Halin's Infinite Ray Theorems: Complexity and Reverse Mathematics: Version E*

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Abstract

Halin [1965] proved that if a graph has n many pairwise disjoint rays for each n then it has infinitely many pairwise disjoint rays. We analyze the complexity of this and other similar results in terms of computable and proof theoretic complexity. The statement of Halin's theorem and the construction proving it seem very much like standard versions of compactness arguments such as König's Lemma. Those results, while not computable, are relatively simple. They only use arithmetic procedures or, equivalently, finitely many iterations of the Turing jump. We show that several Halin type theorems are much more complicated. They are among the theorems of hyperarithmetic analysis. Such theorems imply the ability to iterate the Turing jump along any computable well ordering. Several important logical principles in this class have been extensively studied beginning with work of Kreisel, H. Friedman, Steel and others in the 1960s and 1970s. Until now, only one purely mathematical example was known. Our work provides many more and so answers Question 30 of Montalbán's Open Questions in Reverse Mathematics [2011]. Some of these theorems including ones in Halin [1965] are also shown to have unusual proof theoretic strength as well.

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1 Introduction

In this paper we analyze the complexity of several results in infinite graph theory. These theorems are said to be ones of Halin type or, more generally, of ubiquity theory. The classical example is a theorem of Halin [11]: If a countable graph G contains, for each n, a sequence $\langle R_0, \ldots, R_{n-1}\rangle$ of disjoint rays (a ray is a sequence $\langle x_i \mid i \in N\rangle$ of distinct vertices such that there is an edge between each x_i and x_{i+1}) then it contains an infinite such sequence of rays. (Note: As will be described in Definition 3.2, when we talk about disjoint rays we always mean pairwise disjoint.) Halin actually deals with arbitrary graphs and formulates the result differently. The uncountable cases, however, are essentially just counting arguments. We deal only with countable structures but discuss his formulation in §6. This standard formulation of his theorem seems like a typical compactness theorem going from arbitrarily large finite collections of objects to an infinite collection. The archetypical example here is König's Lemma: If a finitely branching tree has paths of length n for every n then it has a branch, i.e. an infinite path. In outline, a modern proof of Halin's theorem for countable graphs (due to Andreae, see [5, Theorem 8.2.5(i)]) seems much like that of König's Lemma (and many others in infinite graph theory). The construction of the desired sequence of rays proceeds by a recursion through the natural numbers in which each step is a simple procedure. While the procedure is much more delicate than for König's Lemma, it is basically of the same complexity. It uses Menger's theorem for finite graphs at each step but this represents a computable procedure (for finite graphs). The other parts of the step depend on the same type of information as in König's Lemma. They ask, for example, if various sets (computable in the given graph) are nonempty or infinite. Nonetheless, we prove that the complexity of this construction and theorem are much higher than that for König's Lemma or other applications of compactness. The concepts from graph theory, computability theory and proof theory/reverse mathematics that we need for our analysis are discussed in §3. Basic references for terminology, background and standard results not explicitly stated or otherwise attributed are Diestel [5] for graph theory; Rogers [23] and Sacks [25] for computability theory; and for reverse mathematics Simpson [29] with an approach which is primarily in terms of formal systems and Hirschfeldt [13] with one primarily emphasizing computability.

We follow two well established procedures for measuring the complexity of constructions and theorems. The first is basically computability theoretic. It has its formal beginnings in the 1950s but has much earlier roots in constructive or computable mathematics reaching back to antiquity. (See Ershov et al. [6] for history and surveys of the approach in several areas of combinatorics, algebra and analysis.) The measuring rod here is relative computability. We say a set A of natural numbers is (Turing) computable from a set B, $A \leq_T B$, if there is an algorithm (say on a Turing machine or any other reasonable model of general computation) that, when given access to all membership facts about B (an oracle for B) computes membership in A. The standard hierarchies of complexity here are based on iterations of the Turing jump. This operator takes B to B', the halting problem relativized to B, i.e. the set of programs with oracle for B, Φ_e^B, such that Φ_e^B halts on input e. For example, if the tree of König's Lemma is computable in B then there is a branch computable in the double jump B'' of B.

The second approach is proof theoretic. It measures the complexity of a theorem by the logical strength of the axioms needed to prove it. This approach also has a long history but the formal subject, now called reverse mathematics, starts with H. Friedman's work in the 1970s (e.g. [7, 8]). One compares axiomatic systems S and T by saying that T is stronger than S, $T \vdash S$ (T proves S) if one can prove every sentence $\Theta \in S$ from the axioms of T. Of course, we know what it means for Θ to be provable in S. The goal here is to characterize to the extent possible the axioms needed to prove a given mathematical theorem Θ. To this end, one begins with a weak base theory. Then one wants to find a system S such that not only does $S \vdash \Theta$ but also Θ (with the weak base theory) proves all the axioms of S. Hence the name reverse mathematics as we seek to prove the "axioms" of S from the theorem Θ. Typically, the systems here are formalized in arithmetic with quantification over sets as well as numbers. The standard base theory (RCA_0) corresponds to the axioms needed to do computable constructions. Stronger systems are then usually generated by adding comprehension axioms which assert the existence of specific families of sets. For example, a very important system is ACA_0. It is equivalent in the sense of reverse mathematics just described to König's Lemma. Formally, it asserts that every subset of the natural numbers defined by a formula that quantifies only over numbers (and not sets) exists. This is also equivalent to asserting that for every set B, the set B' exists.

The early decades of reverse mathematics were marked by a large variety of results characterizing a wide array of theorems and constructions as being one of five specific levels of complexity including RCA_0 and ACA_0. Each of these systems (Simpson's "big five") have corresponding specific recursion theoretic construction principles. In more recent decades, there has been a proliferation of results placing theorems and constructions outside the big five. Sometimes these are inserted linearly and sometimes with incomparabilities. They are now collectively often called the "zoo" of reverse mathematics. (See https://rmzoo.math.uconn.edu/diagrams/ for pictures.)

Theorems and constructions in combinatorics in general, and graph theory in particular, have been a rich source of such denizens of this zoo. Almost all of them have fallen below ACA_0 (König's Lemma) and so have the objects they seek constructible computably in finitely many iterations of the Turing jump. Ramsey theory, in particular, has provided a very large class of constructions and theorems of distinct complexity. One example of the infinite version of a classical theorem of finite graph theory that is computationally and reverse mathematically strictly stronger than ACA_0 is König's Duality Theorem (KDT) for countable graphs. (Every bipartite graph has a matching and a cover consisting of one vertex from each edge of the matching.) The proofs of this theorem for infinite graphs (Podewski and Steffens [22] for countable and Aharoni [1] for arbitrary ones) are not just technically difficult but explicitly used both transfinite recursions and well orderings of all subsets of the given graph. These techniques lie far beyond ACA_0. Aharoni, Magidor and Shore [2] proved that this theorem is of great computational strength in that there are computable graphs for which the required matching and cover compute all the iterations of the Turing jump through all computable well-orderings. They also showed that it was strong reverse mathematically as it implied ATR_0, the standard system above ACA_0 used to deal with such transfinite recursions. Some of the lemmas used in each of the then known proofs were shown to be equivalent to the next and final of the big five systems, Π^1_1-CA_0 and of corresponding computational strength. Simpson [28] later provided a new proof of the theorem using logical methods that avoided these lemmas and showed that the theorem itself is equivalent to ATR_0 and so strictly weaker than the lemmas both computationally and in terms of reverse mathematics.

The situation for the theorems of Halin type that we study here is quite different. The standard proofs do not seem to use such strong methods. Nonetheless, as we mentioned above, the theorems are much stronger than ACA_0 with some versions not even provable in ATR_0. We prove that these theorems occupy a few houses in the area of the reverse mathematics zoo devoted to what are called theorems (or theories) of hyperarithmetic analysis, THAs (Definition 3.13). Computationally, for each computable well ordering α, there is a computable instance of any THA which has all of its required objects Turing above $0^{(\alpha)}$, the αth iteration of the Turing jump. On the other hand, they are computationally and proof theoretically much weaker than ATR_0 and so KDT. The point here is that there is a single computable graph such that the matching and cover required by KDT lies above $0^{(\alpha)}$ for all the computable well-orderings α, while for each computable instance of a THA there is a computable well-ordering α such that $0^{(\alpha)}$ computes the desired object. In our cases, the instances are graphs with arbitrarily many disjoint rays and the desired object is an infinite sequence of disjoint rays. (The general usage of terms like instances and solutions of a theorem or principle is described at the end of §3.)

Beginning with work of Kreisel [15], H. Friedman [9], Steel [30] and others in the 1960s and 1970s and continuing into the last decade (by Montalbán [16, 17, 19], Neeman [20, 21] and others), several axiomatic systems and logical theorems were found to be THAs and proven to lie in a number of distinct classes in terms of proof theoretic complexity. Until now, however, there has been only one mathematical but not logical example, i.e. one not mentioning classes of first order formulas or their syntactic complexity. This was a result (INDEC) about indecomposability of linear orderings in Jullien's thesis [14] (see Rosenstein [24, Lemma 10.3]). It was shown to be a THA by Montalbán [16].

The natural quest then became to find out if there are any other THAs in the standard mathematical literature. The issue was raised explicitly in Montalbán's "Open Questions in Reverse Mathematics" [18, Q30]. As our answer, we provide many examples. Most of them are provable in a well known system above ACA_0 gotten by adding on a weak form of the axiom of choice (Σ^1_1-AC_0).

Several of the basic Halin type theorems (the $\mathsf{IRT}_{\mathrm{XYZ}}$ defined after Definition 3.4) have versions (the $\mathsf{IRT}^*_{\mathrm{XYZ}}$ of Definition 6.1) like those appearing in the original papers that show that there are always families of disjoint rays of maximal cardinality which are of the same computational strength as the basic versions (Proposition 6.3 and Corollary 6.4). On the other hand, the $\mathsf{IRT}^*_{\mathrm{XYZ}}$ are strictly stronger proof theoretically than the $\mathsf{IRT}_{\mathrm{XYZ}}$ because they imply more induction than is available in Σ^1_1-AC_0 (Theorem 6.8 and Corollary 6.9). Two of the variations we consider are as yet open problems of graph theory ([3] and Bowler, personal communication). We show that if we restrict the class of graphs to directed forests the principles are not only provable but reverse mathematically equivalent to Σ^1_1-$\mathsf{AC}_0 + \mathsf{I}\Sigma^1_1$. Note that as $\mathsf{ATR}_0 \nvdash \mathsf{I}\Sigma^1_1$ [29, IX.4.7], these theorems are not provable even in ATR_0 or from KDT (Corollary 6.14). We do not know of other mathematical but nonlogical theorems of this strength. Other versions that require maximal sets of rays (Definition 6.17) are much stronger and, in fact, equivalent to Π^1_1-CA_0 (Theorem 6.18).

2 Outline of Paper

Section 3 discusses basic concepts and background information. The first subsection (3.1) provides what we need from graph theory. Almost all the definitions are standard. At times we give slight variations that are equivalent to the standard ones but make dealing with the computability and proof theoretic analysis easier. We also state the theorems of Halin and some variants that are the main targets of our analysis.

The second subsection (3.2), assumes an intuitive view of computability of functions $f : N \to N$ such as having an algorithm given by a program in any standard computer language (possibly with access to an "oracle" providing information about a given set or function). It then gives the standard notions and theorems that can be found in basic texts on computability theory needed to follow our analysis of the computational complexity of the graph theoretic theorems we study. In particular, it notes the Turing jump operator and its iterations along countable well orderings. These are our primary computational measuring rods. The final subsection (3.3) provides the syntax and semantics for the formal systems of arithmetic that are used to measure proof theoretic complexity. It also describes the standard basic axiomatic systems and their connections to the computational measures of the previous subsection. It includes the formal definition of the class of theorems which includes most of our graph theoretic examples, the THAs, Theorems (or Theories) of Hyperarithmetic Analysis. These are defined in terms of the transfinite iterations of the Turing jump and the hyperarithmetic sets of the previous subsection. In addition it defines Σ^1_1-AC_0 a weak version of the axiom of choice that is an early well known example of such theories and plays a crucial role in our analysis.

Section 4 provides the proof that Halin's original theorem IRT (Definition 3.4) is computationally very complicated. For example, given any iteration $0^{(\alpha)}$ of the Turing jump, there is a computable graph satisfying the hypotheses of IRT such that any instance of its conclusion computes $0^{(\alpha)}$. Indeed, IRT is a THA. At times, theorems or lemmas are stated in terms of the formal systems of §3.3, but the proofs rely only on the computational notions of §3.2.

Section 5 studies several variations $\mathsf{IRT}_{\mathrm{XYZ}}$ of Halin's IRT where we consider directed as well as undirected graphs, edge rather than vertex disjointness for the rays and double as well as single rays. (See Definitions 3.1 and 3.2 and the discussion after Definition 3.4.) We provide reductions over RCA_0 between many of the pairs of the eight possible variants. The proofs of these reductions proceed purely combinatorially by providing one computational process that takes an instance of some $\mathsf{IRT}_{\mathrm{XYZ}}$, i.e. a graph satisfying its hypotheses, and produces a graph satisfying the hypotheses of another $\mathsf{IRT}_{\mathrm{X'Y'Z'}}$ and another computable process that takes any solution to the $\mathsf{IRT}_{\mathrm{X'Y'Z'}}$ instance, i.e. any sequence of rays satisfying the conclusion of $\mathsf{IRT}_{\mathrm{X'Y'Z'}}$, and produces a solution to the original instance of $\mathsf{IRT}_{\mathrm{XYZ}}$. (See Propositions 5.3, 5.5 and 5.7 and the associated Lemmas. An additional reduction using a stronger base theory is given in the next section (Theorem 6.15).)

We then show that five of the eight possible variants of IRT are THAs (Theorem 5.1). As mentioned in §1, of the remaining three, two are still open problems in graph theory. We do, however, have an analysis of their restrictions to special classes of graphs in Theorem 5.16 and §6.1. The last of the variations, $\mathsf{IRT}_{\mathrm{UED}}$, has been proven more recently by Bowler, Carmesin, Pott [3] using more sophisticated methods than the other results. We have some lower bounds

(Theorem 5.9) but we have yet to fully analyze the complexity of their construction.

In the next section (§6) we study some variations of IRT that ask for different types of maximality for the solutions. The first sort actually follow the original formulation of IRT in Halin [11]: In any graph there is a set of disjoint rays of maximum cardinality. For uncountable graphs this amounts to a basic counting argument on uncountable cardinals as all rays are countable. When restricted to countable graphs these variations, $\mathsf{IRT}^*_{\mathrm{XYZ}}$, are easily seen to be equivalent to our more modern formulation by induction. Technically, the induction used is for Σ^1_1 formulas ($\mathsf{I}\Sigma^1_1$) which is not available in RCA_0. More specifically we show (Proposition 6.3) that $\mathsf{IRT}_{\mathrm{XYZ}} + \mathsf{I}\Sigma^1_1$ and $\mathsf{IRT}^*_{\mathrm{XYZ}} + \mathsf{I}\Sigma^1_1$ are equivalent (over RCA_0). As the definition of THAs only depends on standard models where full induction holds, if $\mathsf{IRT}_{\mathrm{XYZ}}$ is a THA then so is $\mathsf{IRT}^*_{\mathrm{XYZ}}$.

We then prove that these maximal cardinality variants $\mathsf{IRT}^*_{\mathrm{XYZ}}$ are strictly stronger proof theoretically than the basic $\mathsf{IRT}_{\mathrm{XYZ}}$ (when they are known to be provable in Σ^1_1-AC_0). This is done by showing (Theorem 6.8 and the Remark that follows it) that the relevant $\mathsf{IRT}^*_{\mathrm{XYZ}}$ all imply weaker versions of $\mathsf{I}\Sigma^1_1$ that are analogous to the restrictions of Σ^1_1-AC_0 embodied in weak (or unique)-Σ^1_1-AC_0 and finite-Σ^1_1-AC_0 (Definitions 7.1 and 7.2). In all the cases, it is enough induction to prove (with the apparatus of the basic $\mathsf{IRT}_{\mathrm{XYZ}}$) the consistency of Σ^1_1-AC_0 and so by Gödel's second incompleteness theorem they cannot be proved in Σ^1_1-AC_0 (Corollary 6.9).

As for proving full Σ^1_1 induction from an $\mathsf{IRT}^*_{\mathrm{XYZ}}$ we are in much the same situation mentioned above for Σ^1_1-AC_0 and $\mathsf{IRT}_{\mathrm{XYZ}}$. In particular, $\mathsf{IRT}^*_{\mathrm{DVD}}$ and $\mathsf{IRT}^*_{\mathrm{DED}}$ for directed forests each proves $\mathsf{I}\Sigma^1_1$ as well as Σ^1_1-AC_0 (Theorems 6.12 and 6.13) and so are equivalent to $\mathsf{I}\Sigma^1_1 + \Sigma^1_1$-$\mathsf{AC}_0$. As before, this shows that they are strictly stronger than Σ^1_1-AC_0 (Corollary 6.14). Indeed, as mentioned at the end of §1, they are not even provable in ATR_0. We do not know of any other mathematical theorem with this level of reverse mathematical strength.

The second variation of maximality, $\mathsf{MIRT}_{\mathrm{XYZ}}$, studied in §6.2 is also mentioned in the original Halin paper [11]. It asks for a set of disjoint rays which is maximal in the sense of set containment. Of course, this follows immediately from Zorn's Lemma for all graphs. For countable graphs we provide a reverse mathematical analysis, showing that each of the $\mathsf{MIRT}_{\mathrm{XYZ}}$ is equivalent to Π^1_1-CA_0 (Theorem 6.18).

In §7, we discuss the reverse mathematical relationships between the THAs associated with variations of Halin's theorem and previously studied THAs as well as one new logical THA (finite-Σ^1_1-AC_0 of Definition 7.1). Basically, all the $\mathsf{IRT}^*_{\mathrm{XYZ}}$ (and so $\mathsf{IRT}_{\mathrm{XYZ}} + \mathsf{I}\Sigma^1_1$) imply H. Friedman's ABW_0 (Definition 7.4) by Theorem 7.7 and finite-Σ^1_1-AC_0 (Theorem 7.3). On the other hand, none of them are implied by it (Theorem 7.10) or by Δ^1_1-CA_0 (Definition 7.8 and Theorem 7.9). $\mathsf{ABW}_0 + \mathsf{I}\Sigma^1_1$ does, however, imply finite-Σ^1_1-AC_0 which is not implied by weak (unique)-Σ^1_1-AC_0 (Goh [10]). Figure 4 summarizes many of the known relations with references.

In the penultimate section (§8) we study the only use of Σ^1_1-AC_0 in each of our proofs of $\mathsf{IRT}_{\mathrm{XYZ}}$. It consists of $\mathsf{SCR}_{\mathrm{XYZ}}$ which says we can go from the hypothesis that there are arbitrarily many disjoint rays to a sequence $\langle X_k \rangle_k$ in which each X_k is a sequence of k many disjoint rays. We analyze the strength of the $\mathsf{SCR}_{\mathrm{XYZ}}$ and the weakenings $\mathsf{WIRT}_{\mathrm{XYZ}}$ of $\mathsf{IRT}_{\mathrm{XYZ}}$ which each take the existence of such a sequence $\langle X_k \rangle_k$ as its hypothesis in place of there being arbitrarily many disjoint rays. For example, for all the $\mathsf{IRT}_{\mathrm{XYZ}}$ which are consequences of Σ^1_1-AC_0 and so are THAs, $\mathsf{IRT}_{\mathrm{XYZ}}$ is equivalent to $\mathsf{SCR}_{\mathrm{XYZ}}$ over RCA_0 (Corollary 8.5) and so all of them are also THAs. For the same choices of XYZ, ACA_0 proves $\mathsf{WIRT}_{\mathrm{XYZ}}$ over RCA_0. While a natural strengthening of $\mathsf{WIRT}_{\mathrm{XYZ}}$ does imply ACA_0 and indeed is equivalent to it (Theorem 8.9), we do not know if $\mathsf{WIRT}_{\mathrm{XYZ}}$ itself implies ACA_0. All we can prove is that it is not a consequence of RCA_0 (Theorem 8.10).

In the last section (§9), we mention some open problems.

3 Basic Notions and Background

We begin with basic notions and terminology from graph theory. At times we use formalizations that are clearly equivalent to more standard ones but are easier to work with computationally or proof-theoretically. The following two subsections supply background and basic information about the standard computational and logical/proof theoretic notions that we use here to measure the complexity of the graph theorems and constructions that we analyze in the rest of this paper. Note that we denote the set of natural numbers by N when we may be thinking of them in a model of arithmetic as in §3.3 and by $\mathbb{N}$ when we emphasize that we specifically want the standard natural numbers.

3.1 Graph Theoretic Notions

Definition 3.1. A *graph* H is a pair $\langle V, E\rangle$ consisting of a set V (of *vertices*) and a set E of unordered pairs $\{u, v\}$ with $u \neq v$ from V (called *edges*). These structures are also called *undirected graphs (or here U-graphs)*. A structure H of the form $\langle V, E\rangle$ as above is a *directed graph (or here D-graph)* if E consists of ordered pairs $\langle u, v\rangle$ of vertices with $u \neq v$. To handle both cases simultaneously, we often use X to stand for undirected (U) or directed (D). We then use (u, v) to stand for the appropriate kind of edge, i.e. $\{u, v\}$ or $\langle u, v\rangle$.

An *X-subgraph* of the X-graph H is an X-graph $H' = \langle V', E'\rangle$ such that $V' \subseteq V$ and $E' \subseteq E$. It is an *induced X-subgraph* if $E' = \{(u, v) \mid u, v \in V' \wedge (u, v) \in E\}$.

Definition 3.2. An *X-ray in H* is a pair consisting of an X-subgraph $H' = \langle V', E'\rangle$ of H and an isomorphism f from N with edges $(n, n+1)$ for $n \in N$ to H'. (Note that this implies that the range of f is the set V'.) We say that the *ray begins at* $f(0)$. We also describe this situation by saying that H contains the X-ray $\langle H', f\rangle$. We sometimes abuse notation by saying that the sequence $\langle f(n)\rangle$ of vertices is an X-ray in H. A *tail* of an X-ray is a final segment of said X-ray. Similarly we consider *double X-rays* where the isomorphism f is from the integers $\mathcal{Z} = \{-n, n \mid n \in N\}$ with edges $(z, z+1)$ for $z \in \mathcal{Z}$. A *tail* of a double X-ray is a final segment of said X-ray, or an initial segment of said X-ray considered in reverse order.

We use Z-ray to stand for either a (single) ray ($Z =$ S) or double ray ($Z =$ D) and so we have, in general, Z-X-rays or just Z-rays if the type of graph (U or D) is already established. For brevity, when we describe rays we will often only list their vertices in order instead of defining H' and f explicitly. However the reader should be aware that we always have H' and f in the background.

