Type Omitting Theorems for Fragments of Second Order Logics: Classical and Modal

Richard A. Shore*
Department of Mathematics
Cornell University
Ithaca NY 14853

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Abstract

Type omitting arguments provide conditions under which one can find a model $\mathcal{M}$ of a given theory T which does not have elements $\bar{a}$ which satisfy all the formulas $\varphi(\bar{x})$ in a given type Γ, i.e. a specified collection of formulas. Such theorems are an important feature of first order logic and have been proven for a few more general logics. Harrington, Shore and Slaman [2017] proved a theorem about Σ_1^1 sets which implied such results for omega logic (a classical result) and computable infinitary logic as well as other classical theorems of hyperarithmetic theory. In this paper we show that the same theorem implies analogous type omitting theorems for many other logics including a wide range of fragments of second order logic both classical and modal. That is, for the modal logics we are also allowed to use any one of these fragments of second order logic at each world.

1 Introduction

This work grew out of research that began with Ted Slaman during a meeting at the IMS in Singapore. That work later involved Leo Harrington and led to the joint paper Harrington, Shore and Slaman [2017] (hereafter HSS). We spoke about many of the applications of that work presented here at a later meeting at the IMS (Higher Recursion Theory and Set Theory) held during May and June of 2019 in celebration of the work of both Ted Slaman and Hugh Woodin on the occaision of their sixty-fifth birthdays. We are pleased to present this paper in their honor.

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In classical first order logic we begin with a language $\mathcal{L}$ consisting of (logical) symbols common to all first order languages and then a specified set of (nonlogical) relation, function and constant symbols for this particular language. As usual, one next inductively defines the formulas and sentences of $\mathcal{L}$. Then one defines a semantics for $\mathcal{L}$ by first specifying the structures $\mathcal{M}$ for it as sets M with interpretations of its nonlogical symbols and then defining satisfaction, $\mathcal{M} \vDash \varphi$ for sentences of $\mathcal{L}$ (or, as we prefer, for sentences in a language extending $\mathcal{L}$ by adding constant symbols, say c_a, for every $a \in M$). We will here consider only countable languages $\mathcal{L}$ (whether first order or more general). For our purposes, we may as well also assume that the basic language and so formulas, etc. are recursive as otherwise we can just relativize our results to the language. Note that, for first order languages, the relation $\mathcal{M} \vDash \varphi$ for countable structures is arithmetic in $\mathcal{M}$, i.e. in the list of atomic sentences (with names for its elements) true in in $\mathcal{M}$ uniformly in $\mathcal{M}$ and φ. (Of course, the complexity of the arithmetic definition varies with that of φ.) A set T of sentences of $\mathcal{L}$ is a theory if it has a model, i.e. an $\mathcal{L}$ structure $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ for every $\varphi \in T$. (For first order logic, one also has a proof theory and a completeness theorem establishing the equivalence of having a model with being consistent.)

For a theory T in a language $\mathcal{L}$, classically one defines an *n-type of* $\mathcal{L}$ over T to be a set p of formulas of $\mathcal{L}$ each having free variables precisely $x_1, \ldots, x_n$ which is consistent with T (i.e. realized in some model of T in the sense about to be defined) and complete, i.e. maximal consistent. We say p is *realized in a model* $\mathcal{M}$ *of* T if there is an $\bar{a} = a_1, \ldots a_n$ with each $a_i \in M$ such that $\mathcal{M} \vDash \varphi(\bar{a})$ for every $\varphi \in p$. Otherwise, we say that p is *omitted in* $\mathcal{M}$. By a type omitting theorem we mean one that gives conditions on T and p that guarantee that there is a model $\mathcal{M}$ of T that omits p.

The classical type omitting theorem of first order logic says (in one semantically oriented terminology) that if, for every formula $\varphi(\bar{x})$ of $\mathcal{L}$ consistent with T (i.e. there is a model of T satisfying $\exists \bar{x} \varphi(\bar{x})$), there is a $\gamma \in p$ such that T such that $\varphi \& \neg \gamma$ is consistent with T, then there is a model $\mathcal{M}$ of T which omits p. There are a number of standard variations and generalizations. The classical result (for countable first order languages) follows from compactness and can also be proven by direct applications of the constructions such as that of Henkin used to prove completeness. (For the classical versions see e.g. Chang and Keisler [1990] or Hodges [1993].) We are interested in type omitting theorems for logics more general than first order that may not have compactness or completeness theorems. The classical example is the Gandy-Kreisel-Tait theorem for ω-logic (Gandy, Kreisel and Tait [1960] (hereafter GKT), Theorem 1). Other known examples involve admissible sets, (computable) infinitary logic or the like as in Barwise [1975] and Montalbán [ta]. These examples often have notions of proof (with infinitary rules) and appropriate completeness theorems but not always the usual compactness theorems. While some of these infinitary languages are also treated in HSS, we ignore them here and begin our analysis by recalling the classical theorem for ω-logic.

