THE MAXIMAL LINEAR EXTENSION THEOREM IN SECOND ORDER ARITHMETIC

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ABSTRACT. We show that the maximal linear extension theorem for well partial orders is equivalent over RCA_0 to ATR_0. Analogously, the maximal chain theorem for well partial orders is equivalent to ATR_0 over RCA_0.

1. INTRODUCTION

A *wpo* (well partial order) is a partial order $(P, \leq_P)$ such that for every infinite sequence (x_i) of elements of P we can find $i < j$ with $x_i \leq_P x_j$. This notion emerged several times in mathematics, as reported in [Kru72].

There are many characterizations of wpo's, supporting the claim that this is indeed a very natural notion. Wpo's are exactly the partial orders such that any nonempty subset has a finite set of minimal elements, or those which are well founded and contain no infinite antichains. For the purpose of this paper, the most important characterization of wpo's is the one stating that a partial order is a wpo if and only if all its linear extensions (see Definition 2.4) are well-orders.

We mention here only two major results about wpo's and wqo's (see below for the distinction between these two notions). Fraïssé's conjecture states that embeddability on countable linear orders is a wqo. Laver's proved this in [Lav71] by establishing a stronger statement using Nash-Williams' notion of better-quasi-order ([NW68]). Robertson and Seymour proved in a long list of papers culminating in [RS04] (see

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[Tho95, §5] for an overview) that the minor relation on finite graphs is a wpo.

The characterization of wpo's in terms of linear extensions leads to the following natural definition.

Definition 1.1. If $\mathcal{P}$ is a wpo, its *maximal order type* $o(\mathcal{P})$ is the supremum of all ordinals which are order types of linear extensions of $\mathcal{P}$.

The following theorem was originally proved by de Jongh and Parikh ([dJP77]).

Theorem 1.2. *If $\mathcal{P}$ is a wpo, the supremum in the definition of $o(\mathcal{P})$ is actually a maximum, i.e. there exist a linear extension of $\mathcal{P}$ with order type $o(\mathcal{P})$. Such a well-order is called a* maximal linear extension *of $\mathcal{P}$.*

An exposition of (essentially) the original proof appears in [Har05, §8.4]. A proof of Theorem 1.2 based on the study of the partial order of the initial segments of $\mathcal{P}$ is included in [Fra00, §4.11]. Kříž and Thomas ([KT90, Theorem 4.7]) and Blass and Gurevich ([BG08, Proposition 52]) gave proofs with a strong set-theoretic flavor.

In any well founded partial order (and in particular in a wpo), one can look at chains (i.e. linear suborderings of the partial order) and give the following definition.

Definition 1.3. If $\mathcal{P}$ is a well founded partial order, its *height* $\chi(\mathcal{P})$ is the supremum of all ordinals which are order types of chains in $\mathcal{P}$.

The following theorem is contained in [KT90, Theorem 4.9]. Kříž and Thomas attribute the result and the proof to Wolk ([Wol67, Theorem 9]), whose statement is actually a bit stronger (see Theorem 6.5 below).

Theorem 1.4. *If $\mathcal{P}$ is a wpo, the supremum in the definition of $\chi(\mathcal{P})$ is actually a maximum, i.e. there exist a chain in $\mathcal{P}$ with order type $\chi(\mathcal{P})$. Such a well-order is called a* maximal chain *in $\mathcal{P}$.*

Wolk's result appears also in Harzheim's book ([Har05, Theorem 8.1.7]). The result was extended to a wider class of well founded partial orders by Schmidt ([Sch81]) in the countable case, and by Milner and Sauer ([MS81]) in general.

In this paper we study Theorems 1.2 and 1.4 from the viewpoint of reverse mathematics. The goal of reverse mathematics is to calibrate the proof-theoretic strength of mathematical statements by establishing the subsystem of second order arithmetic needed for their proof. We refer the reader to [Sim09] for background information on reverse mathematics and the relevant subsystems of second order arithmetic. The weakest subsystem is RCA_0, which consists of the axioms of ordered semi-ring, plus Δ^0_1 comprehension and Σ^0_1 induction. Adding set-existence axioms to RCA_0 we obtain WKL_0, ACA_0, ATR_0, and $\mathbf{\Pi}^1_1$-CA_0, completing the so-called "big five" of reverse mathematics. In this paper we deal with RCA_0, ACA_0, and ATR_0. ACA_0 is obtained by adding to RCA_0 the axiom scheme of arithmetic comprehension, while ATR_0 further extends ACA_0 by allowing transfinite iterations of arithmetic comprehension. ATR_0 implies $\mathbf{\Delta}^1_1$ comprehension ([Sim09, Lemma VIII.4.1]) and hence $\mathbf{\Delta}^1_1$ transfinite induction.

The question of the proof-theoretic strength of Theorem 1.2 was raised by the first author in the Open Problems session of the workshop "Computability, Reverse Mathematics and Combinatorics" held at the Banff International Research Station (Alberta, Canada) in December 2008 (a list of those open problems is available at `http://www.math.cornell.edu/~shore/pape`

Denoting by MLE and MC the formal versions (to be defined precisely in Section 2 below) of Theorems 1.2 and 1.4 we can state the main results of the paper.

Theorem 1.5. *Over* RCA_0*, the following are equivalent:*

(1) ATR_0*;*
(2) MLE*;*
(3) MLE *restricted to disjoint unions of two linear orders.*

Theorem 1.6. *Over* RCA_0*, the following are equivalent:*

(1) ATR_0*;*
(2) MC*;*
(3) MC *restricted to disjoint unions of two linear orders.*

Theorem 1.5 is connected to the following results which are due to Antonio Montalbán ([Mon07]).

Theorem 1.7. *Every computable wpo has a computable maximal linear extension, yet there is no hyperarithmetic way of computing (an index for) a computable maximal linear extension from (an index for) the computable wpo.*

Notice that the first part of Theorem 1.7 does not imply that Theorem 1.2 is true in the ω-model of computable sets (in fact Theorem 1.5 implies that this is not the case), as there exists computable partial orders which are not wpo's but that "look" wpo's in that model. The second part of Theorem 1.7 suggests ATR_0 as a lower bound for the strength of Theorem 1.2. However we are not able to use Montalbán's proof (which assumes Theorem 1.2) in our proof of (2) $\implies$ (1) of Theorem 1.5.

Theorem 1.2 obviously suggests explicitly computing the maximal order types of different wpo's. In [dJP77] de Jongh and Parikh already computed the maximal order type of the wpo investigated by Higman ([Hig52]). Immediately afterwards Schmidt studied maximal order types in her Habilitationsschrift ([Sch79]) and she gave upper bounds for the maximal order types of the wpo's investigated by Kruskal ([Kru60]) and Nash-Williams ([NW65]) (although the latter proof is flawed and apparently has not been fixed yet). Much more recently the first author and Montalbán ([MM09]) computed the maximal order type of the scattered linear orders of finite Hausdorff rank under embeddability.

The use of maximal order types to calibrate the strength of statements about wpo's in reverse mathematics is crucial. Harvey Friedman (see [Sim85]) used the maximal order type of the relevant wpo to prove that Kruskal's theorem cannot be proved in ATR_0. Further extensions of Friedman's method were then used to show that Robertson and Seymour's result about graph minors is not provable in $\mathbf{\Pi}^1_1$-CA_0 ([FRS87]). Steve Simpson ([Sim88]) used the maximal order type (computed by way of "reifications") of certain wpo's to establish the strength of the Hilbert basis theorem. In [MM09] the computation of the maximal order type of the scattered linear orders of finite Hausdorff rank is instrumental in the reverse mathematics results about the restriction of Fraïssé's conjecture to those linear orders.

Let us mention that in the literature the notion of wqo is probably more common than that of wpo. Well quasi orders are defined by applying the definition of wpo given above to a quasi order (i.e. a binary relation which is reflexive and transitive, but not necessarily anti-symmetric). Since a quasi order can always be turned into a partial order by taking the quotient with respect to the equivalence relation induced by the quasi order, there is nothing lost in dealing with wpo's rather than wqo's. Moreover, for the purposes of this paper it is more convenient to deal with partial orders (e.g. the definition of linear extension of a quasi order is more cumbersome).

We now explain the organization of the paper. In Section 2 we detail the formalization of partial and linear orders in subsystems of second order arithmetic and define MLE. In Section 3 we begin the proof of Theorem 1.5 by showing that ATR_0 proves MLE. Our proof of MLE is related to the proof of Theorem 1.2 in [KT90] and in some sense simpler than those of [dJP77] and [Har05]. In Section 4 we start the proof of the reversal by showing that RCA_0 + MLE implies ACA_0. The reversal is completed in Section 5 by arguing in ACA_0 that MLE implies ATR_0. In these two sections MLE is applied only to partial orders which are the disjoint union of two linear orders. In Section 6 we prove Theorem 1.6. To show that ATR_0 proves MC we apply the ideas of Section 3 to chains (the resulting proof is similar to the proof of Theorem 1.4 in [Sch81]), while the reversal (in which MC is applied to a disjoint union of two linear orders) is straightforward.

2. Partial and linear orders in subsystems of second order arithmetic

The formalization of the notion of linear order in subsystems of second order arithmetic is straightforward and can be carried out in RCA_0 (see e.g. [Mar05]). We typically write $\mathcal{L} = (L, \leq_L)$ to denote a linear order defined on the set L with order relation $\leq_L$. The corresponding irreflexive relation is denoted by $<_L$. If $x \in L$ we write $L_{(\leq_L x)} = \{ y \in L \mid y \leq_L x \}$. Similarly, $L_{(\geq_L x)} = \{ y \in L \mid y \geq_L x \}$. If $x, y \in L$ we write $[x, y]_{\mathcal{L}}$ to denote the set $\{ z \in L \mid x \leq_L z \leq_L y \}$. A specific linear order is $\omega = (\mathbb{N}, \leq)$.

In RCA_0 we define *well-orders* as the linear orders which have no descending chains. In [Hir05a] Hirst studied the equivalence between this definition of well-order and other possible (classically equivalent) definitions. An element of a well-order is often identified with the restriction of the well-order to the strict predecessors of the element. For a survey of the provability of results about well-orders in subsystems of second order arithmetic see [Hir05b].

An important relation between linear orders is embeddability: $\mathcal{L}_0$ *embeds* into $\mathcal{L}_1$ (and we write $\mathcal{L}_0 \preceq \mathcal{L}_1$) if there exists an order preserving function (also called an *embedding*) from the domain of $\mathcal{L}_0$ to the domain of $\mathcal{L}_1$. We write $\mathcal{L}_0 \equiv \mathcal{L}_1$ when $\mathcal{L}_0 \preceq \mathcal{L}_1 \preceq \mathcal{L}_0$, and $\mathcal{L}_0 \prec \mathcal{L}_1$ when $\mathcal{L}_0 \preceq \mathcal{L}_1$ and $\mathcal{L}_1 \not\preceq \mathcal{L}_0$. The following Theorem shows that ATR_0 is necessary to show that well orders are comparable under embeddability. (The equivalence between (1) and (2) is proved in [FH90], while the equivalence between (1) and (3) was obtained in [Sho93].)

Theorem 2.1. *Over* RCA_0*, the following are equivalent:*

1. ATR_0*;*
2. *if $\mathcal{L}_0$ and $\mathcal{L}_1$ are well-orders then either $\mathcal{L}_0 \preceq \mathcal{L}_1$ or $\mathcal{L}_1 \preceq \mathcal{L}_0$;*
3. *if for every n $\mathcal{L}_n$ is a well-order then there exist $i \neq j$ such that $\mathcal{L}_i \preceq \mathcal{L}_j$.*

An immediate, yet very useful, consequence of comparability of well-orders is the following.