H contains k many Z-X-rays for $k \in N$ if there is a sequence $\langle H_i, f_i\rangle_{i<k}$ such that each $\langle H_i, f_i\rangle$ is a Z-X-ray in H (with $H_i = \langle V_i, E_i\rangle$). *$H$ contains k many disjoint (or vertex-disjoint) Z-X-rays* if the V_i are pairwise disjoint. *H contains k many edge-disjoint*

Z-X-*rays* if the E_i are pairwise disjoint. We often use Y to stand for either vertex (V) or edge (E) as in the following definitions.

An X-graph H *contains arbitrarily many Y-disjoint Z-X-rays* if it contains k many such rays for every $k \in N$.

An X-graph H *contains infinitely many Y-disjoint Z-X-rays* if there is an X-subgraph $H' = \langle V', E' \rangle$ of H and a sequence $\langle H_i, f_i \rangle_{i \in N}$ such that each $\langle H_i, f_i \rangle$ is a Z-X-ray in H (with $H_i = \langle V_i, E_i \rangle$) such that the V_i or E_i, respectively for $Y = V, E$, are pairwise disjoint and $V' = \bigcup V_i$ and $E' = \bigcup E_i$.

Definition 3.3. An X-*path* P in an X-graph H is defined similarly to single rays except that the domain of f is a proper initial segment of N instead of N itself. Thus they are finite sequences of distinct vertices with edges between successive vertices in the sequence. If $P = \langle x_0, \ldots, x_n \rangle$ is a path, we say it is a *path of length* n between x_0 and x_n. Our *notation for truncating and combining paths* $P = \langle x_0, \ldots, x_n \rangle$, $Q = \langle y_0, \ldots, y_m \rangle$ and $R = \langle z_0, \ldots, z_l \rangle$ is as follows: $x_i P = \langle x_i, \ldots, x_n \rangle$, $Px_i = \langle x_0, \ldots, x_i \rangle$, and we use concatenation in the natural way, e.g., if the union of Px, xQy and yR is a path, we denote it by $PxQyR$. We treat rays as we do paths in this notation, as long as it makes sense, writing, for example, $x_i R$ for the ray which is gotten by starting R at an element x_i of R; Rx_i is the path which is the initial segment of R ending in x_i and we use concatenation as for paths as well.

The starting point of the work in this paper is a theorem of Halin [11] that we call the infinite ray theorem as expressed in [5, Theorem 8.2.5(i)].

Definition 3.4 (Halin's Theorem)**.** IRT*, the infinite ray theorem,* is the principle that every graph H which contains arbitrarily many disjoint rays contains infinitely many disjoint rays.

The versions of Halin's theorem which we consider in this paper allow for H to be an undirected or a directed graph and for the disjointness requirement to be vertex or edge. We also allow the rays to be single or double. The corresponding versions of Halin's Theorem are labeled as $\mathsf{IRT}_{\mathrm{XYZ}}$ for appropriate values of X, Y and Z to indicate whether the graphs are undirected or directed ($X =$ U or D); whether the disjointness refers to the vertices or edges ($Y =$ V or E) and whether the rays are single or double ($Z =$ S or D), respectively, in the obvious way. We often state a theorem for several or all XYZ and then in the proof use "graph", "edge" and "disjoint" unmodified with the intention that the proof can be read for any of the cases. This is convenient for minimizing repetition in some of our arguments.

We will also consider restrictions of these theorems to specific families of graphs. We need a few more notions to define them.

Definition 3.5. A *tree* is a graph T with a designated element r called its *root* such that for each vertex $v \neq r$ there is a unique path from r to v. A *branch* on (or in) T is a ray that begins at its root. We denote the set of its branches by $[T]$ and say that T is *well-founded* if $[T] = \emptyset$ and otherwise it is *ill-founded*. A *forest* is an *effective disjoint union* of trees, or more formally, a graph with a designated set R (of vertices called roots) such that for each vertex v there is a unique $r \in R$ such that there is a path from r to v and, moreover, there is only one such path. In general, the *effectiveness* we assume when we take *effective disjoint unions of graphs* means that we can effectively (i.e. computably) uniquely identify each vertex in the union with the original vertex (and the graph to which it belongs) which it represents in the disjoint union.

A *directed tree* is a directed graph $T = \langle V, E \rangle$ such that its *underlying graph* $\hat{T} = \langle V, \hat{E} \rangle$ where $\hat{E} = \{\{u, v\} \mid \langle u, v \rangle \in E \vee \langle v, u \rangle \in E\}$ is a tree. A *directed forest* is a directed graph whose underlying graph is a forest.

Definition 3.6. An X-graph H is *locally finite* if, for each $u \in V$, the set $\{v \in E \mid (u, v) \in E \vee (v, u) \in E\}$ of *neighbors of* u is finite. A locally finite X-tree is also called *finitely branching.* (Note this does not mean there are finitely many branches in the tree.)

Of course, there are many well known equivalent definitions of trees and associated notions. We have given one possible set of graph-theoretic definitions. In the case of undirected graphs our definition is equivalent to all the standard ones. Readers are welcome to think in terms of their favorite definition. Note, however, we are restricting ourselves to what would (in set theory) be called countable trees with all nodes of finite rank. Thus, we typically think of trees as *subtrees of* $N^{<N}$, i.e. the sets of finite strings of numbers (as vertices) with an edge between σ and τ if and only if they differ by one being an extension of the other by one element, e.g. $\sigma^\frown k = \tau$.

It does not seem as if there is a single standard definition for directed graphs being directed trees. We have picked one that seems to be at least fairly common and works for the only situations for which we consider them in Theorems 5.16, 6.12, and 6.13 and Corollary 6.14.

3.2 Computability Hierarchies

While we may cite results about uncountable graphs, all sets and structures actually studied in this paper will be countable. Thus for purposes of defining their complexity, we can think of all of them as being subsets of, or relations or functions on, $\mathbb{N}$.

We do not give a formal stand alone definition of computability for sets or functions but assume an at least intuitive grasp of some model of computation such as by a Turing or Register machine that has unbounded memory and is allowed to run for unboundedly many steps. (We do provide in §3.3 a definition via definability in arithmetic that is equivalent to the formal versions of machine model definitions.) Thus we say a function $f : \mathbb{N} \to \mathbb{N}$ is computable if there is a program for one of these machines that computes $f(n)$ as output when given input n. A set X is computable if its characteristic function $\mathbb{N} \to \{0, 1\}$ is computable. Note that as the alphabets or our languages are finite, there are only countably many programs and as our formation rules are effective, we have a computable list of the programs and hence one, Φ_e, of the partial functions they compute. (They are only partial as, of course, some programs fail to halt on some inputs.)

Fundamental to measuring the relative computational complexity of sets or functions is the notion of machines with oracles and Turing reduction. Given a set X or function f we consider machines augmented by the ability to produce $X(n)$ or $f(n)$ if it has already produced n. We say that such a machine is one with an oracle for X or f. We then say that X is *computable from (or Turing reducible to)* Y if there is a machine with oracle Y which computes X via some *reduction* Φ_e^X. We write this as $X \leq_T Y$. We say X is of the same

(Turing) degree as Y, $X \equiv_T Y$, if $X \leq_T Y$ and $Y \leq_T X$. We use all the same terminology and notations for functions.

The first level beyond the computable in our basic hierarchy of computable complexity is given by the halting problem $H = \{e|\Phi_e(e) \textit{ converges}\}$ that is H is set of e such that the computation of the eth machine Φ_e on input e eventually halts. We then define an operator on sets $X \longmapsto X' = \{e|\Phi_e^X(e) \text{ converges}\}$ that is X' is the set of e such that the computation of the eth machine with oracle X, Φ_e^X on input e eventually halts. (It is easy to see that $H \equiv_T \emptyset'$.) The crucial fact here is the undecidability of the halting problem (for every oracle), i.e. for every X, X' is strictly above X in terms of Turing computability. The other basic fact that we need about $\emptyset'$ is that it is *computably enumerable,* i.e. there is a computable function f whose range is $\emptyset'$. If $f(s) = x$ we say that x is *enumerated in*, or *enters*, $0'$ at (stage) s. If we view H as defined by using the empty oracle $\emptyset$, the procedure that takes us from the halting problem to the Turing jump by replacing $\emptyset$ as oracle by X is an instance of a general procedure called *relativization.* It takes any computable function or proof about computable functions or degrees (i.e. ones with oracle $\emptyset$) to the same function, or proof about functions, computable in X (or degrees above that of X). Almost always this procedure trivially transforms correct proofs with oracle $\emptyset$ to ones with arbitrary oracle X. Typically, this transformation keeps the same programs doing the required work with any oracle. For example, X' is *computably enumerable in* X (or relative to X), i.e. there is a function Φ_e^X whose range is X' and this can be taken to be the same e such that $\Phi_e^\emptyset$ enumerates $\emptyset'$. We also use X_s' to denote *the set of numbers enumerated in (or that have entered)* X' *by stage* s. This phenomena of the procedure or result not depending on the particular oracle or depending in a fixed computable way on some other parameters is described as its being *uniform* in the oracle or other parameters. We describe an important example of uniformity in Remark 3.9.

We can now generate a hierarchy of computational complexity by iterating the jump operator beginning with any set X: $X^{(0)} = X$; $X^{(n+1)} = (X^n)'$. While the finite iterations of the jump capture most construction techniques and theorems in graph theory (and most other areas of classical countable/separable mathematics), we will be interested in ones that go beyond such techniques and proofs. The basic idea is that we continue the hierarchy by iteration into the transfinite while still tying the iteration to computable procedures.

Definition 3.7. We represent well-orderings or *ordinals* α as well-ordered relations on $\mathbb{N}$. Typically such *ordinal notations* are endowed with various additional structure such as identifying 0, successor and limit ordinals and specifying cofinal ω-sequences for the limit ordinals. If we have a representation of α then restricting the well-ordering to numbers in its domain provides representations of each ordinal $\beta < \alpha$. We generally simply work with ordinals and omit concerns about translating standard relations and procedures to the representation. An ordinal is recursive (in a set X) if it has a recursive (in X) representation. For a set X and ordinal (notation) α computable from X, we define the transfinite iterations $X^{(\beta)}$ of the Turing jump of X by transfinite induction on $\beta \leq \alpha$: $X^{(0)} = X$; $X^{(\beta+1)} = (X^\beta)'$ and for a limit ordinal λ, $X^{(\lambda)} = \oplus\{X^{(\beta)}|\beta < \lambda\} = \cup\{\beta \times X^{(\beta)}|\beta < \lambda\}$ (or as the effective disjoint sum over the $X^{(\beta)}$ in the specified cofinal sequence in λ).

Definition 3.8. HYP(X), the collection of all sets *hyperarithmetic in* X consists of those sets recursive in some $X^{(\alpha)}$ for α an ordinal recursive in X. We say that Y is *hyperarithmetic* *in* X or *hyperarithmetically reducible to* X , $Y \leq_h X$ if $Y \in \mathrm{HYP}(X)$.

These sets too, will be characterized by a definability class in arithmetic in §3.3. For now we just note that they clearly go far beyond the sets computable from the finite iterations of the jump.

The computational strength of our graph theoretic theorems such as IRT is measured by this hierarchy as we will show that, for every set X and every set Y hyperarithmetic in X, there is a graph G computable from X which satisfies the hypotheses of IRT but for which any collection of rays satisfying its conclusion computes Y. On the other hand, placing an upper bound on the strength of IRT requires analyzing its proof and the principles used in it. The relevant one is a form of the axiom of choice. We define it in the next subsection along with a general class of such principles, the theorems/theories of hyperarithmetic analysis which are, computationally, the primary objects of our analysis in this paper.

We note one important well known basic fact relating the jumps of X to trees computable from X. We will need it for our proofs that IRT and its variants are computationally complex enough to compute all the sets hyperarithmetic in any given set X (as the instances of the graphs range over graphs computable from X).

Remark 3.9. For any set X and any ordinal α computable from X, there is a sequence $\langle T_\beta | \beta < \alpha \rangle$ computable from X of trees (necessarily) computable from X such that each tree has exactly one branch P_β and P_β is of the same complexity as $X^{(\beta)}$, i.e. $P_\beta \equiv_T X^{(\beta)}$. The procedure for computing this sequence is uniform in X and the index for the program computing the well ordering α from X, i.e. there is one computable function that when given an oracle for X, an *index for* α (i.e. the i such that Φ_i^X is the well ordering α) and a β in the ordering, computes the whole sequence $\langle T_\beta | \beta < \alpha \rangle$ and the indices for the reductions between P_β and $X^{(\beta)}$. (See, e.g. [26, Theorem 2.3]). We may also easily assure that the T_β are effectively disjoint so that their union is a forest.

Some versions of the variations on IRT (see §6.2) that call for types of maximality for the infinite set of disjoint rays are stronger both computationally and proof theoretically than the $\mathsf{IRT}_{\mathrm{XYZ}}$ described above. Their computational strength is captured by a kind of jump operator that goes beyond all the hyperarithmetic ones. It captures the ability to tell if a computable ordering is a well-ordering.

Definition 3.10. The *hyperjump of* X, $\mathcal{O}^X$, is the set $\{e | \Phi_e^X$ is (the characteristic function of) a subtree of $\mathbb{N}^{<\mathbb{N}}$ which is well-founded$\}$.

This operator also corresponds to a syntactically defined level of comprehension as we note in §3.3.

3.3 Logical and Axiomatic Hierarchies

The basic notions from logic that we need here are those of languages, structures and axiomatic systems and proofs. As we will deal only with countable sets and structures, we can assume that we are dealing just with the natural numbers with a way to define and use sets and functions on them. Thus, at the beginning, we have in mind the natural numbers $\mathbb{N}$ along with the usual apparatus of the *language of (first order) arithmetic*, say $+, \times, <, 0$ and

1 along with the syntax of standard first order logic (the Boolean connectives $\vee, \wedge$ and $\neg$; the variables such as x and y ranging over the numbers with the usual quantifiers $\forall x$ and $\exists y$ as well as the standard equality relation $=$). A *structure* for this language is a set N along with elements for 0 and 1, binary functions for $+$ and $\times$ and a binary relation for $<$. We also need a way of talking about subsets of (or functions on) the numbers. We follow the standard practice in reverse mathematics of using sets and defining functions in terms of their graphs. So we expand our language by adding on new classes of (second order) variables such as X and Y and the associated quantifiers $\forall X$ and $\exists Y$ along with a new relation symbol $\in$ between numbers and sets.

A structure for this language is one of the form $\mathcal{N} = \langle N, S, +, \times, <, 0, 1, \in\rangle$ where its restriction $\langle N, +, \times, <, 0, 1\rangle$ is a structure for first order arithmetic and $S \subseteq 2^N$ is a specified nonempty collection of subsets of N disjoint from N, the set of "numbers" of $\mathcal{N}$, over which the second order quantifiers and variables of our language range. It is called the *second order part of* $\mathcal{N}$. The usual membership symbol $\in$ always denotes the standard membership relation between elements of N and subsets of N that are in S and the language only allows atomic formulas using $\in$ which are of the form $t \in X$ for t a term of the first order language and X is a second order variable. So a sentence Θ *is true in* N, $\mathcal{N} \vDash \Theta$, if first order quantification is interpreted as ranging over N, second order quantification ranges over S and the relations and functions of the language are as described. This specifies the semantics for second order arithmetic. Note that, following [29], we do not take equality for sets to be a primitive relation on this structure. The notation for it is viewed as being defined by $A = B \leftrightarrow \forall n(n \in A \leftrightarrow n \in B)$.

Proof theoretic notions deal with all possible structures for the language and axiom systems to specify what we need in any particular argument. For most of our purposes and all of the computational ones, one can restrict attention to *standard models of arithmetic*, i.e. ones $\mathcal{N}$ with $N = \mathbb{N}$ and some $S \subseteq 2^{\mathbb{N}}$ with the usual interpretations of the functions and relations. We generally abbreviate these structures as $\langle \mathbb{N}, S\rangle$ with $S \subseteq 2^{\mathbb{N}}$, or simply S, as all the functions and relations are then fixed.

We view the syntax as one for a two sorted first order logic. So the (first order) variables $x, y, \dots$ range over the first sort (N) and the second order ones $X, Y, \dots$ over the second sort (S). We assume any standard proof theoretic system with the caveat that $=$ is interpreted as true equality with the equality axioms included only for N. For S it is a relation defined as above. This generates the provability notion $\vdash$ used above to define our notion of logical strength and equivalences of theories (sets of sentences often called axioms) as above. We now define the standard weak base theory RCA_0 used to define the logical strength of mathematical theorems as described above. We then define a few other common systems that will be used later. The formal details can be found in [29].

Each axiomatic subsystem of second order arithmetic that we consider contains the standard basic axioms for $+$, $\times$, and $<$ (which say that N is a discrete ordered semiring) and an Induction Axiom:

$$(\mathsf{I}_0) \quad (\forall X)((0 \in X \wedge \forall n\,(n \in X \to n+1 \in X)) \to \forall n\,(n \in X)).$$

Typically axiom systems for second order arithmetic are defined by adding various types of set existence axioms although at times additional induction axioms are used as well. In order to define them we need to specify various standard syntactic classes of formulas determined by quantifier complexity. As usual, we add to our language bounded quantifiers $\forall x < t$ and $\exists x < t$ for first order (i.e. arithmetic) terms t defined in the standard way. We typically denote formulas by capital Greek letters except that the indexed Φ_e and Φ_e^X refer, as above, to our fixed enumeration of the Turing machines and associated partial functions.

Definition 3.11. The Σ_0^0 and Π_0^0 formulas of second order arithmetic are just the ones with only bounded quantifiers but we allow parameters for elements of either N or $S(\mathcal{N})$ when working with a structure $\mathcal{N}$. Proceeding inductively, a formula Φ is Σ_{n+1}^0 (Π_{n+1}^0) if it is of the form $\exists x\Psi$ ($\forall x\Psi$) where Ψ is Π_n^0 (Σ_n^0). We assume some computable coding of all these formulas (viewed as strings of symbols from our language) by natural numbers. We say Φ is *arithmetic* if it is Σ_n^0 or Π_n^0 for some $n \in \mathbb{N}$. It is Σ_1^1 (Π_1^1) if it is of the form $\exists X \Psi$ ($\forall X \Psi$) where Ψ is arithmetic. (One can continue to define Σ_n^1 and Π_n^1 in the natural way but we will not need to consider such formulas here.) We say a set X is in one of these classes Γ relative to A (i.e. with A as a parameter) if there is a formula $\Psi(n, A) \in \Gamma$ such that $n \in X \Leftrightarrow \Psi(n, A)$. If X is both Σ_n^i in A and Π_n^i in A it is called Δ_n^i in A.

We mention a few additional standard connections between the syntactic complexity of the definition of a set X and X's properties in terms of computability and graph theoretic notions. They can all be found in [23].

Proposition 3.12. *The sets $A^{(n)}$ are Σ_n^0 in A. A set X is computable in A if and only if it is Δ_1^0 in A. More generally, it is computable in $A^{(n)}$ if and only if it is Δ_{n+1}^0 in A. It is hyperarithmetic in A if and only if it is Δ_1^1 in A. There is a computable function $f(e, n)$ such that if X is Σ_1^1 in A via the Σ_1^1 formula with code e then for every n, $\Phi_{f(e,n)}^A$ is (the characteristic function of) a tree T such that $n \in X \Leftrightarrow T$ has a branch.*

The first system for analyzing the proof theoretic strength of theorems and theories in reverse mathematics is just strong enough to prove the existence of the computable sets and so supplies us with all the usual computable functions such as pairing $\langle n, m\rangle$ or more generally those coding finite sequences as numbers. In particular, it provides the predicates defining the (codes e of) the partial computable functions Φ_e^X and the relations saying the computation $\Phi_e^X(n)$ halts in s many steps with output y. Thus we have the basic tools to define and discuss Turing reducibility and the Turing jump. It is our weak base theory and is assumed to be included in every system we consider.

(RCA_0) Recursive Comprehension Axioms: In addition to the ones mentioned above, its axioms include the schemes of recursive (generally called Δ_1^0) comprehension and Σ_1^0 induction:

(Δ_1^0-CA) $\forall n\,(\Phi(n) \leftrightarrow \Psi(n)) \rightarrow \exists X\, \forall n\,(n \in X \leftrightarrow \Phi(n))$ for all Σ_1^0 formulas Φ and Π_1^0 formulas Ψ in which X is not free.

($\mathsf{I}\Sigma_1^0$) $(\Phi(0) \wedge \forall n\,(\Phi(n) \rightarrow \Phi(n+1))) \rightarrow \forall n\,\Phi(n)$ for all Σ_1^0 formulas Φ.

Note that these formulas may have free set or number variables. As usual, the existence assertion $\exists X$.... of the axiom is taken to mean that for each instantiation of the free variables (by numbers or sets, as appropriate, called *parameters*) there is an X as described. We take this for granted as well as the restriction that the X is not free in the rest of the formula in all of the set existence axioms of any of our systems.

RCA_0 suffices to define and manipulate the basic notions of computability listed at the beginning of Section 3.2. The standard models of RCA_0 are just those whose second order part is closed under Turing reduction and disjoint union ($X \oplus Y = \{\langle 0, x\rangle \mid x \in X\} \cup \{\langle 1, y\rangle \mid y \in Y\}$). As suggested above what are now often called the computable in A sets which are, as mentioned above, the Δ^0_1 in A sets, were originally called the sets recursive in A. Hence the terminology in RCA_0.

Any axiom system we consider from now on will be assumed to include RCA_0. If we have some axiom scheme or principle ABC we typically denote the system formed by adding it to RCA_0 by ABC_0. We next move up to the arithmetic comprehension axiom and its system.

(ACA) $\exists X\, \forall n\, (n \in X \leftrightarrow \Phi(n))$ for every arithmetic formula Φ.

As mentioned above the $X^{(n)}$ are defined by a Σ^0_n formula with X as a parameter. So one can show that this system is equivalent (over RCA_0) to the totality of the Turing jump operator, i.e. for every X, X' exists. Its standard models are those of RCA_0 whose second order part is also closed under Turing jump. It is also equivalent (in the sense of reverse mathematics) to König's Lemma, which asserts that every finitely branching tree with paths of arbitrarily long length has a branch.

In general, we say one system of axioms S is *logically or reverse mathematically reducible* to another T over one R if $R \cup T \vdash \psi$ for every sentence $\psi \in S$. Note that S and/or T may be a single sentence or theorem. We say that S and T are *equivalent* over R if each is reducible to the other. If no system R is specified we assume that RCA_0 is intended.

As we will not deal with it, we have omitted the formal definition of the usual system WKL_0 which falls strictly between RCA_0 and ACA_0. It is characterized by the restriction of König's Lemma to trees that are subsets of $2^{<N}$, the tree of finite binary strings under extension.

The next system of the five basic ones after ACA_0 is ATR_0. Its defining axiom says that arithmetic comprehension can be iterated along any countable well-order and so implies the existence of the sets hyperarithmetic in X for each X but is computationally stronger than this assumption. As usual the formal definition can be found in [29].

Instead, we formally describe the computationally defined class of theorems/theories that are the main focus of this paper and include several variations of IRT. The definition is semantic, not axiomatic and involves only standard models. (Indeed by Van Wesep [31, 2.2.2], there can be no axiomatic characterization of this class in second order arithmetic.)

