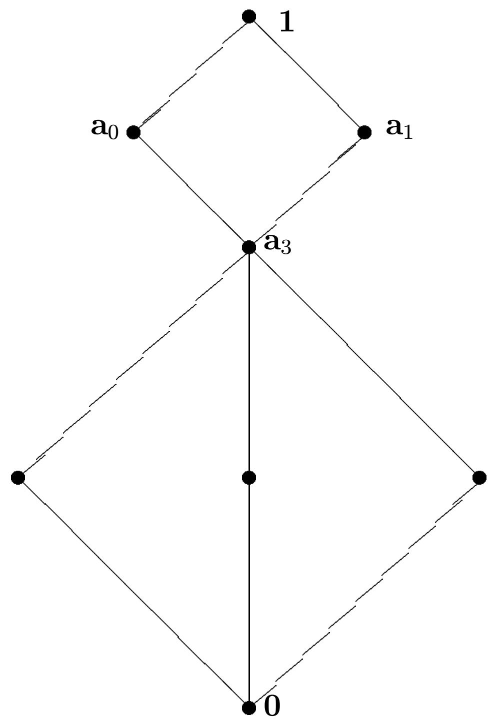

FIGURE 2. The lattice S_8

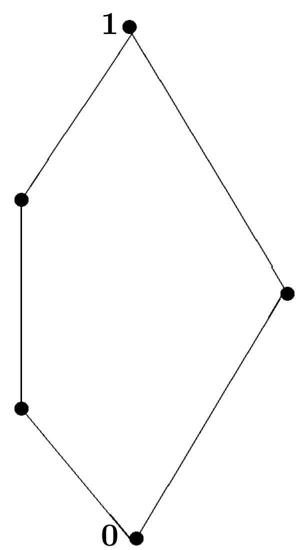

FIGURE 3. The lattice N_5

[n.d.], it is shown in Ambos-Spies and Fejer [1996] that the degrees which are tops of N_5 (see Figure 3) are precisely the "noncontiguous" degrees which are not "locally distributive" in $\mathcal{R}$.

We remark that recently in Lempp and Lerman [n.d.] it is shown that there are lattices without critical triples which are not embeddable into $\mathcal{R}$. But despite the fact that critical triples do not completely capture the nonembeddability picture, it is obvious that their definition plays a central role in our understanding of embeddings into $\mathcal{R}$.

Critical triples and nondistributive lattices also seem to have a connection with a natural operator on the Turing degrees called the jump operator. Informally the jump of a set of natural numbers X, X', is the set of numbers e such that the eth Turing machine (under some standard indexing of all Turing machines) with an oracle for X halts with input e. We use the jump operator to define jump classes: sets of degrees which have the same nth jump for some n. For example, Low_2 is the collection of all computably enumerable Turing degrees $\mathbf{d}$ whose double jump is as low as possible, i.e. $\mathbf{d}'' \equiv \mathbf{0}''$.

Now in Downey and Shore [1996], it is demonstrated that if $\mathbf{d}$ is nonlow_2 then one can embed M_5 below $\mathbf{d}$. The Ambos-Spies and Fejer result proves that if $\mathbf{d}$ is nonlow_2 then $\mathbf{d}$ is the top of an N_5. Because of these results one is led to believe that lattice embeddings and the jump operator are deeply related.

In the back of our minds when we began this project we hoped to show that the low_2 degrees are definable in the computably enumerable Turing degrees (in the language of partial orders) via embedding properties of M_5. There are techniques for working with low_2 computably enumerable degrees [Shore and Slaman, 1991; Downey and Shore, 1995] and nonlow_2 computably enumerable degrees [Downey and Shore, 1996]. Downey and Shore used these techniques to show that a computably enumerable tt degree $\mathbf{d}$ is low_2 iff $\mathbf{d}$ has a minimal cover in the computably enumerable tt degrees [Downey and Shore, 1995] and hence the low_2 computably enumerable tt degrees are definable in the computably enumerable tt degrees. Our plan was to use these techniques to show $\mathbf{d}$ is low_2 iff there is a computably enumerable degree $\mathbf{e}$ such that the lattice M_5 cannot be embedded into either $[\mathbf{0}, \mathbf{e}]$ or $[\mathbf{e}, \mathbf{d}]$.

We had reason to hope that we might be successful: Downey and Shore [1996] had recently shown that the lattice M_5 can be embedded below any nonlow_2 computably enumerable Turing degree and from Cholak and Downey [1993] we knew how to construct intervals of computably enumerable degrees where it is impossible to embed M_5. Hence, we needed to first extend the Downey and Shore result to the following: If $\mathbf{d}$ is nonlow_2 and $\mathbf{e}$ is a low_2 degree below $\mathbf{d}$ (and so $\mathbf{d}$ is nonlow_2 relative to $\mathbf{e}$), then the lattice M_5 can be embedded in the interval $[\mathbf{e}, \mathbf{d}]$. And second, improve the techniques of Cholak and Downey to show that for each low_2 degree $\mathbf{d}$ we could find an $\mathbf{e}$ such that the lattice

M_5 cannot be embedded into the intervals $[\mathbf{0}, \mathbf{e}]$ and $[\mathbf{e}, \mathbf{d}]$. This would have given us a formula defining low$_2$.

However, one of our results implies that there is a computably enumerable degree $\mathbf{e}$ such that the lattice M_5 cannot be embedded into either $[\mathbf{0}, \mathbf{e}]$ or $[\mathbf{e}, \mathbf{0}']$. Thus the proposed formula does not define the low$_2$ computably enumerable Turing degrees.

Before we turn to our own work we should note that Nies, Shore and Slaman [n.d.] have recently announced that the jump classes Low$_n$ and High$_m$ are definable, for $n \geq 2$ and $m \geq 1$. Their methods however, rely on coding models of arithmetic and analyzing the complexity of certain lattice like structures that can be coded below a given degree. We should also mention that there have been at least two other unsuccessful attempts to show that Low$_2$ is "naturally" definable: one by Leonhardi [Leonhardi, 1994] and the other by Cooper and Yi [Cooper and Yi, n.d.]. For more of a discussion of definability and computably enumerable degrees, the reader is directed to Shore [Shore, n.d.].

The rest of this paper is concerned with the construction of intervals of computably enumerable degrees in which M_5 cannot be embedded. Actually, we construct intervals $\mathcal{I}$ of computably enumerable degrees without any weak critical triples (this implies M_5 cannot be embedded in $\mathcal{I}$). The definitions of and the relationship between critical triples and weak critical triples (and the lattice M_5) are isolated in the next section.

The concept of a critical triple first arose implicitly in Ambos-Spies and Lerman [1986] and Ambos-Spies and Lerman [1989]. In Ambos-Spies and Lerman [1989] it is shown that a finite lattice $\mathcal{L}$ can be embedded into $\mathcal{R}$ if there is no (weak) critical triple $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ and no pair $\mathbf{p}$ and $\mathbf{q}$ in $\mathcal{L}$ such that $\mathbf{b}_0 \leq \mathbf{p} \cap \mathbf{q} \leq \mathbf{b}_0 \cup \mathbf{a}$. The concept of a critical triple was first explicitly isolated in Downey [1990]. In Downey [1990], it is shown that there is a degree which does not bound a critical triple. In Cholak and Downey [1993], this work was extended to show that if $\mathbf{a} < \mathbf{b}$ are degrees then there is a degree $\mathbf{e}$ such that $\mathbf{a} < \mathbf{e} < \mathbf{b}$ and there is no critical triple in the interval $[\mathbf{a}, \mathbf{e}]$. The definition of a weak critical triple first appeared in Weinstein [1988] under the name of "pre 1-3-1". He showed that there is a degree below which there is no weak critical triple. Our results further point out the importance of critical triples and weak critical triples in our quest to know what lattices can and cannot be embedded into intervals of computably enumerable degrees.

Our results fall into two groups. The first group of results concerns degrees which do not bound a weak critical triple. We show that every degree can be split into two degrees neither of which bounds a weak critical triple. Hence, the class of degrees which fail to bound a weak critical triple generates $\mathcal{R}$. We also show that there is a properly low_2 degree with no weak critical triple below it. Therefore, the result of Downey and Shore [Downey and Shore, 1996] that the lattice M_5 can be embedded below any non-low_2 degree is the best possible in terms of jump classes. These results are presented in Section 4. The other group of results is presented in Section 6 and concerns degrees above which there is no weak critical triple. We show there is an incomplete degree above which there is no weak critical triple. In addition, we show there is a degree above and below which there is no weak critical triple. By the above result of Downey and Shore such a degree must be low_2.

There are two different types of requirements that reflect the grouping of results: There are the requirements $\mathcal{N}_e$ which ensure that there is no weak critical triple in the desired lower cone and there are requirements $\mathcal{P}_i$ which ensure that there is no weak critical triple in the desired upper cone. Each $\mathcal{N}_e$ is a negative requirement in that it restrains elements from entering the constructed set. The requirements $\mathcal{N}_e$ are presented in Section 3. Each $\mathcal{P}_i$ is an infinite positive requirement; it may add infinitely many elements into the constructed set. The requirements $\mathcal{P}_i$ are presented in Section 5. The two types of requirements and the strategies used to meet them are in a sense duals of each other.

Although demonstrating that there are no weak critical triples in some interval is stronger than the corresponding result for critical triples, it is actually easier to construct intervals of computably enumerable degrees without weak critical triples than it is to construct intervals of computably enumerable degrees without critical triples (at least, as this was done in Downey [1990] and Cholak and Downey [1993]). Fix a triple of computably enumerable degrees, $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ such that $\mathbf{a} \cup \mathbf{b}_0 = \mathbf{a} \cup \mathbf{b}_1$. To show, as in Downey [1990] and Cholak and Downey [1993], that $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ is not a critical triple a computably enumerable degree $\mathbf{d}$ is built such that $\mathbf{d} \leq \mathbf{b}_0, \mathbf{b}_1$ and if $\mathbf{a} \not\leq \mathbf{b}_0$ then $\mathbf{d} \not\leq \mathbf{a}$. Determining whether $\mathbf{a}$ is Turing reducible to $\mathbf{b}_0$ is a Σ_3 question and therefore it is not surprising that these arguments turn out to be $\mathbf{0}'''$ arguments. To show that $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ is not a weak critical triple, we construct a computably enumerable degree $\mathbf{d}$ such that $\mathbf{d} \leq \mathbf{b}_0, \mathbf{b}_1$ and $\mathbf{b}_0 \leq \mathbf{d} \cup \mathbf{a}$. As it turns out, this can be done by a $\mathbf{0}''$ argument. In terms of lattice embeddings, however, the results are equivalent for, as we will see below, a lattice contains a critical triple if and only if it contains a weak critical triple.

A question we tried to answer but could not is whether the lattice M_5 can be embedded above every low degree. For more on this issue, the reader is directed to Section 6.3. It is also open whether one can extend our result that there is an incomplete degree which does not bound a weak critical triple to show such a degree must exist above every nonlow$_2$ degree.