Corollary 2.2. *In* ATR_0 *if* $\mathcal{L}_0$ *and* $\mathcal{L}_1$ *are well-orders the formulas* $\mathcal{L}_0 \preceq \mathcal{L}_1$*,* $\mathcal{L}_0 \prec \mathcal{L}_1$ *and* $\mathcal{L}_0 \equiv \mathcal{L}_1$ *are* $\mathbf{\Delta}^1_1$*.*

Proof. The formula $\mathcal{L}_0 \preceq \mathcal{L}_1$ is clearly $\mathbf{\Sigma}^1_1$. In ATR_0, by Theorem 2.1, if $\mathcal{L}_0$ and $\mathcal{L}_1$ are well-orders $\mathcal{L}_0 \preceq \mathcal{L}_1$ is equivalent to $\mathcal{L}_1 + 1 \not\preceq \mathcal{L}_0$. The latter formula is clearly $\mathbf{\Pi}^1_1$, and hence $\mathcal{L}_0 \preceq \mathcal{L}_1$ is $\mathbf{\Delta}^1_1$.

¿From this and the definitions it follows that $\mathcal{L}_0 \prec \mathcal{L}_1$ and $\mathcal{L}_0 \equiv \mathcal{L}_1$ are also $\mathbf{\Delta}^1_1$. □

In RCA_0 we can define basic operations on linear orders. Suppose $\mathcal{L}_n = (L_n, \leq_{L_n})$ is a linear order for every n. We may also assume that the L_n's are pairwise disjoint. Then we define the linear order $\mathcal{L}_0 + \mathcal{L}_1 = (L_0 \cup L_1, \leq_{L_0+L_1})$ by setting $x \leq_{L_0+L_1} y$ if and only if $x \in L_0$ and $y \in L_1$ or $x, y \in L_n$ and $x \leq_{L_n} y$ for some $n < 2$. The infinitary generalization of this operation $\sum_n \mathcal{L}_n = (\bigcup_n L_n, \leq_{\sum_n L_n})$ is defined similarly. The linear order $\mathcal{L}_0 \cdot \mathcal{L}_1 = (L_0 \times L_1, \leq_{L_0 \cdot L_1})$ is defined by $(x_0, x_1) \leq_{L_0 \cdot L_1} (y_0, y_1)$ iff either $x_1 <_{L_1} y_1$ or $x_1 = y_1$ and $x_0 \leq_{L_0} y_0$. RCA_0 proves that if the $\mathcal{L}_n$'s are well-orders then $\mathcal{L}_0 + \mathcal{L}_1$, $\sum_n \mathcal{L}_n$ and $\mathcal{L}_0 \cdot \mathcal{L}_1$ are also well-orders.

In RCA_0 we can also define the exponentiation ${\mathcal{L}_0}^{\mathcal{L}_1}$ of two linear orders (details are e.g. in [Hir05b]). However RCA_0 cannot prove that when $\mathcal{L}_0$ and $\mathcal{L}_1$ are well-orders ${\mathcal{L}_0}^{\mathcal{L}_1}$ is a well-order. In fact this statement is equivalent to ACA_0 over RCA_0 ([Gir87, p. 299], see [Hir94] for a direct proof).

Using ordinal exponentiation we can define Cantor normal forms, and Jeff Hirst ([Hir94, Theorem 5.2]) proved the following:

Theorem 2.3. *Over* RCA_0*, the following are equivalent:*

(1) ATR_0*;*
(2) *every well order has a* Cantor normal form*, i.e. it is equivalent to a finite sum of exponentials with base* ω *and nonincreasing exponents.*

We now turn to partial orders, which are formalized in a way similar to linear orders. We typically write $\mathcal{P} = (P, \leq_P)$ for a partial order defined on the set P with order relation $\leq_P$. If $\mathcal{P}_0$ and $\mathcal{P}_1$ are partial orders with disjoint domains, $\mathcal{P}_0 + \mathcal{P}_1$ is defined in RCA_0 as in the case of linear orders. We also define the disjoint union $\mathcal{P}_0 \oplus \mathcal{P}_1 = (P_0 \cup P_1, \leq_{P_0 \oplus P_1})$ by setting $x \leq_{P_0 \oplus P_1} y$ if and only if $x, y \in P_n$ and $x \leq_{P_n} y$ for some $n < 2$.

Definition 2.4. Within RCA_0, if $\mathcal{P} = (P, \leq_P)$ is a partial order, *a linear extension of* $\mathcal{P}$ is a linear order $\mathcal{L} = (P, \leq_L)$ such that $x \leq_P y$ implies $x \leq_L y$ for every $x, y \in P$. We denote by $\mathrm{Lin}(\mathcal{P})$ the class of all linear extensions of $\mathcal{P}$ (this is just a convenient shorthand: $\mathrm{Lin}(\mathcal{P})$ does not exist in second order arithmetic).

We will often deal with linear extensions of partial orders which are the disjoint sum of two linear orders.

Definition 2.5. For $\mathcal{I}$ and $\mathcal{J}$ linear orders, we call any element of $\mathrm{Lin}(\mathcal{I} \oplus \mathcal{J})$ a *shuffle of $\mathcal{I}$ and $\mathcal{J}$*.

Now we can formally define the notion of wpo in RCA_0.

Definition 2.6. Within RCA_0, a partial order $\mathcal{P} = (P, \leq_P)$ is a *wpo* if for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leq_P f(j)$.

The different characterizations of wpo have been studied from the viewpoint of reverse mathematics in [Mar05, CMS04]: it turns out that not all equivalences are provable in RCA_0, but that WKL_0 augmented with the chain-antichain principle CAC (i.e. the statement that every infinite partial order has either an infinite chain or an infinite antichain) suffices. (Thus all definitions of wpo are equivalent in, say, ACA_0). In particular we have the following results ([CMS04, Lemma 3.12, Theorem 3.17, Corollary 3.4]).

Lemma 2.7. RCA_0 *proves that every linear extension of a wpo is a well-order.* WKL_0 *proves that if a partial order is such that all its linear extensions are well-orders, then it is a wpo.*

Lemma 2.8. RCA_0 *plus* CAC *(and, a fortiori,* ACA_0*) proves that if $\mathcal{P}$ is a wpo then for every $f : \mathbb{N} \to P$ there exists an infinite $A \subseteq \mathbb{N}$ such that for all $i, j \in A$ with $i < j$ we have $f(i) \leq_P f(j)$.*

We need to make the last statement effective, but for our purposes it suffices to be quite coarse in this effectivization (e.g. we do not use the results of [CJS01] or [HS07]).

Lemma 2.9. ACA_0 *proves that there exists a construction which is uniformly recursive in the double jump of the input and that starting from the wpo $\mathcal{P}$ and $f : \mathbb{N} \to P$ outputs an infinite $A \subseteq \mathbb{N}$ such that for all $i, j \in A$ with $i < j$ we have $f(i) \leq_P f(j)$.*

Proof. Lemma 2.8 is proved using CAC, which is a consequence of Ramsey theorem for pairs. An inspection of the proof of Ramsey Theorem in ACA_0 ([Sim09, Lemma III.7.4]) shows that a homogenous set for a coloring of pairs is computable from any branch in an infinite finitely branching tree which is computable in the coloring. Such a branch is computable in the double jump of the tree. $\square$

We need to formalize Theorem 1.2 within RCA_0. Let $\mathcal{P}$ be a wpo. ¿From [Sim09, Theorem V.6.9] it follows that ATR_0 proves the existence of a well-order $\mathcal{Q}$ such that $\mathcal{R} \preceq \mathcal{Q}$ for all $\mathcal{R} \in \mathrm{Lin}(\mathcal{P})$. From this, in $\mathbf{\Pi}^1_1$-CA_0 we can define $o(\mathcal{P}) = \sup(\mathrm{Lin}(\mathcal{P}))$ as an element of $\mathcal{Q}$. In systems below $\mathbf{\Pi}^1_1$-CA_0 (including ATR_0) it is not clear that we can define $o(\mathcal{P})$ in this way. Therefore we need to state Theorem 1.2 without mentioning $o(\mathcal{P})$. Since the theorem states that wpo's have maximal linear extensions, the following is a natural translation.

Definition 2.10. Within RCA_0 we denote by MLE the following statement: *every wpo $\mathcal{P}$ has a linear extension $\mathcal{Q}$ such that $\mathcal{R} \preceq \mathcal{Q}$ for all $\mathcal{R} \in \mathrm{Lin}(\mathcal{P})$.*

We refer to such a $\mathcal{Q}$ as *a maximal linear extension of $\mathcal{P}$.*

Following the ideas which led to MLE, we now formalize Theorem 1.4.

Definition 2.11. Within RCA_0, if $\mathcal{P} = (P, \leq_P)$ is a partial order, *a chain in $\mathcal{P}$* is a linear order $\mathcal{C} = (C, \leq_P)$ where $C \subseteq P$. We denote by $\mathrm{Ch}(\mathcal{P})$ the class of all chains $\mathcal{P}$ (again, this is just a convenient shorthand).

Definition 2.12. Within RCA_0 we denote by MC the following statement: *every wpo $\mathcal{P}$ has a chain $\mathcal{C}$ such that $\mathcal{C}' \preceq \mathcal{C}$ for all $\mathcal{C}' \in \mathrm{Ch}(\mathcal{P})$.*

We refer to such a $\mathcal{C}$ as *a maximal chain in $\mathcal{P}$.*

3. ATR_0 proves MLE

Before starting with the proof, let us mention that the proofs of Theorem 1.2 in [dJP77], [Har05], and [KT90], when translated into the language of second order arithmetic, require at least $\mathbf{\Sigma}^1_1$ induction, which is not available in ATR_0. The proof of Theorem 1.2 in [Fra00] uses a partial order of sets, and thus cannot be immediately reproduced in second order arithmetic.

We need some preliminaries, starting with the following important tool in the study of wpo's. (Our notation for finite sequences follows [Sim09, Definition II.2.6], although we use Greek letters to denote sequences.)

Definition 3.1. In RCA_0 we define, for a partial order $\mathcal{P} = (P, \leq_P)$, the *tree of bad sequences of $\mathcal{P}$*:

$$\mathrm{Bad}(\mathcal{P}) = \{\, \sigma \in \mathbb{N}^{<\mathbb{N}} \mid (\forall i < \mathrm{lh}(\sigma))(\sigma(i) \in P \wedge (\forall j < i)\, \sigma(j) \not\leq_P \sigma(i)) \,\}.$$

Notice that $\mathcal{P}$ is a wpo if and only if $\mathrm{Bad}(\mathcal{P})$ is *well founded* (i.e. does not have infinite branches). Thus if $\mathcal{P}$ is a wpo we can define by transfinite recursion the *rank function* on $\mathrm{Bad}(\mathcal{P})$ (taking ordinals as values), which we denote by $\mathrm{rk}_{\mathcal{P}}$, by setting

$$\mathrm{rk}_{\mathcal{P}}(\sigma) = \sup\{\, \mathrm{rk}_{\mathcal{P}}(\sigma^\frown\langle x\rangle) + 1 \mid \sigma^\frown\langle x\rangle \in \mathrm{Bad}(\mathcal{P}) \,\},$$

and define the ordinal $\mathrm{rk}(\mathcal{P}) = \mathrm{rk}_{\mathcal{P}}(\emptyset)$ (where $\emptyset$ denotes the sequence of length 0), so that $\mathrm{rk}_{\mathcal{P}} : \mathrm{Bad}(\mathcal{P}) \to \mathrm{rk}(\mathcal{P}) + 1$.

Using transfinite recursion we can mimic this definition in ATR_0 (where ordinals are represented by well-orders), thus obtaining a well-order $\mathrm{rk}(\mathcal{P})$ and a function $\mathrm{rk}_{\mathcal{P}} : \mathrm{Bad}(\mathcal{P}) \to \mathrm{rk}(\mathcal{P}) + 1$.