The definition of ω-logic begins with two (or many, even infinitely many) sorted logic. Here, in addition to the usual syntactic apparatus of first order logic, we have additional kinds of variables x^i in the language for each sort i. In addition, $\mathcal{L}$ may have function and relation symbols ($f^{i,n}$ and $R^{i,n}$) of any arity n whose domain and range are restricted to the sort i. The structures $\mathcal{M}$ for $\mathcal{L}$ have designated subsets N_i for the sorts i which, together with the interpretations of $f^{i,n}$ and $R^{i,n}$ as functions and relations on N_i, form structures $\mathcal{N}_i$. We may write this as $\langle \mathcal{M}, \mathcal{N}_i \rangle$ or the like. Of course, from one point of view this is really just a notational extension of standard first order logic in which one has distinguished unary predicates whose interpretation are taken to be the N_i and one does the appropriate translation to express that the domain or range of a function (relation) may be an N_i. Nonetheless, this approach often has the benefit of providing a more natural presentation of classes of structures such as vector spaces in which one sort is the underlying field and the other the vectors.

The true power and interest of these logics arise when one requires, for example, (some of) the $\mathcal{N}_i$ to be specific first order structures. We call this an $\mathcal{N}$-logic. The classical example of ω-logic has a unary function on N_1 and requires that the associated $\mathcal{N}_1$ is isomorphic to ω with successor. Alternatively, one can include the usual functions and relations of first order arithmetic on $\mathcal{N}_1$ and require of the structures that it is isomorphic to the standard natural numbers $\mathbb{N}$. The structures for such a sorted language in which $\mathcal{N}$ is isomorphic to the standard structure $\mathbb{N}$ are also called ω-models (although $\mathbb{N}$-model might be better in the latter version). One may then consider a language in which there is an $\in$ relation $n \in A$ between members n of N_1 and A of $M - N_1$ (the sets). With some axioms including extensionality for the sets, this gives structures for (varieties of) second order arithmetic. One can extend further and also have other types $\mathcal{N}_i$ ($i > 1$) with membership relations $\in$ between members of N_i and N_{i+1} ($\mathcal{N}_1 \cong \mathbb{N}$). This last version of models of arithmetic of all finite orders with the first bottom sort assumed to be isomorphic to $\mathbb{N}$ is the setting for the primary result of GKT. Another natural example continuing on with vector spaces is requiring that the field be $\mathbb{Q}$.

What GKT explicitly state as their Theorem 1 is that if one has any Π^1_1 set $\mathfrak{A}$ of axioms in the language of ω-logic, then any set $B \subseteq \mathbb{N}$ appearing in every ω-model of $\mathfrak{A}$ is hyperarithmetic (i.e. Δ^1_1). (By $\mathfrak{A}$ being Π^1_1 we mean that the set of number codes for the sentences in $\mathfrak{A}$ is a Π^1_1 set of numbers, i.e. of the form $\{n | (\forall F) \exists x R(F, x, n)\}$ where R is a recursive (or equivalently arithmetic) relation on $\mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$. Remember that we are assuming our language is recursive and so then is the set of Gödel numbers for the formulas etc.) We can see this result as a very specific form of a type omitting argument where the type is one of a variable X over the sort of sets of natural numbers of the form $\{n \in X | n \in B\} \cup \{\neg(n \in X) | n \notin B\}$ for some $B \subseteq \mathbb{N}$. This type is omitted in some ω-model of $\mathfrak{A}$ as long as B is not hyperarithmetic and $\mathfrak{A}$ is a Π^1_1 set of sentences in the extended language. Many strengthenings and extensions of GKT can be found in Grilliot [1972].

We now want to state similar but stronger type omitting theorems in settings as described above that will include also many common (and some uncommon) subsystems of (weak) second order logic as well as logics like ω-logic.

2 $\mathcal{N}$-Logics and Subsystems of Weak Second Order Logic

Many common subsystems of second order logic are (or can be) defined by adding on variables and quantifiers that range over finite subsets of, or relations on, the domain of a structure. Examples include the following:

1. Weak monadic second order logic has new variables X_i ranging over finite subsets of the domain and a syntax that views them as a unary predicate and a semantics that interprets $X_i(t)$ as saying that the interpretation of a term t is a member of X_i.

2. Weak second order logic has new variables $R^{i,n}$ that range over finite relations of arity n on the domain and a syntax that allows us to write $R^{i,n}(t_1, \ldots t_n)$ for terms of the language and interprets this formula as saying the relation holds of the interpretation of the terms

3. Cardinality logic (for $\aleph_0$) has a new quantifier Q_0 where $Q_0 x\varphi(x)$ in interpreted to mean there are infinitely many a such that $\varphi(a)$ is true in the structure.

4. Ancestral logic allows one to talk about the transitive closure of any definable relation by adding a new operator (quantifier) TC and making, for each formula $\varphi(x, y)$, $TC_{x,y}\varphi(x, y)(u, v)$ into a formula with free variables u and v (and x and y bound) where $TC_{x,y}\varphi(x, y)(a, b)$ holds if there is a finite sequence $a = c_0 = \cdots = c_k = b$ such that $\varphi(c_i, c_{i+1})$ holds for every $i < k$.

Clearly we can combine these fragments of second order logic with multisorted logics by allowing restrictions of the new types of variables to the sorts in the natural way. (One could also interpret them as multisorted logics in which some sorts are required to be, e.g. the collection of finite subsets of, or relations on, some other sort.)