Remark 1.1 (Notation). Our notation is standard and generally follows [Soare, 1987] with the following important exceptions: The use of a computation $\Phi(X_s; x)$ is denoted by $\varphi_s(x)$ and similarly for other Greek letters. We assume the uses of all functionals not constructed by us to be nondecreasing in the stage, s, and increasing in the argument, x, for each stage. Furthermore, if the underlying set involved in a computation changes below the use of the computation at some stage s, we will assume that computation diverges at stage s. For example, if we are given Ψ and for some x and s, $\Psi_s(X_s; x)\downarrow$ and $X_s \upharpoonright (\psi_s(x)+1) \neq X_{s+1} \upharpoonright (\psi_s(x)+1)$ then $\Psi_{s+1}(X_{s+1}; x)\uparrow$. When the oracle of a functional is given as the join of sets we assume the use to be computed separately on each set. To make life easier, we will assume that $X \oplus Y \oplus Z$ is defined as $\{3x : x \in X\} \cup \{3y+1 : y \in Y\} \cup \{3z+2 : z \in Z\}$. All other joins are defined normally. When we choose a *large* number we mean a number larger than any other number mentioned or used so far. For any of the functionals (or parameters) which we are building, we will assume that if the underlying set changes at some stage on the use (for this stage) then this functional (or parameter) is undefined at this stage unless otherwise explicitly defined. For example, if we are building Γ and for some x and s, $\Gamma_s(X_s; x)\downarrow$ and $X_s \upharpoonright (\gamma_s(x) + 1) \neq X_{s+1} \upharpoonright (\gamma_s(x) + 1)$ then unless we otherwise explicitly define $\Gamma_{s+1}(X_{s+1}; x)$, $\Gamma_{s+1}(X_{s+1}; x)\uparrow$. Otherwise, all parameters remain the same from stage to stage unless explicitly redefined. We assume the reader is familiar with $\mathbf{0}''$ arguments as in Soare [1987].

2. Definitions and Examples

Definition 2.1. Let $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ be elements in any upper semilattice $\mathcal{L}$ (such as the Turing degrees or the computably enumerable Turing degrees). We say that $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a *critical triple* if $\mathbf{a} \cup \mathbf{b}_0 = \mathbf{a} \cup \mathbf{b}_1$, $\mathbf{b}_0 \not\leq \mathbf{a}$ and for $\mathbf{d} \in \mathcal{L}$, if $\mathbf{d} \leq \mathbf{b}_0, \mathbf{b}_1$ then $\mathbf{d} \leq \mathbf{a}$.

Definition 2.2. Let $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ be elements in any upper semilattice $\mathcal{L}$. We say that $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a *weak critical triple* if $\mathbf{a} \cup \mathbf{b}_0 = \mathbf{a} \cup \mathbf{b}_1$, $\mathbf{b}_0 \not\leq \mathbf{a}$ and for $\mathbf{d} \in \mathcal{L}$, if $\mathbf{d} \leq \mathbf{b}_0, \mathbf{b}_1$ then $\mathbf{b}_0 \not\leq \mathbf{d} \cup \mathbf{a}$.

Any upper semilattice in which the lattice M_5 (for a diagram of the lattice M_5 see Figure 1) can be embedded (as a lattice) contains a

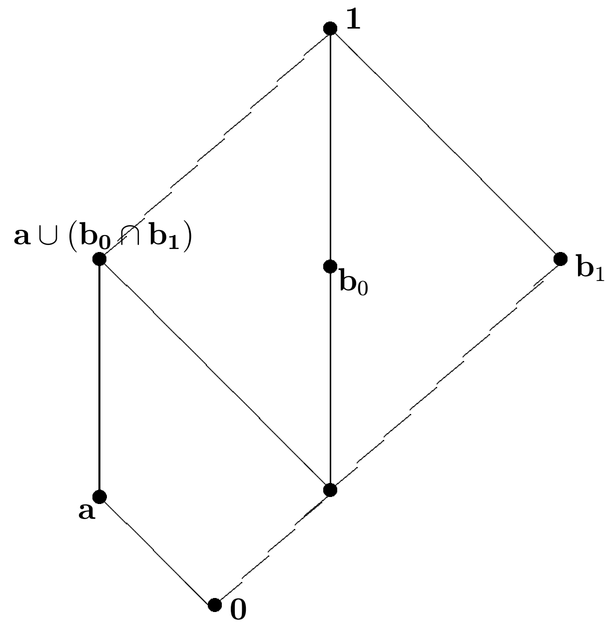

FIGURE 4. A weak critical triple which is not a critical triple

critical triple. If $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a critical triple in some upper semilattice $\mathcal{L}$ then $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a weak critical triple in $\mathcal{L}$. In Figure 4, a weak critical triple $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ is identified within a lattice. Within this lattice, $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ do not form a critical triple but the elements $\mathbf{a} \cup (\mathbf{b}_0 \cap \mathbf{b}_1)$, $\mathbf{b}_0$ and $\mathbf{b}_1$ do. In fact, if $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a weak critical in an upper semilattice $\mathcal{L}$ and $\mathbf{b}_0 \cap \mathbf{b}_1$ exists (for example, this must occur if $\mathcal{L}$ is a lattice), then $\mathbf{a} \cup (\mathbf{b}_0 \cap \mathbf{b}_1)$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a critical triple in $\mathcal{L}$. On the other hand, it is possible to construct an infinite upper semilattice which contains a weak critical triple but no critical triples.

As for $\mathcal{R}$, it is unknown whether there exists an interval $\mathcal{I}$ such that there is a weak critical triple in $\mathcal{I}$ but $\mathcal{I}$ does not contain a critical triple or whether there exits an interval $\mathcal{I}$ such that there is a critical triple in $\mathcal{I}$ but M_5 cannot be embedded into $\mathcal{I}$.

3. THE REQUIREMENTS $\mathcal{N}_e$

Our goal is to build a computably enumerable set E such that the following requirements are met:

$$\mathcal{N} \qquad \begin{array}{c} \text{If } \Lambda(E) = A \oplus B_0 \oplus B_1 \text{ and } \Psi_i(A \oplus B_{\bar{i}}) = B_i \\ \text{then there exists a computably enumerable set } D \\ \text{and functionals } \Delta_i \text{ and } \Gamma \text{ such that} \\ \Delta_i(B_i) = D \text{ and } \Gamma(A \oplus D) = B_0 \end{array}$$

where Λ and Ψ_i $(i = 0, 1)$ are functionals and A, B_0 and B_1 are computably enumerable sets. These six items, the requirement, the set D and the three functionals Δ_i and Γ will later be indexed in some fashion by e; but for now, we will drop the e. If we meet $\mathcal{N}$ then either E does not bound the computably enumerable sets A, B_0 and B_1 or the degrees represented by these sets do not form a weak critical triple. (If $B_0 \leq_T A$ then the requirement can be easily met.)

We will need some auxiliary functions (at first it may seem that we are generating more notation than needed but we will use all these functions later in the construction):

$$L(s) = \max\{x : (\forall y < x)(\forall z \in \{3y, 3y+1, 3y+2\}) [\Lambda_s(E_s; z) = A_s \oplus B_{0,s} \oplus B_{1,s}(z)]\} \tag{3.1}$$

$L(s)$ is the length of agreement function between $\Lambda_s(E_s)$ and $A_s \oplus B_{0,s} \oplus B_{1,s}$. We use the convention that if $\Lambda_s(E_s; y) = A_s \oplus B_{0,s} \oplus B_{1,s}(y)$ and $E_s \restriction \lambda_s(y) + 1$ does not change then no new numbers can enter $A_s \oplus B_{0,s} \oplus B_{1,s} \restriction y + 1$. Let $\{i, \bar{i}\} = \{0, 1\}$.

$$l^{\Psi_i}(s) = \max\{x : (\forall y < x)[\Psi_{i,s}(A_s \oplus B_{\bar{i},s}; y) = B_{i,s}(y) \text{ and } \psi_{i,s}(y) < L(s)]\} \tag{3.2}$$

$l^{\Psi_i}(s)$ is the E_s-correct length of agreement function between $\Psi_{i,s}(A_s \oplus B_{\bar{i},s})$ and $B_{i,s}$. We will use these length of agreement functions to define, by induction, the following length of agreement and use functions:

$$l^0(s) = l^{\Psi_0}(s) \tag{3.3}$$

$$\text{If } x < l^0(s) \text{ let } \rho^0(A,x,s) = \psi_{0,s}(x), \quad \rho^0(B_1,x,s) = \psi_{0,s}(x) \text{ and } \rho^0(B_0,x,s) = x. \tag{3.4}$$

$$l^{2i+1}(s) = \max\{x : (\forall y < x)[y < l^{2i}(s) \text{ and } \rho^{2i}(B_1,y,s) < l^{\Psi_1}(s)]\} \tag{3.5}$$

$$\begin{aligned} &\text{If } x < l^{2i+1}(s) \text{ let } \rho^{2i+1}(A,x,s) = \psi_{1,s}(\rho^{2i}(B_1,x,s)),\\ &\rho^{2i+1}(B_0,x,s) = \psi_{1,s}(\rho^{2i}(B_1,x,s)) \text{ and}\\ &\rho^{2i+1}(B_1,x,s) = \rho^{2i}(B_1,x,s). \end{aligned} \tag{3.6}$$

$$l^{2i+2}(s) = \max\{x : (\forall y < x)[y < l^{2i+1}(s) \text{ and } \rho^{2i+1}(B_0,y,s) < l^{\Psi_0}(s)]\} \tag{3.7}$$

$$\begin{aligned} &\text{If } x < l^{2i+2}(s) \text{ let } \rho^{2i+2}(A,x,s) = \psi_{0,s}(\rho^{2i+1}(B_0,x,s)),\\ &\rho^{2i+2}(B_1,x,s) = \psi_{0,s}(\rho^{2i+1}(B_0,x,s)) \text{ and}\\ &\rho^{2i+2}(B_0,x,s) = \rho^{2i+1}(B_0,x,s). \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\text{If } x < l^i(s), \text{ let } \rho^i(E,x,s) = \max\{\lambda_s(3\rho^i(A,x,s)),\\ &\lambda_s(3\rho^i(B_0,x,s)+1), \lambda_s(3\rho^i(B_1,x,s)+2)\}. \end{aligned} \tag{3.9}$$

$$\text{If } x > l^i(s) \text{ then for all } X \in \{A, B_0, B_1, E\}, \rho^i(X,x,s)\uparrow. \tag{3.10}$$

$$\begin{aligned} &\text{If for almost all stages } s, \rho^i(X,x,s)\downarrow \text{ then let}\\ &\rho^i(X,s) = \lim \rho^i(X,x,s), \text{ for } X \in \{A, B_0, B_1, E\}. \end{aligned} \tag{3.11}$$

If $x < l^i(s)$ then we have an E_s-correct computation involving $i+1$ layers, that is $\Psi_{0,s}(A_s \oplus B_{1,s}) \restriction x+1 = B_{0,s} \restriction x+1, \Psi_{1,s}(A_s \oplus B_{0,s}) \restriction \rho^0(B_1,x,s)+1 = B_{1,s} \restriction \rho^0(B_1,x,s)+1,\ \Psi_{0,s}(A_s \oplus B_{1,s}) \restriction \rho^1(B_0,x,s)+1 = B_{1,s} \restriction \rho^1(B_0,x,s)+1, \ldots$ and $\rho^i(X,x,s)$ is the initial segment of the set X used in this computation, for all $X \in \{A, B_0, B_1, E\}$. Diagram 5 may be helpful at this point. If we preserve E_s below $\rho^i(E,x,s)+1$ then the computations used in defining $l^i(s) > x$ will not change and we will preserve X_s below $\rho^i(X,x,s)+1$, for all $X \in \{A, B_0, B_1\}$.

