The Limits of Determinacy in Second Order Arithmetic: Consistency and Complexity Strength

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Abstract

We prove that determinacy for all Boolean combinations of $F_{\sigma\delta}$ ($\mathbf{\Pi}^0_3$) sets implies the consistency of second-order arithmetic and more. Indeed, it is equivalent to the statement saying that for every set X and every number n, there exists a β-model of $\mathbf{\Pi}^1_n$-comprehension containing X.

We prove this result by providing a careful level-by-level analysis of determinacy at the finite level of the difference hierarchy on $F_{\sigma\delta}$ ($\mathbf{\Pi}^0_3$) sets in terms of both reverse mathematics, complexity and consistency strength. We show that, for $n \geq 1$, determinacy for sets at the nth level in this difference hierarchy lies strictly between (in the reverse mathematical sense of logical implication) the existence of β-models of $\mathbf{\Pi}^1_{n+2}$-comprehension containing any given set X, and the existence of β-models of $\mathbf{\Delta}^1_{n+2}$-comprehension containing any given set X. Thus the nth of these determinacy axioms lies strictly between $\mathbf{\Pi}^1_{n+2}$-comprehension and $\mathbf{\Delta}^1_{n+2}$-comprehension in terms of consistency strength. The major new technical result on which these proof theoretic ones are based is a complexity theoretic one. The *nth* determinacy axiom implies closure under the operation taking a set X to the least Σ_{n+1} admissible containing X (for $n = 1$, this is due to Welch [2012]).

1 Introduction

There are several common ways to calibrate the strength of mathematical or set theoretic assertions. One venerable one is proof theoretic. We say that a theory T is proof

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theoretically stronger (or of higher consistency strength) than one S, $T >_c S$, if T proves the consistency of S. (Here one assumes that the languages and theories being considered are countable, include basic arithmetic (or some natural interpretation of it as in set theory) and are equipped with a standard Gödel numbering of sentences and proofs so that the statement that S is consistent has a natural representation, $Con(S)$, in the language as does $T \vdash Con(S)$.) This ordering is strict by Gödel's incompleteness theorem, i.e. for no reasonable T can $T \vdash Con(T)$.

Another important calibration is that provided by reverse mathematics. Here one works in the setting of second order arithmetic, i.e. the usual first order language and structure $\langle M, +, \times, <, 0, 1\rangle$ supplemented by distinct variables X, Y, Z that range over a collection S of subsets of the domain M of the first order part and the membership relation $\in$ between elements of M and S. Most of countable or even separable classical mathematics can be developed in this setting based on very elementary axioms about the first order part of the model $\mathcal{M}$, an induction principle for sets and various set existence axioms. At the bottom one has a weak system of axioms called RCA_0 that correspond to recursive constructions. One typically then adds additional existence axioms to get other systems P.

The endeavor here is to calibrate the complexity of mathematical theorems by determining precisely which system P of axioms are needed to prove a give theorem Θ. This is done in one direction in the usual way showing that $P \vdash \Theta$. The other direction is a "reversal" that shows that $\mathsf{RCA}_0 + \Theta \vdash P$. The standard text here is Simpson [2009] to which we refer the reader for general background. There is also a brief presentation of the subject and the standard systems in Montalbán and Shore [2012] of which this paper is really a continuation. We henceforth refer to Montalbán and Shore [2012] as [MS] and rely heavily on its proofs and results.

One natural hierarchy of axiomatic (or proof theoretic) systems is given by $\mathbf{\Pi}^1_n$-CA_0 and $\mathbf{\Delta}^1_n$-CA_0: the axioms that say that every set defined by a Π^1_n formula (with set parameters) or, respectively, by both Π^1_n and Σ^1_n formulas exists. (We assume familiarity with the usual hierarchy of formulas and relations Σ^0_n, Π^0_n. Δ^0_n and Σ^1_n, Π^1_n, Δ^1_n that measure the complexity of prenex normal formulas by the number of alternations of first or second order quantifiers, respectively.) The system at the bottom of this hierarchy, $\mathbf{\Pi}^1_1$-CA_0 is the strongest of the systems usually studied in reverse mathematics and so far suffices for almost all mathematical theorems. The systems we consider are all stronger than $\mathbf{\Pi}^1_1$-CA_0 and indeed exhaust the hierarchy with examples provable from $\mathbf{\Pi}^1_n$-CA_0 but not $\mathbf{\Delta}^1_n$-CA_0 for each $n \geq 3$ and even one not provable in full second order arithmetic, $Z_2 = \cup\{\mathbf{\Pi}^1_n\text{-}\mathsf{CA}_0 | n \in \omega\}$.

These systems are taken from the realm of axioms of determinacy. This subject has played an important role historically as an inspiration for increasingly strong axioms (as measured by consistency strength) both in reverse mathematics and set theory. We have given a brief overview of this history in [MS, §1] and refer the reader to that paper for more historical details and other background.

In that paper we analyzed the reverse mathematical strength of a hierarchy of low levels of determinacy axioms, n-Π^0_3-*DET* and ω-Π^0_3-*DET*, which we now define.

Definition 1.1 (Games and Determinacy)**.** Our *games* are played by two players I and II. They alternate *playing* 0 or 1 with I playing first to produce a *play of the game* which is a sequence $x \in 2^\omega$. A *game* G_A *is specified* by a subset A of 2^ω. We say that I *wins a play* x *of the game* G_A *specified by* A if $x \in A$. Otherwise II wins that play. A *strategy* for I (II) is a function σ from binary strings p of even (odd) length into $\{0,1\}$. It is a *wining strategy* if any play x following it (i.e. $x(n) = \sigma(x \upharpoonright n)$ for every even (odd) n) is a win for for I (II). We say that the game G_A *is determined* if there is a winning strategy for I or II in this game. If Γ is a class of sets A, then we say that Γ *is determined* if G_A is determined for every $A \in \Gamma$. We denote the assertion that Γ is determined by Γ *determinacy* or Γ-*DET*.

Recall that the $\mathbf{\Pi}^0_3$ subsets of 2^ω are the $F_{\sigma\delta}$ ones, i.e. the countable intersections of countable unions of closed sets or equivalently the ones definable by formulas of the Π^0_3 form $\forall n \exists m \forall k R(x, n, m, k, Z)$ for some recursive predicate R and $Z \in 2^\omega$. There are a couple of equivalent definitions of the natural hierarchy on the Boolean combinations of these sets (or ones from any class Γ) called the (finite) difference hierarchy. Here is one that generalizes into the transfinite. If carried out through $\aleph_1$ many steps, it exhausts all of the Δ^0_4 sets (those representable by both $\mathbf{\Sigma}^0_4$ and $\mathbf{\Pi}^0_4$ formulas of arithmetic or equivalently in both $G_{\delta\sigma\delta}$ and $F_{\sigma\delta\sigma}$).

Definition 1.2 (Finite Differences on $\mathbf{\Pi}^0_3$)**.** A set $A \subseteq 2^\omega$ is m-$\mathbf{\Pi}^0_3$ if there are $\mathbf{\Pi}^0_3$ sets $A_0, A_1, \ldots, A_{m-1}, A_m = \emptyset$ such that

$$x \in A \Leftrightarrow \text{the least } i \text{ such that } x \notin A_i \text{ is odd.}$$

It is ω-$\mathbf{\Pi}^0_3$ if it is m-$\mathbf{\Pi}^0_3$ for some $m < \omega$.

While we used this standard definition for our proofs of determinacy in [MS] (see Theorem 1.5 below), we there used another one for our negative results (see Theorem 1.6 below). We utilize that representation for our major technical result, Theorem 1.8, here as well.

Definition 1.3. A set $A \subseteq 2^\omega$ is $\mathbf{\Pi}^0_{3,n}$ if there exist $\mathbf{\Pi}^0_3$ sets $A_0, ..., A_n$ such that $A_n = 2^\omega$ and

$$x \in A \Leftrightarrow \text{the least } i \text{ such that } x \in A_i \text{ is even.}$$

We say that *the sequence* $\langle A_i | i \leq m \rangle$ *represents* A (as a $\mathbf{\Pi}^0_{3,n}$ set).

For the purposes of establishing or contradicting determinacy the two hierarchies are equivalent and so we may use either class in our proofs about n-$\mathbf{\Pi}^0_3$-*DET*.

Remark 1.4 ([MS, Corollary 2.7])**.** For every n, determinacy for n-$\mathbf{\Pi}^0_3$ sets is equivalent to that for $\mathbf{\Pi}^0_{3,n}$.

The major reverse mathematical results of [MS] are the following:

Theorem 1.5. *For each* $m \geq 1$, $\mathbf{\Pi}^1_{m+2}$-$\mathsf{CA}_0 \vdash m$-$\mathbf{\Pi}^0_3$-DET.

Theorem 1.6. *For every* $m \geq 1$, $\mathbf{\Delta}^1_{m+2}$-$CA \nvdash m$-$\mathbf{\Pi}^0_3$-DET.

Corollary 1.7. *Determinacy for the class of all finite Boolean combinations of* $\mathbf{\Pi}^0_3$ *classes of sets* (ω-$\mathbf{\Pi}^0_3$-DET) *cannot be proved in second order arithmetic.*

In [MS], we also showed that ω-$\mathbf{\Pi}^0_3$-DET does not prove even $\mathbf{\Delta}^1_2$-CA_0 (or $\mathbf{\Pi}^1_{n+2}$-CA_0 even over $\mathbf{\Delta}^1_{n+2}$-CA_0) and so ω-$\mathbf{\Pi}^0_3$-DET and Z_2 are reverse mathematically incomparable.

When we spoke about these results at Berkeley, John Steel asked the natural question of whether we could improve our reverse mathematical result that ω-$\mathbf{\Pi}^0_3$-DET is not provable in Z_2 to show that it actually implies the consistency of Z_2. If so, while the two theories are incomparable in the sense of reverse mathematics, ω-$\mathbf{\Pi}^0_3$-DET would be strictly stronger than Z_2 in the proof theoretic one. This appears to be a delicate proof theoretic question falling outside the scope of our usual recursion or set theoretic methods as the results of [MS] show that ω-$\mathbf{\Pi}^0_3$-DET does not prove the existence of an ω-model of Z_2. (See Corollary 1.16.) Nonetheless, in this paper we show that this is indeed the case and that much more is true by proving stronger results at every step along the finite difference hierarchy. Our goals here are thus primarily proof theoretic but we also prove reverse mathematical and recursion (complexity) theoretic results and then use them to establish the proof theoretic ones.

In this setting, all the theories we might consider are formulated in the language of second order arithmetic and include at least RCA_0 although we generally omit explicitly mentioning the inclusion of RCA_0 in our theories. Thus we can take a standard formulation of $Con(T)$.

We also note that, in the context of reverse mathematics, a structure for a first order language is an, of course countable, set and collection of relations and functions as usual but we also assume the elementary diagram is given as well. If the theory is one of second order arithmetic we add on a countable set S of subsets of the domain M of the model $\mathcal{M}$ as the range of the second order quantifiers. (Simpson [2009] calls these countably coded models as its collection of allowed sets must be coded into a single set.) Such a model $\mathcal{M} = \langle M, S, \in, +, \times, <, 0, 1\rangle$ is an ω-model if $M = \mathbb{N}$ (the "true" natural numbers). It is a β-model if, in addition, every $\mathbf{\Pi}^1_1$ sentence (with parameters in M) is true in $\mathcal{M}$ if and only if it is "true". (One makes sense of the notions of ω and β submodels in the obvious way.)

In fact, all the theories we actually consider imply $\mathbf{\Pi}^1_1$-CA_0 (over RCA_0) and so we use the available interpretations between second order arithmetic and the language of set theory and the development of L and its fine structure in $\mathbf{\Pi}^1_1$-CA_0 in Simpson [2009] and

[MS] as well as some standard fine structure facts. In particular, for $n \geq 2$, if α_n is the least Σ_n admissible ordinal greater than ω then L_{α_n} is the least β-model of $\mathbf{\Delta}^1_{n+1}$-CA_0. Similarly, if ρ_n is the least Σ_n nonprojectable ordinal greater than ω, then L_{ρ_n} is the least β-model of $\mathbf{\Pi}^1_{n+1}$-CA_0 and if β_0 is the least ordinal which is Σ_n admissible for every n (or equivalently Σ_n nonprojectable for every n), then L_{β_0} is the least β-model of Z_2, (In all these situations when we refer to an L_γ as a structure for second order arithmetic we mean the ω-model with sets S taken to be $L_\gamma \cap \mathbb{R}$.)