Definition 3.2. In RCA_0 we define, for a partial order $\mathcal{P} = (P, \leq_P)$ and $\sigma \in \mathrm{Bad}(\mathcal{P})$, $P_\sigma = \{\, p \in P \mid (\forall i < \mathrm{lh}(\sigma))\, \sigma(i) \not\leq_P p \,\}$. We also write $\mathcal{P}_\sigma = (P_\sigma, \leq_P)$.

Notice that $P_\sigma = \{\, p \in P \mid \sigma^\frown\langle p\rangle \in \mathrm{Bad}(\mathcal{P}) \,\}$. Actually, for every sequence τ we have $\tau \in \mathrm{Bad}(\mathcal{P}_\sigma)$ if and only if $\sigma^\frown\tau \in \mathrm{Bad}(\mathcal{P})$. ¿From this it follows that $\mathrm{rk}_{\mathcal{P}}(\sigma) = \mathrm{rk}(\mathcal{P}_\sigma)$. Notice also that $\mathcal{P} = \mathcal{P}_\emptyset$.

Lemma 3.3. ATR_0 *proves that if $\mathcal{L}$ is a well-order then $\mathcal{L} \equiv \mathrm{rk}(\mathcal{L})$.*

Proof. By transfinite induction on $\mathrm{rk}_{\mathcal{L}}(\sigma)$ for $\sigma \in \mathrm{Bad}(\mathcal{L})$ show that $\mathcal{L}_\sigma \equiv \mathrm{rk}_{\mathcal{L}}(\sigma)$. This formula is $\mathbf{\Delta}^1_1$ in ATR_0 by Corollary 2.2. So we can carry out the induction in ATR_0. All cases of the induction are immediate. □

Lemma 3.4. ATR_0 *proves that if $\mathcal{P}$ is a wpo and $\mathcal{L} \in \mathrm{Lin}(\mathcal{P})$ then $\mathcal{L} \preceq \mathrm{rk}(\mathcal{P})$.*

Proof. By Lemma 2.7 $\mathcal{L}$ is a well-order. Notice that $\mathrm{Bad}(\mathcal{L})$ is a subtree of $\mathrm{Bad}(\mathcal{P})$ and obviously $\mathrm{rk}_{\mathcal{L}}(\sigma) \preceq \mathrm{rk}_{\mathcal{P}}(\sigma)$ for every $\sigma \in \mathrm{Bad}(\mathcal{L})$, so that $\mathrm{rk}(\mathcal{L}) \preceq \mathrm{rk}(\mathcal{P})$. Therefore, using the previous Lemma, $\mathcal{L} \preceq \mathrm{rk}(\mathcal{P})$. □

Notice that the above result implies that if $\mathcal{P}$ is a computable wpo then $o(\mathcal{P})$ is a computable ordinal, as it is at most $\mathrm{rk}(\mathcal{P})$ and $\mathrm{Bad}(\mathcal{P})$ is a computable tree. This formally answers a question of [Sch79], but a real answer and much more information is provided by Montalbán in Theorem 1.7.

Lemma 3.4 suggests our strategy for proving MLE within ATR_0: define, for each wpo $\mathcal{P}$, an $\mathcal{L} \in \mathrm{Lin}(\mathcal{P})$ such that $\mathrm{rk}(\mathcal{P}) \preceq \mathcal{L}$ (so that actually $\mathcal{L} \equiv \mathrm{rk}(\mathcal{P})$).

Our last preliminary result (Lemma 3.7 below) shows that ATR_0 proves a special case of MLE, and indeed computes the maximal order type of a linear extension of the disjoint union of two well-orders. (In Lemma 5.3 we will obtain a much weaker result in ACA_0.)

Before stating the Lemma, we need to adapt the definition of natural (also called Hessenberg, or commutative) sum of ordinals to well-orders.

By Theorem 2.3 ATR_0 proves that every well-order has a Cantor Normal Form: this is what is needed for the definition of natural sum.

Definition 3.5. In ATR_0, suppose $\mathcal{I} \equiv \sum_{i \leq m} \omega^{\mathcal{K}_i}$ and $\mathcal{J} \equiv \sum_{j \leq n} \omega^{\mathcal{L}_j}$ are well-orders with $\mathcal{K}_{i+1} \preceq \mathcal{K}_i$ and $\mathcal{L}_{j+1} \preceq \mathcal{L}_j$ for $i < m$ and $j < n$. Order the set $\{ \mathcal{K}_i \mid i \leq m \} \cup \{ \mathcal{L}_j \mid j \leq n \}$ as $\{ \mathcal{M}_k \mid k \leq m+n \}$ so that $\mathcal{M}_{k+1} \preceq \mathcal{M}_k$ for $k < m+n$. Then we let $\mathcal{I} \# \mathcal{J}$ be $\sum_{k \leq m+n} \omega^{\mathcal{M}_k}$.

The precise definition of the well-order $\mathcal{I} \# \mathcal{J}$ obviously depends on the well-orders $\mathcal{K}_i$ and $\mathcal{L}_j$ used in the Cantor Normal Forms of $\mathcal{I}$ and $\mathcal{J}$. It is therefore to be considered as a definition "up to equivalence". Notice that $\#$ is obviously commutative. The following Lemma states another basic property of the natural sum.

Lemma 3.6. ATR_0 *proves that if* $\mathcal{I}$ *and* $\mathcal{J}$ *are well-orders and* $\mathcal{I} \prec \mathcal{I}'$ *then* $\mathcal{I} \# \mathcal{J} \prec \mathcal{I}' \# \mathcal{J}$*.*

Lemma 3.7. ATR_0 *proves that if* $\mathcal{I}$ *and* $\mathcal{J}$ *are well-orders there exists* $\mathcal{Q} \equiv \mathcal{I} \# \mathcal{J}$ *which is a maximal shuffle of* $\mathcal{I}$ *and* $\mathcal{J}$ *(i.e.* $\mathcal{Q}$ *is a maximal linear extension of* $\mathcal{I} \oplus \mathcal{J}$*).*

Proof. Let $\mathcal{P} = \mathcal{I} \oplus \mathcal{J}$. It is easy to define $\mathcal{Q} \in \mathrm{Lin}(\mathcal{P})$ with $\mathcal{Q} \equiv \mathcal{I} \# \mathcal{J}$: using the notation of the previous definition, elements of $I \cup J$ are identified in the obvious way with elements of $\sum_{k \leq m+n} \omega^{\mathcal{M}_k}$.

To prove that $\mathcal{Q}$ is a maximal linear extension of $\mathcal{P}$, by Lemma 3.4, it suffices to show that $\mathrm{rk}(\mathcal{P}) \preceq \mathcal{I} \# \mathcal{J}$.

For $\sigma \in \mathrm{Bad}(\mathcal{P})$ we let $I_\sigma = P_\sigma \cap I$ and $J_\sigma = P_\sigma \cap J$ and denote by $\mathcal{I}_\sigma$ and $\mathcal{J}_\sigma$ the corresponding linear orders. We use transfinite induction on $\mathrm{rk}_{\mathcal{P}}(\sigma)$ to prove that $\mathrm{rk}_{\mathcal{P}}(\sigma) \preceq \mathcal{I}_\sigma \# \mathcal{J}_\sigma$ for every $\sigma \in \mathrm{Bad}(\mathcal{P})$ (this is again a $\mathbf{\Delta}^1_1$ transfinite induction in ATR_0). Fix $\sigma \in \mathrm{Bad}(\mathcal{P})$. For every $p \in I_\sigma$ and $q \in I_{\sigma^\frown\langle p \rangle}$ we have $q <_{\mathcal{I}} p$, and thus $\mathcal{I}_{\sigma^\frown\langle p \rangle} \prec \mathcal{I}_\sigma$. In this case we also have $\mathcal{J}_{\sigma^\frown\langle p \rangle} = \mathcal{J}_\sigma$. When $p \in J_\sigma$ the situation is symmetric. Thus, for every $p \in P_\sigma$, either $\mathcal{I}_{\sigma^\frown\langle p \rangle} \prec \mathcal{I}_\sigma$ and $\mathcal{J}_{\sigma^\frown\langle p \rangle} = \mathcal{J}_\sigma$, or $\mathcal{J}_{\sigma^\frown\langle p \rangle} \prec \mathcal{J}_\sigma$ and $\mathcal{I}_{\sigma^\frown\langle p \rangle} = \mathcal{I}_\sigma$. In both cases, by Lemma 3.6, we have $\mathcal{I}_{\sigma^\frown\langle p \rangle} \# \mathcal{J}_{\sigma^\frown\langle p \rangle} \prec \mathcal{I}_\sigma \# \mathcal{J}_\sigma$, i.e. $(\mathcal{I}_{\sigma^\frown\langle p \rangle} \# \mathcal{J}_{\sigma^\frown\langle p \rangle}) + 1 \preceq \mathcal{I}_\sigma \# \mathcal{J}_\sigma$. Thus, using the induction hypothesis,

$$\begin{aligned}
\mathrm{rk}_{\mathcal{P}}(\sigma) &= \sup\{ \mathrm{rk}_{\mathcal{P}}(\sigma^\frown\langle p \rangle) + 1 \mid p \in P_\sigma \} \\
&\preceq \sup\{ (\mathcal{I}_{\sigma^\frown\langle p \rangle} \# \mathcal{J}_{\sigma^\frown\langle p \rangle}) + 1 \mid p \in P_\sigma \} \\
&\preceq \mathcal{I}_\sigma \# \mathcal{J}_\sigma.
\end{aligned}$$

When $\sigma = \emptyset$ we have $\mathrm{rk}_{\mathcal{P}}(\emptyset) \preceq \mathcal{I} \# \mathcal{J}$ and thus $\mathrm{rk}(\mathcal{P}) \preceq \mathcal{I} \# \mathcal{J}$. □

We can now prove the main result of this section.

Theorem 3.8. ATR_0 *proves* MLE*.*

Proof. Let $\mathcal{P} = (P, \leq_P)$ be a wpo. Using arithmetical transfinite recursion on rank we will define, for each $\sigma \in \mathrm{Bad}(\mathcal{P})$, a linear order $\mathcal{L}_\sigma$. We will then prove by $\mathbf{\Delta}^1_1$ transfinite induction on rank that $\mathcal{L}_\sigma \in \mathrm{Lin}(\mathcal{P}_\sigma)$ and $\mathrm{rk}_\mathcal{P}(\sigma) \preceq \mathcal{L}_\sigma$. Since $\mathcal{P}_\emptyset = \mathcal{P}$, we have $\mathcal{L}_\emptyset \in \mathrm{Lin}(\mathcal{P})$ and $\mathrm{rk}(\mathcal{P}) \preceq \mathcal{L}_\emptyset$. By Lemma 3.4, $\mathcal{L}_\emptyset$ is a maximal linear extension of $\mathcal{P}$ and the proof is complete.

To define the $\mathcal{L}_\sigma$'s we need some preliminaries. Let

$$S = \{\, \sigma \in \mathrm{Bad}(\mathcal{P}) \mid \mathrm{rk}_\mathcal{P}(\sigma) \text{ is a successor} \,\} \quad \text{and}$$
$$L = \{\, \sigma \in \mathrm{Bad}(\mathcal{P}) \mid \mathrm{rk}_\mathcal{P}(\sigma) \text{ is a limit} \,\}.$$

In ATR_0 we can define a function $p : S \to P$ such that $p(\sigma) \in P_\sigma$ and $\mathrm{rk}_\mathcal{P}(\sigma) = \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle p(\sigma)\rangle) + 1$ for every $\sigma \in S$. We also need, for every $\sigma \in L$, a sequence $\langle x_i \rangle$ of elements of P_σ such that $\mathrm{rk}_\mathcal{P}(\sigma) = \sup\{\, \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle x_i\rangle) \mid i \in \mathbb{N} \,\}$. However we want $\langle x_i \rangle$ to enjoy further properties, so we are going to describe its construction in detail.

