Combinatorial Principles Weaker than Ramsey's Theorem for Pairs

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Abstract

We investigate the complexity of various combinatorial theorems about linear and partial orders, from the points of view of computability theory and reverse mathematics. We focus in particular on the principles ADS (Ascending or Descending Sequence), which states that every infinite linear order has either an infinite descending sequence or an infinite ascending sequence, and CAC (Chain-AntiChain), which states that every infinite partial order has either an infinite chain or an infinite antichain. It is well-known that Ramsey's Theorem for pairs (RT^2_2) splits into a stable version (SRT^2_2) and a cohesive principle (COH). We show that the same is true of ADS and CAC, and that in their cases the stable versions are strictly weaker than the full ones (which is not known to be the case for RT^2_2 and SRT^2_2). We also analyze the relationships between these principles and other systems and principles previously studied by reverse mathematics, such as WKL_0, DNR, and $\mathrm{B}\Sigma_2$. We show, for instance, that WKL_0 is incomparable with all of the systems we study. We also prove computability-theoretic and conservation results for them. Among these results are a strengthening of the fact, proved by Cholak, Jockusch, and Slaman, that COH is Π^1_1-conservative over the base system RCA_0. We also prove that CAC does not imply DNR which, combined with a recent result of Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman, shows that CAC does not imply SRT^2_2 (and so does not imply RT^2_2). This answers a question of Cholak, Jockusch, and Slaman.

Our proofs suggest that the essential distinction between ADS and CAC on the one hand and RT^2_2 on the other is that the colorings needed for our analysis are in some way transitive. We formalize this intuition as the notions of transitive and semitransitive colorings and show that the existence of homogeneous sets for such colorings is equivalent to ADS and CAC, respectively. We finish with several open questions.

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1 Introduction

In this paper we investigate the complexity of various combinatorial theorems about linear and partial orders. We are interested in both computational (computability theoretic) and proof theoretic (reverse mathematical) calibrations and in the interplay between them. The theorems of interest in such investigations are typically of the form $\forall A\,(\Theta(A) \to \exists B\, \Phi(A,B))$ where Θ and Φ are arithmetic and $A, B \in 2^\omega$. Thus, from the computability theoretic point of view,we want to bound or characterize the computational complexity of B given an A satisfying Θ (typically in terms of Turing degree or place in one of the standard arithmetic/analytic definability or jump hierarchies). From the reverse mathematics point of view we want to determine the axiom systems in which the theorem is provable (typically subsystems of second order arithmetic determined by the amount of comprehension assumed). Here, characterizations correspond to reversals in the sense that one proves (over some weak system) the axioms of one of the subsystems of second order arithmetic from the statements of the mathematical theorems being investigated.

We briefly review the five standard systems of reverse mathematics. For completeness, we include systems stronger than arithmetical comprehension, but these will play no part in this paper. Details, general background, and results, as well as many examples of reversals, can be found in Simpson [1999], the standard text on reverse mathematics. Each of the systems is given in the language of second order arithmetic, that is, the usual first order language of arithmetic augmented by set variables and the membership relation $\in$. Each contains the standard basic axioms for $+$, $\cdot$, and $<$ (which say that $\mathbb{N}$ is an ordered semiring). In addition, they all include a form of induction that applies only to sets (that happen to exist):

($\mathbf{I_0}$) $\quad (0 \in X \wedge \forall n\,(n \in X \to n+1 \in X)) \to \forall n\,(n \in X)$.

We call the system consisting of I_0 and the basic axioms of ordered semirings P_0. All the five standard systems are defined by adding various types of set existence axioms to P_0. They also correspond to classical construction principles in computability theory.

($\mathbf{RCA_0}$) Recursive Comprehension Axioms: This is a system just strong enough to prove the existence of the computable sets but not of $0'$ nor indeed of any noncomputable set. In addition to P_0 its axioms include the schemes of Δ^0_1 comprehension and Σ^0_1 induction:

($\mathbf{\Delta^0_1}$-$\mathbf{CA_0}$) $\quad \forall n\,(\varphi(n) \leftrightarrow \psi(n)) \to \exists X\, \forall n\,(n \in X \leftrightarrow \varphi(n))$ for all Σ^0_1 formulas φ and Π^0_1 formulas ψ in which X is not free.

($\mathbf{I\Sigma_1}$) $\quad (\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1))) \to \forall n\, \varphi(n)$ for all Σ^0_1 formulas φ.

The next system says that every infinite binary tree has an infinite path. It is connected to the Low Basis Theorem (Jockusch and Soare [1972]) of computability theory, which says that every such tree has an infinite path whose jump is computable in that of the tree itself.

($\mathbf{WKL_0}$) Weak König's Lemma: This system consists of RCA_0 plus the statement that every infinite subtree of $2^{<\omega}$ has an infinite path.

We next move up to arithmetic comprehension.

($\mathbf{ACA_0}$) Arithmetic Comprehension Axioms: This system consists of RCA_0 plus the axioms $\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$ for every arithmetic formula φ in which X is not free.

In computability theoretic terms, ACA_0 proves the existence of $0'$ and by relativization it proves, and in fact is equivalent to, the existence of X' for every set X.

The next system corresponds to the existence of all (relativized) H-sets, i.e. the existence, for every set X, of the H_e^X (and so the hyperarithmetic hierarchy up to e) for each $e \in \mathcal{O}^X$, the hyperjump of X. It says that arithmetic comprehension can be iterated along any countable well order.

($\mathbf{ATR_0}$) Arithmetical Transfinite Recursion: This system consists of RCA_0 plus the following axiom. If X is a set coding a well order $<_X$ with domain D and Y is a code for a set of arithmetic formulas $\varphi_x(z, Z)$ (indexed by $x \in D$) each with one free set variable and one free number variable, then there is a sequence $\langle K_x \mid x \in D\rangle$ of sets such that if y is the immediate successor of x in $<_X$, then $\forall n\,(n \in K_y \leftrightarrow \varphi_x(n, K_x))$, and if x is a limit point in $<_X$, then K_x is $\bigoplus\{K_y \mid y <_X x\}$.

The systems climbing up to full second order arithmetic (i.e. comprehension for all formulas) are classified by the syntactic level of the second order formulas for which we assume a comprehension axiom.

($\mathbf{\Pi^1_n}$-$\mathbf{CA_0}$) Π^1_n Comprehension Axioms: $\exists X\,\forall k\,(k \in X \leftrightarrow \varphi(k))$ for every Π^1_n formula φ in which X is not free.

The computability theoretic equivalent of the simplest of these systems, Π^1_1-CA_0, is the existence of $\mathcal{O}^X$ for every set X. Together with the four systems listed above, it makes up the standard list of the axiomatic systems of reverse mathematics. Almost all theorems of classical mathematics whose proof theoretic complexity has been determined have turned out to be equivalent to one of them.

The early connections between computability theoretic ideas and methods on the one hand and reverse mathematics in the other typically involved computable mathematics, diagonalization or finite injury arguments, and coding. Consider, for example, a theorem of the form $\forall A\,(\Theta(A) \to \exists B\,\Phi(A, B))$ where Θ and Φ are arithmetic. We call B a *solution* for the instance of the theorem specified by A if $\Theta(A) \to \Phi(A, B)$.

A construction of computable mathematics that shows that there is a solution B computable in any given A generally shows that the theorem is provable in RCA_0. One that shows that B can be obtained arithmetically in A usually shows that the theorem is provable in ACA_0. Standard forms of applications of computability theoretic diagonalization or finite injury results to reverse mathematical calibrations are as follows:

1. If there is no solution B computable in some given A then the theorem is not provable in RCA_0.
2. If there is no solution B that is low over A, i.e. $(B \oplus A)' \equiv_{\mathrm{T}} A'$, then the theorem is not provable in WKL_0.
3. If there is no solution B arithmetic in A then the theorem is not provable in ACA_0.

Coding methods tend to give reversals.

1. If, for any computable tree T, there is a computable instance of the theorem such that any solution codes a path through T then the theorem usually implies WKL_0.
2. If there is a computable instance of the theorem such that any solution computes $0'$ then the theorem usually implies ACA_0.
3. If there is a computable instance of the theorem such that any solution computes $\mathcal{O}$ then the theorem usually implies Π^1_1-CA_0.

We are particularly interested in relations derived from more complex computability theoretic results and constructions, and in theorems that do not correspond to any of the standard systems. In this paper we deal with ones that are strictly below ACA_0. A special inspiration for our investigations here comes from the work of Cholak, Jockusch, and Slaman [2001] (hereafter CJS). They deal with various versions of Ramsey's Theorem and some of their consequences. (Their paper is also a good source of background information and history.) We will summarize some of their results to set the stage for ours, which deal with other consequences of Ramsey's Theorem applied to linear and partial orders. We begin with the statements of some of the theorems they analyze and basic results about their strength. We view each of these theorems as a subsystem of second order arithmetic by adding it to RCA_0. For a map f and a subset X of its domain, we denote the image of X under f by $f``X$.

Definition 1.1. An *n-coloring* (*partition*) of $[\mathbb{N}]^k$, the unordered k-tuples $(n_1, \dots, n_k)$ of natural numbers (listed by convention in increasing order), is a map $f : [\mathbb{N}]^k \to n$. A subset H of $\mathbb{N}$ is *homogeneous* for the coloring f if H is infinite and $|f``[H]^k| = 1$. Unless otherwise stated all colorings will be 2-colorings of $[\mathbb{N}]^2$.

($\mathbf{RT^2_2}$) Ramsey's Theorem for pairs: Every 2-coloring of $[\mathbb{N}]^2$ has a homogeneous set.

It is easy to show that the number of colors does not matter, in the sense that for each $n > 2$, Ramsey's Theorem for 2-colorings of pairs is equivalent to Ramsey's Theorem for n-colorings of pairs. As we will see, however, the size of the tuples does matter.

Ramsey's Theorem, its variants and their consequences have long been a subject of great interest in terms of characterizing their complexity in both computability theoretic and proof theoretic terms. Specker [1971] constructed a computable coloring of $[\mathbb{N}]^2$

with no computable homogeneous set. Thus $\mathrm{RCA}_0 \nvdash \mathrm{RT}^2_2$. A now classic early result concerns the analog for colorings of triples. Jockusch [1972] constructed a computable coloring of $[\mathbb{N}]^3$ such that every homogeneous set computes $0'$. This construction can be carried out in RCA_0, and thus shows that Ramsey's Theorem for triples implies ACA_0. (All such statements about implications will be understood to be over the base theory RCA_0 unless some other system is specifically mentioned.) In fact, one of the standard proofs of Ramsey's Theorem for k-tuples works in ACA_0 (for any fixed k), and so for each $k > 2$, Ramsey's Theorem for k-tuples is equivalent to ACA_0. Whether RT^2_2 itself also implies ACA_0 remained open for twenty years. Seetapun (see Seetapun and Slaman [1995]) proved a degree theoretic cone avoiding theorem (for every set Z, coloring f of $[\mathbb{N}]^2$ computable in Z, and sets $C_i \not\leq_{\mathrm{T}} Z$, there is a homogeneous set not computing any C_i) that implies that RT^2_2 does not imply ACA_0 (even over WKL_0).