These relations between admissible fragments of L and our strong proof theoretic systems are really an extension of the usual correspondences between the standard weaker proof theoretic systems and standard recursion theoretic constructions. In particular, the system ACA_0 corresponds to closure under the Turing jump; ATR_0, to closure under transfinite iterations of the Turing jump; and $\mathbf{\Pi}^1_1$-CA_0 to closure under the hyperjump. (See Shore [2010] for an exposition of the correspondences between the recursion theoretic structures and ω-models of the proof theoretic systems.) The natural extensions of the jump and hyperjump operators to larger ordinals are given by the master code hierarchy for countable initial segments of L (see, for example, Hodes [1980]). In particular, the jumps corresponding to $\mathbf{\Delta}^1_{n+1}$-CA_0 and $\mathbf{\Pi}^1_{n+1}$-CA_0 for $n \geq 2$ are the closures under the operators taking X to the least Σ_n admissible and Σ_n nonprojectable sets containing X, respectively. Here the correspondence moves from ω-models to β-models. In this language, our results (Corollary 1.11) show that the natural operator taking X to the least model of n-$\mathbf{\Pi}^0_3$-DET containing X lies strictly between the closure under the next Σ_{n+1} admissible and under the next Σ_{n+1} nonprojectable. We do not know of any other similar natural operators.

We use this correspondence to achieve our primary goal of locating the consistency strength of n-$\mathbf{\Pi}^0_3$-DET for each n and of ω-$\mathbf{\Pi}^0_3$-DET with respect to the standard fragments of Z_2. To be precise, we will prove (Corollaries 1.14 and 1.16) that, $\mathbf{\Pi}^1_{n+2}$-$\mathsf{CA}_0 >_c n$-$\mathbf{\Pi}^0_3$-$DET >_c \mathbf{\Delta}^1_{n+2}$-$\mathsf{CA}_0$ for each $n \geq 1$ and that ω-$\mathbf{\Pi}^0_3$-$DET >_c Z_2$ (and so ω-$\mathbf{\Pi}^0_3$-$DET >_c ZFC^-$ as well). (ZFC^- is ZFC without the power set axiom and, as is pointed out in [MS Proposition 1.4], is a Π^1_4 conservative extension of Z_2.) In fact, we will prove that, in each case, the distance between each side of the inequality is much greater than simple $>_c$. The main technical result we need will be recursion theoretic in the sense just described:

Theorem 1.8. *For $n \geq 1$, n-$\mathbf{\Pi}^0_3$-DET $\vdash \alpha_{n+1}$ exists.*

The case $n = 1$ is due to Welch [2011, 2012]. We prove this result for $n \geq 2$ in §2. To facilitate our proof theoretic goals, it is also helpful to introduce an operation on theories T of second order arithmetic that significantly increases consistency strength.

Definition 1.9. If T is a theory in the language of second order arithmetic, then $\beta(T)$ is the theory which says that for every set X there is a β-model of T containing X.

Note that for any (at least reasonably definable theory) T, not only is $\beta(T) >_c T$ but it is significantly stronger than T in terms of consistency strength. Indeed, while the provability of the existence of a model of T always implies consistency, and is equivalent to the provability of $Con(T)$ (even over the theory WKL_0), the provability of the existence of even an ω-model of T implies not only $Con(T)$ but also, for example, $Con(T+Con(T))$ and iterations of this operation into the transfinite. The point here is that for any, say arithmetic, T, $Con(T)$ is a sentence of arithmetic and so, if provable from S, S also proves that it is true in any ω-model of T. Thus if S proves that there is an ω-model of T then it also proves that there is one of $T+Con(T)$ etc. Of course, by Gödel's incompleteness theorem, it can never be the case that $T \vdash \beta(T)$. With this terminology we can state our main reverse mathematical theorem in which we include for convenience the main results of [MS] and some simple facts and consequences.

Theorem 1.10. *For every $n \geq 1$ we have the following provability relations none of which can be reversed.*

1. $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0 \vdash n\text{-}\mathbf{\Pi}^0_3\text{-DET}$.
2. $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0 \vdash \beta(n\text{-}\mathbf{\Pi}^0_3\text{-DET})$.
3. $\beta(\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0) \vdash \beta(n\text{-}\mathbf{\Pi}^0_3\text{-DET})$.
4. $\beta(n\text{-}\mathbf{\Pi}^0_3\text{-DET}) \vdash n\text{-}\mathbf{\Pi}^0_3\text{-DET}$.
5. $n\text{-}\mathbf{\Pi}^0_3\text{-DET} \vdash \beta(\mathbf{\Delta}^1_{n+2}\text{-}\mathsf{CA}_0)$.
6. $\mathbf{\Delta}^1_{n+2}\text{-}\mathsf{CA}_0 \vdash \beta(\mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0)$.

$$\begin{tikzcd}
\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0 \arrow[ddd, Rightarrow] \arrow[dr, Rightarrow] & \beta(\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0) \arrow[d, Rightarrow] \\
 & \beta(n\text{-}\mathbf{\Pi}^0_3\text{-DET}) \arrow[d, Rightarrow] \\
 & n\text{-}\mathbf{\Pi}^0_3\text{-DET} \arrow[d, Rightarrow] \\
\mathbf{\Delta}^1_{n+2}\text{-}\mathsf{CA}_0 \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \beta(\mathbf{\Delta}^1_{n+2}\text{-}\mathsf{CA}_0) \arrow[d, Rightarrow] \\
\mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0 & \beta(\mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0)
\end{tikzcd}$$

Proof. When we want to establish $\beta(T)$ for some theory T, we simply show that there is a β-model of T and note that $\beta(T)$ always follows by a straightforward relativization. Similarly, when proving some assertion with set parameters from $\beta(T)$, we ignore the set parameter and argue from the existence of a β-model of T and leave the insertion of parameters to relativization.

The first implication is Theorem 1.1 of [MS]. For (2), we note that the proof of Theorem 6.1 of [MS] can be carried out in $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0$. Given (1) applying that theorem to $n\text{-}\mathbf{\Pi}^0_3\text{-}DET$, produces a δ which is a limit of admissibles such that L_δ is a model of $n\text{-}\mathbf{\Pi}^0_3\text{-}DET$. Recall that if δ is a limit of admissibles, L_δ is a β-model. That (1) cannot be reversed follows from (2) as for all of these theories T, $T \nvdash \beta(T)$. That (2) cannot be reversed follows from noticing that the proof of (2) above shows that, from $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0$, one can prove the existence of a δ such that L_δ is a β-model of $n\text{-}\mathbf{\Pi}^0_3\text{-}DET$. Iterating this construction ω many times and taking the limit produces an ordinal γ such that L_γ is a model (even a β-model) of $\beta(n\text{-}\mathbf{\Pi}^0_3\text{-}DET)$. The fact that this iteration is possible follows, for example, from $\Sigma^1_2\text{-}\mathsf{DC}_0$ which is a consequence of $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0$ by Simpson [2009, Theorem VII.6.9].

Now (3) follows from (1) by applying it inside the β-model of $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0$ given by the hypothesis of (3). That it is not reversible follows from (2) and the fact that $\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0 \nvdash \beta(\mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0)$.

The next implication, (4), follows from the definition of being a β-model: Each instance of n-$\mathbf{\Pi}^0_3$-*DET* is a Σ^1_2 sentence which is true in some β-model and so has a witness in that model. Being a witness is a $\mathbf{\Pi}^1_1$ fact and so truth in the β-model implies truth. This one is clearly non-reversible.

The next implication, (5), is the proof theoretic heart of this paper. It follows from our main recursion or complexity theoretic result, Theorem 1.8. It gives the existence of the least Σ_{n+1} admissible ordinal α_{n+1} and so of $L_{\alpha_{n+1}}$ which as noted above is the smallest β-model of $\mathbf{\Delta}^1_{n+2}$-CA_0. Of course, this argument relativizes to any X to give the result required in (5). That (5) cannot be reversed takes some work, and we will do this in Section 3, where we show that n-$\mathbf{\Pi}^0_3$-*DET* does not hold in $L_{\alpha^*_n}$ where α^*_n is the first limit of n-admissibles.

The final implication follows from a standard fact about admissible ordinals: The Σ_n-nonprojectables are cofinal in the first $(n+1)$-admissible. So, by the remarks above, L inside any model of $\mathbf{\Delta}^1_{n+2}$-CA_0 is $n+1$-admissible and within it the Σ_n-nonprojectables are β-models of $\mathbf{\Pi}^1_{n+1}$-CA_0. $\square$

One recursion/complexity theoretic corollary of these results is placing the "jump" operator taking a set X to the least ordinal δ such that $L_\delta[X]$ is a (necessarily β) model of n-$\mathbf{\Pi}^0_3$-*DET* among the more usual operators.

Corollary 1.11. *For every X, the least δ such that $L_\delta[X] \vDash n$-$\mathbf{\Pi}^0_3$-DET is a limit of admissible ordinals strictly between $\alpha^*_n[X]$, the first limit of Σ_n-admissibles containing X, and $\rho_n[X]$, the least Σ_n-nonprojectable containing X.*

Proof. That δ is between these two ordinals follows from (5), i.e. Theorem 1.8, and (1). That the ordering is strict follows from the proof of nonreversability of (5) in Theorem 3.1 and (2). $\square$

Another corollary of (the uniformities in the proofs of) implications (1), (3), (4) and (5) of Theorem 1.10 and standard absoluteness properties are reverse mathematical and recursion theoretic characterizations of ω-$\mathbf{\Pi}^0_3$-*DET*. The reverse mathematical characterization is as being equivalent to another $\mathbf{\Pi}^1_3$ sentence closely tied to Z_2. The recursion/complexity theoretic one is in terms of the least ordinal δ such that $L_\delta \vDash \omega$-$\mathbf{\Pi}^0_3$-*DET*.

Corollary 1.12. *Over RCA_0, the following are equivalent:*

- ω-$\mathbf{\Pi}^0_3$-DET.
- $\forall n(\beta(\mathbf{\Pi}^1_n\text{-}\mathsf{CA}_0))$, *that is, for every n and every X there is a β-model of $\mathbf{\Pi}^1_n$-CA_0 containing X.*

Corollary 1.13. *The least ordinal δ such that $L_\delta \vDash \omega\text{-}\mathbf{\Pi}^0_3\text{-DET}$ is $\cup\alpha_n$, the supremum of the least Σ_n admissibles over $n \in \omega$. As can be seen from the proof of Theorem 1.8, this ordinal is also the least such that L_δ contains winning strategies for all light-faced $\omega\text{-}\mathbf{\Pi}^0_3$ games.*

Our basic assertions about consistency strength along the hierarchies follow immediately from the numbered implications (2), (5) and (6) of Theorem 1.10.

Corollary 1.14. *For every $n \geq 1$ we have the following chain of consistency strength relations:*

$$\cdots \quad \mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0 \quad <_c \quad \mathbf{\Delta}^1_{n+2}\text{-}\mathsf{CA}_0 \quad <_c \quad n\text{-}\mathbf{\Pi}^0_3\text{-DET} <_c \quad \mathbf{\Pi}^1_{n+2}\text{-}\mathsf{CA}_0 \quad \cdots.$$

An immediate corollary of this chain (and compactness) is an equiconsistency result for Z_2.

Corollary 1.15. *(RCA_0) The following are equiconsistent:*

1. *Z_2, i.e., the scheme which contains, for each $n \in \mathbb{N}$, the axiom $\mathbf{\Pi}^1_n\text{-}\mathsf{CA}_0$.*
2. *The scheme which contains, for each $n \in \mathbb{N}$, the axiom $n\text{-}\mathbf{\Pi}^0_3\text{-DET}$.*
3. *ZFC^-, i.e. ZFC without the power set axiom.*

Proof. The equiconsistency of the first two is our immediate corollary. That for Z_2 and ZFC^- follows from the usual interpretation of ZFC^- in Z_2 as in Simpson [2009, Ch. VII] and in particular Theorem VII.5.10 and VII.5.17 there as well as remarks in [MS]. □

We can say more at level ω but not too much.

Corollary 1.16. $Z_2 <_c Z_2 + Con(Z_2) <_c Z_2 + Con(Z_2) + Con(Z_2 + Con(Z_2)) <_c \cdots <_c \omega\text{-}\mathbf{\Pi}^0_3\text{-DET}$. *However, $\omega\text{-}\mathbf{\Pi}^0_3\text{-DET}$ does not prove that there is an ω-model of Z_2. These same relations hold between $\omega\text{-}\mathbf{\Pi}^0_3\text{-DET}$ and ZFC^- in place of Z_2.*

Proof. The chain of consistency strengths is immediate. As for $\omega\text{-}\mathbf{\Pi}^0_3\text{-}DET$, the proof of Theorem 1.10(5) shows that it proves (over RCA_0, or equivalently, $\Pi^1_1\text{-}\mathsf{CA}_0$) that, for each n, there is a β-model of $\mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0$, uniformly in n. Compactness now directly gives a model of Z_2, i.e. $\omega\text{-}\mathbf{\Pi}^0_3\text{-}DET \vdash Con(Z_2)$. As $Con(Z_2)$ is a sentence of first order arithmetic it is true in every β-model (indeed every ω-model). In particular, it is true in all the β-models of $\mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0$ proven to exist by $\omega\text{-}\mathbf{\Pi}^0_3\text{-}DET$. Thus $\omega\text{-}\mathbf{\Pi}^0_3\text{-}DET$ proves that, for every n, there is a β-model of $\mathbf{\Pi}^1_{n+1}\text{-}\mathsf{CA}_0 + Con(Z_2)$ and so $Con(Z_2 + Con(Z_2))$. We can now iterate this argument by induction.