Definition 3.13. A sentence (theory) T is a *theorem (theory) of hyperarithmetic analysis (THA)* if

1. For every $X \subseteq \mathbb{N}$, $\langle \mathbb{N}, \mathrm{HYP}(X)\rangle \vDash T$ and
2. For every $S \subseteq 2^{\mathbb{N}}$, if $\langle \mathbb{N}, S\rangle \vDash T$ and $X \in S$ then $\mathrm{HYP}(X) \subseteq S$.

It is worth pointing out some of the relations between THAs and ATR_0. THAs are defined by only using standard models and iterations of the jump over true well orderings. ATR_0 talks about all models of RCA_0 and asserts the existence of iterates of the jump over all orderings that appear well-founded in the model. Thus for standard models it implies the second clause of the definition of THAs (Definition 3.13). On the other hand, there is a recursive linear order with no hyperarithmetic infinite descending sequence and so it seems well-founded in HYP but it has a well-founded part longer than every recursive ordinal. Thus iterating the jump along this ordering would yield a set strictly Turing above every hyperarithmetic set. In particular, HYP is not a model of ATR_0 which therefore is not a THA. (See e.g. [29, V.2.6].)

The last of the standard axiomatic systems, Π^1_1-CA_0, is characterized by the comprehension axiom for Π^1_1 formulas:

(Π^1_1-CA) $\exists X\, \forall k\, (k \in X \leftrightarrow \Phi(k))$ for every Π^1_1 formula $\Phi(k)$.

Remark 3.14. The hyperjump, T^X, is clearly a Π^1_1 set with parameter X. In fact, every Π^1_1 set with parameter X is reducible to T^X. Indeed, there is a computable function $f(e, n)$ such that for every index e for a Π^1_1 formula $\Psi(n)$ with parameter X and every n, $\Psi(n) \Leftrightarrow f(e, n) \in T^X$ [23, Corollary 16.XX(b)]. Thus Π^1_1-CA_0 corresponds to closure under the hyperjump. We will see it appear as equivalent to a version of IRT where we ask for a maximal set of disjoint rays in Theorem 6.18.

For this paper, the most important other existence axiom is a restricted form of the axiom of choice.

(Σ^1_1-AC) $\forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n \Phi(n, X^{[n]})$ where Φ is arithmetic and $X^{[n]} = \{m \mid \langle n, m\rangle \in X\}$ is the *nth column of X*.

A more common but clearly equivalent version of this axiom allows Φ to be Σ^1_1. A variant commonly called weak-Σ^1_1-AC (introduced in Definition 7.1 as unique-Σ^1_1-AC) requires the corresponding Φ to be arithmetic. To make these and other choice axioms uniform we have adopted the format required elsewhere and equivalent here to be used for all the variations. The system Σ^1_1-AC_0 is well known to be a THA (essentially in Kreisel [15]). Thus it is strictly stronger than ACA_0. On the other hand, it is strictly weaker than ATR_0. (It is known that $\mathsf{ATR}_0 \vdash \Sigma^1_1$-$\mathsf{AC}_0$ [29, V.8.3] but the converse fails as Σ^1_1-AC is true in HYP while ATR_0, as we have pointed out, is not.) This choice axiom plays a crucial role in our analysis because we provide the upper bound on the strength of most of our theorems by showing that they follow from Σ^1_1-AC_0. This provides the computational upper bound for being a THA as any consequence of a THA must also satisfy Definition 3.13(1). Thus the bulk of our proofs for the computational complexity of the theorems we study consist of showing that they imply Definition 3.13(2), i.e. closure under "hyperarithmetic in".

Over the past fifty years, several other logical axioms have been shown to be THA. We will discuss some of them in §7. However, as we discussed in §1, only one somewhat obscure purely mathematical theorem was previously known to be a THA. We provide several more in this paper (Theorem 5.1, Corollary 6.4 and Theorem 6.13). We also introduce a new logical axiom, finite-Σ^1_1-AC (Definition 7.2) which is a THA as well.

For those interested in the proof theory and so nonstandard models, we also at times explicitly consider the induction axiom at the same Σ^1_1 level.

($\mathsf{I}\Sigma^1_1$) $(\Phi(0) \wedge \forall n(\Phi(n) \rightarrow \Phi(n+1))) \rightarrow \forall n \Phi(n)$ for every Σ^1_1 formula Φ.

This axiom does not imply the existence of any infinite sets and is, of course, true in every standard model. Thus the readers interested only in the computational complexity of the Halin type theorems can safely ignore these considerations.

It is in the nature of reverse mathematics that sentences and sets of sentences of second order arithmetic are often viewed in several different ways. In different contexts they may be seen as mathematical or logical principles, axioms, axiom schemes, theories, theorems or the like. We point out what may be a less familiar terminology that is currently popular. What might be seen as a typical axiom or theorem asserting that for every X of some sort there is a Y with some relation to X, i.e. a sentence of the form $\forall X(\Phi(X) \to \exists Y \Psi(X, Y))$ may be called a principle. With this terminology come the notions of an instance of the principle, i.e. an X satisfying Φ and a solution for X, i.e. a Y such that $\Psi(X, Y)$ holds.

4 IRT and Hyperarithmetic Analysis

We devote this section to the proof of

Theorem 4.1. IRT *is a theorem of hyperarithmetic analysis.*

In this section we consider only vertex-disjoint single rays in undirected graphs, as in the statement of IRT. The proof of Theorem 4.1 will be split into two parts. The first part verifies that IRT satisfies the second clause of Definition 3.13.

Theorem 4.2. *Every standard model of* $\mathsf{RCA}_0 + \mathsf{IRT}$ *is closed under hyperarithmetic reduction.*

Proof. Fix a standard model $\mathcal{M}$ of $\mathsf{RCA}_0 + \mathsf{IRT}$. First, we show that $\mathcal{M}$ contains $\emptyset'$. By relativizing the proof, it follows that $\mathcal{M}$ is closed under Turing jump.

For each n, consider the tree $T_n \subseteq N^{<N}$ consisting of all strings of the form $s^\frown 0^t$ such that some number below n is enumerated into $\emptyset'$ at stage s, and either $t \leq s$ or $\emptyset'_s \upharpoonright n = \emptyset'_t \upharpoonright n$. Observe that T_n has a unique computable branch $\{s^\frown 0^t \mid t \in N\}$, where s is the smallest number such that $\emptyset' \upharpoonright n = \emptyset'_s \upharpoonright n$.

Consider the disjoint union $\bigsqcup_n T_n$. Observe that $\bigsqcup_n T_n$ satisfies the premise of IRT (in $\mathcal{M}$), because each T_n has a computable branch. Apply IRT to $\bigsqcup_n T_n$ to obtain a sequence $\langle R_i \rangle_i$ of disjoint rays in $\bigsqcup_n T_n$. Each R_i is contained in some T_n. We can, uniformly in i, extend or truncate R_i to the unique branch P_n of T_n. Hence $\langle R_i \rangle_i$ computes a sequence of infinitely many distinct branches P_n, which in turn computes longer and longer initial segments of $\emptyset'$. This proves that $\mathcal{M}$ contains $\emptyset'$.

Next we show that, for each computable limit ordinal λ, if $\mathcal{M}$ contains $\emptyset^{(\alpha)}$ for every $\alpha < \lambda$ then $\mathcal{M}$ contains $\emptyset^{(\lambda)}$. (Again the desired result follows by relativization.) By Proposition 3.9, there is a computable sequence $\langle T_\beta \rangle_{\beta<\lambda}$ of trees such that each tree has exactly one branch $P_\beta \equiv_T \emptyset^{(\beta)}$ with these reductions computed uniformly. Fix an increasing computable sequence $\langle \alpha_n \rangle_n$ which is cofinal in λ and consider the disjoint union $\bigsqcup_n T_{\alpha_n}$. Observe that $\bigsqcup_n T_{\alpha_n}$ satisfies the premise of IRT (in $\mathcal{M}$): for each n, $\emptyset^{(\alpha_n)}$ computes the branches P_{α_m} for $m \leq n$. Apply IRT to $\bigsqcup_n T_{\alpha_n}$ to obtain a sequence $\langle R_i \rangle_i$ of disjoint rays in $\bigsqcup_n T_{\alpha_n}$. As before, $\langle R_i \rangle_i$ computes a sequence of infinitely many distinct branches P_{α_n}, and hence a sequence of infinitely many distinct $\emptyset^{(\alpha_n)}$. Each $\emptyset^{(\alpha_n)}$ uniformly computes $\emptyset^{(\alpha_m)}$ for $m \leq n$, so we conclude that $\langle R_i \rangle_i$ computes $\bigoplus_m \emptyset^{(\alpha_m)}$ as desired. □

It follows that IRT is not provable in ACA_0, despite the apparent similarity between IRT and a compactness result. (Indeed, IRT is not even provable in Δ^1_1-CA_0 (Theorem 7.9).)

Next, we present essentially the proof of IRT attributed to Andreae (see [5, Theorem 8.2.5 and bottom of pg. 275]) and then analyze it with an eye to the axioms which can be used to formalize it. We then use this analysis to complete the proof of Theorem 4.1.

The key combinatorial lemma implicit in Andreae's proof is:

Lemma 4.3. *Given disjoint rays $\langle R_i \rangle_{i<n}$ and disjoint rays $\langle S_j \rangle_{j<n^2+1}$ there are $n+1$ disjoint rays $R'_0, \dots, R'_n$ such that for each $i < n$, R_i and R'_i start at the same vertex.*

Before proving Lemma 4.3, let us use it to prove IRT.

Proof of IRT *assuming Lemma 4.3.* Given a graph which has arbitrarily many disjoint rays, we build by recursion on $n \geq 1$ sequences $\langle R^n_i \rangle_{i<n}$ of disjoint rays with initial segments P^n_i of length n such that P^{n+1}_i is P^n_i followed by one more vertex for $i < n$. The required infinite sequence of disjoint rays will then be given by $R_i = \bigcup \{P^n_i \mid n > 0\}$.

Suppose that we have $\langle R^n_i \rangle_{i<n}$ and $\langle P^n_i \rangle_{i<n}$. By assumption, let $S_0, \dots, S_{2n^2}$ be a sequence of disjoint rays. Discard all rays S_j which contain a vertex of some P^n_i. There are at most n^2 many of them, so by discarding and renumbering if necessary we are left with $S_0, \dots, S_{n^2}$.

For each $i < n$, let x_i denote the first vertex on R^n_i after P^n_i. Apply Lemma 4.3 to $\langle x_i R^n_i \rangle_{i<n}$ and $S_0, \dots, S_{n^2}$. We obtain $n+1$ disjoint rays $R'_0, \dots, R'_n$ such that for each $i < n$, R'_i begins at vertex x_i. Now let $R^{n+1}_i = P^n_i x_i R'_i$ for $i < n$ and $R^{n+1}_n = R'_n$. These are disjoint by construction. This completes the inductive step of the construction of the $R^n_0, \dots, R^n_{n-1}$ and so provides the required witnesses for IRT. □

It remains to prove Lemma 4.3. The key ingredient is Menger's theorem for finite graphs. If A and B are disjoint sets of vertices in a graph, we say that P is an *A-B path* if P starts with some vertex in A and ends with some vertex in B. A set of vertices S *separates A and B* if any A-B path contains at least one vertex in S.

Theorem 4.4 (Menger, see [5, Theorem 3.3.1])**.** *Let G be a finite graph. If A and B are disjoint sets of vertices in G, then the minimum size of a set of vertices which separate A and B is equal to the maximum size of a set of disjoint A-B paths.*

We now present the proof of Lemma 4.3.

Proof of Lemma 4.3. Suppose we are given n disjoint rays $R_0, \dots, R_{n-1}$ and $n^2 + 1$ disjoint rays $S_0, \dots, S_{n^2}$. First, define the set

$$\{\langle i, q \rangle \mid R_i \text{ intersects } S_q\}.$$

Then we perform the following recursive procedure. At each step, check if there is some $i < n$ such that R'_i has not been defined and R_i intersects at most n many rays S_q which have not been discarded. If there is no such i, we end the procedure. Otherwise, find the least such i and do the following:

1. discard all rays S_q which intersect R_i;

2. define $R'_i = R_i$.

After the procedure is complete, let I be the set of $i < n$ for which R'_i has not been defined. Let $\mathcal{S}$ be the set of rays S_q which have not been discarded. Let $m = |I|$. We observe that $|\mathcal{S}| \geq m^2 + 1$, because

$$(n^2 + 1) - (n - m)n = mn + 1 \geq m^2 + 1.$$

Next, for each $i \in I$, let z_i be the first vertex on R_i such that $R_i z_i$ meets exactly m many rays in $\mathcal{S}$. (Each z_i exists by construction of I.)

Observe that the finite set $\bigcup_{i\in I} R_i z_i$ meets at most m^2 many rays in $\mathcal{S}$. Since $|\mathcal{S}| \geq m^2+1$, we may pick some ray in $\mathcal{S}$ which does not meet $\bigcup_{i\in I} R_i z_i$. We define R'_n to be said ray. Then, discard all rays in $\mathcal{S}$ which do not meet $\bigcup_{i\in I} R_i z_i$.

Finally, we use Menger's theorem to define R'_i for each $i \in I$. For each $i \in I$, let x_i denote the first vertex of R_i. For each q such that S_q remains in $\mathcal{S}$, let y_q be the first vertex on S_q such that $y_q S_q$ and $\bigcup_{i\in I} R_i z_i$ are disjoint. Then consider the following finite sets of vertices:

$$\begin{aligned}
X &= \{x_i \mid i \in I\} \\
Y &= \{y_q \mid S_q \in \mathcal{S}\} \\
H &= \bigcup_{i\in I} R_i z_i \cup \bigcup_{S_q \in \mathcal{S}} S_q y_q.
\end{aligned}$$

We want to apply Menger's theorem to $X, Y \subseteq H$. Towards that end, we claim that X cannot be separated from Y in H by fewer than m vertices.

Suppose that $A \subseteq H$ and $|A| < m$. Since $|I| = m$ and $\{R_i \mid i \in I\}$ is disjoint, there is some $i \in I$ such that R_i does not meet A. Next, since $R_i z_i$ meets m many disjoint rays in $\mathcal{S}$, there is some q such that $S_q \in \mathcal{S}$ and $R_i z_i$ meets S_q, but S_q does not meet A. Let z be any vertex in both $R_i z_i$ and S_q. Then $R_i z S_q y_q$ is a path in H from x_i to y_q which does not meet A. This proves our claim.

By Menger's theorem, there are m many disjoint X-Y paths in H. Then, for each $i \in I$, define R'_i by starting from x_i, then following the X-Y path given by Menger's theorem to some y_q, and finally following S_q.

We have constructed a collection $R'_0, \ldots, R'_n$ of rays. It is straightforward to check that they are disjoint, and that for each $i < n$, R_i and R'_i start at the same vertex. □

We now analyze these proofs from a reverse mathematical perspective to show that IRT follows from the THA Σ^1_1-AC_0 and so also satisfies the first clause of Definition 3.13.

Theorem 4.5. *i)* IRT *and* Σ^1_1-AC_0 *each implies* ACA_0.

ii) Σ^1_1-AC_0 implies IRT. Hence for every $Y \subseteq \mathbb{N}$, $\mathrm{HYP}(Y)$ satisfies IRT.

Proof. i) The proof of the first part of Theorem 4.2 essentially shows that IRT implies ACA_0. One point to note is that for each n, RCA_0 proves there is some s such that $\emptyset' \upharpoonright n = \emptyset'_s \upharpoonright n$ (using e.g. [29, II.3.9]). Now note that the same argument also proves the fact (essentially in [15]) that Σ^1_1-AC_0 implies ACA_0 as it directly supplies the sequence of branches P_n in T_n.

ii) The proof of IRT presented above is easily seen to be one in ACA_0, except for two points. First, each step n of the induction assumed we had available a sequence $S_0, \ldots, S_{n^2}$ of disjoint rays in our graph. However, all we know is that for each n, there is some collection of disjoint rays of size n. To access this information for each n in a recursive construction, we require that there is a single sequence R_n such that the nth entry of the sequence is a collection of disjoint rays of size n. Such a sequence can be obtained using the axiom of choice. In this case, since the predicate "there exists n many disjoint rays" is Σ^1_1, such a sequence can be obtained using Σ^1_1-AC_0. Therefore we may assume we have a sequence $S = \langle\langle S^n_j\rangle_{j<n}\rangle_{n>0}$ with each $\langle S^n_j\rangle_{j<n}$ a sequence of disjoint rays and the graph G consisting of all the vertices and edges occurring in any S^n_j. We begin our construction with $R^1_0 = S^1_0$.

The second point is the iterated application of Lemma 4.3. The proof of that Lemma can be done in ACA_0. (In particular, one can check that Menger's theorem is provable in RCA_0 by following the first proof for it given in [5, Theorem 3.3.1].) However, here we need a bit more information as generally we cannot, in ACA_0, carry out recursive constructions that increase complexity (even by some fixed number of jumps) at each step. We can only carry out recursions where each step is done computably in some set of the model. So it suffices to show that the constructions for all the instances of Lemma 4.3 needed in its iterations in the proof can be done computably in $(G \oplus S)'$.

The crucial observation is that, by an induction starting with $R^1_0 = S^1_0$, we may take each R^n_i to be of the form PQ where P is a finite path in G and Q is a tail of one of the S_j. To see this simply note that each ray $R'_k \neq R_k$ constructed in the Lemma starts with an initial segment of some R_i; it continues with a finite path in G and ends with a tail of some $S^{2n^2+1}_j$. Thus computably in $(G \oplus S)'$ we can at every stage find one of these descriptions that provides the disjoint rays needed for the next stage of the recursion. Thus the sequences $\left\langle\langle R^n_i\rangle_{i<n}\right\rangle_{n>0}$ and $\left\langle\langle P^n_i\rangle_{i<n}\right\rangle_{n>0}$ are computable from $(G \oplus S)'$. The sequence $\langle\bigcup\{P^n_i \mid n > 0\}\rangle_{i\in N}$ is then also computable from $(G \oplus S)'$ and computing the formal presentation of infinitely many disjoint rays then takes only one more jump. This completes the analysis of our proof of IRT in Σ^1_1-AC_0. □

Theorem 4.1 follows from Theorems 4.2 and 4.5.

We will establish some implications and nonimplications between IRT and other THAs in §7.

5 Variants of IRT and Hyperarithmetic Analysis

In this section, we show that at least five of the eight principles $\mathsf{IRT}_{\mathrm{XYZ}}$ are THAs:

Theorem 5.1. *All single-ray variants of* IRT *(i.e.,* $\mathsf{IRT}_{\mathrm{XYS}}$*) and* $\mathsf{IRT}_{\mathrm{UVD}}$ *are theorems of hyperarithmetic analysis.*

$\mathsf{IRT}_{\mathrm{UVS}}$ and $\mathsf{IRT}_{\mathrm{UVD}}$ were proved by Halin [11, 12]. $\mathsf{IRT}_{\mathrm{UES}}$ is an exercise in [5, 8.2.5(ii)]. $\mathsf{IRT}_{\mathrm{DVS}}$ and $\mathsf{IRT}_{\mathrm{DES}}$ may be folklore.

Of the other three variants, $\mathsf{IRT}_{\mathrm{DED}}$ and $\mathsf{IRT}_{\mathrm{DVD}}$ are open problems of graph theory ([3] and Bowler, personal communication). We do, however, have interesting results about these principles when restricted to directed forests (Theorem 6.13, Corollary 6.14). The other one, $\mathsf{IRT}_{\mathrm{UED}}$, was proved by Bowler, Carmesin, Pott [3] using structural results about ends. We hope to analyze its strength in future work.

The proof of Theorem 5.1 consists of several variations of the proof of Theorem 4.1. One of which ($\mathsf{IRT}_{\mathrm{DES}}$) requires some additional ideas.

In order to minimize repetition, we establish some implications between some variants of IRT over RCA_0. The proofs of each of these reductions follow the same basic plan. To deduce $\mathsf{IRT}_{\mathrm{XYZ}}$ from $\mathsf{IRT}_{\mathrm{X'Y'Z'}}$ we provide computable maps g, h and k which, provably in RCA_0, take X-graphs G to X'-graphs G', Y-disjoint Z-rays or sets of Y-disjoint Z-rays in G to Y'-disjoint Z'-rays or sets of Y'-disjoint Z'-rays in G', and Y'-disjoint Z'-rays or sets of Y'-disjoint Z'-rays in G' to Y-disjoint Z-rays or sets of Y-disjoint Z-rays in G, respectively. These functions are designed to take witnesses of the hypothesis of $\mathsf{IRT}_{\mathrm{XYZ}}$ in G to witnesses of the hypothesis of $\mathsf{IRT}_{\mathrm{X'Y'Z'}}$ in G' and witnesses to the conclusion of $\mathsf{IRT}_{\mathrm{X'Y'Z'}}$ in G' to witnesses to the conclusion of $\mathsf{IRT}_{\mathrm{XYZ}}$ in G. Clearly it suffices to provide such computable maps to establish the desired reduction in RCA_0. (Those familiar with Weihrauch reducibility will recognize that these arguments establish Weihrauch reductions between certain problems corresponding to variants of IRT.)

Unless otherwise noted all definitions and proofs in this section are in RCA_0.

Lemma 5.2. *Given an undirected graph G, we can uniformly compute a directed graph G' and mappings between Z-rays in G and Z-rays in G' which preserve Y-disjointness.*

Proof. We define a computable map g from undirected graphs G to directed graphs G' as follows. The set of vertices of G' consists of the vertices of G, together with two new vertices $x = x(u, v)$ and $y = y(u, v)$ for each edge $\{u, v\}$ in G. The set of edges of G' consists of five edges $\langle u, x\rangle$, $\langle v, x\rangle$, $\langle x, y\rangle$, $\langle y, u\rangle$, $\langle y, v\rangle$ for each edge $\{u, v\}$ in G.

Next we define a computable map h_{S}: given a ray $u_0, u_1, \ldots$ in G, h_{S} maps it to the ray $u_0, x(u_0, u_1), y(u_0, u_1), u_1, \ldots$ in G'. Conversely, we define a computable map k_{S} from rays R' in G' into rays R in G as follows. Observe that exactly one of the first three vertices in R' is a vertex in G, because the only outgoing edges from a vertex $y(u, v)$ lead to u or to v, and the only outgoing edge from a vertex $x(u, v)$ leads to $y(u, v)$. We take this vertex (say u_0) to be the first vertex of R. Every outgoing edge from u_0 leads to some $x(u_0, v)$. Combining the above observations, we deduce that the tail u_0R' has the form $u_0, x(u_0, u_1), y(u_0, u_1), u_1, \ldots$. Then k_{S} maps R' to the ray $R = u_0, u_1, \ldots$ in G.

Similarly, given a double ray $\ldots, u_{-1}, u_0, u_1, \ldots$ in G, h_{D} maps it to the double ray $\ldots, u_{-1}, x(u_{-1}, u_0), y(u_{-1}, u_0), u_0, x(u_0, u_1), y(u_0, u_1), u_1, \ldots$ in G'. We can show that every double ray in G' has this form by considering the incoming edges to each vertex in G'. Therefore we can define a computable map from double rays in G' to double rays in G by $k_{\mathrm{D}} = h_{\mathrm{D}}^{-1}$.