Another view of the proof theoretic strength of such theorems and systems is provided by the analysis of their first order consequences and conservativity results. For example, even though WKL_0 is strictly stronger than RCA_0, the former is Π^1_1-conservative over the latter (Harrington, see Simpson [1999, §IX.2]), i.e. any Π^1_1 sentence provable in WKL_0 is already provable in RCA_0. Thus, in particular, the first order consequences and consistency strengths of the two systems are the same. While it is a major open question whether RT^2_2 actually implies WKL_0, RT^2_2 is known to be strictly stronger than WKL_0 in this proof theoretic sense. The relevant principle is Σ_2-bounding. We state the bounding principle as well as the related induction principle for an arbitrary class Γ of formulas.

- **(BΓ)** $(\forall i < n)(\exists x)\varphi(x) \to (\exists u)(\forall i < n)(\exists x < u)\varphi(x)$ for every formula $\varphi(x) \in \Gamma$.
- **(IΓ)** $(\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1))) \to \forall n\,\varphi(n)$ for every formula $\varphi(x) \in \Gamma$.

Now these principles produce a strict hierarchy in both first and second order arithmetic, with $\mathrm{B}\Sigma_{n+1}$ caught strictly between $\mathrm{I}\Sigma_n$ and $\mathrm{I}\Sigma_{n+1}$ (Paris and Kirby [1978], see also Hájek and Pudlák [1998. IV]). Moreover, even in the first order case, $\mathrm{B}\Sigma_{n+1}$ is not Σ^0_{n+2}-conservative over $\mathrm{I}\Sigma_n$ (Paris [1980, p. 331]). Thus, as Hirst [1987] showed that RT^2_2 implies $\mathrm{B}\Sigma_2$, RT^2_2 is not even Σ^0_3-conservative over RCA_0. (By Friedman [1976], any model M of $\mathrm{I}\Sigma_n$ can always be extended to a model of $\mathrm{RCA}_0 + \mathrm{I}\Sigma_n$ with the same first order part by taking the sets to be those with Δ^0_1 definitions over M.) On the other hand, CJS show that RT^2_2 is Π^1_1-conservative over $\mathrm{I}\Sigma_2$ and so far weaker, in this sense, than ACA_0, which obviously implies $\mathrm{I}\Sigma_n$ for each n.

The more recent analyses of RT^2_2, both computability theoretic and proof theoretic, have crucially relied on splitting the theorem/system into two parts. One (COH) is used to simplify the coloring and make it stable. The other (SRT^2_2) restricts the assertion of the existence of homogeneous sets to stable colorings.

Definition 1.2. A coloring f of $[\mathbb{N}]^2$ is *stable* if $(\forall x)(\exists y)(\forall z > y)[f(x,y) = f(x,z)]$.

Definition 1.3. If $\vec{R} = \langle R_i \mid i \in \mathbb{N}\rangle$ is a sequence of sets, an infinite set S is *$\vec{R}$-cohesive* if $(\forall i)(\exists s)[(\forall j > s)(j \in S \to j \in R_i) \vee (\forall j > s)(j \in S \to j \notin R_i)]$.

($\mathbf{SRT^2_2}$) Stable Ramsey's Theorem for pairs: Every *stable* coloring of $[\mathbb{N}]^2$ has a homogeneous set.

(COH) Cohesive Principle: For every sequence $\vec{R} = \langle R_i \mid i \in \mathbb{N}\rangle$ there is an $\vec{R}$-cohesive set.

Note that COH easily implies the following principle, which asserts that for every coloring of $[\mathbb{N}]^2$ there is a set such that the coloring restricted to that set is stable (in the obvious sense).

($\mathbf{CRT^2_2}$) Cohesive Ramsey's Theorem for pairs: For every coloring f of $[\mathbb{N}]^2$ there is an infinite set S such that $(\forall x \in S)(\exists y)(\forall z \in S)[z > y \to f(x, y) = f(x, z)]$.

Proposition 1.4 (CJS). $\mathrm{RCA}_0 \vdash \mathrm{COH} \to \mathrm{CRT}^2_2$.

Proof. If f is a coloring of $[\mathbb{N}]^2$ and we set $R_i = \{j > i \mid f(i, j) = 0\}$ then any $\vec{R}$-cohesive set S is as required in CRT^2_2. □

By the theorem and system RT^2_2 splitting into SRT^2_2 and COH we mean the fact that $\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \leftrightarrow \mathrm{COH} \wedge \mathrm{SRT}^2_2$. It is immediate from the above proposition that $\mathrm{COH} \wedge \mathrm{SRT}^2_2$ implies RT^2_2, and RT^2_2 obviously implies SRT^2_2. The fact that RT^2_2 implies COH is harder to show. CJS provide (in the proof of Theorem 12.5) an easy proof that requires $\mathrm{I}\Sigma_2$. Their proof in RCA_0 (in Theorem 7.11) is not correct but a more complicated argument that dispenses with $\mathrm{I}\Sigma_2$ is given by Mileti [2004] (and was independently discovered by Jockusch and Lempp). The converse of Proposition 1.4 is clearly related to this issue as well as to alternate versions of other principles related to $\mathrm{B}\Sigma_2$ that we consider in §4. For now we note that there are natural stronger forms of each of these principles (StCRT^2_2 and StCOH, respectively) that are equivalent to each other and to $\mathrm{CRT}^2_2 + \mathrm{B}\Sigma_2$ and $\mathrm{COH} + \mathrm{B}\Sigma_2$, respectively, and so $\mathrm{CRT}^2_2 + \mathrm{B}\Sigma_2$ implies COH (Proposition 4.8).

Principles lying below WKL_0 have also been studied. A function f is *diagonally noncomputable* relative to A if $\forall e\,(f(e) \neq \Phi^A_e(e))$, where Φ_e is the eth Turing functional.

(DNR) Diagonally Nonrecursive Principle: For every set A there is a function f that is diagonally noncomputable relative to A.

Clearly, DNR fails in COMP, the ω-model of RCA_0 whose sets are precisely the computable sets. (An *ω-model* is a structure in the language of second order arithmetic whose first order part is standard.) Giusto and Simpson [2000] showed that DNR follows from the system WWKL_0, which states that if a binary tree T has no infinite path then $\lim_n |\{\sigma \in T \mid |\sigma| = n\}|/2^n = 0$, and has played a role in studying the proof theoretic content of theorems from analysis, as it is strong enough to develop a fair amount of measure theory. The system WWKL_0 was shown to be strictly weaker than WKL_0 by Simpson and Yu [1990] but strictly stronger than DNR by Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [2004]. Recently, Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta] have shown that DNR follows from SRT^2_2 over RCA_0.

We now summarize the known degree theoretic results (primarily from CJS) about the above principles related to Ramsey's Theorem. Some of them directly give reverse mathematical results and the proofs of others play a crucial role in conservativity results. We also include previous results about WKL_0 to help place it with respect to these systems.

Definition 1.5. A degree $\mathbf{a}$ is *PA over* $\mathbf{b}$, written $\mathbf{a} \gg \mathbf{b}$, if $\mathbf{a}$ computes a separating set for any pair of disjoint sets c.e. in $\mathbf{b}$; or, equivalently, an infinite path in any infinite tree computable in $\mathbf{b}$; or a total extension for any $\{0,1\}$-valued function partial computable in $\mathbf{b}$. (See Simpson [1977, pp. 648-649] and CJS, §4.8.) In this case, if $A \in \mathbf{a}$ and $B \in \mathbf{b}$, we also write $A \gg B$.

COH: Every uniformly computable sequence $\vec{R}$ has a Δ^0_2 cohesive set, but for some such $\vec{R}$ the only Δ^0_2 cohesive sets S are high, i.e. $S' \equiv_{\mathrm{T}} 0''$ (Cooper [1972], Jockusch and Stephan [1993]). Moreover, for any degree $\mathbf{d} \gg \mathbf{0}'$ every uniformly computable sequence $\vec{R}$ has a cohesive set S with $S' \leqslant_{\mathrm{T}} \mathbf{d}$. As there are $\mathbf{d} \gg \mathbf{0}'$ with $\mathbf{d}' = \mathbf{0}''$ we may always take S to be low_2, i.e. $S'' \equiv_{\mathrm{T}} 0''$. There is also a single such $\vec{R}$ (the sequence of primitive recursive sets) such that for any S cohesive for it, $S' \gg 0'$ (Jockusch and Stephan [1993]). COH is also Π^1_1-conservative over RCA_0 (CJS).

SRT^2_2: It is easy to see that every computable stable coloring of $[\mathbb{N}]^2$ has a Δ^0_2 homogeneous set, but there are such colorings that have no low homogeneous set (Downey, Hirschfeldt, Lempp and Solomon [2001]). On the other hand, every such coloring has a low_2 homogeneous set (CJS). SRT^2_2, like RT^2_2, proves $\mathrm{B}\Sigma_2$ and so is not even Σ^0_3-conservative over RCA_0 (CJS).

RT^2_2: There are computable colorings of $[\mathbb{N}]^2$ with no Δ^0_2 homogeneous sets (Jockusch [1972]) but, by combining the above results for COH and SRT^2_2, all have low_2 ones (CJS). Indeed, for any degree $\mathbf{d} \gg \mathbf{0}'$ and any computable coloring of $[\mathbb{N}]^2$ there is a homogeneous set H with $H' \leqslant_{\mathrm{T}} \mathbf{d}$ (CJS). Moreover, there is a computable coloring of $[\mathbb{N}]^2$ such that $H' \gg 0'$ for any homogeneous set H. As RT^2_2 proves $\mathrm{B}\Sigma_2$, it is not Σ^0_3-conservative over RCA_0.

WKL_0: As mentioned before, WKL_0 is Π^1_1-conservative over RCA_0 (Harrington). The degree theoretic precursor to this conservation result is the Low Basis Theorem of Jockusch and Soare [1972], which says that every infinite computable tree has a low infinite path. Indeed, there is a single low degree $\mathbf{d}$ that computes an infinite path through any computable infinite tree, i.e. a low $\mathbf{d} \gg \mathbf{0}$ (this follows from the Low Basis Theorem and the equivalence of various standard characterizations of $\mathbf{a} \gg \mathbf{0}$ as in CJS, Lemma 8.17).

The following diagram summarizes our state of knowledge about the relations among these four systems (as well as DNR and $\mathrm{B}\Sigma_2$) as presented primarily in CJS. Double arrows indicate a strict implication and single ones an implication that is not known to be strict. Negated arrows indicate known nonimplications.