To the extent that formalizations are desired for the syntax and semantics of these languages, they are straightforward and left to the reader. (One can also describe proof systems and completeness theorems for these systems but in general, like ω-logic, they require infinitely branching proofs and determining if the proof tree is well founded. Thus provability is often at the Π^1_1 level rather than Σ^0_1.) We do, however, want to point out two crucial facts. The first is, we think, fairly well known to those who work with such logics: They all have the downward Skolem-Löwenheim property. In particular, if $\mathcal{M}$ is a structure for any of them (in the appropriate sense) then it has a countable elementary substructure (in the same sense). Note that this relies on the assumption that the only sorts that we specify up to isomorphism are required to be countable. Otherwise, the proof is of the same kind as for first order logic. We use this fact below just to equate a theory having a model with its having a countable model. The second even more crucial fact, is that the property of a countable sequence of countable sets with countable sequences of functions and relations (on some of them) being a structure for one of these logics is Σ_1^1 in $\mathcal{M}$ and the list of countable structures $\mathcal{N}_j$ specified as being the sort $\mathcal{M}_{i_j}$ with its specific sublanguage. The only requirement that pushes the definition beyond recursive conjunctions of arithmetic conditions are those that each $\mathcal{M}_{i_j}$ (with designated functions and relations from the list) is isomorphic to a given $\mathcal{N}_j$. This is clearly a Σ_1^1 predicate. Everything else, such as some function or relation being a k-ary one on some domain or some variables ranging over the finite subsets of, or relations on, a particular domain, is clearly arithmetic. Given that $\mathcal{M}$ is a countable structure for one of these languages in the appropriate sense, the satisfaction relation $\mathcal{M} \vDash \varphi(a_1, \ldots a_n)$ (for $a_i \in M$) is, uniformly in $\mathcal{M}$ and φ, clearly arithmetic in $\mathcal{M}$ and so the full satisfaction relation is recursive in $\mathcal{M}^{(\omega)}$ the effective join of the nth Turing jumps of $\mathcal{M}$ for $n \in \mathbb{N}$.

For the rest of this section $\mathcal{L}$ will be a (recursive) multisorted language as above and we will fix a semantics for it extending the usual first order one by interpreting the new variables and quantifiers in the intended manner and imposing restrictions on the acceptable interpretations of the sorts as in the examples. To be definite we use the sorts $\mathcal{N}_{\langle k \rangle}$ as the ones that are required to be isomorphic to some given sequence $\hat{\mathcal{N}}_k$ of countable first order structures. We could as well require of our structures, for example, that for various i and j, $\mathcal{N}_i$ is elementary equivalent or isomorphic to, or an elementary substructure of, $\mathcal{N}_j$. We call any of these an $\mathcal{N}$-logic for $\mathcal{N} = \left\langle \hat{\mathcal{N}}_k \right\rangle$. An $\mathcal{N}$*-theory* in $\mathcal{L}$ is a set of sentences of $\mathcal{L}$ which has an $\mathcal{N}$*-model,* i.e. there is an $\mathcal{N}$-structure which satisfies every $\varphi \in T$. We could generalize further by allowing. e.g. quantification over the sorts, their variables and even formulas with the satisfaction relation. Such versions of language and logic seem perhaps unnatural in this setting. They, will however, be both natural and crucial in the presentation of modal logics that are built on some $\mathcal{N}$-logic in each world and so we leave such examples to the next section.

The only properties of $\mathcal{N}$-logic that we need to prove our type omitting theorem are the following ones that we have already noted:

Property of $\mathcal{N}$-Logic 1: For countable $\mathcal{M}$, being an $\mathcal{N}$-structure is a Σ_1^1 property of $\mathcal{M}$ and $\mathcal{N}$.

Property of $\mathcal{N}$-Logic 2: For sentences φ of $\mathcal{L}$ and countable $\mathcal{N}$-structures $\mathcal{M}$ for $\mathcal{L}$ (even with constants for the elements a of M), the set $\{(\mathcal{M}, \varphi) | \mathcal{M} \vDash \varphi\}$ is Σ_1^1 in $\mathcal{N}$.

Property of $\mathcal{N}$-Logic 3: Any $\mathcal{N}$-theory T has a countable $\mathcal{N}$-model.

We can generalize $\mathcal{N}$- logics in various ways that preserve these three properties. For example, instead of requiring that $\mathcal{N}_{\langle k \rangle}$ is isomorphic to a fixed structure $\hat{\mathcal{N}}_k$ we could require only that it is isomorphic to one of a countable list of countable structures, $\hat{\mathcal{N}}_{k,l}$. A natural example here is for vector spaces requiring that the field be finite. Alternatively, we can require that a sort be infinite (so infinite fields). At a more complex level we could require that some binary relation (definable) in one of the sorts not be well-founded. The only constraint on conditions on the structures is Property 1. Our type omitting theorems rely only on these three principles.

Before stating our type omitting theorem we generalize the notion of a type in a language $\mathcal{L}$ over an $\mathcal{N}$-theory T.

Definition 2.1. Let T be a $\mathcal{N}$-theory in a language $\mathcal{L}$ for some version of $\mathcal{N}$-logic. Let Γ be any set of formulas of $\mathcal{L}$ and $n \in \mathbb{N}$. A Γ-n-type is a subset p of Γ with each element having free variables precisely $x_1, \ldots, x_n$. (If Γ is the set of all formulas of $\mathcal{L}$ we do not mention it. Note that requiring the variables to always range over the entire structure does not involve any loss of generality as one can add formulas of the form $\exists x^j(x^j = x_i)$ to the types as long as one includes them in Γ.) The type p is *realized* in a structure $\mathcal{M}$ for $\mathcal{L}$ if there are elements $a_1, \ldots, a_n$ of M such that $p = \{\varphi \in \Gamma | \mathcal{M} \vDash \varphi(a_1, \ldots, a_n)\}$. If p is not realized in $\mathcal{M}$ we say it is *omitted* in $\mathcal{M}$.