It is straightforward to check that h_{S}, k_{S}, h_{D} and k_{D} preserve Y-disjointness. □

Therefore we have

Proposition 5.3. *The directed variants of* IRT *imply their corresponding undirected variants, i.e.,* $\mathsf{IRT}_{\mathrm{DYZ}}$ *implies* $\mathsf{IRT}_{\mathrm{UYZ}}$ *for each value of Y and Z.*

Lemma 5.4. *Given a directed graph G, we can uniformly compute a directed graph G' and mappings between Z-rays in G and Z-rays in G' which satisfy the following properties: if two Z-rays in G are vertex-disjoint, then the corresponding Z-rays in G' are edge-disjoint, and if two Z-rays in G' are edge-disjoint, then the corresponding Z-rays in G are vertex-disjoint.*

Proof. We define a computable map g from directed graphs $G = \langle V, E\rangle$ to directed graphs G' as follows. The set of vertices of G' is $\{x_i, x_o \mid x \in V\}$, where i and o stand for incoming and outgoing respectively. The set of edges of G' consists of $\langle u_o, v_i\rangle$ for each $\langle u, v\rangle \in E$, and $\langle x_i, x_o\rangle$ for each $x \in V$.

Next we define a computable map h_{S}: given a ray $x^0, x^1, \dots$ in G, h_{S} maps it to the ray $x_i^0, x_o^0, x_i^1, x_o^1, \dots$ in G'. Conversely, we define a computable map k_{S} from rays R' in G' to rays R in G as follows. Given R', the ray R visits the vertex x in G whenever $\langle x_i, x_o\rangle$ appears in R'. (For example, we map $x_i^0, x_o^0, x_i^1, x_o^1, \dots$ to $x^0, x^1, \dots$ and we map $x_o^0, x_i^1, x_o^1, \dots$ to $x^1, x^2, \dots$.)

Similarly, h_{D} maps a given double ray $\dots, x^{-1}, x^0, x^1, \dots$ in G to the double ray

$$\dots, x_i^{-1}, x_o^{-1}, x_i^0, x_o^0, x_i^1, x_o^1, \dots$$

in G'. Every double ray in G' has this form, so we may define $k_{\mathrm{D}} = h_{\mathrm{D}}^{-1}$.

It is straightforward to check that the above mappings have the desired properties. □

Therefore we have

Proposition 5.5. *The directed edge-disjoint variants of* IRT *imply their corresponding directed vertex-disjoint variants, i.e.,* $\mathsf{IRT}_{\mathrm{DEZ}}$ *implies* $\mathsf{IRT}_{\mathrm{DVZ}}$ *for each value of* Z*.*

Lemma 5.6. *Given a directed graph G, we can uniformly compute a directed graph G' and mappings between sets of Y-disjoint rays in G and sets of Y-disjoint double rays in G' which preserve cardinality.*

Proof. We define a computable map g from directed graphs G to directed graphs G' containing G as follows. For each vertex x of G we add new vertices x_n for each $n < 0$ and edges $\langle x_{-1}, x\rangle$ and $\langle x_{n-1}, x_n\rangle$ for all $n < 0$.

Next we define a computable map h from sets of Y-disjoint rays in G to sets of Y-disjoint double rays in G' as follows. Given a set of Y-disjoint rays in G, we first ensure that each ray begins at a different vertex, by replacing it with a tail if necessary. (This is only relevant if the rays are edge-disjoint rather than vertex-disjoint.) Then for each ray $x^0, x^1, \dots$, we consider the double ray $\dots, x^0_{-2}, x^0_{-1}, x^0, x^1, \dots$ in G'. This yields a set of Y-disjoint double rays in G' of the same cardinality.

Finally we define a computable map k from sets of Y-disjoint double rays in G' to sets of Y-disjoint rays in G. Given a double ray $\dots, x^{-1}, x^0, x^1, \dots$ in G', we search for the least $n \geq 0$ such that x^n is a vertex in G. (If none of these vertices were in G then as there are edges between them they would all have to be of the form y_{-n} for a single y in G. The only edges between these y_{-n} make them into a copy of the reverse order on N. This order cannot have any subsequence of order type N.) Now we map the given double ray to the ray $x^n, x^{n+1}, \dots$ in G. It is straight forward to check that this map induces a cardinality-preserving map k from sets of Y-disjoint double rays in G' to sets of Y-disjoint rays in G. □

Therefore we have

Proposition 5.7. *The directed double ray variants of* IRT *imply their corresponding directed single ray variants, i.e.,* $\mathsf{IRT}_{\mathrm{DYD}}$ *implies* $\mathsf{IRT}_{\mathrm{DYS}}$ *for each value of* Y*.*

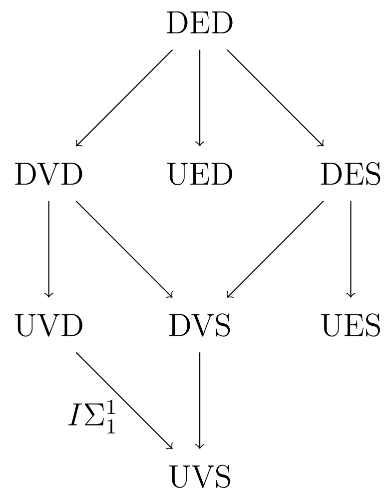

Figure 1: Known implications between variants of IRT. All implications are over RCA_0, except for the implication from UVD to UVS (Theorem 6.15).

Figure 1 summarizes the known implications between variants of IRT over RCA_0. We will show that $\mathsf{IRT}_{\mathrm{UVD}}$ implies $\mathsf{IRT}_{\mathrm{UVS}}$ over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$ (Theorem 6.15).

Remark 5.8. Bowler, Carmesin, Pott [3, pg. 2 l. 3–7] describe an implication from $\mathsf{IRT}_{\mathrm{UVS}}$ to $\mathsf{IRT}_{\mathrm{UES}}$ which appears to use Σ^1_1-AC_0. It turns out that the graph-theoretic principle used to carry out the implication does not imply even ACA_0 over RCA_0 (and is hence much weaker than Σ^1_1-AC_0), but when combined with ACA_0, yields a THA. It and several other principles with the same property (almost theorems/theories of hyperarithmetic analysis) are analyzed in Shore [27].

We return to the goal of proving Theorem 5.1.

Theorem 5.9. *For each choice of XYZ,* $\mathsf{IRT}_{\mathrm{XYZ}}$ *implies* ACA_0. *Furthermore, every standard model of* RCA_0 *and* $\mathsf{IRT}_{\mathrm{XYZ}}$ *is closed under hyperarithmetic reduction.*

Proof. By Proposition 5.3, it suffices to prove the desired result for the undirected variants of IRT. Theorems 4.2 and 4.5(i) together assert the desired result for IRT. We describe how to modify the proofs of Theorems 4.2 and 4.5(i) to prove the desired result for the other variants of IRT.

Observe that in the aforementioned proofs, we only applied IRT to forests such that each of the constituent trees has a unique branch. In such graphs, none of the constituent trees can contain two rays which are edge-disjoint. Hence the aforementioned proofs establish the desired result for $\mathsf{IRT}_{\mathrm{UES}}$ as well.

In order to prove the desired result for $\mathsf{IRT}_{\mathrm{UVD}}$ and $\mathsf{IRT}_{\mathrm{UED}}$, we modify the aforementioned proofs as follows. For each tree, we relabel as needed to then add one computable branch of new vertices (other than the root of the tree) such that the new branches are also disjoint. The resulting trees satisfy the following properties:

- Each tree contains some double ray which is Turing equivalent to the branch on the original tree.

- No two double rays in the tree can be vertex-disjoint or edge-disjoint.
- Given any double ray in the tree, we can uniformly compute the branch on the original tree.

It is straightforward to check that the modified proofs establish the desired result for $\mathsf{IRT}_{\mathrm{UVD}}$ and $\mathsf{IRT}_{\mathrm{UED}}$. □

Henceforth we will not explicitly mention uses of ACA_0 whenever we are assuming any $\mathsf{IRT}_{\mathrm{XYZ}}$ or Σ^1_1-AC_0.

Next, we show that $\mathsf{IRT}_{\mathrm{UVD}}$ and $\mathsf{IRT}_{\mathrm{DES}}$ are provable in Σ^1_1-AC_0 (Theorems 5.10, 5.15). It then follows from Propositions 5.3 and 5.5 that $\mathsf{IRT}_{\mathrm{UES}}$, $\mathsf{IRT}_{\mathrm{DVS}}$ and $\mathsf{IRT}_{\mathrm{UVS}}$ are also provable in Σ^1_1-AC_0, completing the proof of Theorem 5.1.

Theorem 5.10. Σ^1_1-AC_0 *implies* $\mathsf{IRT}_{\mathrm{UVD}}$.

Proof. The mathematical result is due to [11] and is Exercise 42 of Chapter 8 in [5]. Our proof is very similar to that of Theorem 4.5(ii) which follows [5, Theorem 8.2.5(i)] except we need to grow our family in "two directions".

Our plan is to construct, by recursion on n, sequences $\langle R^n_i\rangle_{i<n}$ and $\langle P^n_i\rangle_{i<n}$ such that the R^n_i are disjoint double rays with subpaths P^n_i of length $2n$ such that, for $i<n$, P^{n+1}_i is P^n_i extended by a new vertex at each end. Our required sequence of disjoint double rays is then $\langle\bigcup\{P^n_i \mid n\in N\}\rangle_{i\in N}$.

As we want to reuse the proof of Lemma 4.3, we want to decompose double rays into single rays. To that end we introduce some notation. If R and R_i are double rays we let $R_f=R$ and $R_{i,f}=R_i$ while R_b and $R_{i,b}$ are R and R_i, respectively, but with order reversed. We use d or d' to stand for one of f or b. For single rays R we use ${}^*R$ to denote the reverse sequence of vertices. So, for example, if (c,d) is an edge in a double ray R, we have ${}^*(cR_b)dR_f=R$.

We note the changes needed in the proof of Theorem 4.5. Now our sequence $S=\langle\langle S^n_j\rangle_{j<n}\rangle_{n>0}$ given by Σ^1_1-AC_0 consists of double rays and G is the corresponding graph. Our construction of the desired $\langle R^n_i\rangle_{i<n}$ and $\langle P^n_i\rangle_{i<n}$ again proceeds recursively in $(G\oplus S)'$ and at the end uses one more jump to get the required sequence of double rays. We begin our recursion by setting $R^1_0=S^1_0$ and P^1_0 as any subpath of length 2. Here we note that by construction every R^n_i will be of the form $QaPbR$ where P is a finite path in G while Q and R are each S_f or S_b for S some S^n_j. Again, finding each $\langle R^{n+1}_i\rangle_{i<n}$ and $\langle P^{n+1}_i\rangle_{i<n}$ in turn is recursive in $(G\oplus S)'$ once we prove in ACA_0 that it exists given $\langle R^n_i\rangle_{i<n}$ and $\langle P^n_i\rangle_{i<n}$.

Thus it suffices to prove an analog of Lemma 4.3 in ACA_0: Given $\langle R^n_i\rangle_{i<n}$ and $\langle P^n_i\rangle_{i<n}$ and $\langle S^k_i\rangle_{i<k}$ for $k=11n^2+1$ we can construct $\langle R^{n+1}_i\rangle_{i<n+1}$ and $\langle P^{n+1}_i\rangle_{i<n+1}$ as required. To simplify the notation we omit the superscripts n and k. For $i<n$, let $x_{i,b}$ be the first vertex in R_i before P_i and $x_{i,f}$ be the first vertex in R_i after P_i. We will arrange for P^{n+1}_i to be P_i preceded by $x_{i,b}$ and followed by $x_{i,f}$ which will then also be $x_{i,b}R^{n+1}_i x_{i,f}$.

We first discard from $\{S_j \mid j<11n^2+1\}$ any S_j sharing a vertex with any P_i. As there are $n(2n+1)\le 3n^2$ such vertices we have at least $8n^2+1$ many S_j remaining. We relabel these as S_j for $j<8n^2+1$ and choose an edge $(c_{j,b},c_{j,f})$ in each S_j. We now essentially follow the proof of Lemma 4.3 but for the sets of single rays $\mathcal{R}=\{x_{i,f}R_{i,f},x_{i,b}R_{i,b}\mid i<n\}$ and $\mathcal{S}=\{c_{j,f}S_{j,f},c_{j,b}S_{j,b}\mid j<8n^2+1\}$. Our goal for each $x_{i,d}R_{i,d}\in\mathcal{R}$, $i<n$ and $d\in\{b,f\}$, is to

find a suitable replacement (beginning with the same vertex while maintaining the required disjointness) so that we can assemble the R_i^{n+1} from them for $i < n$. We also want an S_h disjoint from all the double rays R_i^{n+1} with $i < n$ which will be R_n^{n+1}.

We follow the procedure described in the proof of Lemma 4.3 (using $2n$ for n as $\mathcal{R}$ has $2n$ elements) to define single ray replacements $R'_{\langle i,d\rangle}$ for the $x_{i,d}R_{i,d}$ from which we will construct the R_i^{n+1}. When in that procedure we would keep the old ray we do so here as well and let $R'_{\langle i,f\rangle} = x_{i,f}R_{i,f}$. When the procedure is completed we let $I = \{\langle i,d\rangle \mid i < n,\, d \in \{b,f\}$ and we have not defined $R'_{\langle i,d\rangle}\}$ and let $m = |I|$. Clearly we have discarded at most $4n^2$ many more of the rays in $\mathcal{S}$.

Now continue the construction for the $\langle i,d\rangle \in I$ and the first vertices $z_{i,d}$ on $x_{i,d}R_{i,d}$ such that $x_{i,d}R_{i,d}z_{i,d}$ meets exactly m many rays in $\mathcal{S}$. The union F of these finite paths meets at most $m \cdot 2n \le 4n^2$ many rays in $\mathcal{S}$ and so at most $4n^2$ many double rays S_j, $j < 8n^2+1$. Thus there must be at least one S_j such that it does not meet F and no $c_{j,d}S_{j,d}$ was discarded during the process. We let one such be R_n^{n+1} and take any subpath of length $2n+2$ as P_n^{n+1}.

Now discard all single rays S in $\mathcal{S}$ not meeting F. For each $c_{j,d}S_{j,d}$ remaining in $\mathcal{S}$ let $y_{j,d}$ be the first vertex on $c_{j,d}S_{j,d}$ such that $y_{j,d}S_{j,d}$ is disjoint from F. As in the proof of Lemma 4.3, we apply Menger's theorem to X, Y and H where

$$\begin{aligned} X &= \{x_{i,d} \mid \langle i,d\rangle \in I\} \\ Y &= \{y_{j,d} \mid c_{j,d}S_{j,d} \in \mathcal{S}\} \\ H &= F \cup \bigcup_{c_{j,d}S_{j,d}\in\mathcal{S}} c_{j,d}S_{j,d}y_{j,d}. \end{aligned}$$

This produces a set of m disjoint paths $P_{i,d}$ in H from each $x_{i,d}$, $\langle i,d\rangle \in I$, to a $y_{j,d'}$ in $c_{j,d'}S_{j,d'} \in \mathcal{S}$. We can now define $R'_{\langle i,d\rangle}$ for $\langle i,d\rangle \in I$ as the single ray beginning with $P_{i,d}$ and then continuing with $S_{j,d'}$ after $y_{j,d'}$.

We can now define the required double rays R_i^{n+1} for $i < n$ as ${}^*R'_{\langle i,b\rangle}P_i^n R'_{\langle i,f\rangle}$ and check that these have all the desired properties. □

As for $\mathsf{IRT}_{\mathrm{DES}}$, instead of following the proof of Theorem 4.5, we will reduce $\mathsf{IRT}_{\mathrm{DES}}$ to the problem of finding an infinite sequence of *vertex-disjoint* rays in a certain locally finite graph (see [3, pg. 2 l. 3–7]). To carry out this reduction, we define the line graph:

Definition 5.11. The *line graph* $L(G)$ of an X-graph G is the X-graph whose vertices are the edges of G and whose edges are the $((u,v),(v,w))$, where (u,v) and (v,w) are edges in G.

Lemma 5.12. *Let G be an X-graph. There is a computable mapping from rays in G to rays in $L(G)$ such that if two rays in G are edge-disjoint, then their images are vertex-disjoint.*

Proof. Map $x_0, x_1, x_2, \ldots$ to $(x_0,x_1),(x_1,x_2),\ldots$. □

However, vertex-disjoint rays in $L(G)$ do not always yield edge-disjoint rays in G. An extreme counterexample is what is called the *(undirected) star graph* which consists of a single vertex with infinitely many neighbors: It does not contain any rays yet its line graph is isomorphic to the complete graph on N which contains infinitely many vertex-disjoint rays. Nonetheless, if G is locally finite, then vertex-disjoint rays in $L(G)$ do correspond to edge-disjoint rays in G:

Lemma 5.13 (ACA_0)**.** *Let G be a locally finite X-graph. There is a mapping from rays in $L(G)$ to rays in G such that if two rays in $L(G)$ are vertex-disjoint, then their images are edge-disjoint rays in G.*

Proof. Given a ray $R = e_0, e_1, \ldots$ in $L(G)$, we construct a ray $S = y_0, y_1, \ldots$ in G by recursion. Say that $e_i = (u_i, v_i)$. Start by defining y_0 to be u_0. Having defined y_n, we define y_{n+1} as follows. Let k_n be the largest k such that y_n is an endpoint of e_k. Such k exists because G is locally finite and R is a ray. We can find k_n by ACA_0. Then define y_{n+1} to be the endpoint of e_{k_n} other than y_n. This completes the recursion. Note that the k_n are strictly increasing: $e_{k_n} = (y_n, y_{n+1})$ as y_n is not an endpoint of e_{k_n+1} by the maximality of k_n. The next edge then includes y_{n+1} and so $k_{n+1} \geq k_n + 1$. Also note that if the graph is directed the e_i must be of the form $\langle x_i, x_{i+1} \rangle$ and the last occurrence of any x in an e_i must be as its first element. By construction S is infinite and contains no repeated vertices because of the maximality requirement, hence it is a ray. Observe that every edge in S is a vertex in R, so the above mapping maps vertex-disjoint rays in $L(G)$ to edge-disjoint rays in G. □

It remains to show that we can restrict our attention to locally finite graphs. We accomplish this with the help of Σ^1_1-AC_0. Given a directed graph G with arbitrarily many edge-disjoint rays, we can use Σ^1_1-AC_0 to choose a family $\langle \langle R^k_j \rangle_{j<k} \rangle_{k>0}$ where, for each $k > 0$, the rays $\langle R^k_j \rangle_{j<k}$ are edge-disjoint. From this family, we may construct an appropriate locally finite subgraph of G:

Lemma 5.14. *Suppose that G is an X-graph and there is some family $\langle \langle R^k_j \rangle_{j<k} \rangle_{k>0}$ of rays in G such that for each $k > 0$, the rays $\langle R^k_j \rangle_{j<k}$ are edge-disjoint. Then there is some locally finite X-subgraph G' of G and some family $\langle \langle S^k_j \rangle_{j<k} \rangle_{k>0}$ of rays in G' such that for each $k > 0$, the rays $\langle S^k_j \rangle_{j<k}$ are edge-disjoint.*

Proof. Define the vertices of G' to be the vertices of G, say $\{v_i \mid i \in N\}$. We specify the set of edges E' of G' by providing a recursive construction of sets E_i of edges putting in a set of edges at each step. We guarantee that each E_i is a union of finitely many finite sets of edge-disjoint rays in G and that after stage k no edge with a vertex v_i for $i < k$ as an endpoint is ever put into E'.

Begin at stage 0 by putting all the edges in R^1_0 into E_1. Proceeding recursively at stage $k > 0$ we have E_k and consider the edge-disjoint rays R^k_j, $j < k$. For each $j < k$, say $R^k_j = x^k_{j,0}, x^k_{j,1}, \ldots$. Each v_i for $i < k$ appears at most once in R^k_j as R^k_j is a ray. For each $j < k$, since we have access to the set of vertices of R^k_j, we can decide whether R^k_j contains v_i and, if so, find the index n such that $v_i = x^k_{j,n}$. Call it $n^k_{i,j}$. If there is no such n, set $n^k_{i,j} = 0$. Define S^k_j to be the tail of R^k_j after $x^k_{j,\max_{i<k} n^k_{i,j}}$. We put all the edges in S^k_j, $j < k$, into E_{k+1}. Let $E' = \bigcup_k E_k$.

It remains to show that G' is locally finite. Consider any vertex v_k. No edge containing v_k as an endpoint is put in after stage k. On the other hand, E_k is the union of finitely many finite sets of edge-disjoint rays (all of which have been computed uniformly). Each set of edge-disjoint rays in this union has v_k appearing at most once in each of its rays. Thus at most two edges containing v_k appear in each of the finitely many rays in this set. Therefore there are only finitely many edges containing v_k in each of the finite sets of edge-disjoint rays making up E_k. All in all, only finitely many edges in G' contain v_k. □

We are ready to prove

Theorem 5.15. Σ^1_1-AC_0 *implies* $\mathsf{IRT}_{\mathrm{XES}}$ *for each value of* X.

Proof. Given an X-graph G with arbitrarily many edge-disjoint rays, we can use Σ^1_1-AC_0 to choose a family $\langle\langle Q^k_j\rangle_{j<k}\rangle_{k>0}$ such that for each k, the rays Q^k_j, $j<k$, are edge-disjoint. By Lemma 5.14, there is a locally finite subgraph H of G and a family $\langle\langle R^k_j\rangle_{j<k}\rangle_{k>0}$ such that for each k, the R^k_j, $j<k$, are edge-disjoint rays in H. By Lemma 5.12, there is a family $\langle\langle S^k_j\rangle_{j<k}\rangle_{k>0}$ such that for each $k>0$, the S^k_j, $j<k$, are vertex-disjoint rays in $L(H)$. By the second part of the proof of Theorem 4.5(ii) (which can be carried out in ACA_0), $L(H)$ has infinitely many vertex-disjoint rays. Finally by Lemma 5.13, H has infinitely many edge-disjoint rays. Hence G has infinitely many edge-disjoint rays. □

By the discussion before Theorem 5.10, this completes the proof of Theorem 5.1.

Finally, we give a proof of $\mathsf{IRT}_{\mathrm{DED}}$ for directed forests using Σ^1_1-AC_0 (recall that $\mathsf{IRT}_{\mathrm{DED}}$ remains open). We will see that Σ^1_1-AC_0 and $\mathsf{IRT}_{\mathrm{DED}}$ for directed forests are equivalent over $\mathsf{RCA}_0+\mathsf{I}\Sigma^1_1$ but note that Σ^1_1-AC_0 does not imply $\mathsf{I}\Sigma^1_1$. (See Theorem 6.13 and the comment following it).

Theorem 5.16. Σ^1_1-AC_0 *implies* $\mathsf{IRT}_{\mathrm{DED}}$ *for directed forests.*

We first prove two lemmas.