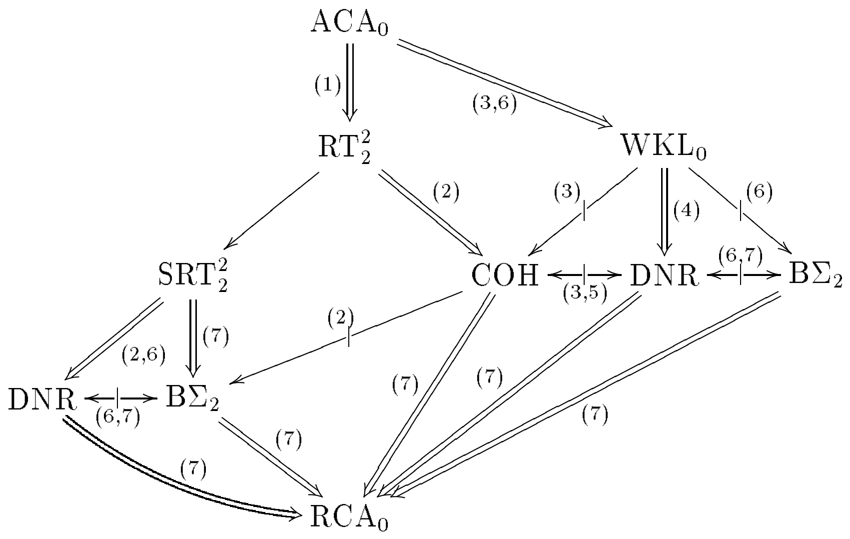

Diagram 1

The reasons for the nonimplications indicated here have all been described and attributed above and are listed below. The implications are all easy except for the following: $RT^2_2 \to$ COH, which was shown by CJS in the presence of $I\Sigma_2$ and in general by Mileti [2004] and Jockusch and Lempp; $SRT^2_2 \to B\Sigma_2$ (CJS); and $SRT^2_2 \to$ DNR (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta]).

1. The degree theoretic cone avoiding argument of Seetapun (in Seetapun and Slaman [1995]) or the conservation result over $I\Sigma_2$ of CJS, Theorem 10.2.

2. COH is Π^1_1-conservative over RCA_0 (CJS, Theorem 9.1) but RT^2_2 is not as it implies $B\Sigma_2$ (Hirst [1987]). In fact, even SRT^2_2 implies $B\Sigma_2$ (CJS).

3. WKL_0, and hence DNR, have a model with only low sets but COH does not.

4. The results of Simpson and Yu [1990] and Giusto and Simpson [2000] mentioned above.

5. That COH does not imply WKL_0 is proved in CJS by a direct forcing argument. (See CJS, §9.5 for both directions.) We supply a stronger, more general conservation result along these lines in Theorem 2.20 and Proposition 2.21, which also applies to DNR. Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta] have also shown that COH does not imply DNR.

6. WKL_0, and hence DNR, are Π^1_1-conservative over RCA_0 (Harrington, see Simpson [1999, §IX.2]) and so do not imply $B\Sigma_2$.

7. None of the five principles RT_2^2, SRT_2^2, COH, WKL_0, and DNR follow from RCA_0 (even with full induction) as they all have computable instances without computable solutions as described above. $\mathrm{B}\Sigma_2$ is not a consequence of RCA_0 either (Paris and Kirby [1978], see also Hájek and Pudlák [1998, Chapter IV]).

The primary combinatorial/reverse mathematical questions left open here are whether SRT_2^2 implies COH (and so RT_2^2) and whether SRT_2^2 or RT_2^2 implies WKL_0. The primary purely proof theoretic one is whether SRT_2^2 or RT_2^2 is arithmetically conservative over $\mathrm{B}\Sigma_2$. In addition, CJS raise the issue of a specific application of RT_2^2 to partial orders and ask about its relation to RT_2^2.

(CAC) Chain-AntiChain: Every infinite partial order $(P, \leqslant_P)$ has an infinite subset S that is either a *chain*, i.e. $(\forall x, y \in S)(x \leqslant_P y \vee y \leqslant_P x)$, or an *antichain*, i.e. $(\forall x, y \in S)(x \neq y \rightarrow (x \nleqslant_P y \wedge y \nleqslant_P x))$.

It is easy to see that RT_2^2 implies CAC just by coloring (x, y) 0 (red) if x and y are comparable in $\leqslant_P$ and 1 (blue) otherwise. Any homogeneous set is the required chain or antichain. It was left as Question 13.8 in CJS whether CAC implies RT_2^2. CAC is degree theoretically similar to RT_2^2, as by Herrmann [2001] there are computable partial orders with no Δ_2^0 infinite chains or antichains. However, we show (Corollary 3.11) that CAC does not imply DNR (over RCA_0). Combining this with the result of Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta] mentioned above, we now know that CAC does not imply even SRT_2^2.

There is a well known analogous principle for linear orders. (We typically use $\leqslant$ to denote the usual ordering on $\mathbb{N}$, though in some cases we use $\leqslant_{\mathbb{N}}$ to avoid possible confusion.)

(ADS) Ascending or Descending Sequence: Every infinite linear order $(L, \leqslant_L)$ has an infinite subset S that is either an ascending sequence, i.e. $(\forall s < t)(s, t \in S \rightarrow s <_L t)$, and so of order type ω, or a descending sequence, i.e. $(\forall s < t)(s, t \in S \rightarrow t <_L s)$, and so of order type ω^*.

This principle also follows easily from RT_2^2 by coloring (x, y) red if $x <_L y$ and blue otherwise, i.e. $x >_L y$. Here it is known that there is a computable linear order (even one of type $\omega + \omega^*$) with no computable ascending or descending sequence (Tennenbaum (see Rosenstein [1982]) and Denisov (see Goncharov and Nurtazin [1973])). Thus $\mathrm{RCA}_0 \nvdash$ ADS. On the other hand, every computable linear order has an ascending or descending sequence computable in $0'$, in fact one that is Π_1^0 (Manaster; see Downey [1998, §5] for this and all the results we mention about these issues).

Our goal is to analyze both CAC and ADS computability theoretically and in the terms of reverse mathematics along the lines followed for RT_2^2 in CJS. Each will be split into a stable version (SCAC; SADS) and a cohesive principle (CCAC; CADS (or related principles that work for both)). In these situations, however, along with various other degree and proof theoretic results, we will be able to show that the splittings are strict (that is, SCAC and SADS do not imply CAC and ADS, respectively, nor do CADS, COH, or CCAC). Moreover, none of these principles will imply DNR or be implied by WKL_0. We will also study the relationship between $\mathrm{B}\Sigma_2$ and these principles and prove some conservation results extending those for COH over RCA_0. Our methods will include priority arguments, forcing constructions, and computable combinatorics.

Our proofs suggest that the essential distinction between ADS and CAC on the one hand and RT^2_2 on the other is that the colorings needed for our analysis are in some way transitive. We formalize this intuition in §5 as the notions of transitive and semitransitive colorings and show that the existence of homogeneous sets for such colorings is equivalent to ADS and CAC, respectively.

We begin with linear orders and some classical computability theoretic results about ADS which motivated our choice of principles and analysis. From now on, all linear or partial orders and chains or antichains within them will be assumed to be infinite unless otherwise stated. When we consider a linear order $\mathcal{L}$, we denote its order relation by $\leqslant_L$. Similarly, we denote the order relation of a partial order $\mathcal{P}$ by $\leqslant_P$.

2 Linear Orders and ADS

An infinite linear order is *computable* if it has domain $\mathbb{N}$ and its ordering relation is computable. There are a number of interesting classical results of computable mathematics about the existence and complexity types of suborders of, and embeddings into, an arbitrary computable linear order. A good reference including proofs is Downey [1998]. The ones relevant to our investigations have to do with suborders of a given order $\mathcal{L}$ of order type ω, ω^*, or $\omega+\omega^*$ (or embeddings of canonical representatives of these types into $\mathcal{L}$). We now state the ones we need.

Theorem 2.1. (Tennenbaum (see Rosenstein [1982]) and Denisov (see Goncharov and Nurtazin [1973]); see also Downey [1998]) *There is a computable linear order of type $\omega+\omega^*$ with no computable suborder of type ω or ω^*.*

The proof involves a finite injury priority argument.

Theorem 2.2. (Lerman [1981]) *There is a computable linear order with no computable suborder of type ω, ω^*, or $\omega+\omega^*$.*

The proof here uses an infinite injury priority argument.

These theorems suggest principles that would be shown by them to be not provable in RCA_0. To state and analyze them in the setting of reverse mathematics we should first be precise about what we mean by having order type ω, ω^*, and $\omega+\omega^*$ in RCA_0.

Definition 2.3. An infinite linear order in which all nonfirst elements have immediate predecessors and all nonlast ones have immediate successors has type

- ω if every element has finitely many predecessors;.
- ω^* if every element has finitely many successors;
- $\omega + \omega^*$ if it is not of type ω or ω^* and every element has either finitely many predecessors or finitely many successors.

Proposition 2.4. ADS *is equivalent over* RCA_0 *to the statement that every linear order* $\mathcal{L}$ *has a suborder of type* ω *or* ω^*.

Proof. An ascending or descending sequence in $\mathcal{L}$ is clearly a suborder of type ω or ω^*, respectively. In the other direction, suppose we are given a subset R of L of order type ω in the order inherited from $\mathcal{L}$ (the ω^* case being symmetric). We can, in RCA_0, define by recursion a function that lists an infinite subset $S = \langle s_i \mid i \in \mathbb{N} \rangle$ of R in increasing natural order (the order of $\mathbb{N}$). Having defined s_n, we need only find an $a \in R$ such that $a >_L s_n$ and $a >_{\mathbb{N}} s_n$, which exists because R has order type ω, and define $s_{n+1} = a$, so this recursion can be performed in RCA_0. The range of S exists provably in RCA_0, because every element b of R has only finitely many predecessors, and hence there is an i such that $b <_L s_i$. If $b \neq s_j$ for $j < i$, then b is not in the range of S. The range of S is the desired ascending sequence in $\mathcal{L}$. □

Corollary 2.5. $\mathrm{RCA}_0 \nvdash$ ADS.

Proof. By Theorem 2.1, COMP, the ω-model of RCA_0 whose sets are precisely the computable sets, is not a model of ADS. □

Every linear order $\mathcal{L}$ of type $\omega + \omega^*$ is stable in a sense analogous to that for colorings, i.e. $(\forall x)(\exists y)(\forall z > y)[x <_L y \leftrightarrow x <_L z]$. Thus the analog of SRT^2_2 is SADS:

(SADS) Stable ADS: Every linear order of type $\omega + \omega^*$ has a subset of order type ω or ω^*.

Similarly, reducing to an order of type $\omega + \omega^*$ corresponds to producing a cohesive set, as in such an order S, we have $(\forall i \in S)(\exists s)[(\forall j > s)(j \in S \rightarrow i <_L j) \vee (\forall j > s)(j \in S \rightarrow i >_L j)]$. This gives us CADS as the analog of COH.

(CADS) Cohesive ADS: Every linear order has a subset S of order type ω, ω^*, or $\omega + \omega^*$.