We can now state our type omitting theorem for logics satisfying Properties 1-3 above. Note that we can omit Property 3 if we require that all theories considered have countable models. All the models we construct to prove any of our type omitting theorems will also be countable

Theorem 2.2. *With the conditions just described on our $\mathcal{N}$-logic, if T is a Π^1_1 (in $\mathcal{N}$) $\mathcal{N}$-theory (i.e. the set of Gödel numbers for the sentences of T is a Π^1_1 (in $\mathcal{N}$) set of numbers); Γ is a Σ^1_1 in $\mathcal{N}$ set of formulas of $\mathcal{L}$ and p is a Γ- n-type which is not Σ^1_1 in $\mathcal{N}$, then there is a countable $\mathcal{N}$-model of T not realizing p.*

This theorem is an immediate corollary of the main theorem of HSS relativized to $\mathcal{N}$.

Theorem 2.3 (HSS Theorem 2.1)**.** *If a real X is Σ^1_1 in every member G of a nonempty Σ^1_1 class $\mathcal{K}$ of reals then X is itself Σ^1_1.*

Here reals are just members of Cantor space, i.e. subsets of $\mathbb{N}$ or functions from $\mathbb{N}$ to $\{0, 1\} = 2$. We note that the proof of this theorem is quite elementary and takes about a page. So the full proofs of our type omitting results are quite short and simple.

Proof of Theorem 2.2. By Properties 1 and 2 above, the class of countable $\mathcal{N}$-models of T (coded as reals in any reasonable way) is a Σ^1_1 in $\mathcal{N}$ class of reals. The Σ^1_1 in $\mathcal{N}$ class of such reals is nonempty by Property 3. As Γ is a Σ^1_1 set of formulas, for any $a_1, \ldots, a_n$ in M, the set $\{\varphi \in \Gamma | \mathcal{M} \vDash \varphi(a_1, \ldots, a_n)\}$ is $\Sigma^1_1(\mathcal{M}, \mathcal{N})$ by Property 2. Thus by Definition 2.1, were p realized in every countable $\mathcal{N}$-model, it would be Σ^1_1 in $\mathcal{N}$ for a contradiction. $\square$

We mention some easy extensions of [HSS Theorem 2.1] and the corresponding ones of Theorem 2.2 that include the ones in Grilliot [1972] for the appropriate logics in his settings.

First, as usual, if one can omit a type, one can omit a countable sequence of types:

Theorem 2.4. *If T is a Π^1_1 (in $\mathcal{N}$) $\mathcal{N}$-theory; Γ_i a uniformly Σ^1_1 in $\mathcal{N}$ sequence of sets of formulas of $\mathcal{L}$ and p_i are Γ_i- n-types none of which is Σ^1_1 in $\mathcal{N}$, then there is a countable $\mathcal{N}$-model of T not realizing any p_i.*

This Theorem follows from the following result of HSS the same way Theorem 2.2 followed from HSS Theorem 2.1.

Theorem 2.5 (HSS Theorem 2.6)**.** *If $\mathcal{K}$ is a nonempty Σ^1_1 class reals and X_n a countable sequence of reals none of which is Σ^1_1, then there is a $G \in \mathcal{K}$ such that no X_n is Σ^1_1 in G.*

As suggested by Grilliot's variations on his omitting types theorems, one can strengthen these results by proving "minimal pair" versions. We do a bit more.

Theorem 2.6. *If $\mathcal{K}_0$ and $\mathcal{K}_1$ are nonempty Σ^1_1 classes of reals then there are $G_0 \in \mathcal{K}_0$ and $G_1 \in \mathcal{K}_1$ such that any X which is Σ^1_1 in both G_0 and G_1 is itself Σ^1_1. In fact, one can prove more: For any $G_0 \in \mathcal{K}_0$ there is a $G_1 \in \mathcal{K}_1$ such that any X which is Σ^1_1 in both G_0 and G_1 is itself Σ^1_1.*

Proof. A construction of a pair of generics satisfying minimal pair type conditions provides a minimal pair G_0 and G_1. A suggestion we received for a direct proof leads to the stronger result: If every Y which is Σ^1_1 in G_0 is Σ^1_1, any $G_1 \in \mathcal{K}_1$ works. If not, let X_n list all the sets which are Σ^1_1 in G_0 but not Σ^1_1. By [HSS, Theorem 2.6], there is a $G_1 \in \mathcal{K}_1$ such that no X_n is Σ^1_1 in G_1. This G_1 is as required. □

The same argument as for Theorem 2.2 now gives the type omitting version.

Theorem 2.7. *If T_0 and T_1 are Π^1_1 $\mathcal{N}$-theories in $\mathcal{L}$ then there are $\mathcal{N}$-models $\mathcal{M}_0$ and $\mathcal{M}_1$ of T_0 and T_1, respectively, such that any Γ-n-type p (with Γ Σ^1_1 in $\mathcal{N}$) realized in both models is Σ^1_1 in $\mathcal{N}$. Indeed, for any model $\mathcal{M}_0$ of T_0 there is an $\mathcal{M}_1 \vDash T_1$ such that any type realized in both models is Σ^1_1.*

Similarly, one can prove countable sequence versions of both minimal pair theorems first getting, for a countable sequence $\mathcal{K}_i$ of nonempty Σ^1_1 classes of reals, a sequence $G_i \in \mathcal{K}_i$ (starting with any $G_0 \in \mathcal{K}_0$) such that any X which is Σ^1_1 in both G_j and G_k for any $j \neq k$ is itself Σ^1_1. Then, on the type omitting side, this tells us that if T_i are Π^1_1 $\mathcal{N}$-theories in $\mathcal{L}$ then there are $\mathcal{N}$-models $\mathcal{M}_i$ of T_i (starting with any $\mathcal{M}_0 \vDash T_0$) such that any Γ-n-type p with Γ Σ^1_1 in $\mathcal{N}$ realized in both both $\mathcal{M}_j$ and $\mathcal{M}_k$ for any $j \neq k$ is Σ^1_1 in $\mathcal{N}$.