Lemma 5.17 (ACA_0)**.** *Let G be a directed forest and let $R_0=\langle x_{0,i}\mid i\in\mathcal{Z}\rangle$, $R_1=\langle x_{1,i}\mid i\in\mathcal{Z}\rangle$ be directed double rays in G. Suppose R_0 and R_1 have an edge $\langle u,v\rangle$ in common. Then there are vertices t and w such $tR_0w=tR_1w$ and R_0, R_1 have no vertices in common outside of those in $tR_0w=tR_1w$. Note that we allow for the possibility that $t=-\infty$ and/or $w=+\infty$ in the sense that $(-\infty)R=R=R(+\infty)$ for any double ray R. We call $tR_0w=tR_1w$ the intersection of R_0 and R_1.*

Proof. Suppose R_0, R_1 provide a counterexample. As they have an edge $\langle u,v\rangle$ in common, they lie in the same directed tree T in G and can be viewed as (undirected) double rays in $\hat{T}$ (the underlying graph for T). As they form a counterexample to the lemma, there must be either a first $t\in R_0$ (i.e. earliest in the double ray R_0) such that $tR_0v=tR_1v$ or a last $w\in R_0$ such that $uR_0w=uR_1w$ but R_0 and R_1 have a vertex z in common outside the common interval. The situations are symmetric and we consider the second. The immediate successors x and y of w in R_0 and R_1, respectively, must be different by our choice of w. Consider now the location of z in R_0. If it is after x then the paths from w to z in R_0 and w to z in R_1 (both considered now as undirected graphs within $\hat{T}$) are different as the immediate successor of w in R_0 is x while in R_1 it is either y or a vertex in vR_1w. Thus there are two different paths in $\hat{T}$ from w to z contradicting $\hat{T}$'s being a tree. If, on the other hand z is before u in R_0, it must be before $t\in R_0$ and a similar argument provides different paths from t to z in R_0 and R_1. □

Lemma 5.18 (ACA_0)**.** *There is a computable function f such that given any sequence $\langle S_i \rangle_{i<n}$ of DED rays with subpaths $\langle P_i \rangle_{i<n}$ of length $2n$ in a directed tree T and sequence $\langle R_j \rangle_{j<f(n)}$ of DED rays in T, we can construct a sequence $\langle S'_i \rangle_{i \leq n}$ of DED rays with subpaths $\langle P'_i \rangle_{i \leq n}$ of length $2n+2$ in T such that P'_i extends P_i at each end for $i < n$. Indeed, we may take $f(n) = 2n^2 + 2^{2n}n! + 1$.*

Proof. First we remove all the R_j that contain an edge in any P_i at cost of at most $2n^2$ many j. Consider any remaining R_j in the second sequence. By Lemma 5.17, its intersections with the S_i are intervals $Q_{j,i}$ of edges in R_j which are disjoint as the S_i are. By our first thinning of the R_j list, none of the $Q_{j,i}$ intersect any of the P_i so each $Q_{j,i}$ must lie entirely above or entirely below P_i. We associate to each R_j a label consisting of the set $C_j = \{i < n \mid Q_{j,i} \neq \emptyset\}$; the elements i of C_j in the order in which the $Q_{j,i}$ (for $i \in C_j$) appear in R_j (in terms of the ordering of $\mathcal{Z}$) along with a $+$ or $-$ depending on which side of P_i it falls in S_i. We write $Q^s_{j,i}$ for the starting vertex of $Q_{j,i}$ and $Q^e_{j,i}$ for the ending one. As above, we allow the values $\pm\infty$ for these endpoints if the intervals are infinite. Now there are, of course, at most finitely many such labels. In particular, there are at most $2^n n! 2^n$ such labels. Thus if we have $2^{2n}n! + 1$ many R_j left at least two of them, say R_a and R_e have the same label say with set C.

Claim: $|C| < 2$.

For the sake of a contradiction, assume we have $k \neq l$ in C with k preceding l in the ordering of C in the label. Say R_a is the ray such that $Q_{a,k}$ is before $Q_{e,k}$ in S_k. Note that $Q_{a,k}$ and $Q_{e,k}$ are edge disjoint as R_a and R_e are. We consider two cases: (1) $Q_{a,l}$ is before $Q_{e,l}$ in S_l and (2) $Q_{e,l}$ is before $Q_{a,l}$ in S_l. We now produce, for each case, two vertices with two distinct sequences (i) and (ii) of adjacent edges in T connecting these two vertices. These sequences are illustrated in Figures 2 and 3. Note that by our assumptions on the orderings of the intervals $Q_{c,d}$ (for $c \in \{a, e\}$ and $d \in \{k, l\}$) as displayed, all of the starting or ending points of $Q_{c,d}$ that appear in our sequences are vertices in one of the rays (i.e. none are $\pm\infty$):

(1i) Start at $Q^e_{a,k}$ in R_a and go to $Q^e_{a,l}$ then in S_l go to $Q^s_{e,l}$.

(1ii) Start at $Q^e_{a,k}$ in S_k and go to $Q^s_{e,k}$ then go in R_e to $Q^s_{e,l}$.

(2i) Start at $Q^s_{a,k}$ in S_k and go to $Q^e_{e,k}$ then in R_e go to $Q^e_{e,l}$ then in S_l go to $Q^e_{a,l}$.

(2ii) Start at $Q^s_{a,k}$ in R_a and go to $Q^e_{a,l}$.

To see that the two sequences of vertices are different, note for (1) that (1i) contains an edge in $Q_{a,l}$ but (1ii) does not. For (2) note that (2i) contains an edge in $Q_{e,k}$ but (2ii) does not. We now, in each case, view the associated two distinct sequences of vertices with the same endpoints in the underlying (undirected) tree $\hat{T}$. The only way one can have such sequences in a tree is for one of the sequences to contain some vertices uvu in order. However, any three successive vertices in any of these sequences lie within one of the R_j or S_i or both and so cannot have two instances of the same vertex. The crucial point is that each $Q_{j,i}$ is in both R_j and S_i and has at least two vertices. Any transition along the sequence between an R_j and an S_i (in either order) goes through $Q_{j,i}$ and so any three consecutive vertices are all contained in one R_j or one S_i (or both).

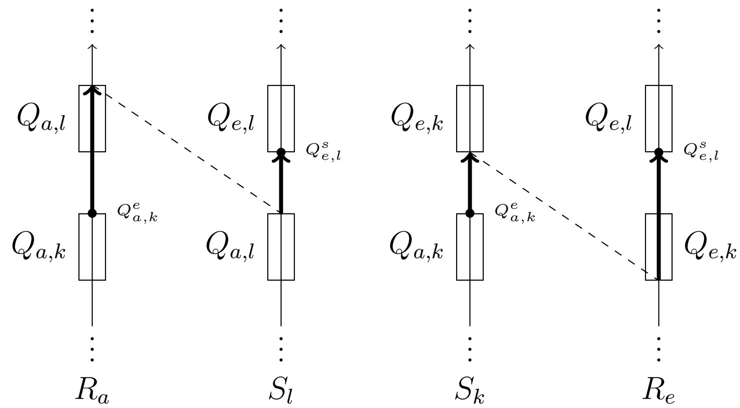

Figure 2: (1i) follows the thick arrows in R_a and S_l. (1ii) follows the thick arrows in S_k and R_e.

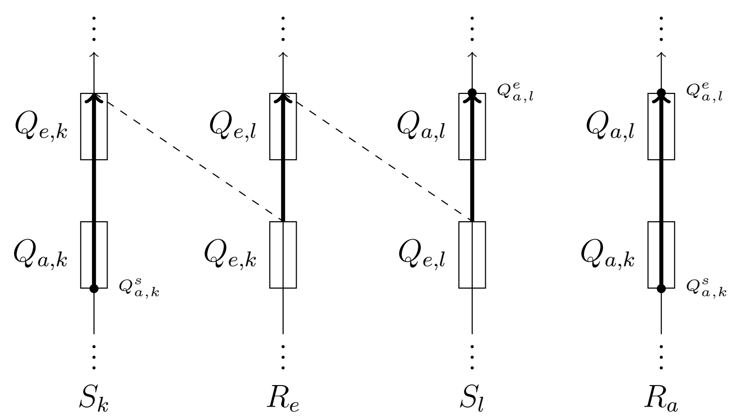

Figure 3: (2i) follows the thick arrows in S_k, R_e and S_l. (2ii) follows the thick arrow in R_a.

Knowing now that $|C|$ is 0 or 1, we complete the proof of the Lemma. If $|C| = 0$, then both R_a and R_e are disjoint from all the S_i and so we may add on either one of them as S'_n with P'_n an arbitrary subpath of length $2n + 2$ while keeping $S_i = S'_i$ and extending the P'_i appropriately for all $i < n$. Otherwise say $C = \{i\}$. Let c be the one of a or e such that $Q_{c,i}$ is closer to P_i. (Remember that they are both on the same side of this interval in S_i by our fixing the label.) Now replace the tail of S_i starting with $Q_{c,i}$ and going away from P_i by the tail of R_c starting with $Q_{c,i}$ and going in the same direction. Let this ray be S'_i. Note that it is disjoint from all the S_j, $j \neq i$ as it contains only edges that are in S_i or R_c neither of which share any edges with such S_j. It is also disjoint from R_d where d is the one of a, e which is not c since all the edges of S'_i are either in R_c or in S_i outside of $Q_{d,i}$ by our choice of $Q_{c,i}$ as closer to P_i. As R_d is also disjoint from all the S_j for $j \neq i$ by our fixing the label, we may define $S'_j = S_j$ and P_j appropriately for $j < n$, $j \neq i$ and $S'_n = R_d$ and choosing P'_n of length $2n + 2$ arbitrarily so as to get the sequence required in the Lemma. □

Lemma 5.18 provides the inductive step for the following proof:

Proof of Theorem 5.16. Assume we are given a directed forest G with arbitrarily many DED rays. By Σ^1_1-AC_0 we may take a sequence $\langle\langle R_{k,i}\rangle_{i<k}\rangle_{k\in N}$ such that, for each k, $\langle R_{k,i}\rangle_{i<k}$ is a sequence of k many DED rays in G. If there are infinitely many of the trees making up G each of which contains some $R_{k,i}$ then we are done. So we may assume that all of them are in one directed tree T. We now wish to define $\langle\langle S_{k,s}\rangle_{k<s}\rangle_{s\in N}$ by recursion such that, for each s, $\langle S_{k,s}\rangle_{k<s}$ is a sequence of s many DED rays with subpaths $P_{k,s}$ of length $2s + 1$ such that for each $s \in N$, $P_{k,s+1}$ extends $P_{k,s}$ at each end so that the $\lim_s P_{k,s}$ form an infinite sequence of DED rays in T as required. Lemma 5.18 provides precisely the required inductive step for the construction since we have the required sequences of DED rays $\left\langle R_{f(n),i} \mid i < f(n)\right\rangle$ at each step n of the construction. Once again we just have to note that Lemma 5.18 provides witnesses for the double rays that are composed of finite paths in T and final segments in one direction or the other of some of the $R_{k,i}$ and so they can be found recursively in $(T \oplus \langle\langle R_{k,i}\rangle_{i<k}\rangle_{k\in N})'$. □

6 Variations on Maximality

In this section, we consider variants of IRT whose solutions are required to be maximal in terms of cardinality or maximal in terms of set inclusion.

6.1 Maximum Cardinality Variants of IRT

Definition 6.1. Let $\mathsf{IRT}^*_{\mathrm{XYZ}}$ be the statement that every X-graph G has a set of Y-disjoint Z-rays of maximum cardinality, or more formally, the statement that for every X-graph G:

- there is no Z-ray in G, or
- there is some $n > 0$ and some R such that $\langle R^{[i]} \mid i < n\rangle$ is a sequence of Y-disjoint Z-rays in G, and there is no R such that $\langle R^{[i]} \mid i < n + 1\rangle$ is a sequence of Y-disjoint Z-rays in G, or

- there is some R such that $\langle R^{[i]} \mid i \in N \rangle$ is a sequence of Y-disjoint Z-rays in G.

When we talk about the cardinality of a (possibly empty) finite sequence $\langle R^{[i]} \mid i < n \rangle$ we mean the number n (which may be 0). Of course a sequence $\langle R^{[i]} \mid i \in N \rangle$ is said to have infinite cardinality.

$\mathsf{IRT}^*_{\mathrm{UVS}}$ was proved by Halin [11], who also proved the corresponding statement for uncountable graphs.

Remark 6.2. The notation in Definition 6.1 is inspired by the well known version ACA^*_0 of ACA_0 which, for every A, asserts the existence of $A^{(n)}$, the nth jump of A, for every n:

$$\mathsf{ACA}^*_0: \quad (\forall A)(\forall n)(\exists W)(W^{[0]} = A \wedge (\forall i < n)(W^{[i+1]} = W^{[i]\prime})).$$

This asserts (in addition to ACA_0) particular instances of $\mathsf{I}\Sigma^1_1$. So too (in addition to $\mathsf{IRT}_{\mathrm{XYZ}}$) do the $\mathsf{IRT}^*_{\mathrm{XYZ}}$ as we are about to see.

Proposition 6.3. *For each choice of XYZ,* $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *implies* $\mathsf{IRT}_{\mathrm{XYZ}}$ *over* RCA_0 *and* $\mathsf{IRT}_{\mathrm{XYZ}}$ *implies* $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$. *Therefore* $\mathsf{IRT}_{\mathrm{XYZ}}$ *and* $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *are equivalent over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$. *In particular, they have the same standard models.*

Proof. The first implication holds because if an X-graph has arbitrarily many Y-disjoint Z-rays, then any sequence of Y-disjoint Z-rays in the graph of maximum cardinality must be infinite. To prove the second implication, let G be an X-graph and $\Phi(n)$ be the Σ^1_1 formula which says that there is a sequence of length n of Y-disjoint Z-rays in G. If $\forall n(\Phi(n) \to \Phi(n+1))$, then by $\mathsf{I}\Sigma^1_1$, $\forall n \Phi(n)$ holds and so by $\mathsf{IRT}_{\mathrm{XYZ}}$ there is a sequence $\langle R_i \rangle_{i \in N}$ of Y-disjoint Z-rays in G as required. On the other hand, if there is an n such that $\Phi(n)$ holds but $\Phi(n+1)$ fails, then this n witnesses $\mathsf{IRT}^*_{\mathrm{XYZ}}$. □

It follows from Proposition 6.3 and Theorem 5.1 that

Corollary 6.4. $\mathsf{IRT}^*_{\mathrm{XYS}}$ *and* $\mathsf{IRT}^*_{\mathrm{UVD}}$ *are theorems of hyperarithmetic analysis.*

It follows from Proposition 6.3 and Theorem 5.9 that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ implies ACA_0, so we will not explicitly mention uses of ACA_0 whenever we are assuming any $\mathsf{IRT}^*_{\mathrm{XYZ}}$.

Using Lemmas 5.2, 5.4 and 5.6, we can prove

Proposition 6.5. $\mathsf{IRT}^*_{\mathrm{DYZ}}$ *implies* $\mathsf{IRT}^*_{\mathrm{UYZ}}$, $\mathsf{IRT}^*_{\mathrm{DEZ}}$ *implies* $\mathsf{IRT}^*_{\mathrm{DVZ}}$ *and* $\mathsf{IRT}^*_{\mathrm{DYD}}$ *implies* $\mathsf{IRT}^*_{\mathrm{DYS}}$.

Next, we show that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ proves sufficient induction in order to transcend Σ^1_1-AC_0. This implies that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ is strictly stronger than $\mathsf{IRT}_{\mathrm{XYZ}}$ for certain choices of XYZ (Corollary 6.10). The connection between Σ^1_1-AC_0 and graphs is obtained by viewing the set of solutions of an arithmetic predicate as the set of (projections of) branches on a subtree of $N^{<N}$. In detail:

Lemma 6.6 ([29, V.5.4])**.** *If* $A(X)$ *is an arithmetic formula,* ACA_0 *proves that there is a tree* $T \subseteq N^{<N}$ *such that*

$$\forall X(A(X) \leftrightarrow \exists f(\langle X, f \rangle \in [T]))$$
$$\textit{and}\ \forall X(\exists \textit{ at most one } f)(\langle X, f \rangle \in [T]).$$

In fact, as the proof in [29, V.5.4] shows, the required functions f are what are called the minimal Skolem functions and are arithmetically defined uniformly in X and the formula A.

The following easy corollary will be useful.

Lemma 6.7. *If $A(n, X)$ is an arithmetic formula, ACA_0 proves that there is a sequence of subtrees $\langle T_n \rangle_n$ of $N^{<N}$ such that for each $n \in N$,*

$$\forall X(A(n, X) \leftrightarrow \exists f(\langle X, f \rangle \in [T_n]))$$
$$\textit{and}\, \forall X(\exists \textit{ at most one } f)(\langle X, f \rangle \in [T_n]).$$

Proof. Say that $B(Y)$ holds if and only if $A(Y(0), X)$ holds, where X is such that $Y = Y(0)^\frown X$. Apply Lemma 6.6 to the arithmetic formula $B(Y)$ to obtain a tree $T \subseteq N^{<N}$. For each $n \in N$, define T_n to be the set of all σ such that $n^\frown\sigma \in T$. It is straightforward to check that $\langle T_n \rangle_n$ satisfies the desired properties. □

Theorem 6.8. $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *proves* ACA^*_0.

Before proving the above theorem, we derive some corollaries:

Corollary 6.9. $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *proves the consistency of* Σ^1_1-AC_0*. Therefore it is not provable in* Σ^1_1-AC_0.

Proof. Simpson [29, IX.4.6] proves that $\mathsf{ACA}_0 + \mathsf{I}\Sigma^1_1$ implies the consistency of Σ^1_1-AC_0. The only use of $\mathsf{I}\Sigma^1_1$ in Simpson's proof is to establish ACA^*_0, so Simpson's proof shows that ACA^*_0 implies the consistency of Σ^1_1-AC_0. The desired result then follows from Theorem 6.8 and Gödel's second incompleteness theorem. □

Corollary 6.10. $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *is strictly stronger than* $\mathsf{IRT}_{\mathrm{XYZ}}$ *for the following choices of XYZ: XYS and UVD.*

Proof. We showed in §5 that the specified variants of IRT are provable in Σ^1_1-AC_0. On the other hand, none of the IRT^* are provable in Σ^1_1-AC_0 (Corollary 6.9). □

We now prove Theorem 6.8:

Proof that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *implies* ACA^*_0. By Proposition 6.5, it suffices to prove the desired result for $\mathsf{IRT}^*_{\mathrm{UYZ}}$. To prove ACA^*_0 from $\mathsf{IRT}^*_{\mathrm{UYS}}$, begin by using Lemma 6.7 to define for each A a sequence of trees $\langle T_n \rangle_n$ such that for each n and W there is at most one f such that $\langle W, f \rangle \in [T_n]$ and

$$\begin{aligned}&(\exists f)(\langle W, f \rangle \in [T_n]) \\ \leftrightarrow\ & W^{[0]} = A \ \wedge\ (\forall i \le n)((W^{[i]})' = W^{[i+1]}) \ \wedge\ (\forall i > n)(W^{[i]} = \emptyset).\end{aligned}$$

We want to show that each T_n is ill-founded. Note that if $m < n$ and T_n is ill-founded, then so is T_m. Therefore it suffices to show that for cofinally many n, T_n is ill-founded.

Apply $\mathsf{IRT}^*_{\mathrm{UYS}}$ to the disjoint union $\bigsqcup_n T_n$ to obtain a collection C of Y-disjoint rays of maximum cardinality. We prove that C is infinite. Suppose not. Then there is some maximum m such that C contains a ray in T_m. A ray in T_m can be computably truncated or extended to a branch on T_m, so T_m is ill-founded. Hence T_{m+1} is ill-founded as well (by ACA_0). But then there is a collection of Y-disjoint rays in $\bigsqcup_n T_n$ which has cardinality greater than that of C, contradiction.

We have proved that C is infinite. Next we prove that each T_n has at most one branch. That would imply that each T_n contains at most one ray in C, so C contains rays in cofinally many T_n, as desired.

If T_n has two distinct branches $\langle W_0, f_0\rangle$ and $\langle W_1, f_1\rangle$, then $W_0 \neq W_1$ by the "at most one" condition in the definition of the T_n. Consider the least i such that $W_0^{[i]} \neq W_1^{[i]}$. Such i exists by ACA_0. Note that $0 < i \leq n$ because $W_0^{[0]} = A = W_1^{[0]}$ and $W_0^{[i]} = \emptyset = W_1^{[i]}$ for $i > n$. But then $W_0^{[i-1]} = W_1^{[i-1]}$ and $(W_0^{[i-1]})' \neq (W_1^{[i-1]})'$, contradiction.

This proves that $\mathsf{IRT}^*_{\mathrm{UYS}}$ implies ACA_0^*. In order to prove that $\mathsf{IRT}^*_{\mathrm{UYD}}$ implies ACA_0^*, we modify the above proof by adding to each T_n a computable branch consisting of new vertices to form a tree S_n. Apply $\mathsf{IRT}^*_{\mathrm{UYD}}$ to $\bigsqcup_n S_n$ to obtain a collection C of Y-disjoint double rays of maximum cardinality. Following the above proof, we may prove that C is infinite and each S_n contains at most one double ray in C. So C contains double rays in cofinally many S_n, as desired. □

Remark 6.11. The same proof shows that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ implies the following induction scheme: Suppose $\langle T_n\rangle_n$ is a sequence of trees such that

1. T_0 has a unique branch;
2. for all n, the number of branches on T_{n+1} is the same as the number of branches on T_n.

Then for all n, there is a sequence $\langle P_m\rangle_{m<n}$ such that for each $m < n$, P_m is the unique branch on T_m. It also shows that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ implies ACA_0^+ (i.e. closure under the ω-jump) and much more. Indeed, similar ideas prove in Theorem 7.3 that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ implies unique-Σ^1_1-AC_0 (Definition 7.1). $\mathsf{IRT}^*_{\mathrm{XYZ}}$ also implies a similar induction scheme analogous to finite-Σ^1_1-AC_0 (Definition 7.2).

We can prove that even fragments of $\mathsf{IRT}^*_{\mathrm{DVD}}$ give more induction than the specific instances derived in Theorem 6.8.

Theorem 6.12. $\mathsf{IRT}^*_{\mathrm{DVD}}$ *(even for directed forests) implies* $\mathsf{I}\Sigma^1_1$ *over* RCA_0.

Proof. Suppose $\Psi(n)$ is a Σ^1_1 formula such that $\Psi(0)$ and $\forall n(\Psi(n) \to \Psi(n+1))$ hold. Let $\langle S_i\rangle_{i\in N}$ be a sequence of subtrees of $N^{<N}$ such that S_i is ill-founded if and only if $\Psi(i)$ holds. Let $\langle T_n\rangle_{n\in N}$ be a sequence of subtrees of $N^{<N}$ such that for each n, T_n consists of all sequences $\langle \sigma_i : i \leq n\rangle$ where for each $i \leq n$, σ_i is (a code for) a string in S_i. We order these sequences by component-wise extension. It is clear that T_n is ill-founded if and only if $(\forall i \leq n)\Psi(i)$ holds.