Of course, CADS says that every linear order has a stable suborder the same way that CRT^2_2 or COH say that for every coloring there is a set on which the coloring is stable. Theorems 2.1 and 2.2 now prove that these principles do not hold in COMP and so do not follow from RCA_0.

Corollary 2.6. $\mathrm{RCA}_0 \nvdash$ SADS*;* $\mathrm{RCA}_0 \nvdash$ CADS.

As with RT^2_2, it is immediate that these principle split ADS.

Proposition 2.7. $\mathrm{RCA}_0 \vdash \mathrm{ADS} \leftrightarrow \mathrm{SADS} + \mathrm{CADS}$.

We show that they also follow from SRT^2_2 and COH, respectively.

Proposition 2.8. $\mathrm{RCA}_0 \vdash \mathrm{SRT}^2_2 \rightarrow \mathrm{SADS}$.

Proof. Given a linear order $\mathcal{L}$ of type $\omega + \omega^*$, color (m, n) blue if $m <_L n$; otherwise, i.e. if $n <_L m$, color (m, n) red. This is a stable coloring by the definition of order type $\omega + \omega^*$. A blue homogeneous set has order type ω. A red one has order type ω^*. □

Proposition 2.9. $\mathrm{RCA}_0 \vdash \mathrm{CRT}^2_2 \rightarrow \mathrm{CADS}$ *and so* $\mathrm{RCA}_0 \vdash \mathrm{COH} \rightarrow \mathrm{CADS}$.

Proof. Given a linear ordering $\mathcal{L}$, color (m, n) blue if $m <_L n$; otherwise color (m, n) red. Let S be a set given by CRT^2_2 for this coloring. Then the coloring restricted to S is stable, which means that each element of S has either finitely many $<_L$-predecessors in S or finitely many $<_L$-successors in S, as required. □

In addition, we have one more combinatorial connection between ADS and COH.

Proposition 2.10. $\mathrm{RCA}_0 \vdash \mathrm{ADS} \rightarrow \mathrm{COH}$.

Proof. Given $\vec{R} = \langle R_i \mid i \in \mathbb{N} \rangle$ define a linear order $\mathcal{L}$ on $\mathbb{N}$ by setting $x <_L y \leftrightarrow \langle R_i(x) \mid i \leqslant x \rangle <_{lex} \langle R_i(y) \mid i \leqslant y \rangle$, where $<_{lex}$ is the lexicographic order on $2^{<\omega}$. This order is just the lexicographic order on a subset of $2^{<\omega}$ that contains exactly one string of each length. We claim that any infinite ascending or descending sequence S in this order is $\vec{R}$-cohesive. Suppose S is ascending (the descending case being symmetric) and consider any $j \in \mathbb{N}$. Let $\sigma \in 2^{j+1}$ be the lexicographically largest element of 2^{j+1} such that $\sigma \leqslant_{lex} \langle R_i(x) \mid i \leqslant x \rangle$ for some $x \in S$. (Such a σ exists by Σ^0_1 induction.) Fix such an x. Since S is ascending in $\mathcal{L}$, for every element $y \geqslant_{\mathbb{N}} x$ of S, we have $\sigma \leqslant$

lex$\langle R_i(y) \mid i \leqslant j \rangle$. But no such sequence can be to the right of σ by our choice of σ. Thus $\langle R_i(y) \mid i \leqslant j \rangle = \sigma$ for every $y \geqslant_{\mathbb{N}} x$ in S. In particular, $R_j(y)$ has the same value for cofinitely many $y \in S$, and so S is $\vec{R}$-cohesive as desired. □

We now turn to a degree theoretic analysis of these principles that will provide proofs that most of the above implications cannot be reversed. For the first one we provide a priority argument here and a forcing one for a related result that could be adapted to a proof of this theorem in Proposition 3.4. The forcing argument is simpler and probably easier to understand than this one, so readers who prefer that kind of argument may wish to skip this proof. On the other hand, this argument is relevant to issues of conservativity, and in particular contrasts with a similar argument in an upcoming paper by Hirschfeldt, Shore, and Slaman on the Atomic Model Theorem, in which a blocking technique is employed that does not work in this case. See §6 for further discussion of conservativity issues in relation to the following proof.

Theorem 2.11. *Every X-computable linear order of type $\omega + \omega^*$ contains a suborder A of type ω or ω^* that is low over X, i.e. $(A \oplus X)' \leqslant_T X'$.*

Proof. We consider the case that X is computable. The general one follows by relativization. Let $\prec$ be a computable linear order of type $\omega + \omega^*$. Let U be the ω part of $\prec$. Clearly, U is Δ^0_2. We build an infinite Δ^0_2 subset A of U such that if A is not low then $\prec$ contains an infinite computable descending sequence. Of course, from A we can computably obtain an infinite ascending sequence in $\prec$.

We have two kinds of requirements:

$$P_e : (\exists n > e)(n \in A)$$

and

$$N_e : \exists^\infty s\,(\Phi^A_e(e)[s]\downarrow) \to \Phi^A_e(e)\downarrow .$$

We assign priorities to these requirements in the usual way. We denote the use function of Φ_e by φ_e.

The idea for satisfying N_e is that, whenever $\Phi^A_e(e)[s]\downarrow$, we try to preserve $A[s] \upharpoonright \varphi^A_e(e)[s]$ to the extent allowed by the need to ensure that $A \subseteq U$. If we fail, then $U[s] \upharpoonright \varphi^A_e(e)[s]$ must contain an element not in U. Because U is an initial segment of $\prec$, this means that the $\prec$-last element of $U[s] \upharpoonright \varphi^A_e(e)[s]$ must be in $\overline{U}$. So if $\exists^\infty s\,(\Phi^A_e(e)[s]\downarrow)$ but $\Phi^A_e(e)\uparrow$, which means that all our attempts at preservation fail, then we have a computable list of elements of $\overline{U}$, from which we can obtain a computable infinite descending sequence in $\prec$.

We adopt the convention that if a computation changes then it becomes undefined for at least one stage. We also assume we have speeded up the computable approximation to U so that for any s there is an element of $U[s]$ greater than every number mentioned in the construction before stage s. We also assume that the approximation obeys the order, i.e. if $n \prec m$ and $m \in U[t]$ then $n \in U[t]$.

At stage s, we say that P_e requires attention if the $\prec$-least element of $U[s]$ larger (in the $\mathbb{N}$ sense) than e and the restraints of stronger priority N-requirements is not in A. We say that N_e requires attention if either

1. $\Phi^A_e(e)[s-1]\uparrow$ and $\Phi^A_e(e)[s]\downarrow$, or
2. $\Phi^A_e(e)[s]\uparrow$ and there is a $t < s$ such that
   - (a) t is greater than the last stage before s at which a requirement stronger than N_e required attention,
   - (b) $\Phi^A_e(e)[t]\downarrow$,
   - (c) $A[t] \upharpoonright \varphi^A_e(e)[t] \subseteq U[s]$, and
   - (d) $A[t] \upharpoonright \varphi^A_e(e)[t] \not\subseteq A[s]$.

We begin the stage by removing from A all elements not in $U[s]$. Then, for the strongest requirement X that requires attention at stage s, we act as follows.

If $X = P_e$ then let n be the $\prec$-least element of $U[s]$ larger (in the $\mathbb{N}$ sense) than e and the restraints of stronger priority N-requirements and put n in A.

If $X = N_e$ then there are two cases. If condition 1 above holds then declare the restraint of N_e to be the maximum of its previous value (if any) and $\varphi_e^A(e)[s]$. If condition 2 holds then let t be the least stage satisfying the condition. For every $n \in A[t] \upharpoonright \varphi_e^A(e)[t]$ such that n is larger (in the $\mathbb{N}$ sense) than the restraints of stronger priority N-requirements, put n in A.

This completes the construction. We now verify its correctness.

Assume by induction that all requirements stronger than P_e are satisfied and eventually stop requiring attention. Then each such N-requirement has a final restraint. Let r be the maximum of these restraints, and let n be the $\prec$-least element of U larger (in the $\mathbb{N}$ sense) than e and r. There is a stage s after which no requirement stronger than P_e requires attention and n does not leave U. If there is a stage $t > s$ such that $n \notin A[t]$ then n enters A at stage t and never leaves A after that. So P_e is satisfied, and hence eventually stops requiring attention.

Now assume by induction that all requirements stronger than N_e are satisfied and eventually stop requiring attention. Let s be the least stage after which no such requirement ever requires attention. Note that A is constant up to the maximum of the restraints imposed by these requirements from stage s onward. There are three cases.

First suppose that for some $t > s$ we have $\Phi_e^A(e)[t]\downarrow$ and $A[t] \upharpoonright \varphi_e^A(e)[t] \subseteq U$. For the least such t, there is a u such that for all $v > u$ we have $A[t] \upharpoonright \varphi_e^A(e)[t] \subseteq U[v]$. If $\Phi_e^A(e)[v]\downarrow$ for all $v > u$ then by convention $\Phi_e^A(e)\downarrow$, and hence N_e is satisfied. Otherwise, at the least stage $v > u$ such that $\Phi_e^A(e)[v]\uparrow$, the requirement N_e will act to put every element of $A[t] \upharpoonright \varphi_e^A(e)[t]$ into A if this is not already the case. Furthermore, no such number will ever leave A, and because of the restraint imposed by N_e, no number will ever enter $A \upharpoonright \varphi_e^A(e)[t]$ after stage t. Thus $A \upharpoonright \varphi_e^A(e)[t] = A[t] \upharpoonright \varphi_e^A(e)[t]$, which implies that $\Phi_e^A(e)\downarrow$, and hence that N_e is satisfied.

Otherwise, for every $t > s$ such that $\Phi_e^A(e)[t]\downarrow$, we have $A[t] \upharpoonright \varphi_e^A(e)[t] \not\subseteq U$. If there are only finitely many such t then N_e is vacuously satisfied, and it is easy to check that in this case N_e eventually stops requiring attention.

So we are left with the case in which $\exists^\infty t\, (\Phi_e^A(e)[t]\downarrow)$, but for every $t > s$ such that $\Phi_e^A(e)[t]\downarrow$, we have $A[t] \upharpoonright \varphi_e^A(e)[t] \not\subseteq U$. In this case we can define a computable infinite descending sequence $n_0 \succ n_1 \succ \cdots$ by recursion as follows. Let n_0 be the last element of $\prec$. Assuming that we have defined n_k so that $n_k \notin U$, search for a $t > s$ such that $\Phi_e^A(e)[t]\downarrow$ and every $n \in A[t] \upharpoonright \varphi_e^A(e)[t]$ is $\prec$-less than n_k. Such a t must eventually be found, since $A[u] \subseteq U[u]$ for all u. Let n_{k+1} be the $\prec$-greatest element of $A[t] \upharpoonright \varphi_e^A(e)[t]$. Since $A[t] \upharpoonright \varphi_e^A(e)[t] \not\subseteq U$, we must have $n_{k+1} \notin U$, and thus the recursion can continue. Since $n_0 \succ n_1 \succ \cdots$ and the sequence of n_k's is defined recursively, we are done. $\square$

Corollary 2.12. *There is an ω-model of* $\mathrm{RCA}_0 + \mathrm{SADS}$ *consisting entirely of low sets.*

Proof. Iterate and dovetail the construction of Theorem 2.11 to produce a sequence of low sets $X_0, X_1, \ldots$ so that for each i, every linear order of type $\omega+\omega^*$ computable in X_i has a suborder of type ω or ω^* computable in some X_j. The class of all sets computable in X_i for some i forms the second order part of an ω-model of RCA_0 + SADS. $\square$

On the other hand, Downey, Hirschfeldt, Lempp and Solomon [2001] have constructed a computable stable coloring of $[\mathbb{N}]^2$ with no low homogeneous set, and so the model of SADS constructed in Corollary 2.12 cannot be a model of SRT^2_2.