Associated with each number x will be another number $i(x)$. We should view $i(x)$ as the number of injuries $\Gamma_s(A_s \oplus D_s; x) = B_{0,s}(x)$ can sustain and still ensure that $\Gamma(A \oplus D; x) = B_0(x)$ (more on this later). The value of $i(x)$ will be determined *before* the construction. In fact, determining the value of $i(x)$ will play a large role in all of our proofs. We will assume that $i(x)$ is a non-decreasing function. We will define the length of agreement $l(s)$ as

$$l(s) = \max\{x : (\forall y < x)[y < l^{i(y)}(s)]\}. \tag{3.12}$$

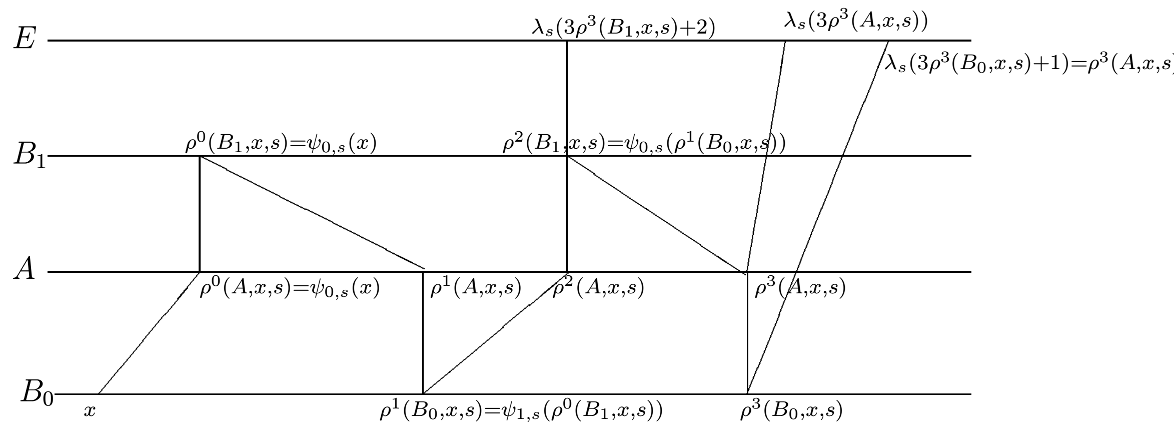

FIGURE 5. 4 layers

If $x < l(s)$ then let $\rho(X, x, s) = \rho^{i(x)}(X, x, s)$ and $\rho(X, x) = \rho^{i(x)}(X, x)$. Think of $\rho(X, x, s)$ as the complete use function for x on X. We say a stage s is expansionary iff $s = 0$ or $t < s$ is the last expansionary stage and $l(s) > l(t)$.

If $\Lambda(E) = A \oplus B_0 \oplus B_1$ and $\Psi_i(A \oplus B_{\bar{i}}) = B_i$ then there will be infinitely many expansionary stages. Hence, to meet $\mathcal{N}$ it is enough to build a computably enumerable set D and the functionals Δ_i and Γ such that if there are infinitely many expansionary stages then $\Delta_i(B_i) = D$ and $\Gamma(A \oplus D) = B_1$. We will do this by splitting $\mathcal{N}$ into infinitely many subrequirements, $\mathcal{N}_x$:

$\mathcal{N}_x$ If there are infinitely many stages s such that $l(s) > x$ then $\Gamma(A \oplus D; x) = B_0(x)$ and $\Delta_i(B_i) \upharpoonright (\gamma(x) + 1) = D \upharpoonright (\gamma(x) + 1)$.

Informally, the strategy to meet $\mathcal{N}_x$ is first to redefine $\Gamma_s(A_s \oplus D_s; x)$ to reflect any change in $B_{0,s}(x)$, if we can. If $\Gamma_s(A_s \oplus D_s; x)\downarrow \neq B_{0,s}(x)$ and for all $i \in \{0, 1\}$, $\Delta_{i,s}(B_{i,s}; \gamma_s(x))\uparrow$ then add $\gamma_s(x)$ to D_{s+1} which will allow us to later redefine $\Gamma(A \oplus D; x)$. Our goal is to prove that this works.

We will now present the formal details. We will have two additional parameters $e^x(s)$ and $p^x(s)$; they will be used to keep track of which layer has been "peeled away" with $e^x(s) \in \{0, 1\}$ indicating which side has changed and $p^x(s)$ at which layer.

Action for $\mathcal{N}_\mathbf{x}$ at stage $s + 1$. Assume $l(s + 1) > x$ (otherwise do nothing). Let $t < s + 1$ be the last stage such that $\Gamma_t(A_t \oplus D_t; x)\downarrow$ and $l(t) > x$, if such a stage exists. Do the first of the following cases which applies:

Case 1: t does not exist.

Action: Let d be large, define $\Gamma_{s+1}(A_{s+1} \oplus D_{s+1}; x) = B_{0,s+1}(x)$ with use $\gamma_{s+1}(x) = d > \rho(A, x, s)$, and define $\Delta_{i,s+1}(B_{i,s+1}; d) = 0$ with use $\delta_{i,s}(d) = \rho(B_i, x, s)$. For all $\gamma_{s+1}(x-1) < y < d$, if $\Delta_{i,s}(B_{i,s}; y)\uparrow$ then define $\Delta_{i,s+1}(B_{i,s+1}; y) = D_{s+1}(y)$ with use 0. If $i(x)$ is even let $e^x(s+1) = 1$, otherwise let $e^x(s+1) = 0$. Let $p^x(s+1) = i(x) + 1$.

Case 2: There is a stage t' such that $t < t' < s+1$ and either Case 3, 4 or 5 applies at stage t' for some $\mathcal{N}_y$, where $y \leq x$.

Action: Same as in Case 1.

Case 3: $A_{s+1} \upharpoonright \rho(A, x, t) \neq A_t \upharpoonright \rho(A, x, t)$.

Action: Do nothing.

Case 4: For $i \in \{0, 1\}$, and $l = p^x(t) - 1$, $B_{i,s+1} \upharpoonright \rho^l(B_i, x, t) \neq B_{i,t} \upharpoonright \rho^l(B_i, x, t)$ (hence $\Delta_{i,s+1}(B_{i,s+1}; \gamma_t(x))$ will diverge).

Action: Add $\gamma_t(x)$ to D_{s+1} (by our convention this will cause $\Gamma_{s+1}(A_{s+1} \oplus D_{s+1}; x)\uparrow$ and will cause Case 2 to apply at the next possible stage).

Case 5: Currently never applies. Will be used in Section 5.

Case 6: For $i = e^x(t)$ and $l = p^x(t)$, $B_{i,s+1} \upharpoonright \rho^l(B_i, x, t) \neq B_{i,t} \upharpoonright \rho^l(B_i, x, t)$ (hence, by Equations (3.6) and (3.8), $B_{i,s+1} \upharpoonright \rho^{l-1}(B_i, x, t) \neq B_{i,t} \upharpoonright \rho^{l-1}(B_i, x, t)$ and therefore $\Delta_{i,s+1}(B_{i,s+1}; \gamma_t(x))$ would diverge except that we now redefine it).

Action: Let $e^x(s+1) = \bar{i}$ and $p^x(s+1) = p^x(s) - 1$. If $\Gamma_s(A_s \oplus D_s; x)\uparrow$ then define $\Gamma_{s+1}(A_{s+1} \oplus D_{s+1}; x) = B_{0,s+1}(x)$ with use $\gamma_t(x)$. Define $\Delta_{i,s+1}(B_{i,s+1}; \gamma_t(x)) = 0$ with large use. For $\gamma_t(x-1) < y < \gamma_t(x)$ and $j \in \{0, 1\}$, let $\Delta_{j,s+1}(B_{j,s+1}; y) = \Delta_{j,t}(B_{j,t}; y)$ with use $\delta_{j,t}(y)$. For $j = \bar{i}$, let $\Delta_{j,s+1}(B_{j,s+1}; \gamma_t(x)) = \Delta_{j,t}(B_{j,t}; \gamma_t(x))$ with use $\delta_{j,t}(\gamma_t(x))$.

Case 7: Otherwise.

Action: If $\Gamma_s(A_s \oplus D_s; x)\uparrow$ then define $\Gamma_{s+1}(A_{s+1} \oplus D_{s+1}; x) = B_{0,s+1}(x)$ with use $\gamma_t(x)$. For $\gamma_t(x-1) < y \leq \gamma_t(x)$ and $j \in \{0, 1\}$, let $\Delta_{j,s+1}(B_{j,s+1}; y) = \Delta_{j,t}(B_{j,t}; y)$ with use $\delta_{j,t}(y)$.

Remark 3.1 (Coordinating the action for different $\mathcal{N}_x$). We will assume that at any stage s we take the action needed for $\mathcal{N}_x$, for $x < s$, in increasing order. Hence since $i(x)$ is a nondecreasing function, $\gamma_s(x)$ is nondecreasing as a function of s and increasing as a function of x. Similarly for the use functions $\gamma_{i,s}$.

Definition 3.2. If Cases 1 or 2 apply or $s+1 = 0$ then we call $s+1$ a *free-clear* stage. If Cases 3, 4 or 5 apply for some $\mathcal{N}_y$, where $y \leq x$, we call $s+1$ an *almost free-clear* stage (assuming that there are infinitely many stages where $l(s) > x$, the next such stage will be a free-clear stage).

Definition 3.3. Let t be a free-clear stage. Let t' be the next almost free-clear stage or ∞ if there is none. We let $s \in I_x^t$ if either $s = t$ or $t < s < t'$ and $p^x(s) < p^x(s-1)$ (i.e. Case 6 holds).