Finally, to bound the complexity of the desired models and reproduce the relevant applications in Grilliot [1972], we note that the proof of Theorem 2.8 of HSS, shows that in all of these cases the desired sets G and so models $\mathcal{M}$ can be constructed hyperarithmetically (recursively) in Kleene's $\mathcal{O}^{\mathcal{N}}$, the complete Π^1_1 in $\mathcal{N}$ set, if the sets X_i (types p_i) are uniformly hyperarithmetic (recursive) in $\mathcal{O}^{\mathcal{N}}$. Moreover, they can be taken to have strictly smaller hyperdegree than $\mathcal{O}^{\mathcal{N}}$.

We mention all of these variations partially to make a comment on a comment of Grilliot [1972] to his Application 2 of his analogous result. Grilliot points out that his theorems give basis results for Σ^1_1 classes. The strongest of them being the countable "minimal pair" version: If S_m is the mth nonempty Σ^1_1 set of functions ($\mathbb{N} \to \mathbb{N}$) in a recursive in $\mathcal{O}$ list, then there are (uniformly) recursive in $\mathcal{O}$ functions $f_{m,n}$ such that $f_{m,n} \in S_m$ for all $m, n \in \mathbb{N}$ and any function hyperarithmetical in more than one $f_{m,n}$ is hyperarithmetical. He says this seems to be about the strongest (basis theorem) that can be obtained while still insisting that the solutions be recursive in $\mathcal{O}$. We point out that our approach strengthens these results not by imposing stronger conditions on the solutions but by moving everything to the realm of sets rather than functions. For functions being Σ^1_1 is the same as being Δ^1_1 (hyperarithmetic). Of course, this is not true for sets. Our approach shows that if one replaces functions ($\mathbb{N} \to \mathbb{N}$) by sets (contained in $\mathbb{N}$) in any of the basis results he proves including the last, one can replace "hyperarithmetical" by "Σ^1_1" in the conclusions.

We conclude this section with a fragment of second order logic stronger than all of the ones considered so far and some natural settings for such a "logic". (We do not know if this particular fragment, which is not really a logic under most definitions, has been previously studied.) The mathematical settings it is intended to capture are extension problems. Given a (countable) model $\mathcal{M}$ of a theory T in one language $\mathcal{L}$ and a language $\mathcal{L}'$ extending $\mathcal{L}$ and a theory T' in $\mathcal{L}'$ extending T, what can we say about the (possible) extensions of $\mathcal{M}$ to an $\mathcal{M}' \vDash T'$? For example, starting with a countable group what can one say about the (possible) extensions to a ordered groups. A similar question studied in logic starts with a model of some fragment of arithmetic of some or all finite types or of set theory and asks what can we say about the (possible) extensions to a model of some stronger version of arithmetic or set theory. Many such questions are studied for subsystems of second order arithmetic such as when can one extend models of RCA_0 (or some other theory) to models of stronger theories perhaps without extending the first order part of the original model. One more unusual example here is the investigation of extensions of models of ZF to ones of KM as in Zygmunt [1981].

We extend any of the logics considered so far by including sentences that are existential second order. That is for a given language $\mathcal{L}$ a theory or type is allowed to be contain sentences of the form $\psi = \exists R^{n_1}_1 \ldots \exists R^{n_k}_k \varphi$ where φ is a sentence in the language extending $\mathcal{L}$ by adding on new n_i-ary relation symbols R_i $(1 \leq i \leq k)$. (Note that, up to logical equivalence, we can close the set of formulas we are allowing under conjunction and disjunction but not negation.) We call this set of formulas $\mathcal{L}^{Ext}$. The semantics is the natural one. A structure $\mathcal{M}$ for $\mathcal{L}$ satisfies $\psi = \exists R^{n_1}_1 \ldots \exists R^{n_k}_k \varphi$ if there are n_i-ary relations $R^{n_i}_i$ on M such that extending $\mathcal{M}$ to a structure $\mathcal{M}'$ by adding on these relations satisfies φ. It is clear that if $\mathcal{M}$ is a countable structure for $\mathcal{L}$, satisfaction for formulas ψ of the required form is Σ^1_1 in $\mathcal{M}$ (and $\mathcal{N}$ if this is an $\mathcal{N}$-language). Moreover, these structures still have the downward Skolem-Löwenheim property. So all of the Properties 1-3 required above hold for $\mathcal{L}^{Ext}$. Thus we have a type omitting theorem for theories and types in this language as well.