Finally, $\omega\text{-}\mathbf{\Pi}^0_3\text{-}DET$ does not prove that there is an ω-model of Z_2 because from (1) of Theorem 1.10 we get that $\omega\text{-}\mathbf{\Pi}^0_3\text{-}DET$ holds in any ω-model of Z_2. That the same assertions hold for ZFC^- follows as in the previous Corollary. □

One more version of the connection between Z_2 and n-$\mathbf{\Pi}^0_3$-*DET* is given by the following Corollary:

Corollary 1.17. *The* Π^1_3 *consequences of* Z_2 *are the same as those of* RCA_0 + $\{n\text{-}\mathbf{\Pi}^0_3\text{-DET} | n \in \omega\}$.

Proof. All the consequences of $\mathsf{RCA}_0 + \{n\text{-}\mathbf{\Pi}^0_3\text{-}DET | n \in \omega\}$ are included in those of Z_2 by Theorem 1.10(1). For the other direction, consider any Π^1_3 sentence $\varphi = \forall X \exists Y \forall Z \psi(X, Y, Z)$ which follows from Z_2, any model $\mathcal{M}$ of RCA_0 + $\{n\text{-}\mathbf{\Pi}^0_3\text{-}DET | n \in \omega\}$ and any set X in $\mathcal{M}$. Now φ is a consequence of Π^1_{n+1}-CA_0 for some n. By Theorem 1.10(5), there is, in $\mathcal{M}$, a β-model $\hat{\mathcal{M}}$ of Π^1_{n+1}-CA_0 containing X. The witness Y in $\hat{\mathcal{M}}$ for $\forall Z \psi(X, Y, Z)$ is a also a witness in $\mathcal{M}$ as $\hat{\mathcal{M}}$ is a β-model in $\mathcal{M}$. Thus φ is true in every model of RCA_0 + $\{n\text{-}\mathbf{\Pi}^0_3\text{-}DET | n \in \omega\}$ as required. $\square$

2 β-models and $\mathbf{\Delta}^1_{n+2}$-CA_0

In this section we prove our main result.

Theorem 1.8. *For* $n \geq 1$, n-$\mathbf{\Pi}^0_3$-*DET* $\vdash \alpha_{n+1}$ *exists.*

Welch [2012] has characterized the complexity of $\mathbf{\Pi}^0_3$-*DET* in terms of the level of L at which strategies must first appear by a condition reminiscent of those used here and in [MS §5]. The case $n = 1$ in the Theorem follows easily from his characterization and a simple fact cited in Welch [2011, Proposition 1] from Burgess [1986] (Welch, personal communication). (The fact that an ordinal ζ for which there is a Σ such that $L_\zeta \preceq_2 L_\Sigma$ is Σ_2 admissible is true even if Σ is a nonstandard element of a model of $V = L$.) Our proof does not work for $n = 1$, so we fix an $n \geq 2$ and proceed by defining a class $\mathcal{G}$ of games G and proving two lemmas.

Lemma 2.1. *If* α_{n+1} *does not exists, then no* $G \in \mathcal{G}$ *is determined.*

Lemma 2.2. *If* α_{n+1} *does not exists, then there is a* n-$\mathbf{\Pi}^0_3$ *game* $G \in \mathcal{G}$.

We begin with the class $\mathcal{G}$ of games. It is formally defined as consisting of all the games whose winning conditions satisfy four properties given below. Informally, we specify the objectives of the game in terms of building models of the theory $T = KP + V = L + \neg\exists\beta[(\omega, (2^\omega)^{L_\beta}) \vDash Z_2]$. Now L_γ is countable inside $L_{\gamma+1}$ for $\gamma < \beta_0$ by [MS, Lemma 3.5]. Thus by the relations described above between nonprojectability and models of $\mathbf{\Pi}^1_{n+1}$-CA_0, T is equivalent to $KP + V = L + \forall\gamma(L_\gamma$ is countable inside $L_{\gamma+1})$. So a model of T looks like an admissible initial segment of L less than β_0. In particular, in it all sets are countable. Of course, if the model is well-founded then it is an L_α for an admissible ordinal $\alpha \leq \beta_0$. Intuitively, the objective of each player in these games is to build a model of T whose well-founded part is longer than the opponent's (with ties going to I). The games are designed so that being determined would force the existence of α_{n+1}. To be precise, we interpret the play of each player in one of our games as the characteristic function of a set of sentences in the language of set theory and specify conditions that, if satisfied, determine a winner of the game. We do this by giving a sequence of conditions with the understanding that the first one to be satisfied tells us which player wins the game. If none of the conditions is satisfied, any determination of a winner places the game in $\mathcal{G}$. (Once we see how to describe the conditions by $\mathbf{\Pi}^0_3$ sets this will fit the $\mathbf{\Pi}^0_{3,n}$ hierarchy of Definition 1.3. Determinacy for that hierarchy is equivalent to that for the n-$\mathbf{\Pi}^0_3$ one level by level as noted in Remark 1.4. We then also have to organize the conditions into a sequence of the required length.)

We begin with easily defined (i.e. simpler even than $\mathbf{\Pi}^0_3$) conditions that set the basic conditions for our models.

G1. Each player has to play a complete consistent theory extending T in the sense that if I fails to do this then II wins while if II fails to do this (while I does play such a theory), then I wins. Next, i.e. assuming neither player has lost yet in this way, we let $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ be the term-models of these theories, i.e. the structures whose members are (equivalence classes) of formulas $\varphi(x)$ which, in the appropriate theory, define unique elements. (See [MS, pp. 241-242]). We next require that these models are ω-models. (In the same sense as before, i.e. if $\mathcal{M}_{\mathrm{I}}$ is not an ω-model then II wins while if $\mathcal{M}_{\mathrm{II}}$ is not an ω-model (but $\mathcal{M}_{\mathrm{I}}$ is), then I wins.)

We are now faced with the problem of comparing the two models $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. Intuitively we want to compare them in the sense of containment but we only have them as term models so all we can hope for is an isomorphism from one to an initial segment of the other. If they were both well-founded, this is not hard to understand as both are then admissible initial segments of L. However, the models may be ill-founded. We need a definition that makes sense in all cases and does the right thing when at least one is well-founded. The crucial idea is to reduce everything to subsets of ω by the countability of every L_β inside each model. As both are ω-models, there is an easily definable isomorphism between $\omega^{\mathcal{M}_{\mathrm{I}}}$ and $\omega^{\mathcal{M}_{\mathrm{II}}}$ (the identity on the terms $x = 1+1\cdots+1$). This isomorphism then carries over to the whole model using the definable countings. The following definition captures this notion and slightly abuses notation by naming the relation between models as containment. We use notations such as $On^{\mathcal{M}_{\mathrm{I}}}$ and $\mathbb{R}^{\mathcal{M}_{\mathrm{I}}}$ to denote the obvious sets in $\mathcal{M}_{\mathrm{I}}$ (the ordinals in $\mathcal{M}_{\mathrm{I}}$ and the subsets of $\omega^{\mathcal{M}_{\mathrm{I}}}$ in $\mathcal{M}_{\mathrm{I}}$, respectively) and similarly for $\mathcal{M}_{\mathrm{II}}$. We will use $\mathcal{M}_{::}$ when we want to consider either or both of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ ambiguously and will use similar notation for other objects subscripted by I and II.

Definition 2.3. We say that $\mathcal{M}_{\mathrm{I}}$ is contained in $\mathcal{M}_{\mathrm{II}}$, $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$, if

$$\begin{aligned} \forall\alpha &\in On^{\mathcal{M}_{\mathrm{I}}}\, \exists x \in \mathbb{R}^{\mathcal{M}_{\mathrm{I}}}, y \in \mathbb{R}^{\mathcal{M}_{\mathrm{II}}}, \beta \in On^{\mathcal{M}_{\mathrm{II}}}\big[\big(\mathcal{M}_{\mathrm{I}} \vDash x \text{ codes } L_\alpha\big) \\ \wedge\big(\mathcal{M}_{\mathrm{II}} &\vDash y \text{ codes } L_\beta\big) \wedge \forall n \in \omega\big(\mathcal{M}_{\mathrm{I}} \vDash n \in x \iff \mathcal{M}_{\mathrm{II}} \vDash n \in y\big)\big]. \end{aligned} \tag{1}$$

By a real x coding a structure such as $<$ on an ordinal or $\in$ on an L_β (in $\mathcal{M}_{::}$), we mean that when we view x as a set of pairs of natural numbers the relation defined by that set of pairs is isomorphic to the one in $\mathcal{M}_{::}$. (We note that while not the definition given in [MS, §5] this one, with β in place of L_β, in the form of Claim 5.3 of [MS], is what is actually used there and should there be taken to be the definition.) We here code all of L_β (and so, of course, β recursively in that code) simply because we primarily need the L_β and this version eliminates the need to think about going from β to L_β in the codes.

Armed with this definition we can state the remaining winning conditions

G2. If $\mathcal{M}_{\rm II} = \mathcal{M}_{\rm I}$ (in the sense that $\mathcal{M}_{\rm I} \subseteq \mathcal{M}_{\rm II}$ and $\mathcal{M}_{\rm II} \subseteq \mathcal{M}_{\rm I}$) then I wins.

G3. If $\mathcal{M}_{\rm I}$ is well-founded, then I wins if

(a) $\mathcal{M}_{\rm II}$ is contained in $\mathcal{M}_{\rm I}$, or

(b) $\mathcal{M}_{\rm II}$ is ill-founded and $\mathcal{M}_{\rm I}$ contains the well-founded part of $\mathcal{M}_{\rm II}$.

G4. If $\mathcal{M}_{\rm II}$ is well-founded, then II wins if

(a) $\mathcal{M}_{\rm I}$ is contained in $\mathcal{M}_{\rm II}$, or

(b) $\mathcal{M}_{\rm I}$ is ill-founded and $\mathcal{M}_{\rm II}$ contains the well-founded part of $\mathcal{M}_{\rm I}$.

We next turn to Lemma 2.1. Its proof is fairly short if somewhat tricky. Most of the ideas in the proof appear in the hint for Exercise 1.4.2 in a draft of a book by Martin [n.d.]. In that exercise one is asked to prove H. Friedman's result [1971] that Σ_4-*DET* implies the existence of β_0.

Proof of Lemma 2.1. Suppose $G \in \mathcal{G}$ and assume that α_{n+1} does not exist. We show that no player has a winning strategy for G.

We first claim that the set $Y = \{\alpha |\ L_\alpha \models T$ and every member of L_α is definable in $L_\alpha\}$ is unbounded. If not, let $\delta = \sup Y$, and let α be the least admissible ordinal greater than δ. Let $\mathcal{M}$ be the elementary submodel of L_α consisting of all its definable elements. Then $\delta \in \mathcal{M}$. Since α_{n+1} does not exist, every ordinal is countable, and hence there is a bijection between ω and δ and the $<_L$-least such bijection belongs to $\mathcal{M}$. Thus $\delta \subseteq \mathcal{M}$, indeed $\delta + 1 \subseteq \mathcal{M}$. Since the Mostowski collapse of $\mathcal{M}$ is admissible and contains $\delta + 1$, it must be L_α. It follows that every member of L_α is definable in L_α and hence that $\alpha \in Y$ for the desired contradiction.

Suppose now that player I has a winning strategy σ for G. Let $\alpha \in Y$ be such that $\sigma \in L_\alpha$. We claim, for our desired contradiction, that if I follows σ and II plays Th_α, the theory of L_α, II wins. First, player I will not play Th_α, because if he did, then following σ would produce exactly the same moves as playing against a strategy for II that just copies I's moves. Hence σ could compute the full play of this game and so Th_α. However, $Th_\alpha \notin L_\alpha$ because every member of L_α is definable in L_α and truth for L_α is not.

So, $\mathcal{M}_{\rm I} \neq \mathcal{M}_{\rm II}$. Now we have to consider conditions G3 and G4. As we are assuming $\mathcal{M}_{\rm II} = L_\alpha$ it is well-founded. (The term model of $Th(L_\alpha)$ is (isomorphic to) L_α for $\alpha < \alpha_{n+1}$.) Since $\sigma \in L_\alpha$ and $Th_\alpha \in L_{\alpha+2}$, $Th(\mathcal{M}_{\rm I}) \in L_{\alpha+2}$, and hence the reals in $\mathcal{M}_{\rm I}$ all belong to $L_{\alpha+2}$. So if $\mathcal{M}_{\rm I}$ is well-founded it is an admissible proper initial segment of $L_\alpha = \mathcal{M}_{\rm II}$ and G3 does not apply but G4(a) does and II wins G. If $\mathcal{M}_{\rm I}$ is ill-founded, then the well-founded part of $\mathcal{M}_{\rm I}$ is included in L_α as otherwise it would go beyond L_α and so contain reals beyond $L_{\alpha+2}$. So II wins again. Thus, in any case, we have contradicted the assumption that I has a winning strategy for G.