Lemma 3.4. *Let t be a free-clear stage. Suppose for all $s \in I_x^t$, $p^x(s) \geq 0$. Then*
(i) For all $l < p^x(s)$ and for all $X \in \{A, B_0, B_1, E\}$, $X_s \restriction (\rho^l(X,x,t)+1) = X_t \restriction (\rho^l(X,x,t)+1)$ (by induction on s, this implies that $\rho^l(X,x,s) = \rho^l(X,x,t)$).
(ii) For $l = p^x(s)$ and $i = e^x(s)$, $B_{i,s} \restriction (\rho^l(B_i,x,t)+1) = B_{i,t} \restriction (\rho^l(B_i,x,t)+1)$ (again, by induction, this implies that $\rho^l(B_i,x,s) = \rho^l(B_i,x,t)$).

Proof. By induction. Clearly the lemma holds for $s = t$. Let $j = e^x(s-1) = \bar{i}$. Since s is in I_x^t and $\rho^l(B_j,x,t) = \rho^{l+1}(B_j,x,t)$ (see Equations (3.6) and (3.8)), for some $k \leq l$, $B_{j,s-1} \restriction (\rho^k(B_j,x,t)+1) \neq B_{j,t} \restriction (\rho^k(B_j,x,t)+1)$. Let k be as small as possible. Suppose $k < l$. Then, by the layering and the induction hypothesis, for some $X \in \{A, B_i\}$, $X_s \restriction (\rho^k(X,x,t)+1) \neq X_t \restriction (\rho^k(X,x,t)+1)$. So either Case 3 or 4 applies at stage s and hence $s \notin I_x^t$. Hence $k = l$ and (i) holds. $B_{i,s} \restriction (\rho^l(B_i,x,t)+1) = B_{i,t} \restriction (\rho^l(B_i,x,t)+1)$, otherwise s is an almost free-clear stage. □

We have just shown that, between (almost) free-clear stages, we only peel back the layers one at a time with changes alternating between B_0 and B_1. The idea of peeling back the layers is called the "top-down approach" in Weinstein [1988]. We say that it is possible to *peel back* (or away) the $p^x(s)^{\text{th}}$ layer of x at stage s.

It may be helpful to refer to Diagram 6 this point. This Diagram is a sub-diagram of Diagram 5. The numbered areas show where the changes in these sets must occur to peel back all 4 layers. The numbers refer to the sequence of events; i.e. there must be a change in first region followed by a change in the second, etc.

Lemma 3.5. *Either $\Gamma(A \oplus D; x)\downarrow$ and for all $y \leq \gamma(x)$, $\Delta_i(B_i; y)\downarrow$ (i.e. the functionals are well-defined) or it is not the case that $\Lambda(E) = A \oplus B_0 \oplus B_1$ and $\Psi_i(A \oplus B_{\bar{i}}) = B_i$.*

Proof. By induction on x. We can assume that there are infinitely many stages s where $l(s) > x$ (otherwise we are done). Case 1 can only apply once. If Case 2 applies infinitely often then it is not the case that $\Lambda(E) = A \oplus B_0 \oplus B_1$ and $\Psi_i(A \oplus B_{\bar{i}}) = B_i$. Case 6 can apply at

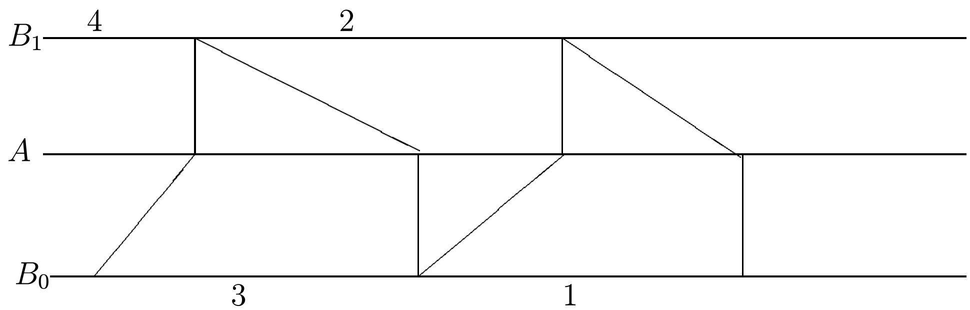

FIGURE 6. Where the changes must occur, in the required order, to peel back all 4 layers

most $i(x) - 2$ times between stages where Case 1 or 2 apply. Hence we can assume Case 7 applies almost always. Therefore $\lim_s \gamma_s(x)$ exists and $\Gamma(A \oplus D; x)\downarrow$. Furthermore, for all $\gamma(x-1) < y \leq \gamma(x)$, $\delta_i(y) = \lim_s \delta_{i,s}(y)$ exists and $\Delta_i(B_i; y)\downarrow$. □

Lemma 3.6. *Suppose* $\Lambda(E) = A \oplus B_0 \oplus B_1$ *and* $\Psi_i(A \oplus B_{\bar{i}}) = B_i$, *there is a* last *free-clear stage* t *and* $|I^t_x| \leq i(x) + 1$. *Then* $\mathcal{N}_x$ *is met.*

Proof. By induction on x. By the above lemma, $\Gamma(A \oplus D; x)\downarrow$ and for all $y \leq \gamma(x)$, $\Delta_i(B_i; y)\downarrow$. Since t is the last free-clear stage, $\Gamma(A \oplus D; x) = \Gamma_t(A_t \oplus D_t; x)$ and for all $y \leq \gamma(x)$, $\Delta_i(B_i; y) = \Delta_{i,t}(B_{i,t}; y)$.

Let s be the greatest stage in I^t_x. By Lemma 3.4 and the above hypothesis, $B_{0,s} \upharpoonright (\rho^0(B_0, x, t)+1) = B_{0,t} \upharpoonright (\rho^0(B_0, x, t)+1)$. $\rho^0(B_0, x, s) = \rho^0(B_0, x, t) = x$. Therefore $B_{0,s}(x) = B_{0,t}(x) = B_0(x)$ (otherwise s is not the greatest stage in I^t_x). Hence $\Gamma(A \oplus D; x) = \Gamma_t(A_t \oplus D_t; x) = B_{0,t}(x) = B_0(x)$.

Let $\gamma(x-1) < y \leq \gamma(x)$ (by Remark 3.1, we know $\gamma(x)$ is an increasing function). By the induction hypothesis, y can only enter D at some stage s iff $y = \gamma_{t'}(x')$, $l(s) > x$, and for both i, $\Delta_{i,s+1}(B_{i,s+1}; y)\uparrow$, for some x' and t', where $t' \leq t$ (see Case 4 or 5). By stage t we will have to redefine Δ_i to reflect this change in D with use 0. Hence $\Delta_i(B_i) \upharpoonright (\gamma(x)+1) = D \upharpoonright (\gamma(x)+1)$. □

Corollary 3.7. *Suppose for every free-clear stage* t, $|I^t_x| \leq i(x) + 1$. *Then* $\mathcal{N}_x$ *is met.*

Definition 3.8. Let s be a stage such that we act for $\mathcal{N}_x$ as above at stage s and let t be the greatest stage less than s such that we act for $\mathcal{N}_x$ as above at stage t (see Remark 3.13). Then $s \in I_x$ iff $E_t \upharpoonright (\rho(E, x, t)\downarrow +1) \neq E_s \upharpoonright (\rho(E, x, t)\downarrow +1)$ (if t does not exist then $s \notin I_x$).

Corollary 3.9. *If* $|I_x| \leq i(x) + 1$ *then* $\mathcal{N}_x$ *is met.*

Proof. If $s \in I_x^t$ then $s \in I_x$. (Note that the converse of the last statement does not hold.) □

Hence the easy way to met $\mathcal{N}_x$ is to ensure that $|I_x| \leq i(x) + 1$. In this case $\mathcal{N}_x$ applies finite restraint. We will close this section with a few remarks:

Remark 3.10 (The priority ordering). We will *always* assume that $\mathcal{N}_x$ has higher priority than $\mathcal{N}_y$ if and only if $x < y$.

Remark 3.11 (Initializing the strategy used for $\mathcal{N}_x$). We cannot initialize $\mathcal{N}_x$ without initializing $\mathcal{N}_y$ for all y. The strategies used by these requirements are all helping construct the same sets and functionals. Hence we will only allow initialization of all $\mathcal{N}_x$, in which case we can restart the construction of the needed sets and functionals and redefine the function $i(x)$. A requirement $\mathcal{R}$ will be allowed to initialize (cancel) the $\mathcal{N}_x$'s iff $\mathcal{R}$ has higher priority than $\mathcal{N}_0$ (and hence all the $\mathcal{N}_x$'s, by the above remark).

Remark 3.12 (Outcomes of $\mathcal{N}_x$). $\mathcal{N}_x$ has $i(x) + 2$ outcomes which we will use when needed. If Case 1, 3, 4 or 5 applies at stage $s + 1$ then this strategy has outcome 0 at stage $s + 1$. Otherwise this strategy has outcome $p^x(s + 1) + 1$ at stage $s + 1$. The final outcome of the strategy is the lim inf of the outcomes. Since our hope is to meet $\mathcal{N}_x$, by Corollary 3.7, we can assume that $p^x(s) \geq 0$, for all s. So if this strategy has outcome 0, then $\Gamma(A \oplus D; x)\uparrow$ (the converse is not true).

Remark 3.13 (Stages). In one of the following proofs we will need to restrict the stages that action for $\mathcal{N}_x$ can take place to *accessible* stages (see Section 4.3). In another we will need to restrict the action for $\mathcal{N}_x$ to *expansionary* stages (see Section 6.2). The above lemmas and discussion still hold true as long as we restrict all stages used above to appropriate stages.

Remark 3.14 (Indexing). All the sets A, B_0 and B_1 and all the functionals Λ, Ψ_0 and Ψ_1 should be indexed in some uniform fashion. Let e be an index for the sets A, B_0 and B_1 and the functionals Λ, Ψ_0 and Ψ_1. The above requirement and everything associated with it should be indexed by e. In particular, the subrequirement $\mathcal{N}_x$ should be $\mathcal{N}_{e,x}$ and $i(x)$ should be $i_e(x)$.

4. Lower Cones

In this section we present three theorems. All of these theorems involve the construction of a degree(s) which does not bound a weak critical triple. This section is split into three subsection each containing a theorem and its proof.

4.1. **Nonbounding.**

Theorem 4.1 ([Weinstein, 1988; Downey, 1990])**.** *There is a noncomputable, computably enumerable degree* **e** *which does not bound a weak critical triple in the computably enumerable degrees.*

It is enough to build a computably enumerable set E which meets all the requirements $\mathcal{N}_{e,x}$ and the requirements

$$\mathcal{R}_e \qquad \overline{E} \neq W_e.$$

We meet $\mathcal{R}_e$ by using the following procedure:

Action for $\mathcal{R}_{\mathbf{e}}$ at stage $s+1$. Do the first of the following cases which applies:

- **Case 1:** If there is no witness w for $\mathcal{R}_e$ and $E_s \cap W_{e,s} = \emptyset$, choose a large witness w.
- **Case 2:** If a witness w for $\mathcal{R}_e$ exists, $E_s \cap W_{e,s} = \emptyset$ and $w \in W_{e,s}$, add w to E_{s+1}. (If this case occurs then we say $\mathcal{R}_e$ acts.)