For each n, orient each edge in T_n towards its root and add a computable D-ray of new vertices which starts at its root. This forms a directed tree G_n. If G_n contains a double ray, T_n is ill-founded and $(\forall i \leq n)\Psi(i)$ holds. Furthermore, no two disjoint double rays can lie in the same G_n.

Let G be the directed forest $\bigsqcup_n G_n$. By $\mathsf{IRT}^*_{\mathrm{DVD}}$, there is a sequence $\langle R_i \rangle_i$ of disjoint double rays in G of maximum cardinality, so the sequence may be for $i < k$ for some k or $i \in N$. Since $\Psi(0)$ holds, $\langle R_i \rangle_i$ is nonempty. If $\langle R_i \rangle_i$ is finite, let n be maximal such that G_n contains some R_i. Then $\Psi(n)$ holds, so $\Psi(n+1)$ holds as well. It follows that G_{n+1} contains some double ray, which we can then add to $\langle R_i \rangle_i$ to obtain a larger sequence of disjoint double rays in G for the desired contradiction. Therefore $\langle R_i \rangle_i$ is infinite. Since each G_n contains at most one R_i, infinitely many G_n contain some R_i. Therefore $\Psi(n)$ holds for all n. □

In fact, we have the following equivalences:

Theorem 6.13. *The following are equivalent (over* RCA_0*):*

1. $\Sigma^1_1\text{-}\mathsf{AC}_0 + \mathsf{I}\Sigma^1_1$;
2. $\mathsf{IRT}_{\mathrm{DED}}$ *for directed forests* $+ \mathsf{I}\Sigma^1_1$;
3. $\mathsf{IRT}^*_{\mathrm{DED}}$ *for directed forests;*
4. $\mathsf{IRT}^*_{\mathrm{DVD}}$ *for directed forests;*
5. $\mathsf{IRT}_{\mathrm{DVD}}$ *for directed forests* $+ \mathsf{I}\Sigma^1_1$.

Proof. (1) → (2) follows from Theorem 5.16. (2) → (3) follows from the proof of Proposition 6.3. (3) → (4) follows from the observation that the mapping of graphs defined in Lemma 5.4 sends a directed forest to a directed forest. (4) → (5) follows from Theorem 6.12 and the proof of Proposition 6.3.

To prove (5) → (1), suppose $A(n, X)$ is an arithmetic formula such that $\forall n \exists X \, A(n, X)$. By Lemma 6.7, there is a sequence $\langle T_n \rangle_n$ of subtrees of $N^{<N}$ such that

$$\forall n \forall X (A(n, X) \leftrightarrow \exists f (\langle X, f \rangle \in [T_n]))$$
$$\text{and } \forall X (\exists \text{ at most one } f)(\langle X, f \rangle \in [T_n]).$$

By assumption on $A(n, X)$, each T_n is ill-founded. We use $\langle T_n \rangle_n$ to construct a sequence $\langle G_n \rangle_n$ of directed trees as we did in the proof of Theorem 6.12 to construct T_n from the S_i.

By $I\Sigma^1_1$, the directed forest $\bigsqcup_n G_n$ contains arbitrarily many disjoint double rays. Therefore $\bigsqcup_n G_n$ contains infinitely many disjoint double rays $\langle R_k \rangle_k$, by $\mathsf{IRT}_{\mathrm{DVD}}$. Note that any double ray in any G_n must contain the computable ray we added, so any two double rays in the same G_n must intersect. This implies that each R_k belongs to some distinct G_n. Therefore for every m, there is some k and some $n > m$ such that R_k is a double ray in G_n. When we remove the added computable ray from R_k we are left with a branch in T_n which is of the form $\langle X, f \rangle$ where X consists of witnesses X_i for $i < n$. □

Since $\Sigma^1_1\text{-}\mathsf{AC}_0$ (ATR_0, even) does not prove $\mathsf{I}\Sigma^1_1$ [29, IX.4.7], it follows that

Corollary 6.14. $\mathsf{IRT}^*_{\mathrm{DYD}}$ *(even for directed forests) is not provable in* ATR_0*, and strictly implies* $\Sigma^1_1\text{-}\mathsf{AC}_0$ *over* RCA_0*.*

Next, we show that $\mathsf{IRT}^*_{\mathrm{UVD}}$ implies $\mathsf{IRT}_{\mathrm{UVS}}$ over RCA_0 (see Figure 1).

Theorem 6.15. $\mathsf{IRT}^*_{\mathrm{UVD}}$ *implies* $\mathsf{IRT}_{\mathrm{UVS}}$ *over* RCA_0. *Therefore (1)* $\mathsf{IRT}_{\mathrm{UVD}}$ *implies* $\mathsf{IRT}_{\mathrm{UVS}}$ *over* $\mathsf{RCA}_0+\mathsf{I}\Sigma^1_1$*; (2) if any standard model of* RCA_0 *satisfies* $\mathsf{IRT}_{\mathrm{UVD}}$*, then it satisfies* $\mathsf{IRT}_{\mathrm{UVS}}$ *as well.*

Proof. Let G be a graph which contains arbitrarily many disjoint single rays. By $\mathsf{IRT}^*_{\mathrm{UVD}}$, there is a sequence of disjoint double rays in G of maximum cardinality. If this sequence is infinite, then there are infinitely many disjoint single rays in G as desired. Otherwise, suppose that $\langle R_i\rangle_{i<j}$ is a sequence of disjoint double rays in G of maximum cardinality j. Let $\mathcal{R}$ be the subgraph of G consisting of the union of all R_i. Let H be the induced subgraph of G consisting of all vertices which do not lie in $\mathcal{R}$. Note that H does not contain any double ray, otherwise G would contain $j+1$ many disjoint double rays. Next, we expand H to the graph H', defined below.

Decompose H into its connected components $\langle H_i\rangle_i$ (there may only be finitely many). Any two single rays in the same H_i must intersect, because if S_0 and S_1 are disjoint single rays in the same H_i, then we can construct a double ray in H_i by connecting them (start with a path between S_0 and S_1 of minimum length, then connect it to the tails of S_0 and S_1 which begin at the endpoints of the path).

For each i, define H'_i by adding a computable ray of new vertices to H_i, which begins at the $<_N$-least vertex in H_i. Define H' to be the disjoint union $\bigsqcup_i H'_i$.

By $\mathsf{IRT}^*_{\mathrm{UVD}}$, there is a sequence of disjoint double rays in H' of maximum cardinality.

<u>Case 1.</u> If this sequence is infinite, then G contains infinitely many disjoint single rays because each double ray in the sequence has a tail which lies in H. In this case we are done.

<u>Case 2.</u> Otherwise, H' does not contain arbitrarily many disjoint double rays. Since any two single rays in the same H_i must intersect, we can transform any collection of disjoint single rays in H into a collection of disjoint double rays in H' of equal cardinality by connecting each single ray to the $<_N$-least vertex in its connected component H_i and then following the computable ray we added. It follows that H does not contain arbitrarily many disjoint single rays. Fix l such that H does not contain $l+1$ many disjoint single rays.

Towards a contradiction, we construct a collection of $(j+1)$-many disjoint double rays in G as follows. Fix a collection $\mathcal{S}$ of $l+2j+4j(j+1)$ many disjoint single rays in G. First, at most $2j$ of these single rays lie in $\mathcal{R}$. In fact at most $2j$ of these single rays can have finite intersection with H, because given a collection of disjoint single rays each of which has finite intersection with H, we can obtain a collection of disjoint single rays in $\mathcal{R}$ of the same cardinality by replacing each ray with an appropriate tail. Second, by reasoning analogous to the above, at most l of these single rays can have finite intersection with $\mathcal{R}$. Therefore, there are at least $4j(j+1)$ many disjoint single rays in $\mathcal{S}$ each of which have infinite intersection with both $\mathcal{R}$ and H.

Next, choose an edge (u_i,v_i) in each R_i and split R_i into two single rays $u_iR_{i,b}$ and $v_iR_{i,f}$. By the pigeonhole principle, there is some single ray R of the form $u_iR_{i,b}$ or $v_iR_{i,f}$, and at least $2(j+1)$ many disjoint single rays in $\mathcal{S}$, each of which have infinite intersection with both R and H. Call these rays $S_0,S_1,\dots,S_{2(j+1)-1}$. Discard all the other rays in $\mathcal{S}$. Below we describe how to connect pairs of single rays S_k using segments of R to form a collection of $(j+1)$-many disjoint double rays in G.

Let $x_0,x_1,\dots$ denote the vertices of R. Since each single ray S_k has infinite intersection with R, by the pigeonhole principle, there is a pair of disjoint rays S_{k_0} and S_{l_0} such that for each tail R' of R, there is a vertex in $S_{k_0} \cap R'$ and a vertex in $S_{l_0} \cap R'$ such that no S_k intersects R between these two vertices. (Formally, we justify this by defining the following coloring recursively. Start from the first vertex in R which is also in some S_k. Search for the next vertex on R which intersects some S_l, $l \neq k$. Then we color 0 with the unordered pair $\{k, l\}$. Then we search for the next vertex on R which intersects some S_m, $m \neq l$ and color 1 with $\{l, m\}$, and so on. Some color $\{k_0, l_0\}$ must appear infinitely often.) Then we commit to connecting S_{k_0} and S_{l_0} (but we do not do so just yet). Applying the pigeonhole principle again, there is a pair of disjoint rays S_{k_1} and S_{l_1} (with k_1, l_1, k_0, l_0 all distinct) such that for each tail R' of R, there is a vertex x in $S_{k_1} \cap R'$ and a first vertex y in $S_{l_1} \cap R'$ (after x in R) such that no S_k, except perhaps S_{k_1}, S_{k_0} or S_{l_0}, intersects R between these two vertices. We may eliminate any elements of S_{k_1} by changing x (if necessary) to the last element of Ry in S_{k_1}. Again we commit to connecting S_{k_1} and S_{l_1}. Repeat this process until we have obtained $j+1$ pairs of single rays. That is, when we have S_{k_i} and S_{l_i} for an $i < j$, we find $S_{k_{i+1}}$ and $S_{l_{i+1}}$ with k_{i+1} and l_{i+1} distinct from all previous k_m and l_m such that for each tail R' of R there is a vertex $x \in S_{k_{i+1}} \cap R$ and a first $y \in S_{l_{i+1}} \cap R$ after x in R such that no S_k, except perhaps S_{k_m} or S_{l_m} for $m \leq i$, intersects R between these two vertices. This process stops when we define k_j and l_j.

Finally, we connect these pairs of single rays in the opposite order in which we defined them: Start by picking some $x^j \in S_{k_j} \cap R$ and some $y^j \in S_{l_j} \cap R$. Then we define a double ray D_j by following $^*S_{k_j}$ until x^j, then following R until y^j, and finally following S_{l_j}, i.e., $D_j := {}^*(x^j S_{k_j}) R y^j S_{l_j}$. Having defined $D_j, D_{j-1}, \ldots, D_{i+1}$, define $D_i := {}^*(x^i S_{k_i}) R y^i S_{l_i}$, where $x^i \in S_{k_i} \cap R$ and $y^i \in S_{l_i} \cap R$ are chosen as follows: Consider a tail R' of R such that the union of $x^j R y^j, \ldots, x^{i+1} R y^{i+1}$ is disjoint from (1) R'; (2) $x S_{k_i}$ for each $x \in S_{k_i} \cap R'$; (3) $y S_{l_i}$ for each $y \in S_{l_i} \cap R'$. By choice of k_i and l_i, there are vertices $x^i \in S_{k_i} \cap R'$ and $y^i \in S_{l_i} \cap R'$ such that none of $S_{k_j}, \ldots, S_{k_{i+1}}$ or $S_{l_j}, \ldots, S_{l_{i+1}}$ intersect $x^i R y^i$.

It is straightforward to check that each of $D_j, D_{j-1}, \ldots, D_{i+1}$ is disjoint from D_i. This process yields disjoint double rays $D_j, D_{j-1}, \ldots, D_0$ in G, contradicting the maximality of j. □

Using some of the ideas in the previous proof, we can prove

Theorem 6.16. $\mathsf{IRT}^*_{\mathrm{UYD}}$ *for forests implies* $\mathsf{IRT}^*_{\mathrm{UYS}}$ *for forests over* RCA_0*. Therefore* $\mathsf{IRT}_{\mathrm{UYD}}$ *for forests implies* $\mathsf{IRT}_{\mathrm{UYS}}$ *for forests over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$*.*

This result will be used in the proofs of Theorems 7.3, 7.7 and 7.10.

Proof. Let G be a forest. If G happens to have arbitrarily many disjoint double rays, then by $\mathsf{IRT}^*_{\mathrm{UYD}}$, G has infinitely many disjoint double rays. Therefore there is an infinite sequence of disjoint single rays in G. Such a sequence has maximum cardinality, so we are done in this case.

Suppose G does not have arbitrarily many disjoint double rays. By $\mathsf{IRT}^*_{\mathrm{UYD}}$ for forests, there is a sequence $\langle R_i \rangle_{i<j}$ of disjoint double rays in G of maximum cardinality. Following the proof of Theorem 6.15, define the forests $\mathcal{R}$, H, and H'. There, we proved that no two single rays in the same connected component H_i of H can be disjoint.

By $\mathsf{IRT}^*_{\mathrm{UYD}}$ for forests, there is a sequence of disjoint double rays in H' of maximum cardinality. If this sequence is infinite, then there is an infinite sequence of disjoint single rays in H because each double ray in the sequence has a tail which lies in H. This is a sequence of disjoint single rays of maximum cardinality in G, so we are done in this case.

Otherwise, suppose $\langle S_k \rangle_{k<l}$ is a disjoint sequence of double rays in H' of maximum cardinality. Consider the following disjoint sequence of single rays in G. First, for each $k < l$, consider the single ray formed by intersecting H and the double ray S_k. Second, for each $i < j$, we can split the double ray R_i into a pair of disjoint single rays in G. This yields a finite sequence $\langle Q_m \rangle_{m<n}$ of disjoint single rays in G.

We claim that $\langle Q_m \rangle_{m<n}$ is a sequence of disjoint single rays in G of maximum cardinality. Suppose there is a larger sequence of disjoint single rays in G. Since G is a forest, any two single rays in G which share infinitely many edges or vertices must share a tail. Therefore there is a single ray Q in this larger sequence which only shares finitely many edges and vertices with each Q_m. Then some tail of Q, say xQ, is vertex-disjoint from each Q_m. In particular, xQ is vertex-disjoint from each R_i, i.e. xQ lies in H. Extend xQ to a double ray in H' by first connecting x to the $<_N$-least vertex in its connected component H_i, then following the computable ray which we added. The resulting double ray is disjoint from every S_k, because no S_k can lie in the same H'_i as xQ (for xQ is vertex-disjoint from $S_k \cap H$ by construction). This contradicts the maximality of l. □

6.2 Maximal Variants of IRT

Instead of sets of disjoint rays of maximum cardinality, we could consider sets of disjoint rays which are maximal with respect to set inclusion. For uncountable graphs, Halin [11] observed that any uncountable maximal set of disjoint rays is in fact of maximum cardinality (because rays are countable). This suggests another variant of IRT, which we call *maximal* IRT:

Definition 6.17. Let $\mathsf{MIRT}_{\mathrm{XYZ}}$ be the statement that every X-graph G has a (possibly finite) sequence $(R_i)_i$ of Y-disjoint Z-rays which is maximal, i.e., for any Z-ray R in G, there is some i such that R and R_i are not Y-disjoint.

$\mathsf{MIRT}_{\mathrm{XYZ}}$ immediately follows from Zorn's Lemma. It is straightforward to show that $\mathsf{MIRT}_{\mathrm{XYZ}}$ implies Π^1_1-CA_0 (see the proof of Theorem 6.18 below), hence $\mathsf{MIRT}_{\mathrm{XYZ}}$ is much stronger than $\mathsf{IRT}_{\mathrm{XYZ}}$ or even $\mathsf{IRT}^*_{\mathrm{XYZ}}$. We show below that $\mathsf{MIRT}_{\mathrm{XYZ}}$ is equivalent to Π^1_1-CA_0. This situation is reminiscent of König's duality theorem for countable graphs. Aharoni, Magidor, Shore [2] proved that the theorem implies ATR_0 and that Π^1_1-CA_0 suffices to prove the required existence of a König cover. Simpson [28] later proved that ATR_0 actually suffices. The covers produced in [2], and indeed in all then known proofs of this duality theorem actually had various maximality properties. Aharoni, Magidor, Shore proved that the existence of covers with any of a variety of maximality properties actually implies Π^1_1-CA_0.

Theorem 6.18. *Π^1_1-CA_0 is equivalent to $\mathsf{MIRT}_{\mathrm{XYZ}}$.*

Proof that $\mathsf{MIRT}_{\mathrm{XYZ}}$ implies Π^1_1-CA_0. We first prove that $\mathsf{MIRT}_{\mathrm{XYZ}}$ implies ACA_0 by adapting the proof of Theorem 5.9: If we apply $\mathsf{MIRT}_{\mathrm{XYZ}}$ instead of $\mathsf{IRT}_{\mathrm{XYZ}}$ to any of the forests constructed in that proof, we obtain a sequence containing a Z-ray in each tree which constitutes the forest. This is more than sufficient for carrying out the remainder of the proof of Theorem 5.9.

To prove that $\mathsf{MIRT}_{\mathrm{UVS}}$ implies Π^1_1-CA_0, suppose we are given a set A. Consider the disjoint union of all A-computable trees (this exists, by ACA_0). Any maximal sequence of Y-disjoint rays in this forest must contain a ray in each ill-founded A-computable tree. Hence its jump computes the hyperjump T^A. This shows that $\mathsf{MIRT}_{\mathrm{UYS}}$ implies Π^1_1-CA_0. To prove that the other $\mathsf{MIRT}_{\mathrm{XYZ}}$ imply Π^1_1-CA_0, it suffices to exhibit a computable procedure which takes trees $T \subseteq N^{<N}$ to X-graphs T' such that T is ill-founded if and only if T' contains a Z-ray. For $\mathsf{MIRT}_{\mathrm{UYD}}$, it suffices to modify each tree by adding a computable branch which is not already on the tree (as we did in the proof of Theorem 5.9). For $\mathsf{MIRT}_{\mathrm{DYZ}}$, it suffices to orient each of the graphs we constructed above in the obvious way. □

Proof that Π^1_1-CA_0 *implies* $\mathsf{MIRT}_{\mathrm{XYZ}}$. First, we give a mathematical proof for $\mathsf{MIRT}_{\mathrm{XVZ}}$ that is a direct construction not relying on Zorn's Lemma or the like. We will then explain how to modify it to apply to the other cases and then how to get it to work in Π^1_1-CA_0.

Suppose we are given an X-graph G whose vertices are elements of N. We build a sequence of disjoint Z-rays in G by recursion. If there are none we are done. Otherwise start with R_0 as any Z-ray in G. Suppose at stage n we have constructed disjoint Z-rays $\langle R_i \rangle_{i<m}$ for some $m \leq n$ such that for each i, R_i begins at x_i. If there is a Z-ray beginning at n which is disjoint from the R_i for $i < m$, choose one as R_{m+1}, if not move on to stage $n+1$. This construction produces a (possibly finite) sequence $R_i, R_i, \ldots$ of disjoint Z-rays in G. We show that this sequence is maximal. If R is a Z-ray which is disjoint from every R_i, then go to stage n of the construction, where n is the first vertex of R. If we did insert some R_m during stage n, then R_m would not be disjoint from R. Hence we did not insert any Z-ray during stage n. But R is a Z-ray that begins at n and is disjoint from $\langle R_{i_j} \rangle_{i_j<n}$, contradiction.

To prove $\mathsf{MIRT}_{\mathrm{XEZ}}$, we modify the above construction as follows. At stage $n = (u, v)$, we search for a Z-ray R which is disjoint from the previous rays and has (u, v) as its first edge. The rest of the proof proceeds as above.

The only real obstacle in formalizing the above proofs even in ACA_0 is being able to find out at stage n if there is an R as requested and, if so, choosing one. The question is Σ^1_1 and so in Π^1_1-CA_0 we can answer it and then perhaps use some construction or choice principle to produce it. As we only get a yes answer some of the time, Σ^1_1-AC_0 does not seem sufficient. Also later choices depend on previous ones. A computability argument using the Gandy basis theorem and its uniformities works but requires more background development. A choice principle that returns an element if there is one satisfying a Σ^1_1 property (but may act arbitrarily otherwise) and that can be iterated in a recursion is *strong* Σ^1_1-DC_0 which consists of the scheme

$$(\exists W)(\forall n)(\forall Y)\left(\Phi\left(n, \bigoplus_{i<n} W^{[i]}, Y\right) \to \Phi\left(n, \bigoplus_{i<n} W^{[i]}, W^{[n]}\right)\right),$$

for any Σ^1_1 formula $\Phi(n, X, Y)$. It is known that strong Σ^1_1-DC_0 and Π^1_1-CA_0 are equivalent [29, VII.6.9]. This clearly has the right flavor and the only issue is defining the required $\Phi(n, X, Y)$ with parameter G. This is slightly fussy but not problematic. We provide the details: To define Φ, first recursively define a finite sequence $i_0, \ldots, i_k < n$. If $i_0, \ldots, i_{j-1}$ have been defined, define i_j to be the least number (if any) above i_{j-1} and below n such that $X^{[i_j]}$ is a Z-ray in G which is disjoint from $X^{[i_0]}, \dots, X^{[i_{j-1}]}$. It is clear that there is an arithmetic formula with parameter G which defines $i_0, \dots, i_k$ from n and X. Next, we say that $\Phi(n, X, Y)$ holds if Y is a Z-ray in G which begins with n, and Y is disjoint from $X^{[i_0]}, \dots, X^{[i_k]}$.

Apply strong Σ^1_1-DC_0 for the formula Φ to obtain some set W. By Σ^0_1-comprehension with parameter $G \oplus W$, we may inductively define a (possibly finite) sequence $i_0, i_1, \dots$, just as we did in the definition of Φ. Clearly $\langle W^{[i_j]} \rangle_j$ is a sequence of disjoint Z-rays in G. We claim that it is maximal.

Suppose that R is a Z-ray in G which is disjoint from every $Z^{[i_j]}$. Suppose that R begins with vertex n. Then R is disjoint from $W^{[i_0]}, \dots, W^{[i_k]}$, where i_k is the largest i_j below n. It follows that $\Phi(n, \bigoplus_{i<n} W^{[i]}, R)$ holds. So $\Phi(n, \bigoplus_{i<n} W^{[i]}, W^{[n]})$ holds, i.e., $W^{[n]}$ is a Z-ray which begins with n and $W^{[n]}$ is disjoint from $W^{[i_0]}, \dots, W^{[i_k]}$. By definition of i_{k+1}, that means that $n = i_{k+1}$. But then R and $W^{[i_{k+1}]}$ are not disjoint, contradiction.