Corollary 2.13. SADS $\nvdash \mathrm{SRT}^2_2$.

Carl Jockusch has pointed out a surprising, purely degree theoretic consequence of Theorem 2.11:

Corollary 2.14. (Jockusch)*: Every* $\mathbf{c} \leqslant_{\mathbf{T}} \mathbf{0}'$ *is c.e. in a low degree.*

Proof. By Harizanov [1998] there is a $C \in \mathbf{c}$ and a computable linear order $\mathcal{L}$ of type $\omega+\omega^*$ such that C is the ω part of $\mathcal{L}$ and $\bar{C}$ is the ω^* part. By the Theorem there is a low A which is an infinite ascending (descending) sequence in $\mathcal{L}$. Thus $C = \{n | (\exists a \in A)(n \leqslant_{\mathcal{L}} a)\}$ $(\bar{C} = \{n | (\exists a \in A)(a \leqslant_{\mathcal{L}} n)\})$ as required. $\square$

We next show that neither SADS nor WKL_0 implies CADS, as each of them has an ω-model with only low sets while CADS does not.

Proposition 2.15. *There is a computable linear order with no low suborder of type* ω*,* ω^**, or* $\omega+\omega^*$*.*

Proof. Start with a computable infinite partial order $\mathcal{P}$ of $\mathbb{N}$ with no chain or antichain computable in $0'$, as given by Herrmann [2001]. Extend $\mathcal{P}$ to a computable linear order $\mathcal{L}$, i.e. $x \leqslant_P y \rightarrow x \leqslant_L y$. (That there is such an $\mathcal{L}$ is cited as a folklore version of the classical theorem of Szpilrajn [1930] and proven in Downey [1998, Observation 6.1].) We claim $\mathcal{L}$ is the desired linear order. Suppose not. Then, by Theorem 2.11, $\mathcal{L}$ has a low ascending or descending sequence $\mathcal{K} = \{k_i \mid i \in \mathbb{N}\}$. Suppose that $\mathcal{K}$ is ascending; the other case is symmetric. We construct a chain or antichain in $\mathcal{P}$ computably in $\mathcal{K}' = 0'$ for the required contradiction. For each $i < j$, either $k_i <_P k_j$ or $k_i \,|_P\, k_j$. Computably in $\mathcal{K}'$ build a series of subsequences $l_{i,n}$ as follows. Let $l_{0,0} = k_0$. Given $l_{i,n}$, let $l_{i,n+1}$ be k_j for the least j such that $l_{i,n} <_P k_j$ if there is such a j; otherwise terminate the i^{th} subsequence with $k_{n_i} = l_{i,n}$ and begin the $(i+1)^{\text{st}}$ with $l_{i+1,0} = k_{n_i+1}$. Note that if k_{n_i} is defined then $k_{n_i} \nless_P k_j$ for any $j > n_i$ and so all the k_{n_i} are incomparable in $\mathcal{P}$. Thus, if one of the subsequences $l_{i,n}$ is infinite it is the desired chain in $\mathcal{P}$, and if not, then the k_{n_i} form the desired antichain in $\mathcal{P}$. $\square$

Corollary 2.16. SADS $\nvdash$ CADS*;* $\mathrm{WKL}_0 \nvdash$ CADS.

We next show that WKL_0 does not imply SADS, by proving two technical computability theoretic theorems involving the following ad-hoc notion. We write $X \leqslant_{\mathrm{T}}^{\mathrm{b}} Y \oplus Z$ if there is a functional Φ and a computable function f such that $\Phi^{Y \oplus Z} = X$ and for each n the computation of $\Phi^{Y\oplus Z}(n)$ queries at most the first $f(n)$ many bits of Y.

Theorem 2.17. *Let A and B be c.e. sets. There is a computable linear order of type $\omega + \omega^*$ with the following property: For any ascending or descending suborder C, either $B \leqslant_T^b A \oplus C$ or $A \leqslant_T^b B \oplus C$.*

Proof. Let A and B be infinite, coinfinite c.e. sets. We make it a convention that at each stage s at most one element k enters at most one of A or B, and $k \leqslant s$.

We build a computable linear order $\prec$ while simultaneously defining Δ^0_2 sets U and D. The idea is that $\prec$ will have order type $\omega + \omega^*$, with U being the elements in the ω part and D those in the ω^* part. We will also have movable markers M_i and N_i for $i \in \omega$. The positions of M_i and N_i at stage s of the construction will be denoted by m_i^s and n_i^s, respectively, and the limits $m_i = \lim_s m_i^s$ and $n_i = \lim_s n_i^s$ will exist. Furthermore, m_i will be computable from $B \restriction i+1$ and $A \restriction i$, and n_i will be computable from $A \restriction i+1$ and $B \restriction i$. Let s be the least stage such that $B \restriction i+1$ and $A \restriction i$ have settled. Then we will have $i \in A$ if and only if either $i \in A[s]$ or $m_i \in D$. Similarly, letting s be the least stage such that $A \restriction i+1$ and $B \restriction i$ have settled, we will have $i \in B$ if and only if either $i \in B[s]$ or $n_i \in U$. As we will argue below, these facts will be enough to establish the theorem.

At stage s, we proceed as follows. First we provide the needed positions for markers:

1. Let $j_0 < \cdots < j_n$ be all numbers $j \leqslant s$ such that M_j does not currently have a position. Let $c_0, \ldots, c_n$ be fresh large numbers. Declare $l \prec c_0 \prec \cdots \prec c_n$ for all l currently in U, put each c_i in U, and give each M_{j_i} the position $m^s_{j_i} = c_i$.

2. Let $j_0 < \cdots < j_n$ be all numbers $j \leqslant s$ such that N_j does not currently have a position. Let $c_0, \ldots, c_n$ be fresh large numbers. Declare $c_n \prec \cdots \prec c_0 \prec r$ for all r currently in D, put each c_i in D, and give each N_{j_i} the position $n^s_{j_i} = c_i$.

If a number k (by convention unique) enters A or B (again, necessarily only one of them) at this stage, then we act to reflect this enumeration. (If not, we proceed to the next stage.)

1. If k enters A then for each $l \in U$ that is equal to, or to the right of, m^s_k, put l in D. Cancel the position of all markers N_i with $i \geqslant k$ and all markers M_i with $i > k$.

2. If k enters B then for each $r \in D$ that is equal to, or to the left of, n^s_k, put r in U. Cancel the position of all markers M_i with $i \geqslant k$ and all markers N_i with $i > k$.

This completes the construction. Clearly $\prec$ is a computable linear order. We now verify that it has the desired properties.

An element x of $U[s]$ can move to D at a stage $t > s$ only if it is equal to, or to the right of, m_k^t for k entering A at stage t. If this happens then x will henceforth be to the right of every marker N_i with $i \geqslant k$, so it will return to U at a stage $u > t$ only if an element $k' < k$ enters B. Repeating this argument, we see that x will once again move to D only if an element $k'' < k'$ enters A, and so forth. So x can shift between U and D only finitely often. Thus U and D are Δ_2^0, and the domain of $\prec$ is $U \cup D$.

Let $j \notin A$ and $k \notin B$, and let $l = \max(j, k)$. If s is a stage such that $A \restriction l+1$ and $B \restriction l+1$ have settled then m_j^s is permanently in U and n_k^s is permanently in D. So U and D are infinite, and hence, by the way elements are put into the order, $(U, \prec)$ and $(D, \prec)$ have order types ω and ω^*, respectively, and every element of U is $\prec$ every element of D. So $\prec$ has order type $\omega + \omega^*$.

Let s be the least stage such that $A \restriction j+1 = A[s] \restriction j+1$ and $B \restriction j = B[s] \restriction j$. Then N_j is never moved from its position n_j^s at a later stage, so $n_j = \lim_t n_j^t$ exists and is computable from $A \restriction j+1$ and $B \restriction j$. If j enters B after stage s then n_j is put into U, and it cannot be moved to D, since that would require either that a number less than or equal to j enter A or that a number less than j enter B. On the other hand, if $j \notin B$ then n_j is initially put into D, and cannot be moved to U, since that would require a number less than or equal to j to enter either A or B. Thus we have $j \in B$ if and only if either $j \in B[s]$ or $n_j \in U$ (or both).

Similarly, let s be the least stage such that $B \restriction j+1 = B[s] \restriction j+1$ and $A \restriction j = A[s] \restriction j$. The same argument as in the previous paragraph shows that $m_j = \lim_t m_j^t$ exists and is computable from $B \restriction j+1$ and $A \restriction j$, and $j \in A$ if and only if either $j \in A[s]$ or $m_j \in D$ (or both).

Now let C be a descending suborder of $\prec$ (the ascending case being symmetric). We show that $B \leqslant_{\mathrm{T}}^{\mathrm{b}} A \oplus C$ by induction. Suppose that we have already computed $B \restriction j$ using $A \restriction j$ and C. To compute $B(j)$ using $A \restriction j+1$ and C, find the least stage s such that $A \restriction j+1 = A[s] \restriction j+1$ and $B \restriction j = B[s] \restriction j$. If $j \in B[s]$ then we are done. Otherwise, we can use $A \restriction j+1$ and $B \restriction j$ to compute n_j, and we know that $j \in B$ if and only if $n_j \in U$. But $x \in D$ if and only if x is to the right of some element of C, so D is c.e. in C. Thus, to compute $B(j)$, we simultaneously enumerate B and D. Either j enters B or n_j enters D, in which case $j \notin B$. □

Theorem 2.18. *Given any low set X there are c.e. sets A_0 and A_1 such that $A_0 \nleqslant_T^b A_1 \oplus X$ and $A_1 \nleqslant_T^b A_0 \oplus X$.*

Proof. We have requirements

$$R_{e,i,k} : \Phi_e^{A_k \oplus X} \text{ total} \wedge \Phi_i \text{ total} \wedge \forall n\,(\text{the computation } \Phi_e^{A_k \oplus X}(n) \text{ queries at most the first } \Phi_i(n) \text{ many bits of } A_k) \rightarrow \exists n\,(\Phi_e^{A_k \oplus X}(n) \neq A_{1-k}(n)),$$

arranged in a priority order in the usual way.