This requirement is positive; it wants to add elements to E. $\mathcal{R}_e$ acts at most once. To initialize this strategy means to discard the current witness. This strategy will met $\mathcal{R}_{\mathbf{e}}$ as long as it is not initialized infinitely often.

Choose some appropriate computable ω-ordering of all the requirements. Given a requirement $\mathcal{N}_{e,x}$ we will let $i_e(x)$ be the number of positive requirements of higher priority.

The Construction at stage $s+1$. For the first s requirements take the action described above (in order of increasing priority). For all e and x, if $\rho_e^s(E,x,s)\downarrow \neq \rho_e^s(E,x,s-1)$ then initialize all positive requirements which have lower priority than $\mathcal{N}_{e,x}$.

Lemma 4.2. *(i)* $|I_{e,x}| \leq i_e(x)$. *Hence* $\mathcal{N}_{e,x}$ *is met. Furthermore, either* $\rho_e(E,x)$ *exists or for almost all* s, $\rho_e(E,x,s)\uparrow$.

(ii) $\mathcal{R}_e$ *is only initialized finitely often. Hence* $\mathcal{R}_e$ *is met.*

Proof. By induction on priority.

(i) By the above initialization, only those $\mathcal{R}_{e'}$ with higher priority can put s into $I_{e,x}$ and each one of these $i_e(x)$ requirements can act at most once. By the induction hypothesis, there is a stage t by which all higher priority $\mathcal{R}_{e'}$ which ever act have acted. If there is a stage $s \geq t$ such that $\rho_e(E,x,s)\downarrow$ then $\rho_e(E,x) = \rho_e(E,x,s)$.

(ii) Wait for a stage s where all higher priority $\mathcal{R}_{e'}$ which are going to act have acted and for all higher priority $\mathcal{N}_{e,x}$ if $\rho_e(E,x)$ exists then $\rho_e(E,x) = \rho_e(E,x,s)$. $\mathcal{R}_e$ is never initialized after stage s. $\square$

4.2. **Splitting.** Clearly there are computably enumerable degrees which bound a weak critical triple in the computably enumerable degrees. The next theorem implies that the set of computably enumerable degrees which do not bound a weak critical triple generates the computably enumerable degrees (under join) and so does not form an ideal in the computably enumerable degrees.

Theorem 4.3. *Every degree* $\mathbf{g}$ *is the join of two degrees* $\mathbf{e_0}$ *and* $\mathbf{e_1}$ *neither of which bound a weak critical triple in the computably enumerable degrees.*

It is enough to build a pair of computably enumerable sets E_0 and E_1 and a functional Θ such that the requirements $\mathcal{N}^j_{e,x}$ (replace "E" with "E_j", for $j \in \{0,1\}$, and index everything with an additional superscript "j" in the discussion in Section 3) are met and the requirements

$$\Theta_y \qquad \Theta(E_0 \oplus E_1; y) = G(y)$$

are met. To meet Θ_y, we will use the following simple scheme: For all y, if $\Theta_s(E_{0,s} \oplus E_{1,s}; y)\downarrow \neq G_s(y)$ then add some $z \le \theta_s(y)$ to one of the E_i's. Θ_y is a positive requirement.

Choose some appropriate computable ω-ordering of all the requirements such that Θ_y has higher priority than Θ_z iff $y < z$. Assume $\mathcal{N}^{j_k}_{e_k,x_k}$ is the kth negative requirement. Let z^*_k be the least number such that $\Theta_{z^*_k}$ has lower priority than $\mathcal{N}^{j_{k-1}}_{e_{k-1},x_{k-1}}$. We define two functions p and $\tilde{p}$ by induction as follows: $p(1) = \tilde{p}(1) = 1$, $p(k+1) = p(k) + \tilde{p}(k+1)$, and

$$\tilde{p}(k+1) = z^*_{k+1} + \sum_{k' \le k} \tilde{p}(k'), \tag{4.1}$$

Let $i^{j_k}_{e_k}(x_k) = p(k)$. We make no claims that this function is a tight bound on the number of injuries. In the future we will drop the k from z^* since the requirement to which the z^*_k refers will always be clear from the context.

The Construction at stage $s+1$. For the first s requirements take the action described above (i.e for $\mathcal{N}^i_{e,x}$) or below (i.e. for Θ_y) in order of increasing priority.

Action for Θ_y **at stage** $s+1$. Do the first of the following cases which applies, if any:

Case 1: Suppose $\Theta_s(E_{0,s} \oplus E_{1,s}; y)\uparrow$. Then define $\Theta_{s+1}(E_{0,s+1} \oplus E_{1,s+1}; y) = G_{s+1}(y)$ with large use.

Case 2: Suppose $\Theta_s(E_{0,s} \oplus E_{1,s}; y)\downarrow \neq G_s(y)$. Let $\mathcal{N}^j_{e,x}$ be the highest priority negative requirement such that if $\rho^l(E_j, x, s)\downarrow$ then

$\rho^l(E_j, x, s) \geq \theta_s(y)$, where $l = p_e^{j,x}(s) - 1$. If $\mathcal{N}_{e,x}^j$ exists, then add $\theta_s(z^*)$ and $\theta_s(y)$ to $E_{s+1}^{\bar{j}}$ (in this case we say that $\mathcal{N}_{e,x}^j$ takes action for Θ). If $\mathcal{N}_{e,x}^j$ does not exist, then add $\theta_s(y)$ to E_{s+1}^0.

Clearly all the positive requirements are met. The following lemma shows that the negative requirements are also met.

Lemma 4.4. *$\mathcal{N}_{e,x}^j$ only takes action for Θ at most $\tilde{p}(k)$ times, where $\mathcal{N}_{e,x}^j$ is the kth negative requirement. $|I_{e,x}^j| \leq i_e^j(x)$. Hence $\mathcal{N}_{e,x}^j$ is met.*

Proof. By induction on priority order. Assume $\rho_e^l(E_j, x, s)\downarrow$, where $l = p_e^{j,x}(x) - 1$ and $\theta_s(z) \leq \rho_e^l(E_j, x, s)$ enters E_j at stage s. Then for all $z' \geq z$ and all stage $t > s$, if $\theta(z')\downarrow$, $\theta(z') > \rho_e^{l-1}(E_j, x, s)$. Unless some $\theta_t(z')$ later enters E_j, where $z' < z$, $\mathcal{N}_{e,x}^j$ will never take action for Θ again. By induction, the higher priority negative requirements can only cause such a z' to later enter E_j, $\sum_{k'<k} \tilde{p}(k')$ many times. There are $z^* - 1$ many positive requirements which also could cause such a z' to later enter E_j. Hence Equation 4.1 bounds the number of times $\mathcal{N}_{e,x}^j$ can take action. □

4.3. **Nonlow nonbounding.**

Theorem 4.5. *There is a nonlow computably enumerable degree* **e** *which does not bound a weak critical triple in the computably enumerable degrees.*

To show E is not low it is enough to build E and a set, W^E, which is computably enumerable in E, such that the following requirements are met:

$$\mathcal{R}_e \qquad w \in W^E \text{ iff } \lim_l \varphi_e(w, l) = 0, \text{ for some witness } w.$$

(If E is low then every set which is computably enumerable in E can be computably approximated. Hence W^E witnesses the fact that E is not low.) We meet $\mathcal{R}_e$ by using the following procedure:

Action for $\mathcal{R}_e$ at stage $s + 1$. Do the first of the following cases which applies, if any:

Case 1: A witness w does not exist. Choose a large witness w and a large use $u_{e,s+1}(w)$ and let $w \in W_{s+1}^{E_{s+1}}$ iff $\varphi_{e,s}(w, s) \neq 0$. If $\varphi_{e,s}(w, s)\downarrow$ then let $l(s + 1) = s + 1$, otherwise let $l(s + 1) = s$.

Case 2: A witness w exists, $\varphi_{e,s}(w, l(s)) = 0$ and $w \notin W_s^{E_s}$. Add $u_s(w)$ to E_{s+1}, add w to $W_{s+1}^{E_{s+1}}$, let $u_{e,s+1}(w)$ be large and let $l(s + 1) = l(s) + 1$.

Case 3: A witness w exists, $\varphi_{e,s}(w, l(s)) = 1$ and $w \in W_s^{E_s}$. Add $u_s(w)$ to E_{s+1}, remove w from $W_{s+1}^{E_{s+1}}$, let $u_{e,s+1}(w)$ be large and let $l(s+1) = l(s) + 1$.

Case 4: A witness w exists and $\varphi_{e,s}(w, l(s))\downarrow$. Let $l(s+1) = l(s)+1$.

This is a positive requirement; it wants to add elements to E. $\mathcal{R}_e$ can act infinitely many times. We say this strategy has outcome 0 if Case 2 or 3 applies infinitely often and outcome 1, otherwise. To initialize this strategy means to discard the current witness. This strategy will meet $\mathcal{R}_{\mathbf{e}}$ as long as it is not initialized infinitely often.

We will meet all the negative requirements $\mathcal{N}_{e,x}$ and all the positive requirements $\mathcal{R}_e$ by using a priority tree. Let $T = \{\omega\}^{<\omega}$ (the subtree of nodes which are accessible at some stage will be a finitely branching tree). Choose some appropriate computable ω-ordering of all the requirements $\mathcal{N}_{e,x}$. Assume $\mathcal{N}_{e_k,x_k}$ is the kth negative requirement. Let $i_{e_k}(x_k) = 2^{2k-1}$ (as always we make no claims this bound is tight). *All* the nodes of length $2k$ will work on meeting $\mathcal{N}_{e_k,x_k}$ using the *same* strategy as described in Section 3. Each node, α, of length $2e+1$ will work on meeting $\mathcal{R}_e$ using the above procedure on stages where α is accessible with its *own* witness w_α.

The Construction of E at stage $s+1$. By induction on k. Let $\beta_{s+1,0} = \lambda$. Let $\beta = \beta_{s+1,k}$. There are two cases:

Case 1: Suppose $|\beta| = 2m$. Use the above procedure for $\mathcal{N}_{e_m,x_m}$. Let o be the outcome of this strategy as determined in Remark 3.12. If $k < s+1$ then let $\beta_{s+1,k+1} = \beta_{s+1,k}\widehat{\ }o$.

Case 2: Suppose $|\beta| = 2m+1$. Use the above procedure for $\mathcal{R}_m$ (using the witness w_β). If $k < s+1$ then if either Case 2 or Case 3 applies (for the strategy based at the node β), let $\beta_{s+1,k+1} = \beta_{s+1,k}\widehat{\ }0$, otherwise let $\beta_{s+1,k+1} = \beta_{s+1,k}\widehat{\ }1$.