A slicker proof suggested by the referee requires perhaps more background in the metamathematics of Reverse Mathematics. It uses a "countable coded β-model" with the given graph G as an element. By Π^1_1-CA_0, for any set G there is a set X such that $X^{[0]} = G$ and $B(X) = \langle N, \{X^{[i]} \mid i \in N\} \rangle$ is a β-model, i.e. any Σ^1_1 formula Φ with parameters from among the $X^{[i]}$ is true if and only if it is true in $B(X)$ [29, VII.2.10]. Now one carries out the mathematical proof above but whenever one asks if there is a ray with some property one asks if there is one such among the $X^{[i]}$. This converts the Σ^1_1 questions to ones of fixed arithmetic complexity in X. Then, if the answer is yes, finding an appropriate i is also arithmetic in X with fixed complexity. This converts the entire construction to one arithmetic in X (and so certainly in Π^1_1-CA_0). Consider now the claim that the sequence of rays given by this construction over $B(X)$ is actually maximal. The existence of a counter-example R with first element n gives the same contradiction as before. The point is that, as $B(X)$ is a β-model, the answer to the question asked at stage n of the construction of whether there is a ray in $B(X)$ with first vertex n disjoint from the finite sequence is the same as the question as to whether there exists one at all. The only other point to note is that even though the sequence of rays R_i is constructed outside of $B(X)$ each finite initial segment is in $B(X)$ by an external induction and the fact that $B(X)$ is obviously a model of ACA_0. The argument with the adjustments for $\mathsf{MIRT}_{\mathrm{XEZ}}$ is then the same. □

7 Relationships Between IRT and Other Theories of Hyperarithmetic Analysis

In this section, we establish implications and nonimplications between variants of IRT and THAs other than Σ^1_1-AC_0. One such standard theory is as follows:

Definition 7.1. The theory *unique*-Σ^1_1-AC_0 consists of RCA_0 and the principle

$$(\forall n)(\exists! X) A(n, X) \to (\exists Y)(\forall n) A(n, Y^{[n]})$$

for each arithmetic formula $A(n, X)$.

The above theory is typically known as weak-Σ^1_1-AC_0 (e.g., [29, VIII.4.12]). We deviate from this terminology to introduce a new choice principle where the requirement for unique solutions is replaced by one for finitely many solutions.

Definition 7.2. The theory *finite*-Σ^1_1-AC_0 consists of RCA_0 and the principle

$$(\forall n)(\exists \text{ nonzero finitely many } X)A(n,X) \to (\exists Y)(\forall n)A(n,Y^{[n]})$$

for each arithmetic formula $A(n,X)$. Formally, "$(\exists$ nonzero finitely many $X)A(n,X)$" means that there is a nonempty sequence $\langle X_i\rangle_{i<j}$ such that for each X, $A(n,X)$ holds if and only if $X = X_i$ for some $i < j$.

Similarly to Σ^1_1-AC, each of these two choice principles are equivalent to ones where A is allowed to be of the form $(\exists! Y)B(n,X,Y)$ or $(\exists$ nonzero finitely many $Y)B(n,X,Y)$, respectively. However, unlike Σ^1_1-AC neither of these two principle is equivalent to the version where A is allowed to be Σ^1_1. Not only would those versions fail to capture the idea that we are dealing with unique or finitely many witnesses and paths through trees but they should be stronger than the stated principles. It is easy to see, for example, that even the unique version with A Σ^1_1 implies Δ^1_1-CA (Definition 7.8) which is stronger than unique-Σ^1_1-AC by Van Wesep [31].

Since the THA Σ^1_1-AC_0 implies finite-Σ^1_1-AC_0 which in turn implies unique-Σ^1_1-AC_0 whose models are closed under hyperarithmetic reducibility by Proposition 3.9, it follows that finite-Σ^1_1-AC_0 is a THA (as is unique-Σ^1_1-AC_0). Goh [10] shows that finite-Σ^1_1-AC_0 is strictly stronger than unique-Σ^1_1-AC_0. We were led to study this version of choice by realizing that a variant of our original proof that $\mathsf{IRT}^*_{\mathrm{UVS}}$ implies unique-Σ^1_1-AC_0 worked for the finite version.

Theorem 7.3. $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *implies finite-*Σ^1_1-AC_0 *over* RCA_0. *(It follows that* $\mathsf{IRT}_{\mathrm{XYZ}}$ *implies finite-*Σ^1_1-AC_0 *over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$*, but this is superseded by Theorem 7.7 below.)*

Proof. We first prove that $\mathsf{IRT}^*_{\mathrm{UYS}}$ for forests implies finite-Σ^1_1-AC_0. By Lemma 6.7, it suffices to prove that for any sequence $\langle T_n\rangle_n$ of subtrees of $N^{<N}$ such that each T_n has finitely many branches, a sequence $\langle P_n\rangle_n$ exists with each $P_n \in [T_n]$. As in the proof of Theorem 6.12, we construct a sequence of trees $\langle S_n\rangle_n$ such that for each n, the branches on S_n are precisely those of the form $P_0 \oplus \cdots \oplus P_n$ where P_i is a branch on T_i for $i \le n$.

By $\mathsf{IRT}^*_{\mathrm{UYS}}$ (for forests) there is a sequence $\langle R_k\rangle_k$ of Y-disjoint rays in $\bigsqcup_n S_n$ of maximum cardinality. We claim that $\langle R_k\rangle_k$ is infinite. If not, let m be least such that there is no R_k in S_m. Then we can increase the cardinality of $\langle R_k\rangle_k$ by adding any ray R from S_m while maintaining disjointness by our choice of n. The point here is that if $R \cap R_k \neq \emptyset$ for any k then they are both in S_m as the trees S_n are disjoint and there are no edges between them. Therefore $\langle R_k\rangle_k$ has a ray in infinitely many S_n. Thus we may construct the desired sequence $\langle P_n\rangle_n$ recursively by searching at stage n for an R_k in S_m for some $m > n$ and take P_n to be the branch in T_n which shares a tail with the nth coordinate of R_k.

Now, by Theorem 6.16, it follows that $\mathsf{IRT}^*_{\mathrm{UYD}}$ for forests implies finite-Σ^1_1-AC_0. By Proposition 6.5, it follows that $\mathsf{IRT}^*_{\mathrm{DYZ}}$ implies finite-Σ^1_1-AC_0 as well and so we are done. □

Another theory of hyperarithmetic analysis which follows from $\mathsf{IRT}^*_{\mathrm{XYZ}}$ is *arithmetic Bolzano-Weierstrass* (ABW_0):

Definition 7.4. The theory ABW_0 consists of RCA_0 and the following principle: If $A(X)$ is an arithmetic predicate on 2^N, either there is a finite sequence $\langle X_i \rangle_i$ which contains every X such that $A(X)$ holds or there is an X such that every one of its neighborhoods has two Y such that $A(Y)$ holds. Such an X is called an accumulation point of the class $\{X \mid A(X)\}$.

Friedman [8] introduced ABW_0 and asserted that it follows from Σ^1_1-AC_0 (with unrestricted induction). Conidis [4] proved Friedman's assertion and established relationships between ABW_0 and most then known theories of hyperarithmetic analysis. Goh [10] shows that $\mathsf{ABW}_0 + \mathsf{I}\Sigma^1_1$ implies finite-Σ^1_1-AC_0. We do not know if ABW_0 is strictly stronger than finite-Σ^1_1-AC_0.

The following two lemmas will be useful in deriving ABW_0 from $\mathsf{IRT}^*_{\mathrm{XYZ}}$. The first lemma describes a connection between sets of solutions of arithmetic predicates and disjoint rays in trees.

Lemma 7.5 (ACA_0)**.** *Suppose $A(X)$ is an arithmetic predicate. Then there is a tree $T \subseteq N^{<N}$ such that if there is a sequence of distinct solutions of $A(X)$, then there is a sequence of Y-disjoint single rays in T of the same cardinality, and vice versa.*

Proof. By Lemma 6.6, there is a tree $T \subseteq N^{<N}$ such that

$$\forall X(A(X) \leftrightarrow \exists f(\langle X, f\rangle \in [T])$$
$$\text{and } \forall X(\exists \text{ at most one } f)(\langle X, f\rangle \in [T]).$$

If $\langle X_i \rangle_i$ is a sequence of distinct solutions of $A(X)$, then, as the required f_i are arithmetic uniformly in the X_i, there is a sequence of distinct branches $\langle\langle X_i, f_i\rangle\rangle_i$ on T of the same cardinality.

By taking an appropriate tail of each branch, we obtain a sequence $\langle R_i \rangle_i$ of vertex-disjoint (hence edge-disjoint) single rays in T of the same cardinality with each one being a tail of $\langle X_i, f_i\rangle$: As no two distinct branches in a tree can have infinitely many vertices in common, simply take R_n to be the tail of $\langle X_n, f_n\rangle$ starting after all vertices it has in common with any $\langle X_i, f_i\rangle$, $i < n$.

Conversely, suppose there is a sequence $\langle R_i \rangle_i$ of Y-disjoint single rays in T. For each R_i, we define a branch on T which corresponds to it as follows. Let x be the vertex in R_i which is closest to the root of T. Then we can extend xR_i to the root to obtain a branch $\langle X_i, f_i\rangle$ on T. We claim that $\langle X_i \rangle_i$ is a sequence of distinct solutions of $A(X)$. For each $i \neq j$, since R_i and R_j are Y-disjoint, they cannot share a tail. So $\langle X_i, f_i\rangle$ and $\langle X_j, f_j\rangle$ must be distinct. Since for each X, there is at most one f such that $\langle X, f\rangle$ is a branch on T, it follows that $X_i \neq X_j$ as desired. □

The second lemma is essentially the well-known fact that the Bolzano-Weierstrass theorem is provable in ACA_0:

Lemma 7.6 ([29, III.2.7])**.** ACA_0 *proves that if $\langle X_n \rangle_n$ is a sequence of distinct elements of 2^N, then there is some Z which is an accumulation point of $\{X_n \mid n \in N\}$.*

Theorem 7.7. $\mathsf{IRT}^*_{\mathrm{XYZ}}$ *implies* ABW_0 *over* RCA_0*. Therefore* $\mathsf{IRT}_{\mathrm{XYZ}}$ *implies* ABW_0 *over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$.

Proof. By Proposition 6.5, it suffices to show that the undirected variants of IRT^* imply ABW_0.

Suppose $A(X)$ is an arithmetic predicate on 2^N such that no finite sequence $\langle X_i \rangle_i$ contains every X such that $A(X)$ holds. By Lemma 7.5, there is a tree $T \subseteq N^{<N}$ such that for any sequence of distinct solutions of $A(X)$, there is a sequence of Y-disjoint single rays in T of the same cardinality, and vice versa.

By $\mathsf{IRT}^*_{\mathrm{UYS}}$, or by $\mathsf{IRT}^*_{\mathrm{UYD}}$ and Theorem 6.16, there is a sequence of Y-disjoint single rays in T of maximum cardinality. This yields a sequence of distinct solutions of $A(X)$ of the same cardinality.

If this sequence is finite, then there is a solution Y of $A(X)$ not in the sequence by our assumption. Hence there is a sequence of distinct solutions of $A(X)$ of larger cardinality, which yields a sequence of Y-disjoint single rays in T of larger cardinality for the desired contradiction.

Thus there is an infinite sequence $\langle X_n \rangle_n$ of distinct solutions of A. By Lemma 7.6, there is an accumulation point of $\{X_n \mid n \in N\}$, which is of course an accumulation point of $\{X \mid A(X)\}$, as desired. □

We now turn our attention to nonimplications. One prominent theory of hyperarithmetic analysis is the scheme of Δ^1_1-*comprehension* (studied by Kreisel [15]):

Definition 7.8. The theory Δ^1_1-CA_0 consists of RCA_0 and the principle

$$(\forall n)(\Phi(n) \leftrightarrow \neg\Psi(n)) \to \exists X(n \in X \leftrightarrow \Phi(n))$$

for all Σ^1_1 formulas Φ and Ψ.

Theorem 7.9. Δ^1_1-$\mathsf{CA}_0 \nvdash \mathsf{IRT}_{\mathrm{XYZ}}, \mathsf{IRT}^*_{\mathrm{XYZ}}$.

Proof. Conidis [4, Theorem 3.1] constructed a standard model which satisfies Δ^1_1-CA_0 but not ABW_0. By Theorem 7.7, this model does not satisfy $\mathsf{IRT}^*_{\mathrm{XYZ}}$. Since standard models satisfy full induction, this model does not satisfy $\mathsf{IRT}_{\mathrm{XYZ}}$ either (by Proposition 6.3). □

Theorem 7.10. $\mathsf{ABW}_0 \nvdash \mathsf{IRT}_{\mathrm{XYZ}}, \mathsf{IRT}^*_{\mathrm{XYZ}}$.

Proof. By Propositions 5.3 and 6.3, it suffices to show that $\mathsf{ABW}_0 \nvdash \mathsf{IRT}_{\mathrm{UYZ}}$. Van Wesep [31, I.1] constructed a standard model $\mathcal{N}$ which satisfies unique-Σ^1_1-AC_0 but not Δ^1_1-CA_0. Conidis [4, Theorem 4.1], using the approach of [20], showed that $\mathcal{N}$ satisfies ABW_0. We show below that $\mathcal{N}$ does not satisfy $\mathsf{IRT}_{\mathrm{UYZ}}$.

In order to define $\mathcal{N}$, van Wesep constructed a tree T^G and branches $\langle f^G_i \rangle_{i \in \mathbb{N}}$ of T^G such that (1) $\mathcal{N}$ contains T^G and infinitely many (distinct) f^G_i (see [31, pg. 13 l. 1–11]); (2) $\mathcal{N}$ does not contain any infinite sequence of distinct branches of T^G (see [31, pg. 12 l. 7–9] and Steel [30, Lemma 7].) Then T^G is an instance of $\mathsf{IRT}_{\mathrm{UYS}}$ in $\mathcal{N}$ which has no solution in $\mathcal{N}$. This shows that $\mathcal{N}$ does not satisfy $\mathsf{IRT}_{\mathrm{UYS}}$ for trees. (The reader who wants to follow the details of the proofs in [4] and [31, I.1] should look at the presentation of the basic methods in [20].)

Since $\mathcal{N}$ is a standard model, it satisfies full induction. By Theorem 6.16, it follows that $\mathcal{N}$ does not satisfy $\mathsf{IRT}_{\mathrm{UYD}}$ for forests. □

Figure 4 illustrates some of our results. In order to simplify the diagram, we have omitted all variants of IRT except $\mathsf{IRT}_{\mathrm{UVS}}$.

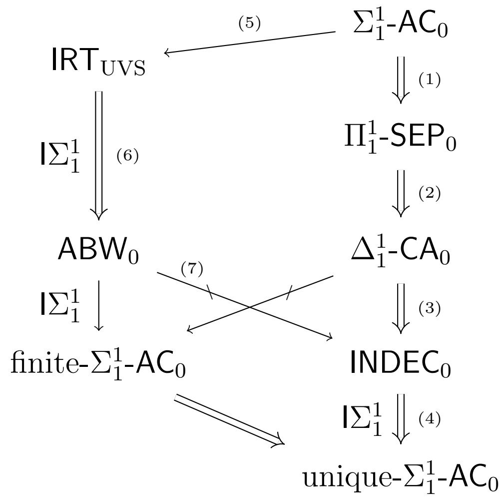

Figure 4: Partial zoo of theories of hyperarithmetic analysis. Single arrows indicate implication while double arrows indicate strict implication. The references for the above results are as follows: (1, 2) Montalbán [17, Theorems 2.1, 3.1]; (3, 4) Montalbán [16, Theorem 2.2], Neeman [20, Theorems 1.2, 1.3, 1.4], see also Neeman [21, Theorem 1.1]; (5) Theorem 4.5; (6) Theorems 7.7, 7.10; (7) Conidis [4, Theorem 4.1]. All results concerning finite-Σ^1_1-AC_0 are in Goh [10].

8 Isolating the Use of Σ^1_1-AC_0 in Proving IRT

We isolate the use of Σ^1_1-AC_0 in our proofs of $\mathsf{IRT}_{\mathrm{XYS}}$ and $\mathsf{IRT}_{\mathrm{UVD}}$ (Theorems 4.5, 5.10, 5.15) by identifying the following principles:

Definition 8.1. Let $\mathsf{SCR}_{\mathrm{XYZ}}$ be the assertion that if G is an X-graph with arbitrarily many Y-disjoint Z-rays, then there is a sequence of sets $\langle X_k \rangle_k$ such that for each $k \in N$, X_k is a set of k Y-disjoint Z-rays in G.

Let $\mathsf{WIRT}_{\mathrm{XYZ}}$ be the assertion that if G is an X-graph and there is a sequence of sets $\langle X_k \rangle_k$ such that for each $k \in N$, X_k is a set of k Y-disjoint Z-rays in G, then G has infinitely many Y-disjoint Z-rays.

SCR stands for Strongly Collecting Rays. WIRT stands for Weak Infinite Ray Theorem.

It is clear that Σ^1_1-AC_0 implies $\mathsf{SCR}_{\mathrm{XYZ}}$ and $\mathsf{SCR}_{\mathrm{XYZ}} + \mathsf{WIRT}_{\mathrm{XYZ}}$ implies $\mathsf{IRT}_{\mathrm{XYZ}}$. The only use of Σ^1_1-AC_0 in our proofs of $\mathsf{IRT}_{\mathrm{XYS}}$ and $\mathsf{IRT}_{\mathrm{UVD}}$ is to prove $\mathsf{SCR}_{\mathrm{XYS}}$ and $\mathsf{SCR}_{\mathrm{UVD}}$ respectively:

Theorem 8.2. ACA_0 *proves* $\mathsf{WIRT}_{\mathrm{XYS}}$ *and* $\mathsf{WIRT}_{\mathrm{UVD}}$.

Proof. For $\mathsf{WIRT}_{\mathrm{UVS}}$, see the proof of Theorem 4.5(ii). In particular, note that the hypothesis of WIRT is exactly the instance of Σ^1_1-AC needed in that proof (and the one referenced there). From then on, the argument proceeds in ACA_0 to give the conclusion of IRT and so of WIRT as desired. Similarly, for $\mathsf{WIRT}_{\mathrm{UVD}}$, see the proof of Theorem 5.10 and for $\mathsf{WIRT}_{\mathrm{XES}}$, see the proof of Theorem 5.15. The desired result for $\mathsf{WIRT}_{\mathrm{DVS}}$ then follows from Lemma 5.4. □

Next we will use the above result to show that $\mathsf{SCR}_{\mathrm{XYS}}$ and $\mathsf{SCR}_{\mathrm{UVD}}$ are equivalent over RCA_0 to $\mathsf{IRT}_{\mathrm{XYS}}$ and $\mathsf{IRT}_{\mathrm{UVD}}$ respectively. First, observe that $\mathsf{IRT}_{\mathrm{XYZ}}$ implies $\mathsf{SCR}_{\mathrm{XYZ}}$ for each choice of XYZ. Second, Lemmas 5.2, 5.4 and 5.6 imply

Proposition 8.3. $\mathsf{SCR}_{\mathrm{DYZ}}$ *implies* $\mathsf{SCR}_{\mathrm{UYZ}}$*,* $\mathsf{SCR}_{\mathrm{DEZ}}$ *implies* $\mathsf{SCR}_{\mathrm{DVZ}}$ *and* $\mathsf{SCR}_{\mathrm{DYD}}$ *implies* $\mathsf{SCR}_{\mathrm{DYS}}$*.*

Proposition 8.4. $\mathsf{SCR}_{\mathrm{XYZ}}$ *implies* ACA_0*.*

Proof. By Proposition 8.3, it suffices to establish the desired result for the undirected variants of SCR. The proofs are almost identical to those of Theorems 4.5(i) and 5.9. There, we applied $\mathsf{IRT}_{\mathrm{UYZ}}$ to forests $G = \bigsqcup_n T_n$, where each T_n contains a Z-ray, and no two Z-rays in T_n can be Y-disjoint. Any infinite sequence of Y-disjoint Z-rays in G must contain a Z-ray in cofinally many graphs T_n. Therefore from such a sequence we can uniformly compute Z-rays in cofinally many graphs T_n, which establishes ACA_0 by the construction of $\bigsqcup_n T_n$. If we assume $\mathsf{SCR}_{\mathrm{UYZ}}$ instead of $\mathsf{IRT}_{\mathrm{UYZ}}$, we only have access to a sequence $\langle X_k \rangle_{k\in N}$ such that for each k, X_k is a set of k Y-disjoint Z-rays in G. From such a sequence we can still uniformly compute Z-rays in cofinally many graphs T_n, because for any k, X_{k+1} must contain a Z-ray in some T_n, $n \geq k$. □

By Theorem 8.2, Proposition 8.4, and the observation that $\mathsf{SCR}_{\mathrm{XYZ}} + \mathsf{WIRT}_{\mathrm{XYZ}} \vdash \mathsf{IRT}_{\mathrm{XYZ}}$, we obtain

Corollary 8.5. $\mathsf{SCR}_{\mathrm{XYZ}}$ *and* $\mathsf{IRT}_{\mathrm{XYZ}}$ *are equivalent over* RCA_0 *for the following choices of* XYZ*: XYS and UVD.*

We now turn our attention to $\mathsf{WIRT}_{\mathrm{XYZ}}$. As usual, Lemmas 5.2, 5.4 and 5.6 imply

Proposition 8.6. $\mathsf{WIRT}_{\mathrm{DYZ}}$ *implies* $\mathsf{WIRT}_{\mathrm{UYZ}}$*,* $\mathsf{WIRT}_{\mathrm{DEZ}}$ *implies* $\mathsf{WIRT}_{\mathrm{DVZ}}$ *and* $\mathsf{WIRT}_{\mathrm{DYD}}$ *implies* $\mathsf{WIRT}_{\mathrm{DYS}}$*.*

Recall that $\mathsf{WIRT}_{\mathrm{XYS}}$ and $\mathsf{WIRT}_{\mathrm{UVD}}$ are provable in ACA_0 (Theorem 8.2). $\mathsf{WIRT}_{\mathrm{DVD}}$ and $\mathsf{WIRT}_{\mathrm{DED}}$ are open, because Σ^1_1-$\mathsf{AC}_0 + \mathsf{WIRT}_{\mathrm{XYZ}}$ implies $\mathsf{IRT}_{\mathrm{XYZ}}$, and $\mathsf{IRT}_{\mathrm{DVD}}$ and $\mathsf{IRT}_{\mathrm{DED}}$ are open (see comments after Theorem 5.1). We do not have an upper bound on the proof-theoretic strength of $\mathsf{WIRT}_{\mathrm{UED}}$ (an upper bound on $\mathsf{WIRT}_{\mathrm{UED}}$ would yield an upper bound on $\mathsf{IRT}_{\mathrm{UED}}$, which we do not currently have).

We do not know if any $\mathsf{WIRT}_{\mathrm{XYZ}}$ is equivalent to ACA_0. In an effort to clarify the situation, we define an apparent strengthening of $\mathsf{WIRT}_{\mathrm{XYZ}}$ and show that it implies ACA_0:

Definition 8.7. Let nonuniform-$\mathsf{WIRT}_{\mathrm{XYZ}}$ be the assertion that if G is an X-graph and there is a sequence of Z-rays $R_0, R_1, \ldots$ in G such that for each k, there are $i_0, \ldots, i_k$ such that $R_{i_0}, \ldots, R_{i_k}$ are Y-disjoint, then G has infinitely many Y-disjoint Z-rays.