To each $R_{e,i,k}$ we associate a number $n_{e,i,k} \in \mathbb{N}^{[\langle e,i,k \rangle]}$, a set $N_{e,i,k}$ (containing values of $n_{e,i,k}$) and a c.e. set of binary strings $W_{e,i,k}$, into which we enumerate strings during the construction. Each time $R_{e,i,k}$ is initialized, a fresh new value is picked for $n_{e,i,k}$, the set $N_{e,i,k}$ is redefined to be $\{n_{e,i,k}\}$, and $W_{e,i,k}$ is redefined to be a new, currently empty, set. By the Recursion Theorem and the lowness of X, there is a computable approximation $w_{e,i,k}(t)$ (which is uniformly computable over all requirements) such that if $R_{e,i,k}$ is initialized only finitely often then $\lim_t w_{e,i,k}(t) = w_{e,i,k}$ exists, and $w_{e,i,k} = 1$ if and only if, for the final version of $W_{e,i,k}$, there is a $\sigma \in W_{e,i,k}$ such that σ is an initial segment of X.

At stage 0, set the restraint imposed by each requirement to 0, pick values for the $n_{e,i,k}$, and let each $N_{e,i,k} = \{n_{e,i,k}\}$.

At stage $s > 0$, proceed as follows. For each $R_{e,i,k}$, let $r_{e,i,k}[s]$ be the maximum of the union of all sets $\{\Phi_{i'}(n)[s] \mid n \in N_{e',i',k'}\}$ taken over all $R_{e',i',k'}$ stronger than $R_{e,i,k}$. If $\min(N_{e,i,k}) < r_{e,i,k}[s]$ then initialize $R_{e,i,k}$.

Say that $R_{e,i,k}$ requires attention at stage s if $w_{e,i,k}(s) = 0$ and for all $n \in N_{e,i,k}$, it is the case that $\Phi_i(n)[s]\downarrow$, that $\Phi_e^{A_k \oplus X}(n)[s]\downarrow = A_{1-k}(n)[s]$, and that the computation $\Phi_e^{A_k \oplus X}(n)[s]$ queries at most $\Phi_i(n)$ many bits of A_k.

For each $R_{e,i,k}$ requiring attention (if any), proceed as follows. Enumerate $X \upharpoonright \varphi_e^{A_k \oplus X}(n_{e,i,k})[s]$ into $W_{e,i,k}$ and wait until either $w_{e,i,k}(t) = 1$ for some $t > s$ or X changes below $\varphi_e^{A_k \oplus X}(n_{e,i,k})[s]$. One of these must happen by the choice of $w_{e,i,k}$. In the first case, enumerate $n_{e,i,k}$ into A_{1-k}, choose a fresh new value for $n_{e,i,k}$, and enumerate this value into $N_{e,i,k}$. This completes the construction. We now verify its correctness.

Assume by induction that for all strategies $R_{e',i',k'}$ stronger than $R_{e,i,k}$, the value of $n_{e',i',k'}$ eventually settles, so that $\lim_s r_{e,i,k}[s]$ exists and hence $R_{e,i,k}$ eventually stops being initialized, say by stage u, which implies that $N_{e,i,k}$ and $W_{e,i,k}$ have final versions. Every time $n_{e,i,k}$ gets redefined at a stage $s > u$ we must have $w_{e,i,k}(s) = 0$ and $w_{e,i,k}(t) = 1$ for some $t > s$. Since $w_{e,i,k} = \lim_s w_{e,i,k}(s)$ exists, this means that $n_{e,i,k}$ eventually settles to a final value.

Now suppose that $\Phi_e^{A_k \oplus X}$ and Φ_i are total and for all n, the computation $\Phi_e^{A_k \oplus X}(n)$ queries at most $\Phi_i(n)$ many bits of A_k. Then $\Phi_i(n_{e,i,k})\downarrow$ and $\Phi_e^{A_k \oplus X}(n)\downarrow$ for all $n \in N_{e,i,k}$. We claim $\Phi_e^{A_k \oplus X}(n)\downarrow \neq A_{1-k}(n)$ for some $n \in N_{e,i,k}$.

First suppose that $w_{e,i,k} = 0$. Then there is a stage u after which $w_{e,i,k}$ has settled and $R_{e,i,k}$ always requires attention. But then at each stage $s > u$ we see X change below $\varphi_e^{A_k \oplus X}(n_{e,i,k})[s]$ (since otherwise we would have $t > s$ with $w_{e,i,k}(t) = 1$). This contradicts the totality of $\Phi_e^{A_k \oplus X}$.

Now suppose $w_{e,i,k} = 1$. Then there is a $\sigma \in W_{e,i,k}$ such that σ is an initial segment of X. But by the definition of $W_{e,i,k}$, at the stage s at which σ enters $W_{e,i,k}$, we have $\Phi_e^{A_k \oplus \sigma}(n)[s]\downarrow = 0$, where n is the value of $n_{e,i,k}$ at that stage, and we put n into A_{1-k}. Furthermore, the computation $\Phi_e^{A_k \oplus \sigma}(n)[s]$ queries at most $\Phi_i(n)$ many bits of A_k, and $A_k \upharpoonright \Phi_i(n)$ never changes after stage s, so in fact $\Phi_e^{A_k \oplus \sigma}(n)\downarrow = 0$, whence $\Phi_e^{A_k \oplus X}(n)\downarrow = 0 \neq A_{1-k}(n)$. $\square$

Corollary 2.19. $\mathrm{WKL}_0 \nvdash \mathrm{SADS}$.

Proof. By Theorems 2.17 and 2.18, no low set can bound (in Turing degree) the sets of an ω-model of SADS. As there are low sets that bound the sets of an ω-model of WKL_0 (see Simpson [1999, proof of Theorem VIII.2.17]), WKL_0 does not imply SADS. $\square$

We remark that Theorem 2.18 cannot be extended by replacing our ad-hoc reducibility $\leqslant^{\mathrm{b}}_{\mathrm{T}}$ with Turing reducibility. This is because Downey (see Arslanov, Cooper, and Li [2004]) has shown that there is a low degree that cups all nonzero c.e. degrees to $\mathbf{0}'$.

Another consequence of Theorems 2.17 and 2.18 is that there is no universal instance of SADS. That is, there is no computable linear order $\mathcal{L}$ of type $\omega+\omega^*$ such that for every infinite ascending or descending suborder C of $\mathcal{L}$ and every computable linear order $\mathcal{L}'$ of type $\omega+\omega^*$, there is an infinite ascending or descending suborder of $\mathcal{L}'$ computable in C.

The next series of results will all be of the form that one can add a solution for an arbitrary instance of one problem by forcing without adding a solution to an instance of another problem that does not have one in the original model. These types of results give various nonimplications among the relevant systems even for ω-models. When it is possible to implement them over arbitrary models, they yield conservation results as well. The basic situation here is that we start with a countable model $\mathcal{M}$ of RCA_0 and a notion of forcing $\mathcal{P}$ defined over $\mathcal{M}$. Moreover, if G is sufficiently generic over $\mathcal{P}$ (usually 1-generic plus satisfying a specific list of requirements that guarantee that if we force an instance of a Σ^0_1 (or Π^0_1) formula then there is a least instance that can be forced and we do so), we can show that $\mathrm{I}\Sigma_1$ holds for formulas over $\mathcal{M}$ with G as an added set parameter. (This typically relies on a sufficiently simple definition of forcing for one quantifier sentences.) In this situation, $\mathcal{M}$ is an ω-submodel of the model $\mathcal{M}[G]$ gotten by adding on to the sets of $\mathcal{M}$ all sets definable over $\mathcal{M}$ by Δ^0_1 formulas with G as an added set parameter, and $\mathcal{M}[G]$ is itself a model of RCA_0 (Friedman [1976], or see Simpson [1999] or CJS §6). (Warning: in CJS, $\mathcal{M}[G]$ is first officially defined as the model of $\mathrm{I}\Sigma_1$ gotten by adding on just G to the sets of $\mathcal{M}$. What we generally want, however, is its extension to a model of RCA_0.)

We begin with COH and Mathias forcing as in CJS §9; see CJS for more on Mathias forcing (including the formal definition of the forcing relation). Conditions are pairs (D, L) where D is $\mathcal{M}$-finite, L is $\mathcal{M}$-infinite and $\max D < \min L$. We say that (D', L') extends (D, L), and write $(D', L') \leqslant (D, L)$, if $D \subseteq D' \subseteq D \cup L$ and $L' \subseteq L$. CJS show that if G is Mathias 1-generic (i.e. every Π^0_1 formula or its negation is forced and the conditions specific to forcing $\mathrm{I}\Sigma_1$ are met) then adding G to $\mathcal{M}$ preserves $\mathrm{I}\Sigma_1$ and so $\mathcal{M}[G] \vDash \mathrm{RCA}_0$ and every sequence $\langle A_i \rangle$ coded by an $A \in \mathcal{M}$ has a cohesive set in $\mathcal{M}[G]$. They note that, by iteration, this proves that COH is Π^1_1-conservative over RCA_0. We strengthen their results to a stronger conservation result that shows that none of the principles that we have discussed follow from COH, except those already shown to follow from it.[1]

Theorem 2.20. *Let $\mathcal{M}$ be a countable model of* RCA_0 *and let $\Phi(A,B)$ be a Σ^0_3 predicate of two set variables such that for some fixed $A \in \mathcal{M}$ there is no $B \in \mathcal{M}$ with $\mathcal{M} \vDash \Phi(A,B)$. If G is Mathias 2-generic over $\mathcal{M}$ then there is no $B \in \mathcal{M}[G]$ with $\mathcal{M}[G] \vDash \Phi(A,B)$.*

Proof. Suppose that $\Phi(A,B) = \exists x\, \forall y\, \exists z\, \varphi(A,B,x,y,z)$ where φ is Δ^0_0 and, for the sake of a contradiction, that there is a $B \in \mathcal{M}[G]$ such that $\mathcal{M}[G] \vDash \Phi(A,B)$. Now $\mathcal{M}$ is an ω-submodel of $\mathcal{M}[G]$, and B is of the form $\Phi_e^{G\oplus C}$ for some Turing functional Φ_e and set C in $\mathcal{M}$, so for some $x \in \mathcal{M}$,

$$\mathcal{M}[G] \vDash \Phi_e^{G\oplus C} \text{ is a total characteristic function and } \forall y\, \exists z\, \varphi(A, \Phi_e^{G\oplus C}, x, y, z).$$

As this whole formula is equivalent (even in RCA_0) to a $\Pi^0_2(G)$ (over $\mathcal{M}$) formula $\forall y\, \exists z\, \Psi(G,C,A,x,y,z)$ and G is 2-generic, there is a condition $(D,L) \in \mathcal{M}$ such that $(D,L) \Vdash \forall y\, \exists z\, \Psi(G,C,A,x,y,z)$.