Fix $\sigma \in L$ and suppose $\mathrm{rk}_\mathcal{P}(\sigma) = \lambda = \sum_{k \leq m} \omega^{\alpha_k}$ with $\alpha_k \geq \alpha_{k+1} > 0$ for every $k < m$. Let $\gamma = \sum_{k<m} \omega^{\alpha_k}$ and look at α_m. If α_m is a successor $\beta + 1$ let $\beta_n = \beta$ for every n. If α_m is a limit, we can compute (from the realization of α_m as a concrete well-order) a sequence (β_n) such that $\beta_n < \beta_{n+1}$ and $\alpha_m = \sup\{\, \beta_n \mid n \in \mathbb{N} \,\}$. In both cases let $\lambda_n = \gamma + \sum_{j<n} \omega^{\beta_j}$, so that $\lambda = \sup\{\, \lambda_n \mid n \in \mathbb{N} \,\}$. Notice also that $\lambda = \gamma + \sum_{i \in \mathbb{N}} \omega^{\beta_{n_i}}$ for any infinite increasing sequence (n_i). We can define by recursion infinite sequences (x_i) and (n_i) such that for all i

(1) $x_i \in P_\sigma$,
(2) $n_i < n_{i+1}$,
(3) $\lambda_{n_i} \leq \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle x_i\rangle) < \lambda_{n_i+1}$.

Lemma 2.8 implies that we can refine the sequence $\langle x_i \rangle$ so that for all i we also have

(4) $x_i \leq_P x_{i+1}$.

Notice that in fact $x_i \neq x_{i+1}$ and hence $x_i <_P x_{i+1}$ and $P_{\sigma^\frown\langle x_i\rangle} \subsetneq P_{\sigma^\frown\langle x_{i+1}\rangle}$ hold.

In the preceding paragraph we showed that for every $\sigma \in L$ there exist the well-orders α_k's representing $\mathrm{rk}_\mathcal{P}(\sigma)$ in Cantor normal form, the sequence (β_n) obtained from α_m (which we use to define the λ_n's), and sequences (x_i) and (n_i) satisfying conditions (1)–(4) above. Using $\mathbf{\Sigma}^1_1$-AC_0, which is provable in ATR_0, we can associate to every $\sigma \in L$ objects satisfying these conditions, which will be used in the definition of $\mathcal{L}_\sigma$.

Before going on, we notice some further properties of the x_i's. First, we have $\gamma = \lambda_0 \leq \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle x_0\rangle)$.

We claim also that $P_\sigma = \bigcup_{i\in\mathbb{N}} P_{\sigma^\frown\langle x_i\rangle}$. In fact if $y \in P_\sigma$ is such that $y \notin P_{\sigma^\frown\langle x_i\rangle}$ for all i, we have $x_i <_P y$ for all i (if $y = x_i$ then $y \in P_{\sigma^\frown\langle x_{i+1}\rangle}$, as $x_i <_P x_{i+1}$). Then $\sigma^\frown\langle y, x_i\rangle \in \mathrm{Bad}(\mathcal{P})$ and $\mathrm{rk}_\mathcal{P}(\sigma^\frown\langle y, x_i\rangle) = \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle x_i\rangle)$ for every i (since $P_{\sigma^\frown\langle y,x_i\rangle} = P_{\sigma^\frown\langle x_i\rangle}$). Therefore $\mathrm{rk}_\mathcal{P}(\sigma^\frown\langle y\rangle) \geq \sup \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle x_i\rangle) = \mathrm{rk}_\mathcal{P}(\sigma)$, which is impossible.

We also let $Q_0 = P_{\sigma^\frown\langle x_0\rangle}$ and $Q_{i+1} = P_{\sigma^\frown\langle x_{i+1}\rangle} \setminus P_{\sigma^\frown\langle x_i\rangle}$. Notice that $P_\sigma = \bigcup_{i\in\mathbb{N}} Q_i$ follows from $P_\sigma = \bigcup_{i\in\mathbb{N}} P_{\sigma^\frown\langle x_i\rangle}$.

We can now define by transfinite recursion the function $\sigma \mapsto \mathcal{L}_\sigma$. When $\mathrm{rk}_\mathcal{P}(\sigma) = 0$ we let $\mathcal{L}_\sigma$ be the empty well-order. When $\sigma \in S$ let $\mathcal{L}_\sigma = \mathcal{L}_{\sigma^\frown\langle p(\sigma)\rangle} + \{p(\sigma)\}$. If $\sigma \in L$ we let

$$\mathcal{L}_\sigma = \sum_{i\in\mathbb{N}} \left(\mathcal{L}_{\sigma^\frown\langle x_i\rangle} \restriction Q_i\right).$$

Here, of course, we are using the x_i's (and hence the resulting Q_i's) fixed in correspondence with σ before the recursion started.

Now we prove by $\mathbf{\Delta}^1_1$ transfinite induction on rank that $\mathcal{L}_\sigma \in \mathrm{Lin}(\mathcal{P}_\sigma)$ and that $\mathrm{rk}_\mathcal{P}(\sigma) \preceq \mathcal{L}_\sigma$ for all $\sigma \in \mathrm{Bad}(\mathcal{P})$.

When $\mathrm{rk}_\mathcal{P}(\sigma) = 0$ we have $P_\sigma = \emptyset$ and the proof is immediate.

When $\sigma \in S$ let $\tau = \sigma^\frown\langle p(\sigma)\rangle$ and recall that $\mathrm{rk}_\mathcal{P}(\sigma) = \mathrm{rk}_\mathcal{P}(\tau) + 1$. First notice that $P_\sigma = P_\tau \cup \{p(\sigma)\}$. In fact, one inclusion is obvious. For the other, observe that if $p' \in P_\sigma \setminus (P_\tau \cup \{p(\sigma)\})$ then $p(\sigma) <_P p'$ and $\tau' = \sigma^\frown\langle p', p(\sigma)\rangle \in \mathrm{Bad}(\mathcal{P})$. Moreover $P_{\tau'} = P_\tau$ and $\mathrm{rk}_\mathcal{P}(\tau') = \mathrm{rk}_\mathcal{P}(\tau)$, which is impossible because $\mathrm{rk}_\mathcal{P}(\sigma) \geq \mathrm{rk}_\mathcal{P}(\tau') + 2$. By the induction hypothesis $\mathcal{L}_\tau \in \mathrm{Lin}(\mathcal{P}_\tau)$ and $\mathrm{rk}_\mathcal{P}(\tau) \preceq \mathcal{L}_\tau$. It is clear that $\mathcal{L}_\sigma = \mathcal{L}_\tau + \{p(\sigma)\}$ is a linear extension of $\mathcal{P}_\sigma$ (if $q \in P_\tau$ then $p(\sigma) \leq_P q$ is impossible) and that $\mathrm{rk}_\mathcal{P}(\sigma) \preceq \mathcal{L}_\sigma$.

When $\sigma \in L$ let γ, (β_n), (λ_n), (x_i), (n_i), and (Q_i) be the objects fixed in correspondence with σ. To simplify the notation we write $\mathcal{Q}_i$ in place of $\mathcal{L}_{\sigma^\frown\langle x_i\rangle} \restriction Q_i$. Notice that since $\mathcal{Q}_0 = \mathcal{L}_{\sigma^\frown\langle x_0\rangle}$ the induction hypothesis implies $\gamma \preceq \mathcal{Q}_0$.

We now claim that $\omega^{\beta_{n_i}} \preceq \mathcal{Q}_{i+1}$. If this is not the case then we have $\mathcal{Q}_{i+1} \prec \omega^{\beta_{n_i}}$. Notice that, by Lemma 3.4, $\mathcal{L}_{\sigma^\frown\langle x_{i+1}\rangle} \restriction P_{\sigma^\frown\langle x_i\rangle} \preceq \mathrm{rk}_\mathcal{P}(\sigma^\frown\langle x_i\rangle) \equiv \lambda_{n_i} + \alpha$ for some $\alpha < \omega^{\beta_{n_i}}$. Since $\mathcal{L}_{\sigma^\frown\langle x_{i+1}\rangle}$ is a shuffle of $\mathcal{L}_{\sigma^\frown\langle x_{i+1}\rangle} \restriction P_{\sigma^\frown\langle x_i\rangle}$ and $\mathcal{Q}_{i+1}$, by Lemma 3.7 we would have

$$\begin{aligned}\mathcal{L}_{\sigma^\frown\langle x_{i+1}\rangle} &\preceq (\mathcal{L}_{\sigma^\frown\langle x_{i+1}\rangle} \restriction P_{\sigma^\frown\langle x_i\rangle}) \mathbin{\#} \mathcal{Q}_{i+1} \\ &\equiv (\lambda_{n_i} + \alpha) \mathbin{\#} \mathcal{Q}_{i+1} \\ &\prec \lambda_{n_i} + \omega^{\beta_{n_i}} \equiv \lambda_{n_i+1} \leq \lambda_{n_{i+1}}.\end{aligned}$$

On the other hand the induction hypothesis implies that $\lambda_{n_{i+1}} \leq \mathrm{rk}_{\mathcal{P}}(\sigma^\frown\langle x_{i+1}\rangle) \preceq \mathcal{L}_{\sigma^\frown\langle x_{i+1}\rangle}$. The contradiction establishes the claim.

Then

$$\lambda = \gamma + \sum_{i\in\mathbb{N}} \omega^{\beta_{n_i}} \preceq \mathcal{L}_\sigma.$$

To check that $\mathcal{L}_\sigma \in \mathrm{Lin}(\mathcal{P}_\sigma)$ recall that $P_\sigma = \bigcup_{i\in\mathbb{N}} Q_i$ and notice that when $x \in Q_i$ and $y \in Q_{j+1}$ with $i \leq j$ we have $x_i \nleq_P x$ and $x_i \leq_P x_j \leq_P y$, which imply $y \nleq_P x$. □

We can also prove MLE in ATR_0 using ideas from Montalbán's proof of the first part of Theorem 1.7. Many modifications are needed, since Montalbán did assume Theorem 1.2. This alternative proof is more complex than the one above, and we have not included it in this paper.

If instead one begins the proof of Theorem 3.8 with the tree of bad sequences with each node labeled with the Cantor normal forms of its rank (in a unified recursive notation system), then the only noneffective step in the transfinite recursion needed for the construction is the extraction of the subsequence to satisfy condition (4) from the sequence satisfying (1)–(3). This step can easily be done computably in the double jump of this labeled tree by Lemma 2.9. Thus relative to the double jump of the labeled tree of bad sequences, the entire construction can be seen as an effective transfinite recursion. This procedure thus provides a uniform construction of a maximal linear extension computable in the double jump of the assignments of ranks and the corresponding Cantor normal forms to the nodes of the tree. So one can compute the level of the hyperarithmetic hierarchy at which one has a uniformly recursive construction of a maximal linear extension. This contrasts with Montalbán's result in 1.7 that while there is always a recursive maximal linear extension, it cannot be computed uniformly even hyperarithmetically.

After we had essentially the proof presented above of MLE in ATR_0 (in its effective form), Harvey Friedman (in response to a lecture given by the second author on some of the material in this paper) informed us that he had a proof of this result using the tree of bad sequences in some handwritten notes that also contained many calculations of the ranks of such trees for many specific partial orders. He dates these notes probably to 1984. We have not seen his proof and do not know if it is the same or different from the one we presented here.

4. MLE implies ACA_0

The first part of Theorem 1.7 suggests that to exploit the strength of MLE within RCA_0 we need to use partial orders which RCA_0 cannot recognize as not being wpo's. Such a partial order will be defined using the linear order supplied by Lemma 4.2. Before stating it, we recall the following definitions from [HS07].

Definition 4.1. Within RCA_0 we say that a linear order $\mathcal{L} = (L, \leq_L)$ *has order type* ω if L is infinite and each element of L has finitely many $\leq_L$-predecessors.

$\mathcal{L}$ has *has order type* $\omega + \omega^*$ if each element of L has either finitely many $\leq_L$-predecessors or finitely many $\leq_L$-successors, and there are infinitely many elements of both types.