Theorem 2.8. *If T is a Π_1^1 set of formulas of $\mathcal{L}^{Ext}$ with a model, Γ a Σ_1^1 set of formulas of $\mathcal{L}^{Ext}$ and $\{p_i\}$ is a set of Γ-n_i-types of $\mathcal{L}^{Ext}$ none of which is Σ_1^1 then there is a model $\mathcal{M}$ of T in which no p_i is realized.*

Essentially, this theorem gives our usual conditions under which we can say that there are models (of a set of sentences in $\mathcal{L}$) which can be extended to relations satisfying additional axioms (as expressed by a theory T in $\mathcal{L}^{Ext}$) involving the new relations which cannot be further extended to one satisfying some one of a collection of sentences p_i in $\mathcal{L}^{Ext}$ and so which cannot be extended with yet additional relations to satisfy a type in the larger extended language.

Sample known results which are immediate consequences (and can be phrased in terms of $\mathcal{N}$-logic alone) are that if some countable model of a theory of arithmetic or set theory can be extended to be one of a stronger theory (with, for example, more types) then no set (or higher type object of a level in the language) not already Σ_1^1 over the model can be realized in every extension to the stronger theory. If the original theory has enough comprehension this condition typically means that no new sets (or higher type object) can be in every model of the stronger theory. There are classical such results for models of arithmetic as in GKT. Less well known generalizations of some of these results for extending models of ZF to ones of KM appear in Ratajczyk [1979]. Our versions say more than this not only because they allow a wider variety of types but they also allow one to consider two step extensions of a given model and to talk about all models of a given theory which have extensions satisfying another theory and give conditions under which one of the models has no further extensions to any of a countable list of theories with associated types realized.

3 Modal Logics: classical and beyond

To set the stage for a wide variety of (as far as we know) generally new modal logics, we begin with a description of basic modal first order logic. A language $\mathcal{L}^{\Box,\Diamond}$ here has a first order language $\mathcal{L}$ (which for simplicity we may (but need not) take to be relational except for some constant symbols as well as modal operators $\Box$ and $\Diamond$. The role of structures are played by Kripke frames $\mathcal{F} = \langle W, S, \{\mathcal{F}(p) | p \in W\}\rangle$ (or more simply $\langle W, S, \mathcal{F}(p)\rangle$) where S is a binary relation on a set W and for each $p \in W$, $\mathcal{F}(p)$ is a (first order) structure on $F(p)$ for $\mathcal{L}$. One then defines the forcing relation $p \Vdash \varphi$ between $p \in W$ and sentences of $\mathcal{L}$, φ being forced in $\mathcal{F}$ in the usual inductive fashion on sentences. (Some sample steps are given below.) There are however, some choices to be made in the definition of the semantics. One important class of variations involves the relationships among the domains $F(p)$. (See for example, Brainer, T. and Ghilardi [2007].) At the core, for ordinary first order modal logic, we adopt one common one, monotonic domains $((\forall p, q)(S(p,q) \to F(p) \subseteq F(q))$ as in Nerode and Shore [1997]. This choice makes many things more straightforward (than say variable domains where the basic structures do not even fit our conventions about classical first order logic). For our purposes, we will see that we can use this setting to get our results for some other choices including, for example, requiring constant domains ($F(p) = F(q)$ for all $p, q \in W$). We can, however, easily allow constants and function symbols with the provisos inherent in monotonicity: If $S(p, q)$ and c is a constant symbol of the language then its interpretation remains the same, i.e. $c^{\mathcal{F}(p)} = c^{\mathcal{F}(q)}$. Similarly, if $\bar{a} \in F(p)^n$ and f is an n-ary function symbol of the language then $f^{\mathcal{F}(q)}(\bar{a})$.

We know of only three examples of type omitting theorems for modal logic. The first two (Mortimer [1974] and Bowen [1979, §15]) are quite tied to first order modal logic and have very restrictive definitions of types. A recent one (Litak et al. [2018]) has been brought to our attention and moves in a much more general direction toward coalgebraic logics. Our goal here is to introduce a wide range of modal logics stronger than first order ones in a variety of ways for which the basic theorems of HHS also supply a number of type omitting theorems.

We begin our generalizations in a well known way by allowing many (pairs of) modal operators $\Box_i$ and $\Diamond_i$ each with their own accessibility relation S_i. (Think that we are representing knowledge for multiple agents.) We also allow the common logic and language used in the possible worlds to be any of the $\mathcal{N}$ ones considered in the previous section such as ω-logic or weak second order logic, etc. (We do not know of any investigations into such modal logics with worlds equipped with these fragments of second order logic and specified sorts. There are however, versions of full second order modal logic presented and studied in Cocchiarella and Freund [2008].) We can also allow the apparatus of common knowledge (Fagin, Halpern, Moses and Vardi [1995]. We can also allow any types of restrictions of the frames $\mathcal{F}$ being considered (say to a class $\mathcal{C}$) as long as the relation of satisfying these restrictions ($\mathcal{F} \in \mathcal{C}$) is a Σ_1^1 in $\mathcal{N}$ property for countable frames $\mathcal{F}$.

As examples of conditions that can define such classes $\mathcal{C}$ of frames $\mathcal{F}$, we can allow any restrictions on the S_i that are Σ_1^1 properties of the W_i if countable, e.g. for all i in some Π_1^1 set the S_i are transitive, reflexive, symmetric or the like. More unusually, one can require that they be isomorphic to any fixed countable binary relations such as $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$ or to any one of a countable collection of binary relations such as all finite ones or infinite ones or ones with a descending chain. Modal logics with some of these restrictions on the allowed frames are often called hybrid logics (Areces and ten Cate [2007]). Some of these restrictions are well known to be equivalent to first order axioms on the binary relation. Others are clearly not. We can impose additional restrictions on the domains such as being constant: $\forall p, q, (F(p) = F(q))$. We can also impose restrictions on the frame as a whole that may involve the forcing relationship. For example, once we see that p forcing φ in a frame $\mathcal{F}$, $p \Vdash_{\mathcal{F}} \varphi$, is at worst a Σ_1^1 in $\mathcal{N}$ relation, we can require that some sentence be forced at some world or at every world or many things in between. Our only constraint is that the analog of Property 1 continues to hold: A countable set $\mathcal{F}$ being an allowed frame for $\mathcal{L}$ (which we take to include any relevant $\mathcal{N}$) is a Σ_1^1 in property of $\mathcal{F}$ and $\mathcal{L}$.