Let $\beta_{s+1} = \beta_{s+1,s+1}$. Initialize all nodes of odd length which are to the right of β_{s+1}.

Lemma 4.6. *Let $f = \liminf \beta_s$. For all k,*

(i) For all stages t, $|I^t_{e_k,x_k}| \le i_{e_k}(x_k)$. Hence $\mathcal{N}_{e_k,x_k}$ is met.

(ii) Let $\alpha = f \upharpoonright (2k+1)$. α is only initialized finitely often. Hence $\mathcal{R}_k$ is met.

Proof. By induction on k.

(i) Let $e = e_k$ and $x = x_k$. We say α injures $\mathcal{N}_{e,x}$ at stage $s+1$ if $s+1$ is not an almost free-clear stage, $s+1$ is not a free-clear stage, $u_s(w_\alpha) < \rho^l_e(E, x, s)$ and $u_s(w_\alpha)$ enters E at stage $s+1$, where $l = p^x_e(s) - 1$. Let $t < s+1$ be the last free-clear stage and $t' < t$ be the last almost free-clear stage (let $t' = t$, if such a stage does not exist). At stage t' all α such that $|\alpha| > 2k$, $|\alpha|$ is odd and $\alpha(2k) \neq 0$ are initialized. Hence no such α can injure $\mathcal{N}_{e,x}$. If α is such that $|\alpha| > 2k$, $|\alpha|$ is odd and $\alpha(2k) = 0$ then α can only act at almost free-clear stages for $\mathcal{N}_{e,x}$. Hence no such α can injure $\mathcal{N}_{e,x}$. Therefore if α injures $\mathcal{N}_{e,x}$ at stage $s+1$, $|\alpha| < 2k$ and α cannot injure again $\mathcal{N}_{e,x}$ until after the next almost free-clear stage (its use is too large). There are less than 2^{2k-1} such nodes.

(ii) Clear. □

5. The requirement $\mathcal{P}_i$

For $X \in \{A, B_0, B_1\}$, let $\widehat{X} = X \oplus E$. Let $\widehat{E} = K$. Our goal is to build a computably enumerable set E such that the following requirements are met:

$$\mathcal{P} \qquad \begin{gathered} \text{If } \widehat{\Lambda}(\widehat{E}) = \widehat{A} \oplus \widehat{B}_0 \oplus \widehat{B}_1 \text{ and } \widehat{\Psi}_i(\widehat{A} \oplus \widehat{B}_{\bar{i}}) = \widehat{B}_i \\ \text{then there exists a computably enumerable set } \widehat{D} \\ \text{and functionals } \widehat{\Delta}_i \text{ and } \widehat{\Gamma} \text{ such that} \\ \widehat{\Delta}_i(\widehat{B}_i) = \widehat{D} \text{ and } \widehat{\Gamma}(\widehat{A} \oplus \widehat{D}) = \widehat{B}_0 \end{gathered}$$

where $\widehat{\Lambda}$ and $\widehat{\Psi}_i$ are functionals and $\widehat{A}, \widehat{B}_0$ and $\widehat{B}_1$ are computably enumerable sets. If we meet $\mathcal{P}$ then the degrees represented by these three sets do not form a weak critical triple.

$\mathcal{P}$ is the dual of $\mathcal{N}$ under the operation of hatting. Hence we can use the dual of the strategy used for $\mathcal{N}$ to meet $\mathcal{P}$ and the dual of everything in Section 3 applies to the requirement $\mathcal{P}$. As in Section 3, we split $\mathcal{P}$ into infinitely many subrequirements:

$$\mathcal{P}_x \qquad \begin{gathered} \text{If there are infinitely many stages } s \text{ such that } \widehat{l}(s) > x \text{ then} \\ \widehat{\Gamma}(\widehat{A} \oplus \widehat{D}; x) = \widehat{B}_0(x) \text{ and } \widehat{\Delta}_i(\widehat{B}_i) \restriction (\widehat{\gamma}(x)+1) = \widehat{D} \restriction (\widehat{\gamma}(x)+1). \end{gathered}$$

The strategy for $\mathcal{P}_x$ is the dual of the strategy used for $\mathcal{N}_x$ with two additional features. First, there is a restraint function $\widehat{r}_x(s)$. $\widehat{r}_x(s)$ will be controlled by the negative requirements and at each stage will be determined *before* the strategy for $\mathcal{P}_x$ acts. $\widehat{r}_x(s)$ will be a nondecreasing function (in s). (Initially, $\widehat{r}_x(0) = 0$.) Second, Case 5 now reads:

Case 5: For $i = \widehat{e}^x(t)$ and $l = \widehat{p}^x(t)$, $\widehat{B}_{i,s+1} \restriction \widehat{\rho}^l(\widehat{B}_i, x, t) \neq \widehat{B}_{i,t} \restriction \widehat{\rho}^l(\widehat{B}_i, x, t)$ and $\widehat{\delta}_{\bar{i},s}(x) > \widehat{r}_x(s+1)$.

Action: Add $\widehat{\delta}_{\bar{i},s}(x)$ to E_{s+1} and $\widehat{\gamma}_t(x)$ to $\widehat{D}_{s+1}$ (by our convention this will make $\widehat{\Delta}_{\bar{i},s}(\widehat{B}_{\bar{i}}, x)\uparrow$, $\widehat{\Gamma}_{s+1}(\widehat{A}_{s+1} \oplus \widehat{D}_{s+1}; x)\uparrow$ and will cause Case 2 to apply at the next possible stage).

By Definition 3.2, we know if Case 5 applies at stage $s+1$, then $s+1$ is an almost free-clear stage. Case 5 adds numbers to E. In fact, this strategy may add infinitely many numbers into E. Hence this strategy is an infinite positive strategy (this makes sense since the dual of a negative strategy should be a positive strategy). Using the dual of the arguments in Section 3, one can prove:

Lemma 5.1 (The Dual of Corollary 3.7)**.** *Suppose for every free-clear stage t, $|\widehat{I}_x^t| \le \widehat{i}(x) + 1$. Then $\mathcal{P}_x$ is met.*

A word of caution: Since $\widehat{E} = K$, we cannot control $|\widehat{I}_x^t|$ ($|\widehat{I}_x|$). Hence while Lemma 5.1 (the dual of Lemma 3.9) holds we cannot ensure that the hypothesis holds. We need some other way to ensure that $\mathcal{P}_x$ is met.

Definition 5.2. Let t be a free-clear stage. Let t' be the next almost free-clear stage or ∞ if there is no such stage. We let $s \in \widehat{R}_x^t$ iff either $s = t$ or $t < s < t'$ and $\widehat{r}_x(s) > \widehat{r}_x(s-1)$ (since $\widehat{r}_x(s)$ is nondecreasing in s, these are the only stages at which the value of $\widehat{r}_x(s)$ changes).

Lemma 5.3. *Suppose for every free-clear stage t, $|\widehat{R}_x^t| \le \frac{1}{2}\widehat{i}(x) - 1$. Then $|\widehat{I}_x^t| \le \widehat{i}(x)$ and so $\mathcal{P}_x$ is met.*

Proof. First note there cannot be three stages, s_0, s_1 and s_2, such that $t \le s_0 \le s_1 < s_2 < t'$, $s_0 \in \widehat{R}_x^t$, $s_1 \in \widehat{I}_x^t$, $s_2 \in \widehat{I}_x^t$ and for all s if $s_0 < s \le s_2$, then $s \notin \widehat{R}_x^t$. (Assume otherwise then at stage s_2 Case 5 will apply rather than Case 6 and hence s_2 is an almost free-clear stage.) So two layers cannot be peeled away between stages when the restraint increases. Therefore $|\widehat{I}_x^t| \le 2(|\widehat{R}_x^t| + 1) \le \widehat{i}(x)$. □

One should think of $\frac{1}{2}\widehat{i}(x) - 1$ as a bound on the number of times the restraint can increase between a free-clear stage and the next almost free-clear stage and still meet $\mathcal{P}_x$.

6. Upper Cones

In this section we present two theorems each of which is presented in its own subsection. Both of these theorems involve the construction of a degree above which there is no weak critical triple (in the computably enumerable degrees). We end this section with a subsection which concerns mixing the requirements $\mathcal{P}_x$ with the standard lowness requirements.

6.1. **Upward nonbounding.** The following theorem follows from Theorem 4.1 and the pseudo-jump theorem in Jockusch and Shore [1983] (see Cholak and Downey [1993] for more details). A direct proof of the theorem using Harrington's "levels" method appears in Weinstein [1988].

Theorem 6.1 ([Weinstein, 1988]). *There is an incomplete computably enumerable degree* $\mathbf{e}$ *above which there is no weak critical triple in the computably enumerable degrees.*

To show E is not complete it is enough to build E and a computably enumerable set, W such that the following requirements are meet:

$\mathcal{R}_e$ $\qquad$ $w \in W$ iff $\Phi_e(E; w) = 0$, for some witnesses w.

(If E is complete then every set which is computably enumerable is computable in E. Hence W witnesses that E is not complete.) We meet $\mathcal{R}_e$ by using the following procedure:

Action for $\mathcal{R}_\mathbf{e}$ at stage $s+1$. Do the first of the following cases which applies, if any:

- **Case 1:** If a witness w does not exist, then choose a large witness w.
- **Case 2:** If a witness w exists and $\Phi_{e,s}(E_s; w) = 0$, add w to W and restrain E below $\varphi_{e,s}(w) + 1$.

This is a negative requirement; it wants to stop elements from entering E. This strategy is injured if $w \in W_s$ and some x later enters E below $\varphi_{e,s}(w) + 1$. When this occurs we will initialize the strategy. To initialize this strategy means to discard the current witness. $\mathcal{R}_e$ can act at most once unless initialized. This strategy will meet $\mathcal{R}_e$ as long as it is not initialized infinitely often.

We will meet all the negative requirements $\mathcal{R}_e$ and all the positive requirements $\mathcal{P}_{e,x}$ by using a tree. Let $T = \{\omega\}^{<\omega}$ (again, the subtree of nodes which are accessible at some stage will be a finitely branching tree). Let f be the true path and β_s be the approximation to the true path at stage s.

Choose some appropriate computable ω-ordering of all the requirements $\mathcal{P}_{e,x}$. Assume $\mathcal{P}_{e_k,x_k}$ is the kth positive requirement. *All* the nodes of length $2k$ will work on meeting $\mathcal{P}_{e_k,x_k}$ using the *same* strategy as described in Section 5 and the *same* restraint $\widehat{r}_{e_k,x_k}$. For shorthand, we will let $\widehat{r}_{e_k,x_k} = \widehat{r}_{2k}$.

A node β of length $2e+1$ will work on meeting $\mathcal{R}_e$ using the above procedure at stages when β is accessible with its *own* witness w_β. It would be nice if no node below β could injure the strategy used at β

but this is not possible. We can *only* restore all needed layers of a positive requirement when there is an almost free-clear stage. Hence we cannot ensure the needed layers are available when the computation converges. These layers may have been used on a previous computation which was later injured or initialized.