The existence of a linear order satisfying the first two conditions of the following lemma is folklore.

Lemma 4.2. RCA_0 *proves that there exists a (computable) linear order* $\mathcal{L} = (L, \leq_L)$ *such that*

(a) $\mathcal{L}$ *has order type* $\omega + \omega^*$*;*
(b) *if there exists a descending sequence in* $\mathcal{L}$ *then* $\emptyset'$ *exists;*
(c) *the formula "x has finitely many $\leq_L$-predecessors" is* $\mathbf{\Sigma}^0_1$ *(and thus "x has finitely many $\leq_L$-successors" is* $\mathbf{\Pi}^0_1$*);*
(d) *for all x with finitely many $\leq_L$-successors and for all $k \in \mathbb{N}$ there exists y with finitely many $\leq_L$-successors such that $|[y, x]_{\mathcal{L}}| > k$ (this means that there exist a one-to-one sequence σ of length k such that $y \leq_L \sigma(i) \leq_L x$ for every $i < k$).*

Proof. Fix a computable total one-to-one function f with range $\emptyset'$. We first define $\mathcal{L}$ satisfying (a), (b) and (c). Then we modify it to satisfy (d) as well.

We let $L = \mathbb{N}$ and define $\leq_L$ by stages: at stage s we have defined $\leq_L$ on $\{0, \dots, s\}$. At stage $s = 0$ there are no decisions to make. At stage $s + 1$ we add $s + 1$ to the order as follows:

- if $f(s + 1) > f(s)$ then $s + 1$ occurs immediately before s;
- if $f(s+1) < f(s)$ then let $t \leq s$ be the $\leq_L$-largest element such that $f(s + 1) < f(t)$, and put $s + 1$ immediately after t.

This completes the definition of $\leq_L$, which is clearly computable.

¿From the construction it is immediate that

(1) if $s < t$ is such that $s <_L t$ then $s <_L r$ for every $r > t$;
(2) if $r < s$ is such that $f(s) < f(r)$ then $r <_L s$.

To check that (a) holds we need to show that each element of L has either finitely many $\leq_L$-predecessors or finitely many $\leq_L$-successors, and there are infinitely many elements of each type.

If s is a true stage for f, i.e. $(\forall t > s)\, f(t) > f(s)$, we have $(\forall t > s)\, t <_L s$. In fact, if $t > s$ were least such that $t >_L s$ there would exist

$r < s$ with $s <_L r <_L t$ such that $f(t) < f(r)$. Since $s <_L r$ and $r < s$, by (2), we have $f(s) > f(r)$, which implies $f(s) > f(t)$. Thus if s is a true stage for f, $L_{(\geq_L s)} \subseteq \{0, \dots, s\}$ is finite.

If s is not a true stage for f, i.e. $(\exists t > s)\, f(t) < f(s)$, let $t_0 + 1$ be the least such t. Then $f(t_0 + 1) < f(s) \leq f(t_0)$ and $s <_L t_0 + 1$. By (1), $L_{(\leq_L s)} \subseteq \{0, \dots, t_0\}$ is finite.

There exist infinitely many true stages for f, otherwise we could easily define a descending sequence in $\mathbb{N}$. There also exist infinitely many nontrue stages for f: otherwise if n_0 is such that all $n \geq n_0$ are true stages, we have $(\exists n)\, f(n) = m$ if and only if $(\exists n \leq n_0 + m)\, f(n) = m$ for every m, which contradicts the incomputability of $\emptyset'$.

We now show that every descending sequence in $\mathcal{L}$ computes $\emptyset'$, establishing (b). If (s_m) is a $<_L$-descending sequence, by the observations above we have that each s_m is a true stage for f and that $s_m < s_{m+1}$, so that $f(s_m) < f(s_{m+1})$. Hence $f(s_m) \geq m$. Therefore

$$(\forall m)((\exists n)\, f(n) = m \iff (\exists n \leq s_m)\, f(n) = m).$$

Thus $\emptyset'$ can be computed from (s_m).

Since "s is a true stage for f" is a $\mathbf{\Pi}^0_1$ statement, (c) holds.

Thus $\mathcal{L}$ satisfies (a), (b) and (c). Notice that proving (d) for $\mathcal{L}$ appears to require $\mathbf{\Sigma}^0_2$ induction, which is not available in RCA_0. We define a linear order $\mathcal{L}' = (L', \leq_{L'})$ satisfying (d) by replacing each $n \in L$ by $n+1$ distinct elements and otherwise respecting the order of $\mathcal{L}$. To be precise, we set

$$L' = \{\, (n, i) \mid n \in L \wedge i \leq n \,\} \quad \text{and}$$
$$(n, i) \leq_{L'} (m, j) \iff n <_L m \vee (n = m \wedge i \leq j).$$

It is easy to check that $\mathcal{L}'$ satisfies (a), (b) and (c).

To prove (d) consider $x = (n, i)$ with finitely many $\leq_L$-successors and a given k. Let $m \in L$ be such that $m <_L n$, $L_{(\geq_L m)}$ is finite, and $m \geq k$. Such an m exists because there exist infinitely many $m \in L$ such that $L_{(\geq_L m)}$ is finite. Let $y = (m, 0) \in L'$: since $[y, x]_{\mathcal{L}'} \supseteq \{\, (m, j) \mid j \leq m \,\}$, we have $|[y, x]_{\mathcal{L}'}| > m \geq k$, as required. $\square$

Theorem 4.3. RCA_0 *proves that* MLE *implies* ACA_0.

Proof. To prove ACA_0 it suffices to show that for every X the jump of X, X', exists. We will do so for $X = \emptyset$, as the obvious relativization extends the proof to every X.

In RCA_0 let $\mathcal{L} = (L, \leq_L)$ be the linear order of Lemma 4.2. We will use the following notation:

$$D = \{\, x \in L \mid L_{(\leq_L x)} \text{ is finite} \,\}, \qquad U = \{\, x \in L \mid L_{(\geq_L x)} \text{ is finite} \,\}.$$

Notice that the existence of D and U as sets is not provable in RCA_0, and expressions such as $x \in D$ should be viewed only as shorthand for more complex formulas. It is immediate that D is downward closed and U is upward closed in $\mathcal{L}$. By (a) U and D are nonempty and form a partition of L. Moreover by (c) the formulas $x \in U$ and $x \in D$ are respectively $\mathbf{\Pi}^0_1$ and $\mathbf{\Sigma}^0_1$.

We will apply MLE to the partial order $\mathcal{P} = \mathcal{L} \oplus \mathcal{L}$. To be precise, $\mathcal{P} = (P, \leq_P)$ where $P = L \times 2$ and

$$(x, i) \leq_P (y, j) \iff i = j \wedge x \leq_L y.$$

For $i < 2$ we write L_i, D_i and U_i for $L \times \{i\}$, $D \times \{i\}$, and $U \times \{i\}$ respectively. $\mathcal{L}_i = (L_i, \leq_P)$ is obviously isomorphic to $\mathcal{L}$.

If $\mathcal{P}$ is not a wpo then, using the pigeonhole principle for two colors in RCA_0, there is a descending sequence in either $\mathcal{L}_0$ or $\mathcal{L}_1$. Hence there exists a descending sequence in $\mathcal{L}$ and, by (b), $\emptyset'$ exists.

We thus assume that $\mathcal{P}$ is a wpo, so that MLE applies and there exists a maximal linear extension $\mathcal{Q} = (P, \leq_Q)$ of $\mathcal{P}$. The proof of the existence of $\emptyset'$ now splits in two cases, depending on the properties of $\mathcal{Q}$.

Case I. For all $i < 2$, $x \in D_i$ and $y \in U_{1-i}$ we have $x <_Q y$.

If for some $i < 2$ there exists $x \in D_i$ such that $y <_Q x$ for all $y \in D_{1-i}$ then notice that $U_{1-i} = \{\, y \in L_{1-i} \mid x <_Q y \,\}$ exists as a set and therefore U exists a set. Then we can define a function which maps each $x \in U$ to some $y \in U$ with $y <_L x$. We can use this function to define a descending sequence in $\mathcal{L}$ and apply (b). Hence $\emptyset'$ exists, so that the proof is complete. The same argument applies if for some $i < 2$ there exists $x \in U_i$ such that $x <_Q y$ for all $y \in U_{1-i}$.

We thus assume that $(\forall i < 2)\, (\forall x \in D_i)\, (\exists y \in D_{1-i})\, x <_Q y$ and $(\forall i < 2)\, (\forall x \in U_i)\, (\exists y \in U_{1-i})\, y <_Q x$. This implies that for every $x \in P$ either $P_{(\leq_Q x)}$ or $P_{(\geq_Q x)}$ is finite. (Thus $\mathcal{Q}$ has order type $\omega + \omega^*$.) Now consider the linear extension $\mathcal{K} = (P, \leq_K)$ of $\mathcal{P}$ defined by

$$(x, i) \leq_K (y, j) \iff i < j \vee (i = j \wedge x \leq_L y).$$

In other words, $\mathcal{K} = \mathcal{L}_0 + \mathcal{L}_1$. Every $x \in U_0 \cup D_1$ is such that both $P_{(\leq_K x)} \supseteq D_0$ and $P_{(\geq_K x)} \supseteq U_1$ are infinite. This implies $\mathcal{K} \not\preceq \mathcal{Q}$, contradicting the maximality of $\mathcal{Q}$.

Case II. There exist $i < 2$, $x \in D_i$ and $y \in U_{1-i}$ such that $y <_Q x$. To simplify the notation, we assume $i = 0$.

Now consider the linear extension $\mathcal{J} = (P, \leq_J)$ of $\mathcal{P}$ defined by

$$(x, i) \leq_J (y, j) \iff x <_L y \vee (x = y \wedge i \leq j).$$

In other words, $\mathcal{J} = 2 \cdot \mathcal{L}$. Notice that it is easily provable in RCA_0 that for all $z, w \in U_i$ with $z \leq_{L_i} w$, we have $|[z,w]_{\mathcal{J}}| = 2 \cdot |[z,w]_{\mathcal{L}_i}| - 1$.

Since $\mathcal{Q}$ is maximal there exists $g : P \to P$ which witnesses $\mathcal{J} \preceq \mathcal{Q}$. The proof splits in two subcases.

Subcase IIa. There exists $w_0 \in U_1$ such that $g(w_0) \leq_Q x$.

We claim that there exists $w \in U_1$ with $w \leq_{L_1} w_0$ satisfying $g(z) \in L_1$ for all $z \in U_1$ such that $z \leq_{L_1} w$. To see this let

$$\begin{aligned} A &= \{\, x' \in L_{0(\leq_{L_0} x)} \mid (\exists w \in L_1)\, g(w) = x' \,\}, \quad \text{and} \\ B &= \{\, x' \in L_{0(\leq_{L_0} x)} \mid (\exists w \in D_1)\, g(w) = x' \,\}. \end{aligned}$$

Since $x \in D_0$, RCA_0 proves the existence of A and B by bounded $\mathbf{\Sigma}^0_1$-comprehension (recall that D_1 is $\mathbf{\Sigma}^0_1$). Let $C = A \setminus B$. Then RCA_0 proves that C exists and is finite. The subcase hypothesis implies that $C \neq \emptyset$, as $g(w_0) \in C$. Let x'_0 be the minimum of C with respect to $\leq_{L_0}$ and let (since $x'_0 \in A$) $w' \in L_1$ be such that $g(w') = x'_0$. Since $x'_0 \notin B$ and g is one-to-one we have $w' \in U_1$. Any $w \in U_1$ such that $w <_{L_1} w'$ has the required property.

Fix w as above, and notice that $g(z) \in U_1$ for any $z \in U_1$ with $z \leq_{L_1} w$. In fact, $P_{(<_J z)} \supseteq L_{1(<_{L_1} z)}$ is infinite while $P_{<_Q x'} \subseteq L_{0(<_{L_0} x)} \cup L_{1(<_{L_1} x')}$ is finite when $x' \in D_1$ (recall that in this case $x' \leq_Q y \leq_Q x$).