Finally, suppose that II has a winning strategy σ for G. Again, let $\alpha \in Y$ be such that $\sigma \in L_\alpha$. We claim that if I plays Th_α, he wins. Note that II cannot copy I's moves, or he would loose by G2. The rest of the argument is analogous to the one above with I winning by either G3(a) or G3(b) depending on whether $\mathcal{M}_{\rm II}$ is well-founded or not. □

We now turn to the new and much more difficult Lemma 2.2. Our proof adapts and extends the machinery developed in §5 of [MS].

2.1 Proof of Lemma 2.2

We begin by describing various $\mathbf{\Pi}^0_3$ conditions (i.e. sets) a Boolean combination of which could be used to specify a game $G \in \mathcal{G}$. We then show how to reorganize the game so as to get one that is $\mathbf{\Pi}^0_{3,n}$ and show that it is in $\mathcal{G}$. (Making G a $\mathbf{\Pi}^0_{3,n}$ game suffices by Remark 1.4.) Note that to show that a game G is in $\mathcal{G}$ it suffices to prove that for every play of the game which has a winner determined by the rules of $\mathcal{G}$, the specification of G determines the same winner.

Our description incorporates variants of most of the conditions of [MS, §5] as well as some new ones. To make our treatment more nearly self-contained, we repeat, at times with modifications, some of the definitions and proofs from [MS].

First we can clearly guarantee that G1 is satisfied by using the following conditions:

($R_{\rm I}0$): II does not play a complete consistent extension of T.

($R_{\rm II}0$): I does not play a complete consistent extension of T.

($R_{\rm I}1$): $\mathcal{M}_{\rm II}$ is not an ω-model.

($R_{\rm II}1$): $\mathcal{M}_{\rm I}$ is not an ω-model.

For most of the quantifier complexity assertions below a crucial fact to keep in mind is that the plays of our game specify $Th(\mathcal{M}_{::})$ and so saying that some sentence of set theory is true in $\mathcal{M}_{::}$ is recursive in the play. Also the map from n to the term denoting n in $\mathcal{M}_{::}$ is recursive.

Claim 2.4. *Conditions* $(R_{\rm I}1)$ *and* $(R_{\rm II}1)$ *are* Σ^0_1. *Conditions* $(R_{\rm I}1)$ *and* $(R_{\rm II}1)$ *are* Σ^0_2.

Proof. For the first assertion, saying that the set of sentences played form a complete and consistent theory is Π^0_1. Saying that the axioms of T_n are included in these sets is also Π^0_1.

For the second, to say that a term model $\mathcal{M}$ is an ω-model, we have to say that for every term t, if $\mathcal{M} \models t \in \omega$ (i.e. in $\mathcal{M}$, the unique element satisfying the associated formula is a member of ω), then there exists a number n such that $\mathcal{M} \models t = 1+1+....+1$, where 1 is added n times. This is a Π^0_2 formula about $\mathcal{M}$. $\square$

Next, we can guarantee that G2 is satisfied using the condition

$(R_{\mathrm{I}}2)$: $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$.

Claim 2.5. *Condition* $(R_{\mathrm{I}}2)$ *is* $\mathbf{\Pi}^0_3$.

Proof. Note that containment in each direction is $\mathbf{\Pi}^0_3$ by inspecting Definition 2.3. $\square$

An inspection of the remaining rules for $\mathcal{G}$ shows that if $\mathcal{M}_{\mathrm{I}} \subsetneq \mathcal{M}_{\mathrm{II}}$ we need to let II win if $\mathcal{M}_{\mathrm{II}}$ is well-founded and let I win if $\mathcal{M}_{\mathrm{II}}$ is not and $\mathcal{M}_{\mathrm{I}}$ is its well-founded part. In §2.2 we introduce a new condition $(R_{\mathrm{I}}new)$ to handle this case. Similarly if $\mathcal{M}_{\mathrm{II}} \subsetneq \mathcal{M}_{\mathrm{I}}$ we need to let I win if $\mathcal{M}_{\mathrm{I}}$ is well-founded and let II win if $\mathcal{M}_{\mathrm{I}}$ is not and $\mathcal{M}_{\mathrm{II}}$ is its well-founded part. We here use an analogous condition $(R_{\mathrm{II}}new)$.

If $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable, then the only cases we are interested in, i.e. for which the definition of $\mathcal{G}$ determines a winner of the game, are ones when one of the models is well-founded. In these cases we will use conditions $(R_{\mathrm{I}}2+k)$ and $(R_{\mathrm{II}}2+k)$ for $k \geq 1$ as defined below to verify that the definition of $\mathcal{G}$ is satisfied.

We now turn to a description of the conditions not yet described. We assume from now on that $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are distinct ω-models of T as will be guaranteed by our final organization of the conditions $(R_{\mathrm{I}}0)$, $(R_{\mathrm{II}}0)$, $(R_{\mathrm{I}}1)$, $(R_{\mathrm{II}}1)$ and $(R_{\mathrm{I}}2)$ in §2.4. As $\mathcal{M}_{\mathrm{I}} \neq \mathcal{M}_{\mathrm{II}}$, the rules for $\mathcal{G}$ only specify a winner when at least one of them is well-founded. Thus we can restrict ourselves to this case. We begin with the cases corresponding to G3a and G4a.

2.2 Comparable models

We define conditions that will be used to recognize if $\mathcal{M}_{\mathrm{II}}$ is ill-founded in the case when $\mathcal{M}_{\mathrm{I}} \subset \mathcal{M}_{\mathrm{II}}$ and if $\mathcal{M}_{\mathrm{I}}$ is ill-founded in the case when $\mathcal{M}_{\mathrm{II}} \subset \mathcal{M}_{\mathrm{I}}$ with an eye toward satisfying the rules given by G3(a) and G4(a).

$(R_{\mathrm{I}}new)$: $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$ and, for every $\beta \in On^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{A}_{\mathrm{II}}$, $\mathcal{M}_{\mathrm{I}} \not\preceq_n L_\beta^{\mathcal{M}_{\mathrm{II}}}$.
$(R_{\mathrm{II}}new)$: $\mathcal{M}_{\mathrm{II}} \subseteq \mathcal{M}_{\mathrm{I}}$ and, for every $\beta \in On^{\mathcal{M}_{\mathrm{I}}} \setminus \mathcal{A}_{\mathrm{I}}$, $\mathcal{M}_{\mathrm{II}} \not\preceq_n L_\beta^{\mathcal{M}_{\mathrm{I}}}$.

Here $\mathcal{A}_{\mathrm{I}}$ is the image inside $\mathcal{M}_{\mathrm{I}}$ of the "intersection" of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$, i.e. the union of all the L_β in $\mathcal{M}_{\mathrm{I}}$ with codes that are in $\mathcal{M}_{\mathrm{II}}$ as reals which, in $\mathcal{M}_{\mathrm{II}}$, also code some L_δ. This makes sense since we are assuming that one of the models is well-founded and so isomorphic to an L_β. The notation for II is analogous and we use $\mathcal{A}$ to denote ambiguously the structure which is (isomorphic to) $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$ and the isomorphism between them, i.e. the set of pairs $\langle a_1, a_2\rangle$ such that the isomorphism takes $a_1 \in \mathcal{A}_{\mathrm{I}}$ to $a_2 \in \mathcal{A}_{\mathrm{II}}$.

Claim 2.6. *The sets $\mathcal{A}$, $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_2$ are Σ^0_2.*

Proof. For $\mathcal{A}$ we have the following Σ^0_2 definition.

$$(z,w) \in \mathcal{A} \iff \exists x \in \mathbb{R}^{\mathcal{M}_{\mathrm{I}}}, y \in \mathbb{R}^{\mathcal{M}_{\mathrm{II}}} \big(\mathcal{M}_{\mathrm{I}} \models x \text{ codes } z \;\; \wedge \;\; \mathcal{M}_{\mathrm{II}} \models y \text{ codes } w \;\; \wedge \\ \forall n \in \omega\big(\mathcal{M}_{\mathrm{I}} \models n \in x \iff \mathcal{M}_{\mathrm{II}} \models n \in y\big)\big).$$

For $\mathcal{A}_{\mathrm{I}}$ we have that $z \in \mathcal{A}_{\mathrm{I}} \iff \exists w\ ((z,w) \in \mathcal{A})$, and analogously for $\mathcal{A}_{\mathrm{II}}$. □

Claim 2.7. *($R_{\mathrm{I}}new$) and ($R_{\mathrm{II}}new$) are $\mathbf{\Pi}^0_3$ conditions.*

Proof. Consider ($R_{\mathrm{I}}new$). As noted in the proof of Claim 2.5, $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$ is a $\mathbf{\Pi}^0_3$ condition. Since $\mathcal{A}_{\mathrm{II}}$ is Σ^0_2 by Claim 2.6, the set $On^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{A}_{\mathrm{II}}$ is $\mathbf{\Pi}^0_2$. Finally, as we are assuming that $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$, $\mathcal{M}_{\mathrm{I}} \not\preceq_n L_\beta^{\mathcal{M}_{\mathrm{II}}}$ means that there is a Σ_n formula $\varphi(x)$ and a pair of parameters $(x_1,x_2) \in \mathcal{A}$ such that $\neg(\mathcal{M}_{\mathrm{I}} \models \varphi(x_1) \iff L_\beta^{\mathcal{M}_{\mathrm{II}}} \models \varphi(x_2))$. As membership in $\mathcal{A}$ is $\mathbf{\Sigma}^0_2$ by Claim 2.6 and satisfaction in each structure or members of it such as L_β is recursive in the theories the whole relation is $\mathbf{\Pi}^0_3$ as required. The argument for ($R_{\mathrm{II}}new$) is analogous. □

Now the crucial fact for our analysis is that certain failures of these conditions produce a well founded Σ_{n+1} admissible and so would contradict our hypothesis that α_{n+1} does not exist.

Lemma 2.8. *Suppose $\mathcal{M}_{\mathrm{I}}$ is the well-founded part of $\mathcal{M}_{\mathrm{II}}$ and*

$$\exists \beta \in On^{\mathcal{M}_{\mathrm{II}}}\ (\mathcal{M}_{\mathrm{I}} \preceq_n L_\beta^{\mathcal{M}_{\mathrm{II}}}).$$

Then $\mathcal{M}_{\mathrm{I}}$ is Σ_{n+1} admissible. The same is true with I *and* II *interchanged.*

Proof. We prove that $\mathcal{M}_{\mathrm{I}}$ is Σ_k admissible by induction on $k \leq n+1$. For $k=1$, this holds by our condition that $\mathcal{M}_{\mathrm{I}}$ is a model of KP. (It would also follow from [MS Lemma 3.4] without this assumption on T.) So assume it is $k-1$ admissible but not k admissible for $k \leq n+1$. Let a function f witnessing the failure of k admissibility be defined by the Σ_k formula $\exists u \varphi(x,y,u)$ where φ is Π_{k-1}. As every set in $\mathcal{M}_{\mathrm{I}}$ is countable we may assume that the domain of f is ω. Of course, f is unbounded in $\mathcal{M}_{\mathrm{I}}$. Consider now the formula $\theta(x,\langle y_0,y_1\rangle) \equiv \varphi(x,y_0,y_1)\ \&\ \forall z <_L y_1 \neg\varphi(x,y_0,z)$. By Σ_{k-1} admissibility the second conjunct is equivalent (in $\mathcal{M}_{\mathrm{I}}$) to a Σ_{k-1} formula saying that there is a function from $\{z | z <_L y_1\}$ to the associated witness that $\varphi(x,y_0,z)$ fails.