To determine $\widehat{i}_{e_k}(x_k)$ we need the following functions:

$$\text{If } k \leq e \text{ then } g(2e+1, k) = 0 \tag{6.1}$$

$$\text{If } k+1 > e \text{ then } g(2e+1, k+1) = 1 + 2g(2e+1, k) \tag{6.2}$$

$$h(k) = \sum_{j<k} (g(2j+1, k) + 1) \tag{6.3}$$

$$\widehat{i}_{e_k}(x_k) = 2(h(k) + 1) \tag{6.4}$$

(We make no claims that our bounds are tight.)

The Construction of E at stage $s+1$. By induction on k. Let $\beta_{s+1,0} = \lambda$. Let $\beta = \beta_{s+1,k}$. There are two cases:

Case 1: Suppose $|\beta| = 2m$. Use the above procedure for $\mathcal{P}_{e_m,x_m}$. We say β injures α at stage $s+1$ if $|\alpha| = 2e+1$, $\varphi_{e,s}(w_\alpha)\downarrow$ and Case 5 applies and adds a number less than or equal to $\varphi_{e,s}(w_\alpha)$ into E at stage $s+1$. Let o be the outcome of this strategy as determined by the dual of Remark 3.12. If $k < s+1$ then let $\beta_{s+1,k+1} = \beta\frown o$.

Case 2: Suppose $|\beta| = 2m+1$. Use the above procedure for $\mathcal{R}_m$ (using the witness w_β). If $k < s+1$ then let $\beta_{s+1,k+1} = \beta\frown 0$. Let $t > 2m$ be the last stage such that $\beta_t <_{\mathrm{L}} \beta$ (if such a stage does not exist then let $t = 2m+1$). If $|\alpha| > t$ then increase $\widehat{r}_{|\alpha|}(s+1)$ to $s+1$. For $|\alpha| \leq t$, let t_α be the last stage such that α injured β at stage t_α. Suppose that t_α exists, $t < t_\alpha$ and for all γ if $|\gamma| > |\alpha|$ then either t_γ does not exist or $t_\gamma < t_\alpha$. Then increase $\widehat{r}_{|\alpha|}(s+1)$ to $s+1$.

Let $\beta_{s+1} = \beta_{s+1,s+1}$. Initialize all nodes of odd length which are to the right of β_{s+1}.

Lemma 6.2. *Let $\beta \subset f$ (the true path on T) and $|\beta| = 2e+1$. Fix t, t' and k. Assume that for all stages s if $t \leq s < t'$, $\beta \leq \beta_s$. Assume no node of length $2k+1$ or greater injures β during those stages s where $t \leq s < t'$. Then β can be injured at most $g(2e+1, k)$ times during those stages s where $t \leq s < t'$.*

Proof. By induction on k. Clearly, the lemma holds for $k \leq e$ (otherwise β is initialized). All the nodes of length $2k$ are using the same strategy. If one of these nodes injures β, they all are later restrained from injuring β. Hence, collectively, these nodes can injure β at most once. But this injury allows all the nodes of length less than $2k$ to reinjure β, if needed. □

Lemma 6.3. *Let $\beta \subset f$ and $|\beta| = 2e+1$. Let $2k > |\beta|$. Then β can only increase the restraint function $\widehat{r}_{2k}$ $(g(2e+1,k)+1)$ times between a free-clear stage (for $\mathcal{P}_{e_k,x_k}$) and the next almost free-clear stage (for $\mathcal{P}_{e_k,x_k}$).*

Proof. Let t be a free-clear stage and t'' be the next almost free-clear stage. Let t' be the least stage such that $t < t' \leq t''$ and either $\beta_{t'} <_{\mathrm{L}} \beta$ or some node γ of length greater than $2k$ injures β. After stage t', β will not increase the restraint function $\widehat{r}_{2k}$ until after stage t''. If such a t' does not exist let $t' = t''$. Now by the above lemma, β can be injured at most $g(2e+1,k)$ times during those stages s where $t \leq s < t'$. □

Lemma 6.4. *Let $2k > |\beta|$.*

(i) The restraint function $\widehat{r}_{2k}$ only increases $h(k)$ times between a free-clear stage (for $\mathcal{P}_{e_k,x_k}$) and the next almost free-clear stage (for $\mathcal{P}_{e_k,x_k}$).

(ii) $|\widehat{R}^t_{e_k,x_k}| \leq \frac{1}{2}\widehat{i}_{e_k}(x_k) - 1$. Hence $\mathcal{P}_{e_k,x_k}$ is met.

Proof. (i) Follows from Lemma 6.3 and the fact that if $|\beta| > 2k$ then β can never increase $\widehat{r}_{2k}$.

(ii) Follows from (i) and Lemma 5.3. □

Lemma 6.5. *Let $\beta \subset f$ and $|\beta| = 2e+1$. β is only injured finitely often. Hence $\mathcal{R}_e$ is met.*

Proof. Let $2t \geq 2e+1$ be the least stage such that for all stages $s \geq 2t$, $\beta \leq_{\mathrm{L}} \beta_s$. No node of length $2e$ or less ever injures β after stage t. No node of length $2t+1$ or greater ever injures β after stage t. Now, by Lemma 6.2, β is injured at most $g(2e+1,t)$ more times. □

6.2. **Upward and downward nonbounding.**

Theorem 6.6. *There is a computably enumerable degree* **e** *above which there is no weak critical triple* and *below which there is no weak critical triple.*

By Downey and Shore [1996] we know that **e** must be low$_2$ since they show that below every nonlow$_2$ degree one can embed a copy of M_5.

We will meet all the negative requirements $\mathcal{N}_{e,x}$ and all the positive requirements $\mathcal{P}_{e,x}$ by using a tree. Let $T = \{\omega\}^{<\omega}$ (again, the subtree of nodes which are accessible at some stage will be a finitely branching tree). Let f be the true path and β_s be the approximation to the true path at stage s.

Choose some appropriate computable ω-ordering of all the requirements $\mathcal{P}_{e,x}$. Assume $\mathcal{P}_{e_k,x_k}$ is the kth positive requirement. If $x_k = 0$ then all nodes of length $2k$ will work on meeting $\mathcal{P}_{e_k,x_k}$; each using a *different* strategy. Such a node is called a *parent* node. If $|\alpha| = 2k$, α's parent node is the substring of length $2k'$, where $e_{k'} = e_k$ and $x_{k'} = 0$ (by the dual of Remark 3.10 such a k' always exists). All nodes α of length $2k$ which share the same parent node will work on meeting $\mathcal{P}_{e_k,x_k}$ using the *same* strategy as described in Section 5 and the *same* restraint $\widehat{r}_\alpha$. Such a node is called a *child* node of its parent. Hence if α and α' are using the same strategy to meet a positive requirement, $\widehat{r}_\alpha = \widehat{r}_{\alpha'}$.

Choose some appropriate computable ω-ordering of all the requirements $\mathcal{N}_{e,x}$. Assume $\mathcal{N}_{e_k,x_k}$ is the kth negative requirement. If $x_k = 0$ then all nodes of the nodes of length $2k+1$ will work on meeting $\mathcal{N}_{e_k,x_k}$; each using a *different* strategy. Such a node is called a *parent* node. If γ is a parent node and $|\gamma|$ is odd then let l_γ be the length of agreement function as defined in Equation 3.12. If $|\alpha| = 2k+1$, α's parent node is the substring of length $2k'+1$, where $e_{k'} = e_k$ and $x_{k'} = 0$ (by Remark 3.10 such a k' always exists). All nodes α of length $2k+1$ which share the same parent node will work on meeting $\mathcal{N}_{e_k,x_k}$ using the *same* strategy as described in Section 3. Such a node is called a *child* node of its parent (parents are children of themselves). We will let $\rho_\alpha(E,s) = \rho_{e_k}(E,x_k,s)$ if α is working on $\mathcal{N}_{e_k,x_k}$. Hence if α and α' are using the same strategy to meet a negative requirement, $\rho_\alpha(E,s) = \rho_{\alpha'}(E,s)$.

We can initialize parent nodes γ (see Remark 3.11) and all of their children. At which point we can redefine the function $i_{e_k}(x_k)$ for those children which are initialized.

We will meet $\mathcal{N}_{e_k,x_k}$ by using Corollary 3.9. Hence, we must be able to count the number of possible injuries. Similarly, since we will use Lemma 5.3, we must also count the number of times the restraint for $\mathcal{P}_{e_k,x_k}$ will increase between free-clear stages.

Informally, the key idea is: Assume β is working on $\mathcal{N}_{e_k,x_k}$. Let γ be β's parent node. Let α be a node of even length. Let $2t^* + j < t$ (for $j \in \{0,1\}$) be the last stage at which γ was initialized (if such a stage does not exist let $t^* = 0$). If $|\alpha| > t^* + |\beta|$ then α can never injure β. Assume $|\alpha| \leq t^* + |\beta|$. If α is started before β, α may injure β once. After this initial injury, α cannot injure β unless some α', where $|\alpha'| > |\alpha|$ injures β first.

Actions and accessibility at stage $s+1$: Assume β is working on $\mathcal{N}_{e_m,x_m}$. Fix a stage $s \geq |\beta|$. Let γ be β's parent node. $s+1$ is γ-expansionary iff $s = 0$ or $l_\gamma(s) > l_\gamma(t)$, for all $t < s$. β will only *act* at accessible γ-expansionary stages $s+1$ where $l_\gamma(s) > x_m$. We say β is *started* by stage $s+1$ if some β' working on $\mathcal{N}_{e_m,x_m}$ using the same strategy as β acts (for the first time) by stage $s+1$.

Injuries at stage $s+1$: Assume that α is working on $\mathcal{P}_{e_m,x_m}$ and that α acts by adding a number n into E at stage $s+1$. Assume β is working on $\mathcal{N}_{e_k,x_k}$, γ is β's parent node, $\gamma \subset \alpha$ and β has not been injured since the last stage t when it acted (defined below). If $n < \rho_\beta(E,t)$ then we say α *injures* β at stage $s+1$. If α injures β at stage $s+1$ and α' is also working on $\mathcal{P}_{e_m,x_m}$ using the same strategy as α, then we say α' injures β. The same thing occurs when β and β' are both using the same strategy to meet $\mathcal{N}_{e_k,x_k}$. Hence a child of γ can be injured at most once between γ-expansionary stages. Let $\mathcal{C} = \{\beta_i\}$ be a finite collection of γ's children. We say this finite collection is injured at stage $s+1$ if the collection has not been injured since the last γ-expansionary stage and some β_i is injured at stage $s+1$.