We now wish to find $z_0 \leq_{L_1} w$ such that $g(z_0) <_Q z_0$ (and hence $g(z_0) <_{L_1} z_0$, because $g(z_0) \in L_1$ by our choice of w). If $g(w) <_{L_1} w$ it suffices to let $z_0 = w$. If $w \leq_{L_1} g(w)$ let, by (d), $z_0 \in U_1$ be such that $z_0 \leq_{L_1} w$ and

$$|[z_0, w]_{\mathcal{L}_1}| > |[w, g(w)]_{\mathcal{L}_1}| + |L_{0(<_{L_0} x)}|.$$

Then, using $g(w) \leq_Q x$, we have

$$\begin{aligned} |[z_0, w]_{\mathcal{J}}| &= 2 \cdot |[z_0, w]_{\mathcal{L}_1}| - 1 \\ &> |[z_0, w]_{\mathcal{L}_1}| + |[w, g(w)]_{\mathcal{L}_1}| + |L_{0(<_{L_0} x)}| - 1 \\ &= |[z_0, g(w)]_{\mathcal{L}_1}| + |L_{0(<_{L_0} x)}| \\ &\geq |[z_0, g(w)]_{\mathcal{Q}}|. \end{aligned}$$

Since g maps the interval $[z_0, w]_{\mathcal{J}}$ injectively into the interval $[g(z_0), g(w)]_{\mathcal{Q}}$, this implies that $g(z_0) <_Q z_0$, as we wanted.

Now recursively define $z_{n+1} = g(z_n)$. By $\mathbf{\Pi}^0_1$ induction on n it is straightforward to show that $z_n \in U_1$ and $z_{n+1} <_{L_1} z_n \leq_{L_1} w$. We have thus defined a descending sequence in $\mathcal{L}_1$ and hence in $\mathcal{L}$. By (b), $\emptyset'$ exists.

Subcase IIb. For every $w \in U_1$ we have $x <_Q g(w)$.

Since for all $w \in U_0$ there exists $w' \in U_1$ such that $w' <_J w$ we also have $x <_Q g(w)$ for every $w \in U_0$.

We claim that $w \in U_0$ and $g(w) \in L_0$ imply $g(w) \in U_0$. To see this, we argue by contradiction and assume that there exists $w \in U_0$ with $g(w) \in D_0$. Then $[x, g(w)]_{\mathcal{Q}} \subseteq L_{0(\leq_{L_0} g(w))} \cup L_{1(>_{L_1} y)}$ is finite. If, by (d), $w' \in U_0$ is such that $|[w', w]_{\mathcal{L}_0}| \geq |[x, g(w)]_{\mathcal{Q}}|$ then $g(w') \leq_Q x$, which contradicts what we noticed above.

Notice also that $w \in L_{0(\geq_{L_0} x)}$ and $g(w) \in L_1$ imply $y <_{L_1} g(w)$. Since $L_{1(>_{L_1} y)}$ is finite, this can happen only finitely many times. Thus, arguing as in the previous subcase, we can find $w \in U_0$ such that $g(z) \in U_0$ for all $z \in U_0$ such that $z \leq_{L_0} w$.

We mimic the argument used in the previous subcase, finding $z_0 \in U_0$ with $z_0 \leq_{L_0} w$ such that $g(z_0) <_Q z_0$. When $g(w) <_{L_0} w$ we set $z_0 = w$. If $w \leq_{L_0} g(w)$, we pick, by (d), $z_0 \in U_0$ with $z_0 \leq_{L_0} w$ such that

$$|[z_0, w]_{\mathcal{L}_0}| > |[w, g(w)]_{\mathcal{L}_0}| + |L_{1(>_{L_1} y)}|.$$

Then, using $y <_{\mathcal{Q}} x <_{\mathcal{Q}} z_0$, we have

$$\begin{aligned} |[z_0, w]_{\mathcal{J}}| &= 2 \cdot |[z_0, w]_{\mathcal{L}_0}| - 1 \\ &> |[z_0, w]_{\mathcal{L}_0}| + |[w, g(w)]_{\mathcal{L}_0}| + |L_{1(>_{L_1} y)}| - 1 \\ &= |[z_0, g(w)]_{\mathcal{L}_0}| + |L_{1(>_{L_1} y)}| \\ &\geq |[z_0, g(w)]_{\mathcal{Q}}|. \end{aligned}$$

Since g maps the interval $[z_0, w]_{\mathcal{J}}$ injectively into the interval $[g(z_0), g(w)]_{\mathcal{Q}}$, this implies $g(z_0) <_Q z_0$, as we wanted.

We now define $z_{n+1} = g(z_n)$ for all n. Using again $\mathbf{\Pi}^0_1$ induction, we can show that this is a descending sequence in $\mathcal{L}_0$. By (b), $\emptyset'$ exists. □

5. MLE implies ATR_0

Although most properties of well-orders require ATR_0, some of them (such as the fact that well-orders are closed under exponentiation) can be proved in ACA_0. In this section we will use two of these facts, both due to Jeff Hirst ([Hir94, Theorem 3.5 and Lemma 4.3]).

Theorem 5.1. ACA_0 *proves that if* $\mathcal{L}$ *is a well order, then* $\omega^{\mathcal{L}}$ *is indecomposable, i.e. if* $\omega^{\mathcal{L}} \preceq \mathcal{I} + \mathcal{J}$ *then either* $\omega^{\mathcal{L}} \preceq \mathcal{I}$ *or* $\omega^{\mathcal{L}} \preceq \mathcal{J}$.

Theorem 5.2. ACA_0 *proves that if* $\mathcal{L}_0$ *and* $\mathcal{L}_1$ *are well-orders then* $\mathcal{L}_0 \preceq \mathcal{L}_1$ *if and only if* $\omega^{\mathcal{L}_0} \preceq \omega^{\mathcal{L}_1}$.

We will also need the following Lemma, which is a much weaker version of Lemma 3.7.

Lemma 5.3. ACA_0 *proves that if* $\mathcal{I}$ *and* $\mathcal{J}$ *are well-orders without a maximum element and* $\mathcal{L}$ *is a shuffle of* $\mathcal{I}$ *and* $\mathcal{J}$*, then* $\mathcal{L} \preceq \mathcal{I} \cdot \mathcal{J}$ *or* $\mathcal{L} \preceq \mathcal{J} \cdot \mathcal{I}$*.*

Proof. Let $\mathcal{I} = (I, \leq_I)$, $\mathcal{J} = (J, \leq_J)$ and (assuming I and J are disjoint) $\mathcal{L} = (I \cup J, \leq_L)$. At least one of I and J is cofinal in $\mathcal{L}$. We assume that I is cofinal in $\mathcal{L}$ and we define an embedding f of $\mathcal{L}$ into $\mathcal{J} \cdot \mathcal{I}$ (if J is cofinal we obtain $\mathcal{L} \preceq \mathcal{I} \cdot \mathcal{J}$). Let m be the $\leq_J$-least element of $\mathcal{J}$. Using ACA_0 we can define the operations s_I and s_J mapping each element of I and J to its successor according to $\mathcal{I}$ and $\mathcal{J}$. Similarly, again using ACA_0, we can define the function t which maps $x \in J$ to the $\leq_I$-least $y \in I$ such that $x <_L y$. Define $f : I \cup J \to J \times I$ as follows:

$$f(x) = \begin{cases} (m, s_I(x)) & \text{if } x \in I; \\ (s_J(x), t(x)) & \text{if } x \in J. \end{cases}$$

To see that f preserves order, consider the four possible cases. If $x <_I y$ then $s_I(x) <_I s_I(y)$ and so $(m, s_I(x)) <_{J \times I} (m, s_I(y))$. If $x <_J y$, then $t(x) \leq_J t(y)$ and $s_J(x) <_J s_J(y)$ and so $(s_J(x), t(x)) <_{J \times I} (s_J(y), t(y))$. If $x \in I$, $y \in J$ and $x <_L y$, then $s_I(x) \leq_I t(y)$ and, of course, $m <_J s_J(y)$ and so $(m, s_I(x)) <_{J \times I} (s_J(y), t(y))$. Finally, if $x \in J$, $y \in I$ and $x <_L y$, then $t(x) \leq_I y <_I s_I(y)$ and so $(s_J(x), t(x)) <_{J \times I} (m, s_I(y))$ as required. □

Theorem 5.4. ACA_0 *proves that* MLE *implies* ATR_0*.*

Proof. We work in ACA_0, assume that ATR_0 fails and work toward a contradiction. By Theorem 2.1, the failure of ATR_0 implies the existence of a sequence $\langle \mathcal{I}_n \rangle$ of well-orders that are pairwise mutually nonembeddable. For every n let $\mathcal{J}_n = \omega^{\omega^{\mathcal{I}_n}}$. Using Theorem 5.2 twice we have that the $\mathcal{J}_n$'s are also pairwise mutually nonembeddable. Let $\mathcal{J}_n = (J_n, \leq_{J_n})$: without loss of generality, we may assume that the J_n's are pairwise disjoint.

We claim that if k, m and n are distinct then $\mathcal{J}_n$ is not embeddable in any shuffle of $\mathcal{J}_k$ and $\mathcal{J}_m$. To see this notice that by Lemma 5.3 it suffices to prove that $\mathcal{J}_n \not\preceq \mathcal{J}_m \cdot \mathcal{J}_k$. Suppose the contrary, i.e. that $\omega^{\omega^{\mathcal{I}_n}} \preceq \omega^{\omega^{\mathcal{I}_m}} \cdot \omega^{\omega^{\mathcal{I}_k}} = \omega^{\omega^{\mathcal{I}_m} + \omega^{\mathcal{I}_k}}$ (where the equality, which is really an isomorphism, is provable in RCA_0). Theorem 5.2 implies $\omega^{\mathcal{I}_n} \preceq \omega^{\mathcal{I}_m} + \omega^{\mathcal{I}_k}$. As $\omega^{\mathcal{I}_n}$ is indecomposable (Theorem 5.1) we would then have either $\omega^{\mathcal{I}_n} \preceq \omega^{\mathcal{I}_m}$ or $\omega^{\mathcal{I}_n} \preceq \omega^{\mathcal{I}_k}$, and so, by Theorem 5.2 again, $\mathcal{I}_n \preceq \mathcal{I}_m$ or $\mathcal{I}_n \preceq \mathcal{I}_k$, contrary to our choice of the $\mathcal{I}_i$.

Let $\mathcal{L}_0 = \sum_n \mathcal{J}_{2n}$ and $\mathcal{L}_1 = \sum_n \mathcal{J}_{2n+1}$: these are well-orders by [Hir05b, Theorem 12], and we can define the wpo $\mathcal{P} = (P, \leq_P)$ as $\mathcal{L}_0 \oplus \mathcal{L}_1$. By MLE let $\mathcal{Q} = (P, \leq_Q)$ be a maximal linear extension of $\mathcal{P}$.

For every n let x_n be the least element of $\mathcal{J}_n$ with respect to $\leq_{J_n}$ (and hence also to $\leq_Q$). Notice that, for example, $x_{2n} <_P x_{2n+2}$ and $x_{2n+1} <_P x_{2n+3}$ (and hence also $x_{2n} <_Q x_{2n+2}$ and $x_{2n+1} <_Q x_{2n+3}$) for every n.