Consider now the set S defined in $\mathcal{M}_{\mathrm{II}}$ as $\{\langle y_0,y_1\rangle\,|\,\exists x \in \omega(L_\beta \models \theta(x,\langle y_0,y_1\rangle)\ \&\ L_\beta \models (\forall \langle z_0,z_1\rangle <_L \langle y_0,y_1\rangle)(\neg\theta(x,\langle z_0,z_1\rangle))\}$. It is clear from the definition of S that, for each $x \in \omega$, there is at most one $\langle y_0,y_1\rangle \in L_\beta$ put into S by x. On the other hand, for each $x \in \omega$ there is a unique $\langle y_0,y_1\rangle$ satisfying θ in $\mathcal{M}_{\mathrm{I}}$ (as f is a function y_0 is unique and y_1 was defined to be least such that $\varphi(x,y_0,z)$). This $\langle y_0,y_1\rangle$ must also satisfy θ in $L_\beta^{\mathcal{M}_{\mathrm{II}}}$ as θ is a conjunct of a Π_{k-1} and a Σ_{k-1} formula and $\mathcal{M}_{\mathrm{I}} \preceq_{k-1} L_\beta^{\mathcal{M}_{\mathrm{II}}}$. As $\mathcal{M}_{\mathrm{I}}$ is an initial segment of $L_\beta^{\mathcal{M}_{\mathrm{II}}}$, any $\langle z_0, z_1\rangle <_L \langle y_0, y_1\rangle$ in $L_\beta^{\mathcal{M}_{\mathrm{II}}}$ is in $\mathcal{M}_{\mathrm{I}}$ and so if it satisfied $\theta(x, \langle z_0, z_1\rangle)$ in $L_\beta^{\mathcal{M}_{\mathrm{II}}}$ it would satisfy it in $\mathcal{M}_{\mathrm{I}}$ but, as above, there is only one such pair in $\mathcal{M}_{\mathrm{I}}$. So the only pair put into S by $x \in \omega$ in $\mathcal{M}_{\mathrm{II}}$ is the unique $\langle y_0, y_1\rangle$ such that $\mathcal{M}_{\mathrm{I}} \vDash \theta(x, \langle y_0, y_1\rangle)$. Thus $S \subseteq \mathcal{M}_{\mathrm{I}}$ and, as f was unbounded in $\mathcal{M}_{\mathrm{I}}$, we have that $\cup S = \cup \mathcal{M}_{\mathrm{I}}$ and so we have defined the well founded part of $\mathcal{M}_{\mathrm{II}}$ in $\mathcal{M}_{\mathrm{II}}$ for a contradiction.

The same proof works with I and II interchanged. $\square$

The consequence of this key fact that we need to satisfy G3(a) and G4(b) is the following:

Lemma 2.9. *If $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{I}}$ is well-founded, then ($R_{\mathrm{I}}new$) holds if and only if $\mathcal{M}_{\mathrm{I}}$ is equal to the well-founded part of $\mathcal{M}_{\mathrm{II}}$. The same is true with* I *and* II *interchanged*

Proof. If $\mathcal{M}_{\mathrm{I}}$ is not equal to the well-founded part of $\mathcal{M}_{\mathrm{II}}$, then the well-founded part of $\mathcal{M}_{\mathrm{II}}$ is strictly greater than that of $\mathcal{M}_{\mathrm{I}}$ by our assumptions. Thus there is a $\beta \in On^{\mathcal{M}_{\mathrm{II}}} - \mathcal{A}_{\mathrm{II}}$ such that $\mathcal{M}_{\mathrm{I}} = L_\beta^{\mathcal{M}_{\mathrm{II}}}$. For this β we have $\mathcal{M}_{\mathrm{I}} \preceq_n L_\beta^{\mathcal{M}_{\mathrm{II}}}$ and so the required failure of ($R_{\mathrm{I}}new$).

For the other direction of our if and only if, suppose that $\mathcal{M}_{\mathrm{I}}$ is the well-founded part of $\mathcal{M}_{\mathrm{II}}$, but that ($R_{\mathrm{I}}new$) does not hold. By Lemma 2.8, $\mathcal{M}_{\mathrm{I}}$ is $n+1$-admissible contradicting our basic assumption that α_{n+1} does not exist.

The same proof works with I and II interchanged. $\square$

For the remaining cases, we have to primarily consider those where neither $\mathcal{M}_{\mathrm{I}}$ nor $\mathcal{M}_{\mathrm{II}}$ is contained in the other. In this case, we can assume that exactly one of them is well-founded. (Two well founded models of $V = L$ are always comparable and we already know that one has to be well-founded for the play to be relevant.) In this case, it suffices, by the definition of $\mathcal{G}$, to guarantee the player producing the well-founded model wins.

2.3 Incomparable models

If $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable (as will be guaranteed at some point in our game by the failure of previous conditions) we may assume that exactly one of them is well-founded. We denote the well-founded one by $\mathcal{M}$ and the ill-founded one by $\mathcal{N}$. We want to identify the one which is well founded to determine the winner so as to satisfy G3(b) and G4(b). Each condition in the sequence to be described below asks for a descending chain in one of the models. The failure of each successive condition produces a model with one more level of admissibility. If one condition succeeds then we identify the well founded model and can declare a winner consistent with G3(b) and G4(b). If all of them fail we produce a Σ_{n+1} admissible for the desired contradiction.

The idea in the definition of the remaining conditions ($R_{::}2+k$) is that we want them to provide evidence that one of the models is ill-founded by finding a nonempty set of ordinals without a least element. (The nonemptyness was inadvertently left out in [MS] but this makes no significant difference in the proofs provided there.) To this end we define various sets of ordinals in the two models and establish some of their properties.

Looking ahead to our proof of the nonreversal of Theorem 1.10(5) in §3, it would be useful for the readers of this section to note that all the Lemmas apply to similar games. To be specific, in §3 we have a theory T_n^* in place of T here. This theory implies that there are unboundedly many n-admissibles but that the universe has Σ_1-cofinality ω rather than being admissible. It still satisfies all sets are countable. While some of the conditions defining the game there are different, the general set up is quite similar. When we consider a play of the game there the cases of interest will be that exactly one of the two models is the standard model of the theory ($L_{\alpha_n^*}$) and the other is a nonstandard model of the theory incomparable (in the above sense of containment) with the standard model. It is routine to verify that all the Lemmas of this section apply to such a situation.

Definition 2.10.

$$\begin{aligned} C_{\mathcal{M}_{\mathrm{II}},1} = \{\beta \in On^{\mathcal{M}_{\mathrm{II}}} : \exists (x_1, x_2) \in \mathcal{A},\ \varphi \in \Delta_0, \\ \big(\big(\exists z \in L_\beta^{\mathcal{M}_{\mathrm{II}}}\ \mathcal{M}_{\mathrm{II}} \models \varphi(z, x_2)\big) \wedge \big(\mathcal{M}_{\mathrm{I}} \models \neg\exists y \varphi(y, x_1)\big)\big)\}. \end{aligned}$$

$C_{\mathcal{M}_{\mathrm{I}},1}$ is defined analogously.

Claim 2.11. *The sets $C_{\mathcal{M}_{\mathrm{II}},1}$ and $C_{\mathcal{M}_{\mathrm{I}},1}$ are Σ_2^0.*

Proof. By Claim 2.6, $\mathcal{A}$ is Σ_2^0 while the second line of the definition is computable from the elementary diagrams of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. □

We now have our first level of the $(R_{::}2 + k)$ conditions:

($R_{\mathrm{I}}3$): $C_{\mathcal{M}_{\mathrm{II}},1}$ is not empty and has no least element.

($R_{\mathrm{II}}3$): $C_{\mathcal{M}_{\mathrm{I}},1}$ is not empty and has no least element.

Clearly these conditions are Π_3^0.

At this point in [MS] we could assume that the models were incomparable. Unfortunately, in our game will not be able to know that the models are incomparable until after we consider ($R_{\mathrm{I}}3$), ($R_{\mathrm{II}}3$) and ($R_{\mathrm{I}}4$). This necessitates a real modification in the definition of $C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{II}},k}$ and ($\bigstar_{k-1}$) which are used below to define the $(R_{::}(2+k))$ for $k > 1$. We must now allow for the possibility that β_1 or β_2 might take on the value ∞. (Looking ahead, the advantage of this change is that, if $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$, we might still have that ($\bigstar_1$)(∞, β_2) for some β_2 when it might fail for every $\beta_1 \in \mathcal{M}_{\mathrm{I}}$ (which is well-founded).)

Definition 2.12. For $k \geq 1$ we define the condition ($\bigstar_k$)(β_1, β_2) by

$$\begin{aligned} \beta_1 \in (On^{\mathcal{M}_{\mathrm{I}}} \setminus \mathcal{A}_{\mathrm{I}}) \cup \{\infty\} \wedge \mathcal{M}_{\mathrm{I}} \models \beta_1 \text{ is } (k-1)\text{-admissible} \wedge \\ \beta_2 \in (On^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{A}_{\mathrm{II}}) \cup \{\infty\} \wedge \mathcal{M}_{\mathrm{II}} \models \beta_2 \text{ is } (k-1)\text{-admissible} \wedge \\ L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}. \end{aligned}$$

Here 0-admissibility is taken to be just transitivity and $L_\infty^{\mathcal{M}_{::}}$ is taken to be $\mathcal{M}_{::}$. The symbol $\equiv_{k,\mathcal{A}}$ denotes Σ_k-elementary equivalence with parameters from $\mathcal{A}$. In other words, $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$ means that for every Σ_k formula $\varphi(z)$, and every $x_1 \in \mathcal{A}_{\mathrm{I}}$ and $x_2 \in \mathcal{A}_{\mathrm{II}}$, if x_1 and x_2 represent the same set (i.e. $(x_1, x_2) \in \mathcal{A}$), then $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \models \varphi(x_1) \iff L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}} \models \varphi(x_2)$.

Claim 2.13. *The relation $\bigstar_k$ is a Π^0_2 property of β_1 and β_2.*

Proof. By Claim 2.6, $On^{\mathcal{M}_{::}} \setminus \mathcal{A}_{::}$ are Π^0_2. Whether β_i is $(k-1)$-admissible is just one sentence in the theory of the structure. To check that $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$, we need to say that, for every Σ_k formula $\varphi(x)$, and for every pair of parameters $(x_1, x_2) \in \mathcal{A}$, $\mathcal{M}_{\mathrm{I}} \models \varphi(x_1) \iff \mathcal{M}_{\mathrm{II}} \models \varphi(x_2)$. As $\mathcal{A}$ is Σ^0_2 by Claim 2.6, counting quantifiers shows that $(\bigstar_k)$ is Π^0_2. □

We next define the sets $C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{II}},k}$ and $C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{I}},k}$ (correcting two typos in [MS]).

Definition 2.14. For $k > 1$,

$$C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{II}},k} = \{\beta < \beta_2 : \exists (x_1, x_2) \in \mathcal{A},\ \exists \varphi \in S_{k-1} \\ \big(\exists z \in L_\beta^{\mathcal{M}_{\mathrm{II}}}\ L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}} \models \varphi(z, x_2)\big) \wedge \big(L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \models \neg \exists y \varphi(y, x_1)\big)\}.$$

Here S_{k-1} is the class of formulas that are Boolean combinations of formulas of the form $(\forall x \in z)\psi(x, z, \bar{y})$ where ψ is Σ_{k-1}. Again β_1 or β_2 may be ∞ and $L_\infty^{\mathcal{M}_{::}}$ is taken to be $\mathcal{M}_{::}$. $C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{I}},k}$ is defined analogously.

We note the following property of S_k sentences for any ω-models $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ produced according to the rules of our game so far.

Lemma 2.15. *If $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$, then $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}}$ and $L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$ satisfy the same S_k-sentences with parameters from $\mathcal{A}$ substituted for the free variables z and $\bar{y}$.*

Proof. Let φ be a sentence of the form $(\forall x \in z)\psi(x, z, \bar{y})$ with parameters from $\mathcal{A}$ where ψ is Σ_k. Since φ has no free variables, z and $\bar{y}$ are parameters from $\mathcal{A}$, and so every $x \in z$ is also in $\mathcal{A}$. Since $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$, we have that for every $x \in z$, $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \models \psi(x, z, \bar{y}) \Leftrightarrow L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}} \models \psi(x, z, \bar{y})$. So, of course, $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \models \varphi \Leftrightarrow L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}} \models \varphi$. Finally, if ϕ is a Boolean combination of sentences of this form, then we also have $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \models \phi \Leftrightarrow L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}} \models \phi$. □

We can now define the remaining conditions $R_{::}(2+k)$ for $k > 1$.

Definition 2.16. For $k > 1$,

($R_{\mathrm{I}}(2+k)$): For every β_1, β_2, if $(\bigstar_{k-1})(\beta_1, \beta_2)$, then $C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{II}},k}$ is not empty and has no least element.

($R_{\mathrm{II}}(2+k)$): For every β_1, β_2, if $(\bigstar_{k-1})(\beta_1, \beta_2)$, then $C^{\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{I}},k}$ is not empty and has no least element.

Here again β_1 and β_2 may be ∞ as well as ordinals of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$, respectively.

We must now verify the key properties of the conditions $((R_{::}(2+k)))$. When $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable, as remarked above, one of them, denoted by $\mathcal{M}$, is well-founded and the other, denoted by $\mathcal{N}$, is not. In this case, we only need some modifications of the proofs in [MS] to accommodate the possibility that β_1 or β_2 is ∞.