Every instance of $\mathsf{WIRT}_{\mathrm{XYZ}}$ is also an instance of nonuniform-$\mathsf{WIRT}_{\mathrm{XYZ}}$, so nonuniform-$\mathsf{WIRT}_{\mathrm{XYZ}}$ implies $\mathsf{WIRT}_{\mathrm{XYZ}}$. Conversely, we have

Proposition 8.8. $\mathsf{ACA}_0 + \mathsf{WIRT}_{\mathrm{XYZ}}$ *implies nonuniform-*$\mathsf{WIRT}_{\mathrm{XYZ}}$*.*

Proof. Suppose G is an instance of nonuniform-$\mathsf{WIRT}_{\rm XYZ}$, i.e., G is an X-graph and $\langle R_n\rangle_{n\in N}$ is a sequence of Z-rays in G such that for each k, there are $i_0,\dots,i_k$ such that $R_{i_0},\dots,R_{i_k}$ are Y-disjoint. Then ACA_0 can find such $i_0,\dots,i_k$ uniformly in k. Therefore by ACA_0, G is an instance of $\mathsf{WIRT}_{\rm XYZ}$. By $\mathsf{WIRT}_{\rm XYZ}$, G has infinitely many Y-disjoint Z-rays as desired. □

Theorem 8.9. *Nonuniform-*$\mathsf{WIRT}_{\rm XYZ}$ *implies* ACA_0 *over* RCA_0*. It follows that nonuniform-*$\mathsf{WIRT}_{\rm XYS}$ *and nonuniform-*$\mathsf{WIRT}_{\rm UVD}$ *are both equivalent to* ACA_0 *over* RCA_0*.*

Proof. By Proposition 8.8 and Theorem 8.2, ACA_0 implies nonuniform-$\mathsf{WIRT}_{\rm XYS}$ and nonuniform-$\mathsf{WIRT}_{\rm UVD}$.

Next, we show that nonuniform-$\mathsf{WIRT}_{\rm XYZ}$ implies ACA_0. By Lemma 5.2, it suffices to consider the undirected versions of nonuniform-WIRT. First, we prove that nonuniform-$\mathsf{WIRT}_{\rm UYS}$ implies ACA_0 by constructing a computable instance of nonuniform-$\mathsf{WIRT}_{\rm UVS}$ such that every nonuniform-$\mathsf{WIRT}_{\rm UES}$ solution computes $\emptyset'$. (The desired result follows by relativization.) We use a variation of the graph used in the analogous result in Theorem 4.2.

Construction of $G=(V,E)$: $V=\{0^n \mid n>0\}\cup\{n^\frown s^\frown 0^t \mid n>0$ and some number below n is enumerated into $\emptyset'$ at stage s, and either $t\le s$ or $\emptyset'_s\restriction n=\emptyset'_t\restriction n\}$. $E=\{(0^n,0^{n+1}) \mid n>0\}\cup\{(n^\frown s^\frown 0^t, n^\frown s^\frown 0^{t+1}) \mid n^\frown s^\frown 0^t, n^\frown s^\frown 0^{t+1}\in V\}\cup\{(n^\frown s^\frown 0^t,0) \mid n^\frown s^\frown 0^t\in V$ and $n^\frown s^\frown 0^{t+1}\notin V\}$. G is clearly computable.

Verification: It is clear that there is exactly one ray $R_{\langle n,s\rangle}$ in G beginning with $n^\frown s$ for $n^\frown s\in V$ and the sequence $\langle R_{\langle n,s\rangle}\rangle_{n^\frown s\in V}$ is also computable. Note that if $\emptyset'\restriction n=\emptyset'_s\restriction n$ then this ray is $\langle n^\frown s^\frown 0^t\rangle_{t\in N}$. Otherwise, it is $\langle n^\frown s^\frown 0^t\rangle_{n^\frown s^\frown 0^t\in V}{}^\frown\langle 0^n\rangle_{n>0}$. Next observe that for each $k\in N$, $\{\langle n,i\rangle \mid i<n\le k$ and $i\in\emptyset'\}$ is Σ^0_1 and contained in $k\times k$ and so is a set by bounded Σ^0_1 comprehension [29, II.3.9]. Thus the finite function taking $n>l$ (the least number in $\emptyset'$) to the last stage s_n at which an $i<n$ is enumerated in $\emptyset'$ is also (coded by) a finite set. So we have, for each k, a sequence of V-disjoint rays $\langle R_{n,s_n}\rangle_{l<n\le l+k}$ of length k as required for the hypothesis of nonuniform-$\mathsf{WIRT}_{\rm UVS}$.

Suppose then that S_i is the sequence of rays in a solution for nonuniform-$\mathsf{WIRT}_{\rm UES}$. We wish to compute $\emptyset'$ from this solution. As the S_i are E-disjoint at most one of them contains the edge $(0,00)$. So by eliminating that one, we can assume none of the S_i contain $(0,00)$. If any of the remaining rays contain some edge of the form $(0^j,0^{j+1})$ for $j>0$ then (as it does not contain $(0,00)$) it must contain $(0^k,0^{k+1})$ for every $k\ge j$. Thus there can be at most one such ray among the remaining S_i and so we can discard it and assume there are no such rays in our list. No remaining ray can have 0 as its first vertex as if it did its second vertex would have to be of the form $n^\frown s^\frown 0^t$ with $n^\frown s^\frown 0^{t+1}\notin V$. Any continuation of this sequence would have to follow the $n^\frown s^\frown 0^r$ with r descending from t and so would have to terminate at $n^\frown s$ and not be a ray. Thus all the remaining S_i are of the form $\langle n_i{}^\frown s_i{}^\frown 0^{t+j}\rangle_{j\in N}$ for some t with $n_i\ne n_k$ for $i\ne k$. So the remaining S_i witness the conclusion of nonuniform-$\mathsf{WIRT}_{\rm UVS}$ as desired.

As we can replace the first vertex of S_i by the sequence beginning with $n^\frown s$ and ending with its second vertex, we know that $\emptyset'\restriction n_i=\emptyset'_s\restriction n_i$. Since the sequence S_i and so that of the n_i is infinite, given any m we can find an $n_i>m$ and so compute $\emptyset'\restriction m$ as $\emptyset'_{n_i}\restriction m$ as required.

To show that nonuniform-$\mathsf{WIRT}_{\rm UYD}$ implies ACA_0, define G as above. Consider the graph G' gotten by adding on for each $n^\frown s\in V$ new vertices $x_{n,s,k}$ for $k>0$ and edges $(n^\frown s, x_{n,s,1})$ and $(x_{n,s,k},x_{n,s,k+1})$ for $k>0$. The witnesses for the hypothesis of nonuniform-$\mathsf{WIRT}_{\rm UVS}$ in

G supply ones for nonuniform-$\mathsf{WIRT}_{\mathrm{UVD}}$ by tacking on the $x_{n,s,k}$ before $n^\frown s$ in reverse order. The witnesses for the conclusion of nonuniform-$\mathsf{WIRT}_{\mathrm{UED}}$ can be converted into ones for the conclusion of nonuniform-$\mathsf{WIRT}_{\mathrm{UES}}$ in G by removing the new vertices. So once again we can compute $\emptyset'$. □

We are unable to show that $\mathsf{WIRT}_{\mathrm{XYZ}}$ implies ACA_0, but we can prove the following:

Theorem 8.10. $\mathsf{WIRT}_{\mathrm{XYZ}}$ *is not provable in* RCA_0.

Proof. By Proposition 8.6, it suffices to consider the undirected variants of WIRT. For all these variants it suffices to construct a computable graph G on $\mathbb{N}$ and a computable sequence $\left\langle \langle X_i^k \rangle_{i<k} \right\rangle_{k\in\mathbb{N}}$ such that (1) for each $k \in \mathbb{N}$, the X_i^k for $i < k$ are pairwise vertex-disjoint double rays in G and (2) there is no computable sequence $\langle R_j \rangle_{j\in\mathbb{N}}$ of edge-disjoint single rays in G. It is clear that the G constructed for this C is a counterexample to each $\mathsf{WIRT}_{\mathrm{UYZ}}$ in the standard model of RCA_0 with second order part the recursive sets. Of course, as this model is standard, $\mathsf{WIRT}_{\mathrm{XYZ}}$ is not provable in RCA, RCA_0 plus induction for all formulas.

The computable construction will be a finite injury priority argument. At the end of stage s of our construction, for each $i < k \leq s$, we will have defined a path $P_{i,s+1}^k$ with lengths strictly increasing with s which is intended to be a segment of the double ray X_i^k. We think of these paths P_s as having domain a segment $[u, v]$ of $\mathbb{Z}$ containing $[-s, s]$. Its *endpoints* are $P_s(u)$ and $P_s(v)$. The intention is that the $X_i^k = \bigcup_s P_{i,s}^k$ will be the desired double rays such that, for each k, the X_i^k for $i < k$ will be vertex-disjoint. We will also have put all numbers less than s in as vertices in at least one of these $P_{i,s}^k$. In future stages, we will not add any edges between vertices which are currently in any $P_{i,s}^k$ for $i < k$. Thus G will be a computable graph given by the union of the double rays $X_i^k = \bigcup_s P_{i,s}^k$. We let G_s be the graph defined so far, i.e. $\bigcup\{P_{i,s}^k \mid i < k < s\}$. We let $G_s^{>t}$ be its subgraph defined as the union of the $P_{0,s}^k, \ldots, P_{k-1,s}^k$ for $k > t$ and similarly for $k < t$ and other interval notations.

We say that the *disjointness condition, d.c., holds at stage* s if for every $m < s$ and distinct n and $n' < m$, $P_{n,s}^m$ and $P_{n',s}^m$ are vertex-disjoint. Otherwise we say we have *violated the d.c.* Clearly, if we never violate the d.c. the X_n^m (for fixed m) are pairwise vertex-disjoint. We arrange the construction so that we obviously never violate the d.c.

So it suffices to also meet the following requirements:

> Q_e : If $R_0, R_1, \ldots$ is a computable sequence of single rays in G defined by Φ_e, i.e. $\Phi_e(i, n) = R_i(n)$ then the R_i are not edge-disjoint.

The requirements Q_e are listed in order of priority. During our construction, if all else fails, we will attempt to satisfy each Q_e at some stage s by *merging* certain rays X_i^k and X_j^l *using vertices* x and $y \neq x$ which are endpoints of $P_{i,s}^k$ and $P_{j,s}^l$, respectively. We do this by adding the least new number r (*the merge point of this merger*) as a vertex of G as well as edges (x, r) and (y, r) which are appended to each $P_{r,s}^q$ with x or y, respectively, as an endpoint. We also ensure that $P_{i,t}^k$ and $P_{j,t}^l$ henceforth agree after the vertex r as they grow in the corresponding directions.

Without loss of generality, and to simplify notation later, we make the assumption that if $\Phi_{e,s}(i, u)$ is convergent for any e, i and u then so is $\Phi_{e,s}(i, u')$ for every $u' < u$.

Construction. At stage s of the construction, we are given a finite graph G_s consisting of, for each $k < s$, finite vertex-disjoint paths $P^k_{0,s}, \ldots, P^k_{k-1,s}$ as described above. We let $f_s(e)$ be the final stage before s at which Q_e was initialized. For notational convenience when s and e are specified we simply write f for $f_s(e)$.

First, we act for the requirement Q_e of highest priority with $e < s$ which requires attention as described below. All requirements Q_e are initialized and unsatisfied at stage 0 and are initialized and declared to be unsatisfied whenever we act for a $Q_{e'}$ with $e' < e$.

We say that Q_e *requires attention at stage* s if Q_e is not satisfied and there are $a, u, x, b, v, y < s$ such that $u > 0$, $\Phi_{e,s}(a, u) \downarrow$, $\langle \Phi_{e,s}(a, n) \rangle_{n \leq u}$ is a path in G_s disjoint from $G^{<f}_s$ which can be extended in only one way to a maximal path in G_s and this extension eventually reaches an $x \notin G^{<f}_s$ which is an endpoint of some $P^k_{i,s}$ for $k \geq f$ and similarly for b, v and y for some $P^l_{j,s}$ such that x and y are not both endpoints of the same P^q_r. We also require that the merger using x and y would not violate the d.c. We then let $a(e, s)$ etc. be the associated witnesses for the least such computation. In this case, the actions for Q_e is to perform the merger using $x(e, s)$ and $y(e, s)$ as defined above and declare Q_e to be satisfied.

Finally, for each w, in turn, which is an endpoint of any the paths P^k_i as now defined we extend those paths by taking the least new number z which we append after w in each of these paths (and so add (w, z) as a new edge). For each $i < s$, in turn, we also take the next $2s + 1$ least new numbers and let the $P^s_i(n)$ be these numbers in order for $n \in [-s, s]$. This defines the $P^k_{i,s+1}$ for $i < k < s + 1$ and completes stage s of the construction. As promised we let $X^k_i = \bigcup_s P^k_{i,s}$.

Verification. It is clear that, for each $i < k \in \mathbb{N}$, X^k_i is a double ray and that G is a computable graph consisting of the union of these rays. It is also clear that by construction we never violate the d.c. and so for each k the X^k_i for $i < k$ are pairwise vertex disjoint as required. Thus we only need to prove that we meet each Q_e.

We now state a series of facts about G each of which follows immediately (or by simple inductions) from the construction and previous facts on the list.

Lemma 8.11. *Every vertex r which is a merge point has exactly three neighbors and they are the x and y used in the merger and the z added on after r in the final part of the action at the merger stage. In addition, no endpoint of any $P^k_{i,s}$ is a merge point and every vertex which is not a merge point has exactly two neighbors.*

If $u \in G_s$ and so $u \in P^k_{i,s}$ for some $i < k < s$, then for any $j < l$, $u \in X^l_j \Leftrightarrow u \in P^l_{j,s}$ and if so $l < s$.

If $(u, v) \in G_s$ then not both u and v are merge points. If neither are merge points then $\forall k \forall i < k(v \in P^k_{i,s} \Leftrightarrow u \in P^k_{i,s})$. If one, say u, is a merge point r for a merger at some stage t necessarily less than s using some x and y with (r, z) the edge added on at the end of stage t, then the other (v) is x, y or z; $\forall k \forall i < k(r \in P^k_{i,t+1} \Leftrightarrow x \in P^k_{i,t} \vee y \in P^k_{i,t})$; $\forall k \forall i < k(r \in P^k_{i,t+1} \Leftrightarrow r \in P^k_{i,s} \Leftrightarrow z \in P^k_{i,t+1} \Leftrightarrow z \in P^k_{i,s})$.

Any ray in G which begins in G_s remains in G_s until it reaches an endpoint of some $P^k_{i,s}$.

Each requirement acts at most once after the last time it is initialized. So, by induction, each requirement acts and is initialized only finitely often. Thus there is an infinite sequence w_i such that at each w_i we act for some Q_e and we never act for any $Q_{e'}$ with $e' \leq e$ afterwards. The observation to make here is that if there were a last stage w at which we act for any Q_e then there would be no mergers of any X^k_i and X^l_j for $k, l > w$. In this case all *the X_i^k, $i < k$ for $k > w$ would be disjoint. It is then easy to see from the observations above that at some stage after w we would act for a Q_e with $e > w$ such that $\Phi_e(0, n) = X_i^k(n)$ and $\Phi_e(1, n) = X_j^l(n)$ for $k, l > w$.*

If w is one of the w_i just defined, $z \in G^{<w}$ and $(z, z') \in G$ then $z' \notin G^{\geq w}$. The point to notice here is that by construction no merger at a stage $s > w$ can use any $x \in G^{<w}$ as every Q_e which can act after w is initialized at w by the choice of the w_i. Thus any path in G which starts with a $z \in G^{<w}$ never enters $G^{\geq w}$. □

Suppose now that Φ_e is total and defines a sequence of edge-disjoint single rays $\langle R_i \rangle_{i \in \mathbb{N}}$ in G. By Lemma 8.11, we may choose a $w = w_c$ for some c after which no $Q_{e'}$ for $e' \leq e$ ever acts again.

If Q_e is not satisfied at the end of stage w, we argue that we act for it later to get a contradiction. Once a ray begins in $G^{<w}$, it must stay there by Lemma 8.11 and once beyond all the (finitely many) merge points in $G^{<w}$ (none can be put in after stage w and no ray has repeated vertices) it remains in some X_i^k for $k < w$ with which it shares a tail (as each vertex which is not a merge point has exactly two neighbors). So by the edge-disjointness of the R_i there is an a such that $\Phi_e(a, 0) = R_a(0)$ is not in $G^{<w}$. It starts in $G^{<w'}$ with $w' = w_{c'}$ for some $c' > c$ and so shares a tail with some P_j^k with $w_c \leq k < w_{c'}$. Similarly there is b such that R_b begins in some $G^{\geq w_d}$ with $d > c'$ and shares a tail with some X_j^l with $w_d \leq l$. So eventually we have a stage s such that $\Phi_{e,s}(a, u) \downarrow$ and $\Phi_{e,s}(b, v) \downarrow$ define paths in G_s which go beyond the points by which R_a and R_b share tails with X_i^k and X_j^l, respectively, and after which neither tail contains a merge point and so these paths have unique extensions to paths in G_s (determined by the appropriate tails of X_i^k and X_j^l) ending with the endpoints of $P_{i,s}^k$ and $P_{j,s}^l$, respectively. Finally, note that the merger of X_i^k and X_j^l at s would not violate d.c. as by Lemma 8.11 $P_{i,s}^k$ can share vertices only with $P_{n,s}^m$ with $w_c \leq m < w_{c'}$ and $P_{j,s}^l$ can share vertices only with $P_{n',s}^m$ with $m \geq w_d > w_{c'}$. Thus at stage s we would act to satisfy Q_e for the desired contradiction.

So Q_e is satisfied at w and was satisfied at some $s \leq w$. With the notations as at s, $\Phi_e(a, u)$ and $\Phi_e(b, v)$ define initial segments of R_a and R_b which can each be extended in only one way to maximal paths in G_s eventually reaching the vertices x and y, respectively, as described at s. The merger performed at s puts the merge point r in both $P_{i,s+1}^k$ and $P_{l,s+1}^j$ and so in R_a and R_b (as the successor of x and y, respectively, as by Lemma 8.11 r is the only neighbor in G of x other than its predecessor in R_a and similarly for y and R_b). We then add in (r, z) to G at stage s. By Lemma 8.11, the only neighbors of r are x, y and z so any continuation of R_a and R_b after r must produce a shared edge, as R_a can continue only with either (y, r) which is an edge of R_b or with (r, z) and R_b can continue only with (x, r) which is an edge of R_a or (r, z). This yields the final contradiction. □

9 Open Questions

In addition to the variations of the Halin type theorems investigated here that remain open problems of graph theory ($\mathsf{IRT}_{\mathrm{DVD}}$ and $\mathsf{IRT}_{\mathrm{DED}}$) the most intriguing computational and reverse mathematical questions are about either separating the variants or providing additional reductions or equivalences among the $\mathsf{IRT}_{\mathrm{XYZ}}$ and Σ_1^1-AC_0. Clearly the most important issue is deciding if any (or even all) the $\mathsf{IRT}_{\mathrm{XYZ}}$ which are known to be provable in Σ^1_1-AC_0 are actually equivalent to it. We extend this problem to include the $\mathsf{IRT}^*_{\mathrm{XYZ}}$ and $\mathsf{I}\Sigma^1_1$.

Question 9.1. Can one show that any of the $\mathsf{IRT}_{\mathrm{XYZ}}$ which are provable in Σ^1_1-AC_0 ($\mathsf{IRT}_{\mathrm{XYS}}$ and $\mathsf{IRT}_{\mathrm{UVD}}$) do not imply Σ^1_1-AC_0 over RCA_0 or even over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$? An intermediate result might be that $\mathsf{IRT}^*_{\mathrm{XYZ}}$ (for one of these versions) does not imply Σ^1_1-AC_0 over RCA_0.

Should any of these $\mathsf{IRT}_{\mathrm{XYZ}}$ be strictly weaker than Σ^1_1-AC_0, the question would then be to determine the relations among the $\mathsf{IRT}_{\mathrm{XYZ}}$ and analogously the $\mathsf{IRT}^*_{\mathrm{XYZ}}$.

Question 9.2. Can any additional arrows be added to Figure 1 over RCA_0 or $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$? (This includes the question of whether $\mathsf{RCA}_0 \vdash \mathsf{IRT}_{\mathrm{UVD}} \rightarrow \mathsf{IRT}_{\mathrm{UVS}}$.)

As we noted in Remark 5.8 there is an apparent additional reduction in Bowler, Carmesin, Pott [3, pg. 2 l. 3–7]. They use an intermediate reduction to locally finite graphs in the sense of relying on the fact that if a graph has arbitrarily many disjoint rays it has a locally finite subgraph with arbitrarily many disjoint rays. This is the principle to which that Remark refers. It plus ACA_0 is a THA but over RCA_0 it does not imply ACA_0 and is provably very weak (in the sense of being highly conservative over RCA_0). Shore [27] proves these results and further analyzes this and many similar principles some related to the $\mathsf{IRT}_{\mathrm{XYZ}}$ and others to an array of classical logical principles.

Any reductions in RCA_0 as requested in the Question above would, of course, provide the analogous ones for the $\mathsf{IRT}^*_{\mathrm{XYZ}}$. However, it is possible that other implications can be proven for the $\mathsf{IRT}^*_{\mathrm{XYZ}}$:

Question 9.3. Can any implications of the form $\mathsf{IRT}^*_{\mathrm{XYZ}} \rightarrow \mathsf{IRT}^*_{\mathrm{X'Y'Z'}}$ be proven in RCA_0 other than the ones known to hold for the IRT versions?

Probably more challenging is the problem of separating the principles.

Question 9.4. Can one prove any nonimplication over RCA_0 or over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$ for any pair of the $\mathsf{IRT}_{\mathrm{XYZ}}$?

Of course, any such separation for the $\mathsf{IRT}_{\mathrm{XYZ}}$ of Question 9.1 would answer a case of that question by proving that at least one of these principles is strictly weaker than Σ^1_1-AC_0. In addition, a separation by standard models or even ones over $\mathsf{I}\Sigma^1_1$ for the $\mathsf{IRT}_{\mathrm{XYZ}}$ would give nonimplication for the corresponding $\mathsf{IRT}^*_{\mathrm{XYZ}}$ but it might be that nonstandard models could be used to separate one pair of versions but not the other.

The next natural question looks below ABW_0 in Figure 4.

Question 9.5. Can one prove that finite-Σ^1_1-AC_0 does not imply ABW_0 over RCA_0 or $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$?

The weaker versions, $\mathsf{WIRT}_{\mathrm{XYZ}}$, of the $\mathsf{IRT}_{\mathrm{XYZ}}$, prompt a question about ACA_0.

Question 9.6. Do any of the $\mathsf{WIRT}_{\mathrm{XYZ}}$ (especially the ones provable from ACA_0) imply ACA_0? An easier question might be whether they imply WKL_0?

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