We claim that

(*) if $F \subseteq \mathbb{N}$ is finite and $m \notin F$, then $\mathcal{J}_m \not\preceq \mathcal{Q} \restriction (\bigcup_{i \in F} J_i)$

To prove (*) suppose f witnesses $\mathcal{J}_m \preceq \mathcal{Q} \restriction (\bigcup_{i \in F} J_i)$. Let i and k be such that x_{2i} and x_{2k+1} are the largest of the x_l for $l \in F$, l even and odd respectively, such that $f(x), f(\hat{x}) \in J_l$, respectively, for some $x, \hat{x} \in J_m$ (if the range of f intersects only J_l with l even, or only J_l with l odd, the argument is even simpler). For any $y >_m x, \hat{x}$, we must have $f(y) \in J_{2i}$ or $f(y) \in J_{2k+1}$ as i and k are the largest of their type and $f(y) >_Q f(x), f(\hat{x})$. Now f provides an embedding of a final segment of $\mathcal{J}_m$ (and so, by indecomposability, of $\mathcal{J}_m$ itself) into a shuffle of $\mathcal{J}_{2i}$ and $\mathcal{J}_{2k+1}$, contradicting what we proved earlier and establishing (*).

Now consider the order of the x_n in $\mathcal{Q}$. This is a linear extension of $\omega \oplus \omega$ (and so classically of order type $\omega + \omega$ or ω corresponding to Cases I and II below).

Case I. There exists k such that $x_{2n} <_Q x_{2k+1}$ for all n (the reverse situation, where some x_{2k} is above all the x_{2n+1}, is similar). Notice that for every n and $x \in J_{2n}$ we have $x <_Q x_{2n+2} <_Q x_{2k+1}$. Now consider the linear extension $\mathcal{L} = \sum \mathcal{J}_n$ of $\mathcal{P}$ and suppose f witnesses $\mathcal{L} \preceq \mathcal{Q}$.

Subcase Ia. There exists $x \in P$ such that $x_{2k+1} \leq_Q f(x)$. By the definition of $\mathcal{L}$ we have that for some n we have $x_{2k+1} <_Q f(x_{2n})$. Fix $x \in J_{2n}$: since $f(x) \geq_Q f(x_{2n})$, the case hypothesis implies the existence of $l \geq k$ such that $f(x) \in J_{2l+1}$. Analogously, $f(x_{2n+1}) \in J_{2m+1}$ for some $m \geq k$. Therefore $f \restriction J_{2n}$ witnesses $\mathcal{J}_{2n} \preceq \mathcal{Q} \restriction (\bigcup_{l=k}^{m} J_{2l+1})$, contradicting (*).

Subcase Ib. $f(x) <_Q x_{2k+1}$ for all $x \in P$. If $f(x_{2k+2}) >_Q x_{2n}$ for all n then for every $y \geq_L x_{2k+2}$ we have $f(y) \in J_{2n+1}$ for some $n < k$, so that $f \restriction J_{2k+2}$ witnesses $\mathcal{J}_{2k+2} \preceq \mathcal{Q} \restriction (\bigcup_{n<k} J_{2n+1})$, against (*). Otherwise $f(x_{2k+2}) \leq_Q x_{2m}$ for some m, and $f \restriction J_{2k+1}$ witnesses $\mathcal{J}_{2k+1} \preceq \mathcal{Q} \restriction (\bigcup_{n<m} J_{2n} \cup \bigcup_{n<k} J_{2n+1})$, again violating (*).

Case II. Neither version of Case I holds and so the x_{2n} and x_{2n+1} are cofinal in each other in $\mathcal{Q}$ and each has only finitely many of them preceding it in $\mathcal{Q}$. Consider now the linear extension $\mathcal{K} = \mathcal{L}_0 + \mathcal{L}_1$ of $\mathcal{P}$ and an embedding g witnessing $\mathcal{K} \preceq \mathcal{Q}$. By the cofinality assumption there is a k such that $g(x_1) <_Q x_{2k}, x_{2k+1}$. Thus $P_{(\leq_Q g(x_1))} \subseteq \bigcup_{i<2k} J_i$. Notice that g maps every J_{2n} to $P_{(\leq_Q g(x_1))}$. In particular $g \restriction J_{2k}$ witnesses $\mathcal{J}_{2k} \preceq \mathcal{Q} \restriction (\bigcup_{i<2k} J_i)$, for one more contradiction to (*). □

6. ATR_0 and MC are equivalent

We prove MC in ATR_0 in a fashion similar to the way we proved MLE. For this purpose, we adapt the proof in [Sch81], which translated literally into the language of second order arithmetic requires the use of $\mathbf{\Sigma}^1_1$ induction. We will avoid the use of $\mathbf{\Sigma}^1_1$ induction by using the same approach we took in Section 3. The proof of MC in [Wol67, KT90, Har05] is based on Radó Selection Lemma (a weak form of the Axiom of Choice) and can also be formalized in ATR_0.

Definition 6.1. In RCA_0 we define, for a partial order $\mathcal{P} = (P, \leq_P)$, the *tree of descending sequences of* $\mathcal{P}$:

$$\mathrm{Desc}(\mathcal{P}) = \{\, \sigma \in \mathbb{N}^{<\mathbb{N}} \mid (\forall i < \mathrm{lh}(\sigma))(\sigma(i) \in P \wedge (\forall j < i)\, \sigma(i) <_P \sigma(j)) \,\}.$$

Notice that $\mathcal{P}$ is well founded (as a partial order) if and only if $\mathrm{Desc}(\mathcal{P})$ is well founded as a tree. Thus if $\mathcal{P}$ is well founded we can define by transfinite recursion the rank function on $\mathrm{Desc}(\mathcal{P})$ (taking ordinals as values), which we denote by $\mathrm{ht}_{\mathcal{P}}$, and define the ordinal $\mathrm{ht}(\mathcal{P}) = \mathrm{ht}_{\mathcal{P}}(\emptyset)$. As in Section 3, using transfinite recursion we mimic this definition in ATR_0.

We let, for $\sigma \in \mathrm{Desc}(\mathcal{P})$, $P^c_\sigma = \{\, p \in P \mid (\forall i < \mathrm{lh}(\sigma))\, p <_P \sigma(i) \,\} = \{\, p \in P \mid \sigma^\frown\langle p\rangle \in \mathrm{Desc}(\mathcal{P}) \,\}$, and we write $\mathcal{P}^c_\sigma = (P^c_\sigma, \leq_P)$.

When $\mathcal{L}$ is a well-order we have $\mathrm{Desc}(\mathcal{L}) = \mathrm{Bad}(\mathcal{L})$. Hence from Lemma 3.3 it follows immediately that ATR_0 proves that if $\mathcal{L}$ is a well-order then $\mathcal{L} \equiv \mathrm{ht}(\mathcal{L})$. We can now prove the next Lemma exactly as we proved Lemma 3.4.

Lemma 6.2. ATR_0 *proves that if* $\mathcal{P}$ *is a well founded partial order and* $\mathcal{C} \in \mathrm{Ch}(\mathcal{P})$ *then* $\mathcal{C} \preceq \mathrm{ht}(\mathcal{P})$.

We need the following version of Lemma 2 in [Sch81].

Lemma 6.3. ATR_0 *proves that for each wpo* $\mathcal{P} = (P, \leq_P)$ *and each* $\{\, y^j_i \mid j \leq i \,\} \subseteq P$ *there exists a strictly increasing* $g : \mathbb{N} \to \mathbb{N}$ *satisfying* $y^j_{g(j)} \leq_P y^j_{g(j+1)}$ *for every* j. *Moreover we can require* g *to be uniformly recursive in the* ω*-jump of* $\mathcal{P} \oplus \{\, y^j_i \mid j < i \,\}$.

Proof. We follow Schmidt's proof. Fix the wpo $\mathcal{P}$ and $\{\, y^j_i \mid j \leq i \,\}$. Let $Z = \mathcal{P} \oplus \{\, y^j_i \mid j < i \,\}$. We define recursively a sequence of infinite sets (A_j) so that A_j is computable in $Z^{(2j+2)}$ (the $(2j+2)$th jump of Z) as follows.

By applying Lemma 2.9 to the function $i \mapsto y_i^0$ we can find A_0 infinite, computable in Z'', and such that $y_i^0 \leq_P y_{i'}^0$ for all $i, i' \in A_0$ with $i < i'$. If we have defined A_j infinite and computable in $Z^{(2j+2)}$ we choose $A_{j+1} \subseteq A_j$ infinite and computable in $(Z \oplus A_j)''$ (and hence in $Z^{(2j+4)}$) such that $y_i^{j+1} \leq_P y_{i'}^{j+1}$ for all $i, i' \in A_{j+1}$ with $i < i'$. Again, the existence of A_{j+1} follows from Lemma 2.9.

Let now for all j, h_j be the function enumerating in increasing order A_j. Set $g(j) = h_j(j)$. To prove that g has the desired property notice that, since $A_{j+1} \subseteq A_j$, there exists $i \geq j+1$ such that $g(j+1) = h_{j+1}(j+1) = h_j(i)$. This implies $g(j+1) > g(j)$ and $y^j_{g(j)} = y^j_{h_j(j)} \leq_P y^j_{h_j(i)} = y^j_{g(j+1)}$ for every j. Moreover g is computable in $\bigoplus_j A_j$ and, by the uniformity of our construction, $\bigoplus_j A_j$ is computable in the ω-jump of Z. $\square$

We can now prove the main theorem.

Theorem 6.4. ATR_0 *proves* MC.

Proof. By Lemma 6.2 to prove MC within ATR_0 it suffices to define, for each wpo $\mathcal{P}$, $\mathcal{C} \in \mathrm{Ch}(\mathcal{P})$ such that $\mathrm{ht}(\mathcal{P}) \preceq \mathcal{C}$. We adapt the strategy of the proof of Theorem 3.8. In fact, we define, for each $\sigma \in \mathrm{Desc}(\mathcal{P})$, a set C_σ and a function f_σ. We then prove by $\mathbf{\Delta}^1_1$ transfinite induction on rank that $C_\sigma \subseteq P^c_\sigma$, that C_σ is totally ordered by $\leq_P$ (so that $\mathcal{C}_\sigma = (C_\sigma, \leq_P) \in \mathrm{Ch}(\mathcal{P}^c_\sigma)$), and that f_σ is an isomorphism between $\mathcal{C}_\sigma$ and $\mathrm{ht}_\mathcal{P}(\sigma)$. Since $\mathcal{P}^c_\emptyset = \mathcal{P}$, we have $\mathcal{C}_\emptyset \in \mathrm{Ch}(\mathcal{P})$ and $\mathrm{ht}(\mathcal{P}) \equiv \mathcal{C}_\emptyset$.

As in the proof of Theorem 3.8, but using $\mathrm{Desc}(\mathcal{P})$, $\mathrm{ht}_\mathcal{P}$, and P^c_σ in place of $\mathrm{Bad}(\mathcal{P})$, $\mathrm{rk}_\mathcal{P}$, and P_σ, respectively, we define S, L, $p : S \to P$ and for every $\sigma \in L$ the sequences (λ_n), (x_i) and (n_i). Notice that here we use that $\mathcal{P}$ is a wpo (and not only a well founded order) when we require $x_i <_P x_{i+1}$ for every i.

We now define by arithmetical transfinite recursion on rank C_σ and f_σ. When $\mathrm{ht}_\mathcal{P}(\sigma) = 0$ we let $C_\sigma = \emptyset$ and f_σ be the empty function. When $\sigma \in S$ let $C_\sigma = C_{\sigma^\frown\langle p(\sigma)\rangle} \cup \{p(\sigma)\}$ and, recalling that $\mathrm{ht}_\mathcal{P}(\sigma) = \mathrm{ht}_\mathcal{P}(\sigma^\frown\langle p(\sigma)\rangle) + 1$, let f_σ extend $f_{\sigma^\frown\langle p(\sigma)\rangle}$ by mapping $p(\sigma)$ to $\mathrm{ht}_\mathcal{P}(\sigma^\frown\langle p(\sigma)\rangle)$.