Lemma 2.17. *If G1 does not determine the winner of a play of our game (i.e., $R_{\mathrm{I}}i$ and $R_{\mathrm{II}}i$ fail for $i < 2$), the models $\mathcal{M}$ and $\mathcal{N}$ produced are incomparable (i.e. G2 does not determine the winner of the play) and both $R_{\mathrm{I}}3$ and $R_{\mathrm{II}}3$ fail, then there is a $\beta \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$ such that $\mathcal{A} \preccurlyeq_1 L^{\mathcal{N}}_{\beta}$ and so β_1 and β_2 such that $\bigstar_1(\beta_1, \beta_2)$ holds.*

Proof. Suppose the first assertion fails, that is, for every $\gamma \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$, there is a Σ_1 formula with parameters in $\mathcal{A}$, that is true in L_γ but not in $\mathcal{A}$. Since conditions ($R_{\mathrm{I}}3$) and ($R_{\mathrm{II}}3$) are not satisfied, $C_{\mathcal{N},1}$ is empty or has a least element δ. If $C_{\mathcal{N},1}$ is empty, then, by definition, $\mathcal{A} \preccurlyeq_1 L^{\mathcal{N}}_{\beta}$ for every $\beta \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$. Thus, we may assume that $C_{\mathcal{N},1}$ has a least element δ. Notice that δ cannot be in $\mathcal{A}$ because if the witness z for a Δ_0 formula $\varphi(z, x_2)$ is in $L_{\mathcal{N}_\delta} \subseteq \mathcal{A}$, then $\exists y \varphi(y, x_1)$ also holds in $\mathcal{M}$. Let $\delta > \gamma_0 > \gamma_1 > \gamma_2 > ...$ be a descending sequence in $On^{\mathcal{N}} \smallsetminus \mathcal{A}$, converging down to the cut $(On^A, On^N \smallsetminus \mathcal{A})$. For each i, there is a Δ_0 formula φ_i with parameters from $\mathcal{A}$ and a $<_L$-least witness $z_i \in L_{\gamma_i}$ such that $\mathcal{N} \vDash \varphi_i(z_i)$ but $\mathcal{A} \vDash \neg\exists y \varphi_i(y)$. By thinning out the sequence if necessary, we may assume that $z_i \in L_{\gamma_i}$ but $z_i \notin L_{\gamma_{i+1}}$. So $\{z_i : i \in \omega\}$ is an $<^{\mathcal{N}}_L$-descending sequence. Since $\gamma_i \notin C_{\mathcal{N},1}$, $\mathcal{M} \vDash \exists y \varphi_i(y)$. Let y_i be the $<^{\mathcal{M}}_L$-least such witness. Since $\mathcal{M}$ is well-founded, the sequence $\{y_i : i \in \omega\}$ cannot be a $<^{M}_L$-descending sequence. So, there exists $i < j$ such that $z_j <^{\mathcal{N}}_L z_i$ but $y_i <^{\mathcal{M}}_L y_j$. Therefore, L_{γ_j} is a witness in L_{γ_j+1} for x showing the truth in $\mathcal{N}$ of the Δ_0 formula

$$\psi(x) \equiv \exists z \in x \varphi_j(z) \wedge \forall z \in x \neg\varphi_i(z)$$

but there is no witness for $\psi(x)$ in $\mathcal{M}$. This shows that $\gamma_j + 1$ is in $C_{\mathcal{N},1}$, contradicting our choice of δ as the least element of $C_{\mathcal{N},1}$.

Now choose α such that $L_\alpha = \mathcal{A}$ and so $\alpha \in \mathcal{M}$ and β the ordinal of $\mathcal{N}$ such that $\mathcal{A} \preccurlyeq_1 L^{\mathcal{N}}_{\beta}$. By the definition of $\bigstar_1$, α and β, in the appropriate order, provide the required witnesses for $\bigstar_1$. $\square$

Lemma 2.18. *Suppose that β_1, β_2 satisfy $(\bigstar_k)$. As the definition of $\bigstar_k$ is symmetric we may suppose that $\beta_1 \in \mathcal{M}$ and $\beta_2 \in \mathcal{N}$. Then*

1. *$L_\alpha \preccurlyeq_k L_{\beta_1}$ and $\mathcal{A} \preccurlyeq_k L_{\beta_2}$.*
2. *α is $(k+1)$-admissible.*

3. *There exists a descending sequence of $\mathcal{N}$-ordinals γ converging down to $On^{\mathcal{A}}$ such that $L_\gamma \preccurlyeq_k L_{\beta_2}$.*

Proof. If $\beta_1 \neq \infty$, it is simply an ordinal in the well-founded $\mathcal{M}$ and $L^{\mathcal{M}}_{\beta_1} = L_{\beta_1}$. If $\beta_1 = \infty$ then as usual $L^{\mathcal{M}}_{\beta_1} = \mathcal{M}$ but note that in this case, as $\mathcal{M}$ is well-founded, we may also view β_1 as the (true) ordinal of $\mathcal{M} = L_{\beta_1}$. For $\beta_2 = \infty$, we simply view $L^{\mathcal{N}}_{\beta_2}$ as $\mathcal{N}$.

First, we note that $\{\alpha\}$ is not Σ_k definable in L_{β_1} with parameters from L_α. Suppose otherwise. Since $\alpha \in \mathcal{M} \models T$, every α is countable in $L_{\alpha+1} \in \mathcal{M}$ by [MS Lemma 3.5]. Thus there is a definable over L_α map from ω onto α (possibly with parameters). This defines in L_α, an ordering of ω of order type α. This ordering cannot belong to $\mathcal{N}$ as it would then define its well-founded part. In L_{β_1} we can define this ordering with a Σ_k formula using the Σ_k definition of α and quantification over L_α. Since $L_{\beta_1} = L^{\mathcal{M}}_{\beta_1} \equiv_{k,\mathcal{A}} L^{\mathcal{N}}_{\beta_2}$, this ordering is definable in $L^{\mathcal{N}}_{\beta_2}$ by the same formula, and hence belongs $\mathcal{N}$ for the desired contradiction.

Now for part (1), since β_1 is $(k-1)$-admissible, L_{β_1} has a Σ_k-Skolem function without parameters ([MS Lemma 3.1]). Let H be the Σ_k-Skolem hull of L_α in L_{β_1}. We show that $H = L_\alpha$. Note that this implies that $L_\alpha \preccurlyeq_k L_{\beta_1}$ (as desired) and also $\mathcal{A} \preccurlyeq_k L_{\beta_2}$ since $L^{\mathcal{M}}_{\beta_1} \equiv_{k,\mathcal{A}} L^{\mathcal{N}}_{\beta_2}$ by $\bigstar_k$. Suppose that $H \supsetneq L_\alpha$. Let L_γ be the Mostowski collapse of H, so $\alpha < \gamma \leq \beta_1$. Let α' be the ordinal in H that is sent to $\alpha \in L_\gamma$ by the collapse. As α' is in the image of the Σ_k Skolem function, $\{\alpha'\}$ is Σ_k definable in H with parameters from L_α . The same formula then gives a definition of $\{\alpha\}$ in L_γ as the collapse is the identity on members of L_α. Thus $\{\alpha\}$ is Σ_k definable in L_γ with parameters from L_α but clearly

$$L^{\mathcal{M}}_{\gamma} \equiv_{k,\mathcal{A}} H \equiv_{k,\mathcal{A}} L^{\mathcal{M}}_{\beta_1}.$$

So $\{\alpha\}$ would be Σ_k definable in $L^{\mathcal{M}}_{\beta_1}$ contradicting our first observation.

(2) Suppose α is not $(k+1)$-admissible and so (since every ordinal is countable in $\mathcal{M}$) there is a Π_k/L_α function f from ω onto α. Since $\mathcal{A} \preccurlyeq_k L^{\mathcal{N}}_{\beta_2}$, the same formula defining f over L_α defines a function from ω onto α in $L^{\mathcal{N}}_{\beta_2}$. But then, in $\mathcal{N}$ we could define its well-founded part, L_α, for a contradiction.

(3) Now since α is $(k+1)$-admissible, there are unboundedly many $\gamma < \alpha$ such that $L_\gamma \preccurlyeq_k L_\alpha$ ([MS, Lemma 3.3]). For all these γ we also have $L^{\mathcal{N}}_{\gamma} \preccurlyeq_k L^{\mathcal{N}}_{\beta_2}$. The set of $\gamma < \beta_2$ such that $L^{\mathcal{N}}_{\gamma} \preccurlyeq_k L^{\mathcal{N}}_{\beta_2}$ is definable in $\mathcal{N}$. If for some $\delta \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$, the set of $\gamma < \delta$ such that $L^{\mathcal{N}}_{\gamma} \preccurlyeq_k L^{\mathcal{N}}_{\beta_2}$ had supremum α, then α would be definable in $\mathcal{N}$ and we know it is not. So for every $\delta \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$, there exists $\gamma < \delta$, $\gamma \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$ such that $L^{\mathcal{N}}_{\gamma} \preccurlyeq_k L^{\mathcal{N}}_{\beta_2}$. □

Lemma 2.19. *If in a play of our game the resulting real does not satisfy any condition $(R_{\mathrm{I}}i)$ or $(R_{\mathrm{II}}i)$ for $i \leq 2+k$, then there are β_1 and β_2 satisfying $(\bigstar_k)$.*

Proof. Lemma 2.17 shows that there are β_1 and β_2 satisfying $(\bigstar_1)$. The proof now proceeds by induction, so, without loss of generality, we assume that we have witnesses β_1, β_2 satisfying $(\bigstar_{k-1})$.

We claim that no $\delta \in \mathcal{A}$ is in $C^{\beta_1,\beta_2}_{\mathcal{N},k}$. Consider $\delta \in \mathcal{A}$ and any S_{k-1} formula $\forall x \in z\varphi(z,\bar{y})$ and $z_2, \bar{y}_2 \in L^{\mathcal{N}}_\delta \subseteq \mathcal{A}$ such that $L^{\mathcal{N}}_{\beta_2} \models \forall x \in z_2\varphi(z_2,\bar{y}_2)$. Since L_{β_1} and L_{β_2} satisfy the same Σ_{k-1} formulas with parameters from $\mathcal{A}$ (by $(\bigstar_{k-1})$), $L^{M}_{\beta_1} \models \forall x \in z_1\varphi(z_1,\bar{y}_1)$, where z_1 and $\bar{y}_1$ are the images of $z_2, \bar{y}_2$ in $\mathcal{M}$. Thus $\delta \notin C^{\beta_1,\beta_2}_{\mathcal{N},k}$ by its definition.

Since the conditions $(R_{\mathrm{I}}(2+k))$ and $(R_{\mathrm{II}}(2+k))$ are not satisfied, $C^{\beta_1,\beta_2}_{\mathcal{N},k}$ is empty or has a least element δ, necessarily not in $\mathcal{A}$. By clause (3) of the previous lemma, there is a descending sequence $\gamma_0 > \gamma_1 > \gamma_2 > \ldots$ in $On^{\mathcal{N}}$ (all below δ if it exists), converging down to $\alpha = On^{\mathcal{A}}$, such that for each i $L^{\mathcal{N}}_{\gamma_i} \preccurlyeq_{k-1} L^{\mathcal{N}}_{\beta_2}$. We now argue much as for Lemma 2.17. We claim that, for some i, $L_\alpha \preccurlyeq_k L^{\mathcal{N}}_{\gamma_i}$. If not, there is, for each i, a Π_{k-1} formula $\varphi_i(z)$ with parameters from $\mathcal{A}$ and a $<^{\mathcal{N}}_L$-least $z_i \in L_{\gamma_i}$ such that $L_{\gamma_i} \models \varphi_i(z_i)$ and $z_i \notin \mathcal{A}$. As $L^{\mathcal{N}}_{\gamma_i} \preccurlyeq_{k-1} L^{\mathcal{N}}_{\beta_2}$, $L^{\mathcal{N}}_{\beta_2} \models \varphi_i(z_i)$. By thinning out the sequence if necessary, we may assume that $z_i \in L^{\mathcal{N}}_{\gamma_i}$ but $z_i \notin L^{\mathcal{N}}_{\gamma_{i+1}}$. So $\{z_i : i \in \omega\}$ is an $<^{\mathcal{N}}_L$-descending sequence. Since $\gamma_i \notin C^{\beta_1,\beta_2}_{\mathcal{N},k}$, $L^{\mathcal{M}}_{\beta_1} \models \exists y \varphi_i(y)$. Let y_i be the $<^{\mathcal{M}}_L$-least such witness. Since $\mathcal{M}$ is well-founded, the sequence $\{y_i : i \in \omega\}$ cannot be a $<^{\mathcal{M}}_L$-descending sequence. So, there exist $i < j$ such that $z_j <^{\mathcal{N}}_L z_i$ but $y_i <^{\mathcal{M}}_L y_j$. Therefore, $L^{\mathcal{N}}_{\gamma_j} \in L^{\mathcal{N}}_{\gamma_j+1}$is a witness for x showing the truth in $L^{\mathcal{N}}_{\beta_2}$of the S_{k-1} formula

$$\psi(x) \equiv \exists z \in x\varphi_j(z,x) \wedge \forall z \in x \neg\varphi_i(z,x)$$

while there is no such witness for the corresponding formula in $L^{\mathcal{M}}_{\beta_1}$. This contradicts our choice of δ as the least element of $C^{\beta_1,\beta_2}_{\mathcal{N},k}$. Thus we have an i such that $L_\alpha \preccurlyeq_k L^{\mathcal{N}}_{\gamma_i} \preccurlyeq_{k-1} L^{\mathcal{N}}_{\beta_2}$.