Restraint at stage $s+1$: Assume that β is working on $\mathcal{N}_{e_m,x_m}$ and that β acts at stage $s+1$. Let γ be β's parent. Let $2t^* + j < s+1$ be the last stage at which γ was initialized and $j \in \{0,1\}$ (if such a stage does not exist let $t^* = 0$). Let $t+1 \leq s$ be the last γ-stage. If $\rho_\beta(E,t) \neq \rho_\beta(E,s)$ or β is started by stage $s+1$ then do the following: Assume α is working on $\mathcal{P}_{e_k,x_k}$ and $\gamma \subset \alpha$. If $|\alpha| > 2t^* + |\beta|$ then increase $\widehat{r}_\alpha(s+1)$ to $s+1$. Let t' be the last stage when α injured some child of γ (if t' does not exist let $t' = 0$). Suppose that $2t^* + j < t'$ and β was started by stage t'. For $|\alpha'| > |\alpha|$ let $t_{\alpha'}$ be the last stage such that α' injured some child of γ at stage $t_{\alpha'}$. Suppose for all α', if $|\alpha| < |\alpha'|$ then either $t_{\alpha'}$ does not exist or $t_{\alpha'} < t$. Then increase $\widehat{r}_\alpha(s+1)$ to $s+1$.

Determining the stages $s+1$ at which β restrains α only depends on whether $s+1$ is γ-expansionary, the last stage where γ was initialized, $l_\gamma(s) > x_m$ and $\rho_\beta(E,s)$ has increased since the last γ-expansionary stage. Hence this only depends on γ and the stage. We will assume that α's restraint at stage $s+1$ is determined before α acts at stage $s+1$.

To determine $i_{e_k}(x_k)$ and $\widehat{i}_{e_k}(x_k)$ we need the following functions: Given k, let $p(k)$ be the length of the parent node for all nodes of length k (this is a well-defined function).

$$\text{If } k \leq m \text{ then } g(2m+1,k) = 0 \tag{6.5}$$

$$\text{If } m < k+1 \text{ then}$$

(6.6) $$g(2m+1,k+1) = 2^{2k} + (2^{2k}+1)g(2m+1,k)$$

(6.7) $$h(k) = \sum_{m<k} 2g(2m+1,k-1) + (k-m) + 2$$

(6.8) $$\widehat{i}_{e_k}(x_k) = 2(h(k)+1)$$

If β is working on $\mathcal{N}_{e_k,x_k}$ and β is initialized at stage $2t^* + j$, where $j \in \{0,1\}$, then define $i_{e_k}(x_k) = g(p(2k+1), t^* + k)$ at stage $2t^* + j$. (We make no claim that our bounds are tight.)

The Construction of E at stage $s+1$. By induction on k. Let $\beta_{s+1,0} = \lambda$. Let $\beta = \beta_{s+1,k}$. There are two cases:

- **Case 1:** Suppose $|\beta| = 2m$. Use the above procedure for $\mathcal{P}_{e_m,x_m}$ (in Section 5). Let o be the outcome of this strategy as determined by the dual of Remark 3.12. If $k < s+1$ then let $\beta_{s+1,k+1} = \beta\hat{\ }o$.
- **Case 2:** Suppose $|\beta| = 2m+1$. Let γ be β's parent node. If s is β-expansionary and $l_\gamma(x) > x_m$, use the above procedure for $\mathcal{N}_{e_m,x_m}$ (in Section 3) and impose restraint as described above. If $k < s+1$ then let $\beta_{s+1,k+1} = \beta\hat{\ }0$ (we do not care about the outcome in the sense of Remark 3.12).

Let $\beta_{s+1} = \beta_{s+1,s+1}$. Initialize all nodes whose parent node is to the right of β_{s+1}.

Lemma 6.7. *Let γ be a parent node of length $2m+1$. Fix k and some stages $t < t'$. Assume for all stages s, if $t \le s \le t'$, $\gamma \le \beta_s$. Let $2t^* + j < t$ (for $j \in \{0,1\}$) be the last stage at which γ was initialized (if such a stage does not exist let $t^* = 0$). Let $\mathcal{C} = \{\beta_i\}$ be a finite collection of γ's children such that for all i either β_i has been started by stage t or $2t^* + |\beta_i| < 2k$. Assume no node of length $2k+1$ or greater injures any β_i during those stages s where $t \le s < t'$. Then the collection $\{\beta_i\}$ can be injured at most $g(2m+1,k)$ times by nodes of length $2k$ or smaller during stages s where $t \le s < t'$.*

Proof. By induction on k. Clearly, the lemma holds for $k \le m$ (otherwise γ is initialized). Suppose a node α of length $2k$ injures β_i at some stage between t and t'. Hence α last injured a child of γ before β_i was started and $2t^* + |\beta_i| > 2k$. At the next γ-expansionary stage, α is restrained from injuring any β_i. There are 2^{2k} such nodes and they may act independently. But each one of these injuries allows all the nodes of length less than k to reinjure the collection $\mathcal{C}$, if needed. ☐

Lemma 6.8. *Let $\gamma \subset f$ be a parent node of length $2m+1$. Let $2t^* + j < t$ (for $j \in \{0,1\}$) be the last stage at which γ was initialized (if such a* *stage does not exist let $t^* = 0$). Let β be a child of γ working on $\mathcal{N}_{e_k,x_k}$ of length $2k+1$. Then β can be injured at most $i_{e_k}(x_k) = g(2m+1, t^*+k)$ times after stage $2t^* + j$. Hence for all e and x, $\mathcal{N}_{e,x}$ is meet.*

Proof. First note no node of length greater than $2t^* + 2k + 1 = 2(t^* + k) + 1$ can ever injure β. Now apply Lemma 6.7 to $\mathcal{C} = \{\beta\}$. □

Lemma 6.9. *Let $\gamma \subset f$ be a parent node of length $2m+1$. Fix $k > m$. Let α be a node of length $2k$. Let t be a free-clear stage for $\mathcal{P}_{e_k,x_k}$ and t' be the next almost free-clear stage t' (for $\mathcal{P}_{e_k,x_k}$). Let $2t^* + j < t$ (for $j \in \{0, 1\}$) be the last stage at which γ was initialized (if such a stage does not exist let $t^* = 0$).*

(i) After stage s, the children β of γ such that $2t^* + |\beta| > 2k$ can collectively increase α's restraint at most $g(2m + 1, k - 1) + 1$ times between t and t'.

(ii) After stage s, the children of γ such that $2t^* + |\beta| < 2k$ can collectively increase α's restraint at most $g(2m+1, k-1)+(k-m)+1$ times between t and t'.

(iii) After stage s, the children of γ of any length can collectively increase α's restraint at most $2g(2m + 1, k - 1) + (k - m) + 2$ times between t and t'.

Proof. (i) Let s_i be the stages such that $t \le s_0 < s_1 \ldots < s_j \le t'$, $\widehat{r}_\alpha(s_i) > \widehat{r}_\alpha(s_i - 1)$ and this increase was caused by a child β of γ such that $2t^* + |\beta| > 2k$. Let β_i be a child of γ such that β_i restrained α at stage s_i and $2t^* + |\beta_i| > 2k$. β_i was started by stage t. No α' such that $|\alpha| \le |\alpha'|$ can injure any child of γ between stages t and s_j (otherwise β_j cannot restrain α at stage s_j). γ cannot be initialized between stages t and s_j (otherwise β_j cannot restrain α at stage s_j). Let $i < i'$. Since s_i is a γ-expansionary stage and $\beta_{i'}$ was started by stage t, $\rho_{\beta_{i'}}(E, s_i)\downarrow < s_i$. So once β_i restrains α, $\beta_{i'}$ cannot restrain α unless it is later injured. Now Lemma 6.7 applies to $\mathcal{C} = \{\beta_i\}$. Hence $j \le g(2m + 1, k - 1) + 1$.

(ii) Let s_i be the stages such that $t \le s_0 < s_1 \ldots < s_j \le t'$, $\widehat{r}_\alpha(s_i) > \widehat{r}_\alpha(s_i - 1)$ and this increase was caused by a child β of γ such that $2t^* + |\beta| < 2k$. Let β_i be a child of γ of length such that β_i restrained α at stage s_i and $2t^* + |\beta_i| < 2k$. No node of length $2k$ or greater can ever injure any β_i. γ cannot be initialized between stages t and s_j (otherwise β_j cannot restrain α at stage s_j). Let $i < i'$. Since s_i is a γ-expansionary stage, if $\beta_{i'}$ was started by stage s_i, $\rho_{\beta_{i'}}(E, s_i)\downarrow < s_i$. Hence once β_i restrains α, $\beta_{i'}$ cannot restrain α unless it is later started or injured. After $(k - m)$ γ-expansionary stages, all the β_i are started. Lemma 6.7 applies to $\mathcal{C} = \{\beta_i\}$. Hence $j \le g(2m+1, k-1)+(k-m)+1$.

(iii) Since γ has no child of even length this follows from (i) and (ii). □

Corollary 6.10. *Suppose α is working on $\mathcal{P}_{e_k,x_k}$ of length $2k$. Let t be a free-clear stage for $\mathcal{P}_{e_k,x_k}$ and t' be the next almost free-clear stage t' (for $\mathcal{P}_{e_k,x_k}$).*

(i) α's restraint can increase at most $h(k)$ times between t and t'.

(ii) $|\widehat{R}^t_{e_k,x_k}| \leq \frac{1}{2}\widehat{i}_{e_k}(x_k) - 1$. Hence $\mathcal{P}_{e_k,x_k}$ is met.

Proof. (i) Only those β whose parent node γ is a substring of α can restrain α. Hence the above lemma applies at most k times.

(ii) Follows from (i) and Lemma 5.3. □

6.3. $\mathcal{P}_{e,x}$ **and lowness requirements.** The astute reader might wonder why one could not combine the requirements $\mathcal{P}_{e,x}$ and lowness requirements in the following fashion: First note that one meets the standard lowness requirements by finite restraint. All the negative requirements in this section were met by finite restraint. However one cannot meet the standard lowness requirements by acting independently at different nodes on a tree with infinitary positive requirements. Now turn the infinite positive strategy used for $\mathcal{P}_{e,x}$ into a finite positive strategy by only allowing Case 5 to act when x actually enters B_0 and ignoring the outcomes. Hence a negative requirement can be injured at most once by each positive requirement of higher priority. So hopefully we can have enough layers in the positive requirements of lower priority to handle this fixed number of increases in the restraint.

However, the problem with this approach, is that a positive requirement $\mathcal{P}_{e,x}$ may in fact injure a negative requirement of higher priority: The strategy used for $\mathcal{P}_{e,x}$ may have peeled away all of its layers and be in a position that it *must* add some number n into E iff x enters B_0. Now the computation for the negative requirement converges and wishes to restrain n. Then x enters B_0 which forces n to enter E. This may be repeated infinitely many times for the same negative requirement. This remains true even if we restrict the positive requirements to a single fixed e. Thus the question of whether M_5 can be embedded above every low computably enumerable degree remains open.

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