By the general definition of forcing this means that for every y in $\mathcal{M}$ and every condition $(D',L') \leqslant (D,L)$, there is a $z \in \mathcal{M}$ and a condition $(D'',L'') \leqslant (D',L')$ such that $(D'',L'') \Vdash \Psi(G,C,A,x,y,z)$. By the definition of Mathias forcing for Δ^0_0-formulas, however, this last assertion depends on only a finite part of the forcing condition. More precisely, it depends on certain numbers being in D'' and certain numbers being out of L''. Thus, for every $y \in \mathcal{M}$ and every $D' \supseteq D$ with $D' - D \subseteq L$, there is a finite set $F_y(D') \supseteq D'$ of $\mathcal{M}$ and a number $z \in \mathcal{M}$ such that $F_y(D') - D' \subseteq L$ and $\mathcal{M} \vDash \Psi(F_y(D'),C,A,x,y,z)$. We can now define by recursion the function f such that $f(0) = D$ and $f(n+1) = F_n(f(n))$. As $\mathcal{M}$ is a model of RCA_0, this function is an element of $\mathcal{M}$ and so gives the characteristic function of a set $H \in \mathcal{M}$ which by construction satisfies $\forall y\, \exists z\, \Psi(H,C,A,x,y,z)$. Finally, if $B' = \Phi_e^{H\oplus C}$ then $B' \in \mathcal{M}$ and $\mathcal{M} \vDash \forall y\, \exists z\, \varphi(A,B',x,y,z)$ for the desired contradiction. □

We now have our conservation and nonimplication results. In the terminology of Simpson [1999, Definition VII.2.28], a model $\mathcal{M}$ is a *restricted β-submodel* of a model $\mathcal{M}'$ if $\mathcal{M}$ is an ω-submodel of $\mathcal{M}'$ and for every sentence of the form $\exists X\, \psi$ where ψ is Π^0_2 with parameters in $\mathcal{M}$, we have $\mathcal{M} \vDash \exists X\, \psi$ if and only if $\mathcal{M}' \vDash \exists X\, \psi$. (This condition says that a subtree of $\omega^{<\omega}$ in $\mathcal{M}$ has a path in $\mathcal{M}$ if and only if it has a path in $\mathcal{M}'$, whence the terminology.)

Corollary 2.21. COH *is conservative over* RCA_0 *for sentences of the form $\forall A\, (\Theta(A) \to \exists B\, \Phi(A,B))$ where Θ is arithmetic and Φ is Σ^0_3. Furthermore, every model of RCA_0 is a restricted β-submodel of a model of* COH.

[1] In the aforementioned upcoming joint paper with Slaman, we prove the analogous theorem about Cohen forcing and so conservation and nonimplication results for the principle AMT which says that every atomic theory has an atomic model.

Proof. Consider any sentence of the specified form and any model $\mathcal{M}$ of RCA_0 not satisfying the sentence. Thus there is a set A of $\mathcal{M}$ such that $\mathcal{M} \vDash \Theta(A)$ for which there is no B in $\mathcal{M}$ such that $\mathcal{M} \vDash \Phi(A, B)$. Construct a sequence $\langle G_i \mid i \in \omega \rangle$ of subsets of $\mathbb{N}^{\mathcal{M}}$ such that G_{i+1} is Mathias 2-generic over $\mathcal{M}[G_1] \dots [G_i]$. Let $\mathcal{M}' = \bigcup\{\mathcal{M}[G_0] \dots [G_i] \mid i \in \omega\}$. By the results of CJS mentioned above, $\mathcal{M}[G_0] \dots [G_i] \vDash \mathrm{RCA}_0$ for each i, and every sequence $\vec{R}$ in $\mathcal{M}[G_0] \dots [G_i]$ has a cohesive set in $\mathcal{M}[G_0] \dots [G_i][G_{i+1}]$. Thus $\mathcal{M}' \vDash \mathrm{RCA}_0$ + COH as the first order part of every element of this ascending sequence of models is the same. By induction (on ω) our theorem shows that there is no $B \in \mathcal{M}[G_0] \dots [G_i]$ such that $\mathcal{M}[G_0] \dots [G_i] \vDash \Phi(A, B)$. Again, as each successive model in this list is an ω-extension of the preceding one and $\mathcal{M}'$ is just their union, we see that while $\mathcal{M}' \vDash \Theta(A)$, there is no $B \in \mathcal{M}'$ such that $\mathcal{M}' \vDash \Phi(A, B)$. Thus our original sentence is not a theorem of RCA_0 + COH, as required.

The version of the corollary in terms of restricted β-extensions follows by taking Θ to be empty and Φ to be Π^0_2. Our argument shows that if $\mathcal{M}' \vDash \exists B\, \Phi$ then $\mathcal{M} \vDash \exists B\, \Phi$. The other direction follows from the fact that $\mathcal{M}$ is an ω-submodel of $\mathcal{M}'$. □

We note that this result also literally extends the one of CJS for Π^1_1 sentences by taking Φ to be any vacuously true sentence. Moreover, the result is the best possible one of this form as COH itself is a sentence of the form $\forall A\, (\Theta(A) \to \exists B\, \Phi(A, B))$ with Θ arithmetic and Φ a Π^0_3 formula. It is, however, strong enough to show that any principle that asserts, for example, the existence of an infinite set satisfying some computable condition such as being homogeneous, a path through a tree, or a chain or antichain in a linear or partial order cannot be implied by COH, as all of these statements are ones in which Φ is Π^0_2.

Corollary 2.22. *None of the following principles are implied by* COH *(nor* CADS*, which follows from it):* RT^2_2*,* SRT^2_2*,* WKL_0*,* DNR*,* CAC*,* ADS*,* SADS*.*

Of course, as discussed above, this result was already known for RT^2_2, SRT^2_2, and WKL_0. For DNR, it has also been proved by Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta].

By Corollary 2.16 we now have the desired independence for our splittings of ADS.

Corollary 2.23. CADS *and* SADS *are incomparable over* RCA_0 *and both are strictly weaker than* ADS*.*

We now investigate the relationships with WKL_0 by first showing that adding it to COH does not yield any of the other principles mentioned in Corollary 2.22, except for DNR, of course. It suffices to prove this for SADS as it is implied by all the principles other than WKL_0 and DNR on this list. We then show that ADS does not imply WKL_0 or even DNR. CAC is considered in the next section.

Proposition 2.24. *For any set X, if $\mathcal{L}$ is a linear order computable in X with no ascending or descending sequence computable in X and T is an infinite binary tree computable* *in X, then there is a path f in T such that no ascending or descending sequence in $\mathcal{L}$ is computable in $f \oplus X$.*

Proof. We consider the case that X is computable. The full proposition then holds by relativization. We use forcing (in standard arithmetic) with infinite computable subtrees S of T. Of course, any 1-generic G gives us a path $P = P_G$ through T as the union of the roots of the trees $S \in G$. Suppose G is 2-generic and Φ_e^P is total and an ascending sequence in $\mathcal{L}$. (The case of a descending sequence is symmetric.) Thus we have a tree $S \subseteq T$ with $P \in [S]$ such that $S \Vdash (\Phi_e^P$ is total and $(\forall m)(\forall n < m)(\Phi_e^P(n) <_L \Phi_e^P(m)))$.

As $S \Vdash (\Phi_e^P$ is total), for each n there is an s such that $\Phi_e^\sigma \upharpoonright n$ is defined for every σ on level s of S. (Otherwise, $\{\tau \in S \mid (\exists i < n)\, \Phi_e^\tau(i)\uparrow\}$ is infinite and, in fact, a subtree of S that forces Φ_e^P not to be defined on some $i < n$, contrary to our choice of S.) Consider now the computable subtree $\hat{S}$ of S consisting of all σ such that $(\forall i < |\sigma|)(\Phi_e^\sigma(i)\downarrow \wedge \Phi_e^\sigma(i+1)\downarrow \rightarrow \Phi_e^\sigma(i) <_L \Phi_e^\sigma(i+1))$. If this $\hat{S}$ were not infinite, then the subtree $\tilde{S}$ of S consisting of the downward closure of the nodes where this condition fails would be an infinite computable subtree of S that forces Φ_e^P not to be an ascending sequence in $\mathcal{L}$, contrary to our assumption on S.

We now define an ascending sequence $\langle a_i \mid i \in \omega\rangle$ in $\mathcal{L}$ computably in $\hat{S}$ for our contradiction. To find a_i find the least s_i such that $\Phi_e^\sigma \upharpoonright (i+1)\downarrow$ for every σ on level s_i of $\hat{S}$. Let a_i be the $\mathcal{L}$-least member of $\{\Phi_e^\sigma(i) \mid \sigma \in \hat{S} \wedge |\sigma| = s_i\}$. (All of these values must be in the domain of $\mathcal{L}$ by our definition of $\hat{S}$.) We claim that $\langle a_i\rangle$ is ascending in $\mathcal{L}$, i.e. $a_i <_L a_{i+1}$ for every i. Consider any a_i and a_{i+1}. They were chosen $\mathcal{L}$-least from among the values of $\Phi_e^\sigma(i)$ and $\Phi_e^\tau(i+1)$ for σ on level s_i and τ on level s_{i+1} of $\hat{S}$. Say $a_{i+1} = \Phi_e^\tau(i+1)$ for some τ on level s_{i+1}. There must be a σ on level s_i such that $\sigma \subseteq \tau$. By our definition of $\hat{S}$ and choice of s_i, we have $\Phi_e^\sigma(i)\downarrow = \Phi_e^\tau(i) <_L \Phi_e^\tau(i+1)$. On the other hand, our definition of a_i makes $a_i \leqslant_L \Phi_e^\sigma(i) <_L \Phi_e^\tau(i+1) = a_{i+1}$ as required. □

Corollary 2.25. COH + $\mathrm{WKL}_0 \nvdash$ SADS.

Proof. Start with the standard model COMP with second order part consisting of all computable sets, and a computable linear order $\mathcal{L}$ of type $\omega + \omega^*$ with no computable ascending or descending sequence. Iterate and dovetail the forcing constructions of Theorem 2.20 and Proposition 2.24 to produce an ω-model of COH and WKL_0 in which $\mathcal{L}$ still has no ascending or descending sequence. □

We next work at showing that ADS does not imply WKL_0, or even DNR. We begin with SADS.

Proposition 2.26. *Let $\mathcal{L}$ be a linear order of type $\omega+\omega^*$ computable in a set X such that there is no diagonally noncomputable function computable in X. There is then a subset G of $\mathcal{L}$ of order type ω or ω^* with no diagonally noncomputable function computable in $G \oplus X$.*

Proof. We again treat the case that X is computable and note that the general case holds by relativization. If there is a computable subset G of $\mathcal{L}$ of order type ω^* we are done, so suppose not. Let U be the subset of $\mathcal{L}$ consisting of all elements in the ω part of $\mathcal{L}$, i.e. all elements with only finitely many predecessors in $\mathcal{L}$.