When $\sigma \in L$ let us write $\lambda_{n_{-1}}$ for the least element of $\mathrm{ht}_\mathcal{P}(\sigma)$ and for $j \leq i$ let y_i^j be the element such that $f_{\sigma^\frown\langle x_i\rangle}(y_i^j) = \lambda_{n_{j-1}}$. (To be scrupulous, at this stage we are not sure that such a y_i^j exists and is unique, and we should let y_i^j to be some fixed member of P if this is not the case, an event we will later show never occurs.) By Lemma 6.3 we can find a strictly increasing $g : \mathbb{N} \to \mathbb{N}$ which is uniformly recursive in the ω-jump of $\mathcal{P} \oplus \{ y_i^j \mid j < i \}$ satisfying $y^j_{g(j)} \leq_P y^j_{g(j+1)}$ for every j. (To be precise, this definition of a set computable in the ω-jump can be replaced by $\omega + 1$ arithmetic steps.) For $j > 0$ let

$$D_j = \{ p \in C_{\sigma^\frown\langle x_{g(j)}\rangle} \mid y^{j-1}_{g(j)} \leq_P p <_P y^j_{g(j)} \} \quad \text{and set } C_\sigma = \bigcup_{j>0} D_j.$$

To define f_σ, for every $p \in C_\sigma$ find the least j such that $p \in D_j$ (it will follow that there exists only one such j) and set $f_\sigma(p) = f_{\sigma^\frown\langle x_{g(j)}\rangle}(p)$.

Now we prove by $\mathbf{\Delta}^1_1$ transfinite induction on rank that $C_\sigma \subsetneq P^c_\sigma$, that $\mathcal{C}_\sigma = (C_\sigma, \leq_P) \in \mathrm{Ch}(\mathcal{P}^c_\sigma)$ and that f_σ is an isomorphism between $\mathcal{C}_\sigma$ and $\mathrm{ht}_\mathcal{P}(\sigma)$. When $\mathrm{ht}_\mathcal{P}(\sigma) = 0$ there is nothing to prove. When $\sigma \in S$ it suffices to notice that $p <_P p(\sigma)$ for every $p \in C_{\sigma^\frown\langle p(\sigma)\rangle}$ and apply the induction hypothesis.

Fix now $\sigma \in L$. First, the induction hypothesis implies that $y_i^j \in C_{\sigma^\frown\langle x_i\rangle}$ and $f_{\sigma^\frown\langle x_i\rangle}(y_i^j) = \lambda_{n_{j-1}}$ for every $j \leq i$. Moreover we have $D_j \subseteq C_{\sigma^\frown\langle x_{g(j)}\rangle} \subsetneq P^c_{\sigma^\frown\langle x_{g(j)}\rangle} \subset P^c_\sigma$, and hence $C_\sigma \subsetneq P^c_\sigma$. To check that $\mathcal{C}_\sigma$ is a chain fix $p, p' \in C_\sigma$. If $p, p' \in D_j \subseteq C_{\sigma^\frown\langle x_{g(j)}\rangle}$ for some j, comparability of p and p' follows from the induction hypothesis. If $p \in D_j$ and $p' \in D_{j'}$ for $j < j'$ then $p <_P y^j_{g(j)} \leq_P y^j_{g(j+1)} \leq_P \cdots \leq_P y^{j'-1}_{g(j')} \leq_P p'$ (where the first $\leq_P$ follows from the property of g) and we have $p <_P p'$. This shows also that $\mathcal{C}_\sigma = \sum_{j>0} \mathcal{D}_j$, where $\mathcal{D}_j$ is of course $(D_j, \leq_P)$. Notice also that by the induction hypothesis and the definition of y_i^j, f_σ restricted to D_j is an isomorphism between $\mathcal{D}_j$ and the interval $[\lambda_{n_{j-2}}, \lambda_{n_{j-1}})$ of $\mathrm{ht}_\mathcal{P}(\sigma)$. This means that f_σ is an isomorphism between $\mathcal{C}_\sigma$ and $\sum_{j>0}[\lambda_{n_{j-2}}, \lambda_{n_{j-1}}) = \mathrm{ht}_\mathcal{P}(\sigma)$. □

Our proof of MC in ATR_0 actually shows the following stronger result.

Theorem 6.5. ATR_0 *proves that any wpo* $\mathcal{P}$ *contains a chain* $\mathcal{C}$ *such that*

$$(\forall \alpha < \mathrm{ht}(\mathcal{P}))\, (\exists p \in C)\ \mathrm{ht}_\mathcal{P}(p) = \alpha,$$

where $\mathrm{ht}_\mathcal{P}(p) = \mathrm{ht}_\mathcal{P}(\langle p\rangle) = \mathrm{ht}(\mathcal{P}_{(<_P p)})$.

Proof. In the preceding proof it can be shown inductively that $f_\sigma(p) = \mathrm{ht}_\mathcal{P}(p)$ for every $\sigma \in \mathrm{Desc}(\mathcal{P})$ and $p \in C_\sigma$. □

The statement contained in Theorem 6.5 (let us call it MC^+) is Wolk's original result. One reason for focusing on MC rather than on MC^+ is that stating the latter requires the existence of the function $\mathrm{ht}_\mathcal{P}$ which is defined using ATR_0. Thus we cannot state MC^+ in RCA_0. Another reason for our preference for MC is the strong similarity with MLE.

As mentioned in the introduction, the proof of (3) $\implies$ (1) in Theorem 1.6 is very simple.

Theorem 6.6. RCA_0 *proves that* MC *implies* ATR_0.

Proof. By Theorem 2.1 it suffices to prove that if $\mathcal{L}_0$ and $\mathcal{L}_1$ are well-orders then either $\mathcal{L}_0 \preceq \mathcal{L}_1$ or $\mathcal{L}_1 \preceq \mathcal{L}_0$. Given well-orders $\mathcal{L}_0$ and $\mathcal{L}_1$ let $\mathcal{P} = \mathcal{L}_0 \oplus \mathcal{L}_1$. $\mathcal{P}$ is a wpo and by MC it has a maximal chain $\mathcal{C} = (C, \leq_P)$. It is immediate that either $C \subseteq L_0$ or $C \subseteq L_1$, and we may assume the first possibility holds, so that $\mathcal{C} \preceq \mathcal{L}_0$. Since $\mathcal{L}_1 \in \mathrm{Ch}(\mathcal{P})$ we have $\mathcal{L}_1 \preceq \mathcal{C}$ and thus $\mathcal{L}_1 \preceq \mathcal{L}_0$. □

References

[BG08] Andreas Blass and Yuri Gurevich. Program termination and well partial orderings. *ACM Trans. Comput. Log.*, 9(3):Art. 18, 26, 2008.

[CJS01] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1–55, 2001.

[CMS04] Peter Cholak, Alberto Marcone, and Reed Solomon. Reverse mathematics and the equivalence of definitions for well and better quasi-orders. *J. Symbolic Logic*, 69(3):683–712, 2004.

[dJP77] D. H. J. de Jongh and Rohit Parikh. Well-partial orderings and hierarchies. *Nederl. Akad. Wetensch. Proc. Ser. A* **80**=*Indag. Math.*, 39(3):195–207, 1977.

[FH90] Harvey Friedman and Jeffry L. Hirst. Weak comparability of well orderings and reverse mathematics. *Ann. Pure Appl. Logic*, 47(1):11–29, 1990.

[Fra00] Roland Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000. With an appendix by Norbert Sauer.

[FRS87] Harvey Friedman, Neil Robertson, and Paul Seymour. The metamathematics of the graph minor theorem. In Stephen G. Simpson, editor, *Proceedings of the AMS-IMS-SIAM joint summer research conference on applications of mathematical logic to finite combinatorics held at Humboldt State University, Arcata, Calif., August 4–10, 1985*, pages 229–261. American Mathematical Society, Providence, R.I., 1987.

[Gir87] Jean-Yves Girard. *Proof theory and logical complexity*. Bibliopolis, Naples, 1987.

[Har05] Egbert Harzheim. *Ordered sets*, volume 7 of *Advances in Mathematics (Springer)*. Springer, New York, 2005.

[Hig52] Graham Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3)*, 2:326–336, 1952.

[Hir94] Jeffry L. Hirst. Reverse mathematics and ordinal exponentiation. *Ann. Pure Appl. Logic*, 66(1):1–18, 1994.

[Hir05a] Jeffry L. Hirst. Reverse mathematics and ordinal suprema. In Simpson [Sim05], pages 235–243.

[Hir05b] Jeffry L. Hirst. A survey of the reverse mathematics of ordinal arithmetic. In Simpson [Sim05], pages 222–234.

[HS07] Denis R. Hirschfeldt and Richard A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. *J. Symbolic Logic*, 72(1):171–206, 2007.

[Kru60] J. B. Kruskal. Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. Amer. Math. Soc.*, 95:210–225, 1960.

[Kru72] Joseph B. Kruskal. The theory of well-quasi-ordering: A frequently discovered concept. *J. Combinatorial Theory Ser. A*, 13:297–305, 1972.

[KT90] Igor Kříž and Robin Thomas. Ordinal types in Ramsey theory and well-partial-ordering theory. In *Mathematics of Ramsey theory*, volume 5 of *Algorithms Combin.*, pages 57–95. Springer, Berlin, 1990.

[Lav71] Richard Laver. On Fraïssé's order type conjecture. *Ann. of Math. (2)*, 93:89–111, 1971.

[Mar05] Alberto Marcone. Wqo and bqo theory in subsystems of second order arithmetic. In Simpson [Sim05], pages 303–330.

[MM09] Alberto Marcone and Antonio Montalbán. On Fraïssé's conjecture for linear orders of finite Hausdorff rank. *Ann. Pure Appl. Logic*, 160(3):355–367, 2009.

[Mon07] Antonio Montalbán. Computable linearizations of well-partial-orderings. *Order*, 24(1):39–48, 2007.

[MS81] E. C. Milner and N. Sauer. On chains and antichains in well-founded partially ordered sets. *J. London Math. Soc. (2)*, 24(1):15–33, 1981.

[NW65] C. St. J. A. Nash-Williams. On well-quasi-ordering transfinite sequences. *Proc. Cambridge Philos. Soc.*, 61:33–39, 1965.

[NW68] C. St. J. A. Nash-Williams. On better-quasi-ordering transfinite sequences. *Proc. Cambridge Philos. Soc.*, 64:273–290, 1968.

[RS04] Neil Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[Sch79] Diana Schmidt. *Well-partial Orderings and Their Maximal Order Types.* Habilitationschrift, Heidelberg University, 1979.

[Sch81] Diana Schmidt. The relation between the height of a well-founded partial ordering and the order types of its chains and antichains. *J. Combin. Theory Ser. B*, 31(2):183–189, 1981.

[Sho93] Richard A. Shore. On the strength of Fraïssé's conjecture. In John N. Crossley, Jeffrey B. Remmel, Richard A. Shore, and Moss E. Sweedler, editors, *Logical methods*, volume 12 of *Progress in Computer Science and Applied Logic*, pages 782–813. Birkhäuser Boston Inc., Boston, MA, 1993. Papers from the conference in honor of Anil Nerode's sixtieth birthday held at Cornell University, Ithaca, New York, June 1–3, 1992.

[Sim85] Stephen G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In L. A. Harrington, M. D. Morley, A. Ščedrov, and S. G. Simpson, editors, *Harvey Friedman's research on the foundations of mathematics*, pages 87–117. North-Holland, Amsterdam, 1985.

[Sim88] Stephen G. Simpson. Ordinal numbers and the Hilbert basis theorem. *J. Symbolic Logic*, 53(3):961–974, 1988.

[Sim05] Stephen G. Simpson, editor. *Reverse mathematics 2001*, volume 21 of *Lecture Notes in Logic*. Association for Symbolic Logic, La Jolla, CA, 2005.

[Sim09] Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge, second edition, 2009.

[Tho95] Carsten Thomassen. Embeddings and minors. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of combinatorics, Vol. 1*, pages 301–349. Elsevier Science, Amsterdam, 1995.

[Wol67] E. S. Wolk. Partially well ordered sets and partial ordinals. *Fund. Math.*, 60:175–186, 1967.

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