Now $L^{\mathcal{N}}_{\gamma_i}$ is Σ_{k-1} admissible by [MS Lemma 3.4] as $L^{\mathcal{N}}_{\beta_2}$ is Σ_{k-2} admissible by $(\bigstar_{k-1})$ while α is (even) Σ_k admissible by our induction hypothesis and the previous Lemma and so $(\bigstar_k)(\alpha,\gamma_i)$ as required. □

As we shall see in our final organization of the conditions in the next section, the only situation in which we have to rely on any of these facts without being able to assume that $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable is the argument that $\bigstar_1$ holds for some β_1 and β_2 if $(R_{\mathrm{I}}3)$ and $(R_{\mathrm{II}}3)$ both fail. We analyze this situation when it occurs in the next section.

For now we suggest how the final arguments will produce the desired $(n+1)$-admissible ordinal by combining Lemmas 2.17-2.19.

Corollary 2.20. *If $\mathcal{M}$ and $\mathcal{N}$ are incomparable ω-models of T derived from a play of our game with (in our notation as above) $\mathcal{M} = L_\alpha$ and $(R_{\mathrm{I}}i)$ and $(R_{\mathrm{II}}i)$ both fail for $i \leq 2+k$, then α is $(k+1)$-admissible.*

Proof. For $k = 1$, Lemma 2.17 shows that $\bigstar_1(\beta_1,\beta_2)$ holds for some β_1 and β_2. Lemma 2.19 then shows that there are also β_1 and β_2 such that $\bigstar_k(\beta_1,\beta_2)$ holds if $k > 1$. Finally Lemma 2.18(2) says that α is $(k+1)$-admissible as required. □

2.4 Organizing the Conditions and Verifications

Formally we define our $\mathbf{\Pi}^0_{3,n}$ game G by a list of n many $\mathbf{\Pi}^0_3$ sets $A_0, A_1 \ldots, A_{n-1}$ followed by $A_n = 2^\omega$. Remember that I wins if the first of these sets containing the play of our game has an even index and otherwise, i.e. the first one has an odd index, II wins. The table below contains the definition of the A_i arranged with the even index ones in column I and the odd index ones in column II in alternating blocks.

	win for I	win for II
A_0	$(\neg R_{\mathrm{II}}0 \& \neg R_{\mathrm{II}}1)\&$ $[R_{\mathrm{I}}0 \vee R_{\mathrm{I}}1 \vee$ $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}} \vee$ $(R_{\mathrm{I}} new) \vee$ $(R_{\mathrm{I}}3)]$	$\vdots$
A_1		$R_{\mathrm{II}}0 \vee R_{\mathrm{II}}1 \vee$ $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}} \vee$ $(R_{\mathrm{II}} new) \vee$ $(R_{\mathrm{II}}3) \vee$ $(R_{\mathrm{II}}4)$
A_2	$\mathcal{M}_{\mathrm{II}} \subseteq \mathcal{M}_{\mathrm{I}} \vee$ $(R_{\mathrm{I}}4) \vee$ $(R_{\mathrm{I}}5)$	
A_3		$(R_{\mathrm{II}}5) \vee$ $(R_{\mathrm{II}}6)$
	$\vdots$	
A_{n-1}		$\vdots$

We end this list with A_{n-1} corresponding to $(R_{\mathrm{II}}(1+n)) \vee (R_{\mathrm{II}}(2+n))$ if $n \geq 2$ is even and $(R_{\mathrm{I}}(1+n)) \vee (R_{\mathrm{I}}(2+n))$ if $n > 2$ is odd. (Recall that we are only proving our theorem for $n \geq 2$.) This gives us n many conditions. We finish the description of the game in standard $\mathbf{\Pi}^0_{3,n}$ format by adding on the the full space 2^ω as our $(n+1)st$ and last set A_n. We now argue that the game G specified by this sequence of $\mathbf{\Pi}^0_3$ sets is in $\mathcal{G}$. If not, then there is a play of G that violates the defining conditions of $\mathcal{G}$. We consider any play and the set in our sequence in which it first appears and verify that we have not violated the definition of $\mathcal{G}$.

We begin with a play in A_0. By the definition of A_0, I has played a complete consistent extension of T whose term model $\mathcal{M}_{\mathrm{I}}$ is an ω-model (as $\neg R_{\mathrm{II}}0$ and $\neg R_{\mathrm{II}}1$ hold and so I has not lost the game yet according to G1). We now check each of the disjunctions in the rest of the definition of A_0 to see that a win by I given by satisfying the disjunction is consistent with our definition of $\mathcal{G}$. If $R_{\mathrm{I}}0$ or $R_{\mathrm{I}}1$ holds then I wins by G1. If they fail then II has played a complete consistent extension of T whose term model $\mathcal{M}_{\mathrm{II}}$ is an

ω-model. If $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$ then I wins by G2. Suppose now that $\mathcal{M}_{\mathrm{I}} \neq \mathcal{M}_{\mathrm{II}}$. Remember that from now on we only need to consider plays of the game in which at least one of the models is well-founded to verify that the winner declared by G is consistent with the definition of $\mathcal{G}$. If $(R_{\mathrm{I}}new)$ holds then $\mathcal{M}_{\mathrm{I}} \subset \mathcal{M}_{\mathrm{II}}$ by our assumption and the definition of the condition. As we are assuming that (at least) one of the models is well-founded, $\mathcal{M}_{\mathrm{I}}$ is certainly well-founded. Then by Lemma 2.9, $\mathcal{M}_{\mathrm{I}}$ is the well-founded part of $\mathcal{M}_{\mathrm{II}}$ but, as the models are not equal, $\mathcal{M}_{\mathrm{II}}$ is not well-founded and I wins according to G3b. Next, if $(R_{\mathrm{I}}3)$ holds we only have to verify that the play cannot be a win for II according to the rules for $\mathcal{G}$, i.e. G4a or G4b. Now both of these require that $\mathcal{M}_{\mathrm{II}}$ is well-founded but $(R_{\mathrm{I}}3)$ says that it is not. So neither condition can require a win for II as desired.

We now move to A_1 under the assumption that the play is not in A_0. First if $R_{\mathrm{II}}0$ or $R_{\mathrm{II}}1$ hold then II wins the game according to G1. If not, we again have the two models $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ being distinct and we need only consider the case that at least one is well-founded.

Suppose then that the next condition, $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$, holds and so $\mathcal{M}_{\mathrm{I}}$ must be well-founded. Under these conditions, the failure of $(R_{\mathrm{I}}new)$ implies, by Lemma 2.9, that $\mathcal{M}_{\mathrm{I}}$ is not the well-founded part of $\mathcal{M}_{\mathrm{II}}$ and so neither G3a nor G3b can apply. Thus it is consistent with the definition of $\mathcal{G}$ that we declare a win for II.

The next case is that $(R_{\mathrm{II}}new)$ holds and so in particular, $\mathcal{M}_{\mathrm{II}} \subseteq \mathcal{M}_{\mathrm{I}}$, $\mathcal{M}_{\mathrm{II}}$ at least must be well-founded and, by Lemma 2.9 again, $\mathcal{M}_{\mathrm{II}}$ is the well-founded initial segment of $\mathcal{M}_{\mathrm{I}}$ which cannot be well-founded as $\mathcal{M}_{\mathrm{I}} \neq \mathcal{M}_{\mathrm{II}}$. Once again neither G3a nor G3b can apply and we are safe.

Next we have $(R_{\mathrm{II}}3)$. If it holds then $\mathcal{M}_{\mathrm{I}}$ is not well-founded and once again we can safely declare a win for II. Finally, we have to consider the case that all the previous conditions have failed but $(R_{\mathrm{II}}4)$ holds. If we knew that there were β_1 and β_2 for which $\bigstar_1$ holds, we would again know that $\mathcal{M}_{\mathrm{I}}$ is not well founded and we can declare a win for II. If $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable then Lemma 2.17 tells is that $\bigstar_1$ holds as desired for the α and β described there. Thus we may assume that $\mathcal{M}_{\mathrm{II}} \subset \mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ is well-founded. The failure of $(R_{\mathrm{II}}new)$ then tells us that, by Lemma 2.9, $\mathcal{M}_{\mathrm{II}}$ is not equal to the well-founded part of $\mathcal{M}_{\mathrm{I}}$. Thus there is a $\beta \in On^{\mathcal{M}_{\mathrm{I}}} - \mathcal{A}$ and indeed that $\alpha = \cup On^{\mathcal{M}_{\mathrm{II}}} \in \mathcal{M}_{\mathrm{I}} - \mathcal{A}$. We then have $\bigstar_1(\infty, \alpha)$ as required.

Now if the play of the game is not in A_0 or A_1 then we have our models $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. If both are well founded then they are comparable and, by the failure to get into A_0 or A_1, $\mathcal{M}_{\mathrm{II}} \subset \mathcal{M}_{\mathrm{I}}$ and according to G3a I should win the game. We claim this outcome is always in A_2 as desired. If we are considering $n = 2$ (and the sequence of sets from our table ends with A_1) then this is trivial since $A_2 = 2^\omega$. If $n > 2$ then it is in A_2 by the first clause of its definition. Thus we may assume from now on (in our check of consistency with the definition of $\mathcal{G}$) that exactly one of the models is well-founded.

Next suppose that $\mathcal{M}_{\mathrm{II}} \subseteq \mathcal{M}_{\mathrm{I}}$ so $\mathcal{M}_{\mathrm{II}}$ is well-founded and not $\mathcal{M}_{\mathrm{I}}$. The failure of $(R_{\mathrm{II}}new)$ and Lemma 2.9 then says that $\mathcal{M}_{\mathrm{II}}$ is not the well-founded initial segment of $\mathcal{M}_{\mathrm{I}}$. Thus both G4a and G4b fail and it is safe for us to declare a win for I. From this point on in our argument we may assume that $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are incomparable as well as that exactly one is well-founded. At this point $(R_{\mathrm{I}}i)$ and $(R_{\mathrm{II}}i)$ fail for $i \leq 3$ as does $(R_{\mathrm{II}}4)$. If $(R_{\mathrm{I}}4)$ holds, $\mathcal{M}_{\mathrm{II}}$ is not well-founded since there are, by Lemma 2.17, β_1 and β_2 such that $\bigstar_1$ holds. It is then safe to declare a win for I. If $(R_{\mathrm{I}}4)$ fails, then, by Corollary 2.20, we have a Σ_3 admissible ordinal which is the desired contradiction for the case $n = 2$ and we have finished our verification. If $n > 2$, $(R_{\mathrm{I}}4)$ fails but $(R_{\mathrm{I}}5)$ holds, $\mathcal{M}_{\mathrm{II}}$ is once again not well-founded by Lemma 2.19. Thus we may safely declare a win for I.

Next consider $n = 3$. We already know that our play getting into A_0, A_1 or A_2 is consistent with being in $\mathcal{G}$. If it fails to get into A_0, A_1 or A_2, but satisfies $(R_{\mathrm{II}}5)$ or $\neg(R_{\mathrm{II}}5) \wedge (R_{\mathrm{II}}6)$ then $\mathcal{M}_{\mathrm{I}}$ is not well-founded as in the previous cases and it is safe to declare a win for II. If both fail, then Corollary 2.20 provides a Σ_4 admissible ordinal for the desired contradiction. The analysis for all $n > 3$ is now the same as for $n = 3$ and we have that for every $n \geq 2$ our $\mathbf{\Pi}^0_{3,n}$ game given by the specifications above is in $\mathcal{G}$ to complete the proof of Lemma 1.6.

3 No reversal

We now prove that (5) in Theorem 1.10 does not reverse. That is, for every $n \geq 2$, $\beta(\Delta^1_{n+1}\text{-}\mathsf{CA}_0) \nvdash (n-1)\text{-}\mathbf{\Pi}^0_3\text{-}DET$.

Let α^*_n be the first limit of n-admissibles. Thus, $\mathbb{R} \cap L_{\alpha^*_n} \models \beta(\Delta^1_{n+1}\text{-}\mathsf{CA}_0)$ and is itself a β-model. We will modify the proofs of Theorem 1.8 and Lemma 2.2 to provide a witness to the nonreversal.

Theorem 3.1. *The model given by $L_{\alpha^*_n}$, i.e. $\mathbb{R} \cap L_{\alpha^*_n}$, does not satisfy $(n-1)$-$\mathbf{\Pi}^0_3$-DET.*

The crucial lemma is the following:

Lemma 3.2. *For every $n \geq 2$, there is a game G^*_n that is $(n\text{-}1)$-Π^0_3, such that, if we interpret the play of each player as the characteristic function of a set of sentences in the language of set theory, then*

1. *If* I *plays $Th_{\alpha^*_n}$, he wins.*
2. *If* I *does not play $Th_{\alpha^*_n}$ but* II *does, then* II *wins.*

We first provide the proof of the Theorem from the Lemma.