We note that we also want the basic relation of a sentence φ being forced at a world p to be Σ^1_1 (in a countable frame $\mathcal{F}$ and any $\mathcal{N}$ used in the logic). This is easy to see for the usual inductive definitions of forcing extended to our more general logics. The typical definition for classical first order logic defines $p \Vdash_{\mathcal{F}} \varphi$ for p in a frame $\mathcal{F}$ inductively for sentences φ in the (classical) language $\mathcal{L}(p)$ gotten by expanding the basic language of the frame $\mathcal{L}$ by adding at least constant symbols c_a for every $a \in F(p)$. If, for example, we want to have the logic be weak second order logic as in the previous section, then we want $\mathcal{L}(p)$ to also include new n-ary relation symbols R^n_e for each finite n-ary relation e on $F(p)$ (we write this as $e \in F(n,p) = [F(p)^n]^{<\omega}$) with the natural interpretations. If we have an $\mathcal{N}$ logic then we require the appropriate type in each $\mathcal{F}(p)$ to be isomorphic to $\mathcal{N}$. The definition then proceeds as would be expected by induction. For example, $p \Vdash \varphi$ for atomic sentences φ of $\mathcal{L}(p)$ if and only if $\mathcal{F}(p) \vDash \varphi$; $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$; $p \Vdash \exists x \varphi(x)$ iff $p \Vdash \varphi(c_a)$ for some $a \in F(p)$ where if, for example, the variable x is typed to be say in $\mathcal{N}$ then a must be in the isomorphic copy of $\mathcal{N}$ in $\mathcal{F}(p)$; $p \Vdash \exists R^n \varphi(R^n)$ iff $\exists e \in F(n,p)(p \Vdash \varphi(R^n_e))$; $p \Vdash \Box_i \varphi$ iff $\forall q(S_i(p,q) \rightarrow q \Vdash \varphi)$ and so as expected for $\vee, \rightarrow, \neg, \forall x, \forall R^n$ and $\Diamond_i$ or the operators for common knowledge. (Note that by our assumption of monotonicity, $S_i(p,q) \rightarrow F(p) \subseteq F(q)$, implies that if φ is a formula of $\mathcal{L}(p)$ then it is also one of $\mathcal{L}(q)$ so the induction makes sense.) It is straight-forward to see that such definitions make $p \Vdash \varphi$ uniformly arithmetic in $\mathcal{F}$ (and the $\mathcal{N}$) where the quantifier complexity of the arithmetic definition depends uniformly on the syntactic form of φ. Thus over all $p \in W$ and sentences φ of $\mathcal{L}(p)$, the relation $p \Vdash \varphi$ is clearly Σ^1_1 in $\mathcal{F}$ and $\mathcal{N}$. All of this remains true if we allow as well the apparatus of common knowledge as in Fagin, Halpern, Moses and Vardi [1995] where we view the sequence of groups of agents as part of the logic as well and so can quantify over them to get the operators E_G and iterate the E_G to get the C_G as and keep the relation $p \Vdash \varphi$ arithmetic as before. (A version of common knowledge with infinitely many agents and groups as we would allow here is considered in Halpern and Shore [2004].)

Thus the analog of Property 2 holds for all these modal logics: for sentences φ of $\mathcal{L}$ and countable frames $\mathcal{F}$ for $\mathcal{L}$ the relation $p \Vdash \varphi$ is Σ^1_1 in $\mathcal{L}$ (which, remember, we take to include the relevant $\mathcal{N}$) and $\mathcal{F}$. As usual, a Skolem-Löwenheim type argument (involving W as well as the S_i and the $\mathcal{F}(p)$) shows that we maintain Property 3 as well for frames and theories about them in the expanded languages and associated generalized modal logics.

We next give definitions of theories and types for these modal logics analogous to those for classical logics in §2. We fix one of our generalized $\mathcal{N}$ logics and languages $\mathcal{L}$ as well as a Σ^1_1 (in $\mathcal{L}$ and $\mathcal{N}$) class $\mathcal{C}$ of acceptable frames for $\mathcal{L}$.

Definition 3.1. A $\mathcal{C}$-theory is a set T of sentences φ of $\mathcal{L}$ such that there is a $\mathcal{C}$-frame $\mathcal{F}$ such that $p \Vdash \varphi$ for every $\varphi \in T$ and $p \in W$. (We write this as $\mathcal{F} \Vdash T$.) For Γ a set of formulas of $\mathcal{L}$ and $n \in \mathbb{N}$, a Γ-n-type is a subset P of Γ with each element having free variables precisely $x_1, \ldots, x_n$. The type P is *realized* in a $\mathcal{C}$-frame $\mathcal{F}$ for $\mathcal{L}$ if there are elements $a_1, \ldots, a_n$ of some $\mathcal{F}(p)$ such that $P = \{\varphi \in \Gamma | p \Vdash \varphi(a_1, \ldots, a_n)\}$. In this case, we also say that $\mathcal{F}$ realizes P. If P is not realized in $\mathcal{F}$ we say it is *omitted* in $\mathcal{F}$.