Proof of Theorem 3.1 from Lemma 3.2. As we mentioned above, $\mathbb{R} \cap L_{\alpha^*_n}$ is a β-model of $\beta(\Delta^1_{n+1}\text{-}\mathsf{CA}_0)$ and so it suffices to show that $L_{\alpha^*_n} \nvDash (n\text{-}1)$-$\Pi^0_3$-DET. Let G be as in the Lemma; and suppose it is determined in $L_{\alpha^*_n}$. Player II cannot have a winning strategy for G in $L_{\alpha_n^*}$ because if II has a winning strategy σ in $L_{\alpha_n^*}$, σ would also be a winning strategy in V as $\mathbb{R} \cap L_{\alpha_n^*}$ is a β-model (and σ being a winning stratgey for G is a Π^1_1 property). But, I has a winning strategy for G in V by clause (1) of the Lemma. So, I must have a winning strategy σ for G in $L_{\alpha_n^*}$. Again, as $\mathbb{R} \cap L_{\alpha_n^*}$ is a β-model, σ is truly a winning strategy for I (in V). We claim that if II plays so as to simply copy I's moves, then σ has to play $Th_{\alpha_n^*}$. If not, then at some first point I plays a bit that is different from $Th_{\alpha_n^*}$. At this point II could stop copying I and just continue playing $Th_{\alpha_n^*}$ and he would win (by clause (2) of the Lemma). Thus σ would not be a truly winning strategy for I (in V). We conclude that σ computes $Th_{\alpha_n^*}$ as the sequence of I's plays against II's copying his moves and so $Th_{\alpha_n^*} \in L_{\alpha_n^*}$ for the desired contradiction. $\square$

The proof of Lemma 3.2 is similar to that of Lemma 2.2 and we adopt much of its notation.

We do, however, replace the theory T with the following:

$$T_n^* = V = L + \text{there are unboundedly many } n\text{-admissibles, but only finitely many below each ordinal.}$$

So, $L_{\alpha_n^*}$ is the least well-founded model of T_n^* and is also a β-model. The general set up is as in Lemma 2.2 including the notations such as $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{::}, \mathcal{A}, \mathcal{A}_{\mathrm{I}}, (R_{::}i)$, etc. One new fact that simplifies the conditions needed for our game is a simple one:

Lemma 3.3. *If $\mathcal{M}_{\mathrm{I}} \neq \mathcal{M}_{\mathrm{II}}$ are ω-models of T_n^* then they are incomparable (all in the sense of the notion of containment from Definition 2.3).*

Proof. If $\mathcal{M}_{\mathrm{I}} \subsetneq \mathcal{M}_{\mathrm{II}}$ then there are unboundedly many n-admissibles in $\mathcal{M}_{\mathrm{I}}$ and some $\beta \in On^{\mathcal{M}_{\mathrm{II}}} - \mathcal{A}$ and so infinitely many n-admissibles in $\mathcal{M}_{\mathrm{II}}$ below β contradicting $\mathcal{M}_{\mathrm{II}}$ being a model of T_n^*. $\square$

We also change the conditions $(R_{::}new)$ to reflect the new theory:

$$(R_{\mathrm{I}new*}):$$

$\mathcal{M}_{\mathrm{I}}$ is an ω-model of a complete consistent extension of T_n^* and
$(\forall \beta \in On^{\mathcal{M}_{\mathrm{I}}} \setminus A_{\mathrm{I}}\ \forall m\ \forall \langle \gamma_1, \ldots, \gamma_m \rangle$ an increasing sequence of n-admissibles $\leq \beta$ in $M_{\mathrm{I}})$
$\{[\exists \gamma \in On^{\mathcal{M}_{\mathrm{I}}}(\forall i \leq m(\gamma \neq \gamma_i)$ and $M_{\mathrm{I}} \vDash \gamma \leq \beta$ is n-admissible$]$ or $[\exists i \leq m(L_{\gamma_i}^{\mathcal{M}_{\mathrm{I}}} = A_{\mathrm{I}})]\}$.

$$(R_{\mathrm{II}new*}):$$

$\mathcal{M}_{\mathrm{II}}$ is an ω-model of a complete consistent extension of T_n^* and
$(\forall \beta \in On^{\mathcal{M}_{\mathrm{II}}} \setminus A_{\mathrm{II}}\ \forall m\ \forall \langle \gamma_1, \ldots, \gamma_m \rangle$ an increasing sequence of n-admissibles $\leq \beta$ in $M_{\mathrm{II}})$
$\{[\exists \gamma \in On^{\mathcal{M}_{\mathrm{II}}}(\forall i \leq m(\gamma \neq \gamma_i)$ and $M_{\mathrm{II}} \vDash \gamma \leq \beta$ is n-admissible$]$ or $[\exists i \leq m(L_{\gamma_i}^{\mathcal{M}_{\mathrm{II}}} = A_{\mathrm{II}})]\}$.

Using the Claims above about quantifier complexity, we can easily verify that saying that $L_{\gamma_i}^{\mathcal{M}_{::}} = A_{::}$ is Π^0_3 and that the conditions $(R_{\mathrm{I}new*})$ and $(R_{\mathrm{II}new*})$ are also Π^0_3.

We now replace the old conditions $(R_{::}new)$ on our list by the $(R_{::new*})$. Because of Lemma 3.3 we can omit the conditions $\mathcal{M}_{\mathrm{I}} \subseteq \mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{II}} \subseteq \mathcal{M}_{\mathrm{I}}$ in the A_1 and A_2 groups, respectively. Otherwise, our new game G_n^* is defined by the same sequence of conditions given in §2.4 except that we are producing a $\Pi^0_{3,n-1}$ game rather than a $\Pi^0_{3,n}$ one:

	win for I	win for II
A_0	$(\neg R_{\mathrm{II}}0 \& \neg R_{\mathrm{II}}1)\&$ $[R_{\mathrm{I}}0 \vee R_{\mathrm{I}}1 \vee$ $\mathcal{M}_{\mathrm{II}} = \mathcal{M}_{\mathrm{I}} \vee$ $(R_{\mathrm{I}}new^*) \vee$ $(R_{\mathrm{I}}3)]$	$\vdots$
A_1		$R_{\mathrm{II}}0 \vee R_{\mathrm{II}}1 \vee$ $(R_{\mathrm{II}}new^*) \vee$ $(R_{\mathrm{II}}3) \vee$ $(R_{\mathrm{II}}4)$
A_2	$(R_{\mathrm{I}}4) \vee$ $(R_{\mathrm{I}}5)$	
A_3		$(R_{\mathrm{II}}5) \vee$ $(R_{\mathrm{II}}6)$
	$\vdots$	
A_{n-2}		$\vdots$

Again, we add on 2^ω as the last entry but here in the A_{n-1} slot. We must verify that these conditions define a game G_n^* as required for Lemma 3.2. We begin with characterizing the relevant situations in which $(R_{::new*})$ holds.

Lemma 3.4. *If $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are distinct ω-models of T_n^* then $(R_{::new*})$ holds if and only if*
(i) $\exists\gamma[(\mathcal{M}_{::} \vDash \gamma$ is n-admissible) and $L_\gamma^{\mathcal{M}_{::}} = \mathcal{A}_{::}]$.

Proof. Suppose (i) holds and consider any β and $\langle \gamma_1, \ldots, \gamma_m \rangle$ as in the definition of $(R_{::new*})$. (There is such a β by Lemma 3.3.) If (i) holds with witness γ, then either γ can be added to the list of γ_i or it is already one of them and so $(R_{::new*})$ holds as required. On the other hand, if $(R_{::new*})$ holds and (i) fails then, by Lemma 3.3, there is a $\beta \in \mathcal{M}_{::} - \mathcal{A}_{::}$ and so by the definition of T_n^* there is an m and a sequence $\langle \gamma_1 \ldots \gamma_m \rangle$ containing all the n-admissibles $\leq \beta$ in $\mathcal{M}_{::}$. For this sequence, $(R_{::new*})$ guarantees that, for some $i \leq m$, $L_\gamma^{\mathcal{M}_{::}} = \mathcal{A}_{::}$ as required. □

We must now verify that these conditions cannot cause a loss for the first player to play $Th_{\alpha_n^*}$ as required. If I plays $Th_{\alpha_n^*}$ and so $\mathcal{M}_{\mathrm{I}} = L_{\alpha_n^*}$, then the first worry is that

($R_{\mathrm{I}new*}$) and all the other clauses of the first group fail but ($R_{\mathrm{II}new*}$) holds. (The two preceding disjuncts in group A_1 clearly contradict $\mathcal{M}_{\mathrm{I}} = L_{\alpha_n^*}$ by asserting that it is not an ω-model of $T_{\alpha_n^*}$.) By the failure of the clauses in A_0, $\mathcal{M}_{\mathrm{II}} \neq \mathcal{M}_{\mathrm{I}}$ and so by Lemma 3.4 there is an n-admissible $\gamma \in \mathcal{M}_{\mathrm{II}}$ such that $L_\gamma^{\mathcal{M}_{\mathrm{II}}} = A_{\mathrm{II}}$. As γ is admissible it is not α_n^* and so is strictly less than α_n^*. In this case we would have $\gamma + 1$ in both $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ and so in $\mathcal{A}$ for a contradiction. Now $R_{\mathrm{II}}3$ cannot hold as it implies that $\mathcal{M}_{\mathrm{I}}$ is not well-founded. The next worry in this case is then that $R_{\mathrm{II}}4$ holds.

We recall that the proofs for the Lemmas of §2.3 apply for theories such as T_n^* whose models are limits of admissibles and games of the sort we have here when considering play for which exactly one of the two models constructed is standard and the two are incomparable. That is precisely our situation at this point in our analysis. Thus we may use all the Lemmas of §2.3. This suffices, for example, for our current case as Lemma 2.17 shows that there are β_1 and β_2 such that $\bigstar_1(\beta_1, \beta_2)$ holds and so $R_{\mathrm{II}}4$ would imply that $\mathcal{M}_{\mathrm{I}}$ is not well-founded for a contradiction.

We now show that if II plays Th_{α_n} (and so $\mathcal{M}_{\mathrm{II}} = L_{\alpha_n^*}$) but I does not, then I does not win by group A_0 being true. The first conjunct in A_0 makes $\mathcal{M}_{\mathrm{I}}$ an ω-model of $T_{\alpha_n^*}$. That $\mathcal{M}_{\mathrm{II}} = L_{\alpha_n^*}$ means that $R_{\mathrm{I}}0$, $R_{\mathrm{I}}1$ and $R_{\mathrm{I}}3$ all fail. By Lemma 3.3, $\mathcal{M}_{\mathrm{II}} \neq \mathcal{M}_{\mathrm{I}}$. Finally, if $R_{\mathrm{I}}new^*$ held then we would contradict $\mathcal{M}_{\mathrm{II}} = L_{\alpha_n^*}$ using Lemma 3.4 as in the previous argument.

Suppose now that our play has passed the A_0 and A_1 groups without a winner having been declared. We are still only concerned with plays in which one player plays $L_{\alpha_n^*}$, the other does not, the two models are incomparable (and so the one not equal to $L_{\alpha_n^*}$ is not well-founded) and both $R_{::}new^*$ fail. The Lemmas from §2 and, in particular their Corollary 2.20, show that if we pass through all the $R_{::}2+i$ for $1 \leq i \leq k$ without declaring a winner (i.e. they all fail), then α is Σ_{k+1} admissible. For $k = n-1$ this would contradict the assumed situation and, in particular, the failure of $R_{::}new^*$ by Lemma 3.4. Thus one player must win the game and there is a first $R_{::}2+i$ which holds. Whichever it is, the associated $\bigstar$ has witnesses by Lemma 2.18 and so the other model is not well-founded. Thus only the player playing $L_{\alpha_n^*}$ can win the game as required.

4 Bibliography

Burgess, John P. [1986], The truth is never simple, *Journal of Symbolic Logic* **51**, 663–681.

Friedman, H. [1971], Higher set theory and mathematical practice, *Annals of Mathematical Logic* **2,** 325-357.

Hodes, H. T. [1980], Jumping through the transfinite: The mastercode hierarchy of Turing degrees, *Journal of Symbolic Logic* **45**, 204-220.

Martin, D. A. [n.d.] *Determinacy*, circulated drafts, about 578 pp.

Montalbán, A. and Shore, R. A. [2012], The limits of determinacy in second order arithmetic, *Proceedings of the London Mathematical Society* (3) (2012) **104**, 223-252.

Shore, R. A. [2010], Reverse Mathematics: the Playground of Logic, *Bulletin of Symbolic Logic* **16**, 378-402.

Simpson, S. G. [2009], *Subsystems of Second Order Arithmetic* 2^{nd} ed., *Perspectives in Logic*, ASL and Cambridge University Press, New York.

Welch, P. D.[2011], Weak systems of determinacy and arithmetical quasi-inductive definitions, *Journal of Symbolic Logic* **76**, 418–436.

Welch, P. D. [2012], $\mathbf{G}_{\delta\sigma}$-games, Isaac Newton Institute Preprint Series No. NI12050-SAS, 2012.