We now have a type omitting theorem for our modal $\mathcal{L}$ analogous to Theorem 2.2 for classical logic.

Theorem 3.2. *For a modal $\mathcal{N}$-language and logic $\mathcal{L}$ with a class $\mathcal{C}$ of frames with the properties described above analogous to those of §2, a Π_1^1 (in $\mathcal{L}$) $\mathcal{C}$-theory T, a Σ_1^1 in $\mathcal{L}$ set Γ of formulas and P a Γ- n-type which is not Σ_1^1 in $\mathcal{L}$, there is a countable $\mathcal{C}$-frame $\mathcal{F}$ such that $\mathcal{F} \Vdash T$ which does not realize P.*

Proof. As before we apply Theorem 2.6 of HSS. The only things to note is that once again the $\mathcal{C}$-frames $\mathcal{F}$ such that $\mathcal{F} \Vdash T$ is a Σ_1^1 in $\mathcal{L}$ class and if P is realized in some $\mathcal{C}$-frame $\mathcal{F}$ then it is Σ_1^1 in $\mathcal{L}$ and $\mathcal{F}$ by definition and the assumed properties of $\mathcal{L}$ and $\mathcal{C}$. $\square$

In the same style we can prove the analogs of Theorems 2.4 and 2.7 and their common generalization for types and sequences of types in the modal setting.

We close with some remarks and conjectures about expected proof systems and corresponding soundness and completeness results for the generalized modal logics we have considered. We believe that one can modify standard tableaux style proof systems for classical modal logics (as in, for example, Nerode and Shore [1997]) to get ones for these logics as well. We assume that we are dealing with systems with equality with the associated axioms/tableaux rules. As the intended semantics uses true equality, one expects the usual moding out by an equivalence relation for completeness proofs. Of course, these proof systems cannot be any simpler than those for the underlying generalized classical logics. Thus we expect to define tableaux and systematic tableaux which are infinitely branching trees. A proof of φ should then be as usual a tableau with root labeled $F\varphi$ on which every path is contradictory. Thus being a proof is a Π_1^1 property. (We can also think of terminating paths when they become contradictory. Then proofs are the well founded tableaux.)

To give an indication of how such systems should work we mention a few examples of new types of atomic tableaux. Our basic setting here is that for modal logics (with equality) in Nerode and Shore [1997]. If we have quantification over finite sets S, then we would have one with root $Tp \Vdash \exists S\varphi$ and immediate successors for each $n \in \mathbb{N}$. The successor for n would introduce n many new constants $c_1, \ldots, c_n$ and have label $Tp \Vdash \varphi'$ where each instance in φ of $t \in S$ for terms t (or $S(t)$ depending on the syntax) is replaced by the disjunction over $i \in [1, n]$ of $t = c_i$. If we quantify over finite relations then we would have one with root $Tp \Vdash \exists R^n\varphi$ and infinitely many successors, one for each finite n-ary relations R_e^n with domain $d_1, \ldots, d_m$ of size m which we label with $p \Vdash T\varphi'$ where we introduce m many new constants $c_1, \ldots c_m$ and φ' is gotten by replacing each occurrence in φ of $R^n(t_1, \ldots, t_n)$ for terms t_i by $\displaystyle\bigvee_{j_i, j_2, \ldots j_n \in [1,m] \ \& \ R_e^n(d_{j_1}, \ldots d_{j_n})} \ \bigwedge_{i \in [1,n]} (t_i = c_{j_i})$.
If the base logic is an $\mathcal{N}$ logic requiring, for example, some sort to be isomorphic to a

countable structure $\mathcal{N}$ as in ω-logic,then we list the elements c_i of N intending them to be elements of the appropriate sort in every $\mathcal{F}(p)$ and put them in along every path of the tableau as well as $Tp \Vdash \theta$ for every θ in the atomic diagram of $\mathcal{N}$. We also have, for example, a new atomic tableau with root $Tp \Vdash \exists x \in N\varphi(x)$ which has infinitely many immediate successors each of the form $Tp \Vdash \varphi(c_i)$ for $i \in \mathbb{N}$.

Of course, we also have the expected variations for universal quantifications and the usual definition for $\Box_i\varphi$ and $\Diamond_i\varphi$. Similar moves should work for all the fragments of second order $\mathcal{N}$-logic discussed in §2. We can also handle many restrictions on the class $\mathcal{C}$ of acceptable frames in similar ways. For example if S_i is required to be ω with the usual ordering $<$, then we have worlds p_i in W for $i \in \omega$ on which we specify S_i accordingly. Then, for example, we would introduce a new atomic tableau with root $Tp_i \Vdash \Diamond_i\varphi$ whose immediate successors are $Tp_j \Vdash \varphi$ for $j > i$ and when we get a node α labeled $Tp_i \Vdash \Box\varphi$ we promise to put a node labeled $Tp_j \Vdash \varphi$ for every $j > i$ on every path below α.

Admittedly, this is all quite sketchy but we hope that the project of producing proof systems generating (recursively for recursive $\mathcal{L}$ and theories T) trees whose well founded members can be taken as proofs from T should not be too difficult to carry out. The goal then would be to prove soundness and completeness theorems as one does for classical first order modal logics for as many of the generalized modal logics as possible.

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