We consider the notion of forcing $\mathcal{F}$ in which conditions are finite sequences $\sigma \in \omega^{<\omega}$ that are strictly ascending in both natural order ($<_{\mathbb{N}}$) and the order of $\mathcal{L}$ ($<_L$), with all elements in U (or equivalently with last element in U). Extension is as usual for finite sequences: σ extends τ if $\tau \supseteq \sigma$. Let G be 2-generic for this forcing. Then G is obviously an ascending sequence in $\mathcal{L}$. We claim that no Φ_e^G is a diagonally noncomputable function. If one were then there would be a condition $\sigma \Vdash (\Phi_e^G \text{ is total} \wedge \forall k\, (\Phi_k(k)\downarrow \to \Phi_e^G(k) \neq \Phi_k(k)))$. Fix such a σ to argue for a contradiction.

For each k let $\tau(k)$ be the first string ρ found in a computable search through $\omega^{<\omega}$ that is increasing in both $<_{\mathbb{N}}$ and $<_L$ such that $\rho \supseteq \sigma$ and $\Phi_e^\rho(k)\downarrow$. We do not assume here that $\rho(|\rho|) \in U$ and so τ is a partial computable function. In fact, τ is total as $\sigma \Vdash (\Phi_e^G \text{ is total})$ and so for any k there is actually a $\rho \supseteq \sigma$ in $\mathcal{F}$ such that $\Phi_e^\rho(k)\downarrow$. If there are infinitely many k such that $\Phi_k(k)\downarrow$ and $\Phi_e^{\tau(k)}(k) = \Phi_k(k)$ then we can computably find infinitely many such k. If for such k, $\tau(k) \in \mathcal{F}$ then $\tau(k)$ extends σ and $\tau(k) \Vdash \Phi_e^G(k)\downarrow = \Phi_k(k)\downarrow$, contradicting our choice of σ. By standard conventions $|\tau(k)| \geqslant k$. Thus for each of these k we have effectively produced an element, $\tau(k)(|\tau(k)|)$ of $\bar{U}$ (the ω^* part of $\mathcal{L}$) larger (in natural order) than k. Thus we can produce an infinite computable subset of $\bar{U}$. We can then computably thin out this set to require that the elements be descending in $\leqslant_L$ and ascending in $\leqslant_{\mathbb{N}}$, as for each a_i there are only finitely many a_j that are above it in $\leqslant_L$, since a_i is in the ω^* part of $\mathcal{L}$. This contradicts our assumption that there is no computable descending sequence in $\mathcal{L}$. Thus there are only finitely many k such that $\Phi_k(k)\downarrow$ and $\Phi_e^{\tau(k)}(k) = \Phi_k(k)$ and so a finite variation of the computable function taking k to $\Phi_e^{\tau(k)}(k)$ would be DNR for the desired contradiction. □

Corollary 2.27. SADS $\nvdash$ DNR.

Proof. By the usual iteration and dovetail argument we can use Proposition 2.26 to build an ω-model of SADS in which there is no diagonally noncomputable function and so DNR fails. □

Corollary 2.28. ADS $\nvdash$ DNR.

Proof. We can combine the forcing of Proposition 2.26 with that of Theorem 2.20 to get an ω-model in which both SADS and COH hold but in which there is no diagonally noncomputable function and so DNR fails. By Propositions 2.7 and 2.9 this model is also one of ADS. □

Corollary 2.29. ADS $\nvdash$ SRT^2_2 *and so* ADS $\nvdash$ RT^2_2.

Proof. Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta] show that $\mathrm{SRT}^2_2 \vdash$ DNR and so neither SRT^2_2 nor RT^2_2 can be a consequence of ADS. □

It is interesting to note that we can use the forcing notion of Proposition 2.26 to provide an alternate proof of Theorem 2.11. We provide a stronger version in Theorem 3.4 as part of our analysis of CAC.

We have now shown that ADS does not imply WKL_0, or even DNR, and also does not imply RT^2_2, or even SRT^2_2. We have also shown that it splits into two incomparable principles, SADS (implied by SRT^2_2 but with low solutions) and CADS (implied by COH and almost equivalent to it as will be shown in Theorem 4.4). Moreover, neither of these principles is implied by WKL_0. The results of this section are summarized and depicted in the diagram below, which uses the same notation as the diagram in the previous section. We proceed to do a similar analysis for CAC in the next section.

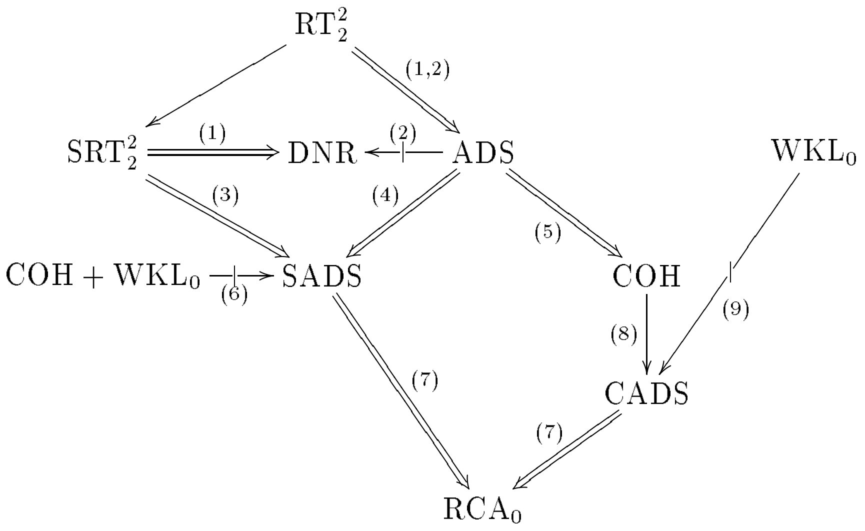

Diagram 2

The references for these results are as follows: (1) Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta]; (2) Corollary 2.28; (3) Proposition 2.8 and Corollary 2.13; (4) Proposition 2.7 and Corollary 2.16; (5) Proposition 2.10 and Corollary 2.22; (6) Corollaries 2.19 and 2.25; (7) Corollary 2.6; (8) Proposition 2.9; (9) Corollary 2.16.

3 Partial Orders and CAC

In this section we will analyze CAC. We begin with a connection to ADS.

Proposition 3.1. $\mathrm{RCA}_0 + \mathrm{CAC} \vdash \mathrm{ADS}$.

Proof. Let $\mathcal{L}$ be a linear order on $\mathbb{N}$. We define a partial order $\mathcal{P}$ on $\mathbb{N}$ by setting $m \leqslant_P n$ if and only if $m \leqslant_L n \wedge m \leqslant_{\mathbb{N}} n$. If C is a chain in $\mathcal{P}$ and we list it in increasing natural order, then it is an ascending sequence in $\mathcal{L}$, as if $x <_{\mathbb{N}} y$ the only way x and y can be comparable in $\mathcal{P}$ is if $x <_L y$. On the other hand, if A is an antichain in $\mathcal{P}$ and we list it in increasing natural order, then it is a descending sequence in $\mathcal{L}$, as if $x <_{\mathbb{N}} y$ and $x <_L y$ then $x <_P y$. □

We next define the appropriate notion of stability for CAC.

Definition 3.2. A partial order $\mathcal{P}$ is *stable* if either

$$(1) \qquad (\forall i \in P)(\exists s)[(\forall j > s)(j \in P \to i <_P j) \vee (\forall j > s)(j \in P \to i \mid_P j)]$$

or

$$(2) \qquad (\forall i \in P)(\exists s)[(\forall j > s)(j \in P \to i >_P j) \vee (\forall j > s)(j \in P \to i \mid_P j)].$$

Now for the associated principle and system:

(SCAC) Stable CAC: Every infinite stable partial order has an infinite chain or antichain.

While this might not be one's first guess at the right notion of stability for a partial order, our analysis (especially Proposition 5.4) will bear out the analogy with the previous notions of stability. We begin with a couple of easy combinatorial facts that place SCAC between SRT^2_2 and SADS.

Proposition 3.3. $\mathrm{RCA}_0 \vdash \mathrm{SRT}^2_2 \to \mathrm{SCAC} \to \mathrm{SADS}$.

Proof. For the first implication, let $\mathcal{P}$ be a stable partial order with domain $\mathbb{N}$. Define a coloring f by $f(x, y) = 0$ if x and y are comparable in $\mathcal{P}$ and $f(x, y) = 1$ if $x \mid_P y$. By the definition of stability for a partial order, f is a stable coloring. If H is homogeneous to 0 then it is a chain in $\mathcal{P}$ and if H is homogeneous to 1 then it is an antichain in $\mathcal{P}$.

For the second, given a linear order $\mathcal{L}$ on $\mathbb{N}$ of type $\omega + \omega^*$ we define a partial order $\mathcal{P}$ on $\mathbb{N}$ as in Proposition 3.1. Consider any m. If m is in the ω part of $\mathcal{L}$, then almost every n is larger than m in both the natural order and that of $\mathcal{L}$, and so above it in $\mathcal{P}$. If m is in the ω^* part of $\mathcal{L}$, then almost every n is above it in the natural order but below it in the order of $\mathcal{L}$. Thus almost every n is incomparable with m in $\mathcal{P}$. So $\mathcal{P}$ is a stable partial order. As shown in Proposition 3.1, a chain in $\mathcal{P}$ is an ascending sequence in $\mathcal{L}$ and an antichain in $\mathcal{P}$ is a descending sequence in $\mathcal{L}$. □

We next show that SCAC has low solutions and does not imply DNR, as we did for SADS in Theorem 2.11, Proposition 2.26, and Corollary 2.27, but now entirely by forcing arguments.

Theorem 3.4. *Every stable partial order $\mathcal{P}$ computable in X has a chain or antichain S that is low over X, i.e. $(S \oplus X)' \leqslant_T X'$.*

Proof. As usual, we assume that X is computable and that the domain of $\mathcal{P}$ is $\mathbb{N}$. We also assume that $\mathcal{P}$ satisfies (1) in the definition of a stable partial order. (The argument if $\mathcal{P}$ satisfies (2) is symmetric.) If $\mathcal{P}$ has a computable antichain, we are done, so we assume not and construct a low chain in $\mathcal{P}$. Let $U = \{i \mid (\exists s)(\forall j > s)(i <_P j)\}$. Note that by the definition of stability U and its complement are both Σ^0_2 and so computable in $0'$. We use the same notion of forcing $\mathcal{F}$ as in Proposition 2.26 with $\mathcal{P}$ replacing $\mathcal{L}$. Thus conditions are sequences $\sigma \in \omega^{<\omega}$ that are ascending in both natural order and in $<_P$ and all of whose elements are in U (or equivalently, whose last elements are in U). Extension is defined as usual.

We construct the low generic G as the union of a sequence $\langle \sigma_i \rangle$ in $\mathcal{F}$ computable in $0'$. For each i, let C_i be the set of $\rho \in \omega^{<\omega}$ such that ρ is strictly ascending in both $\leqslant_{\mathbb{N}}$ and $\leqslant_P$, and $\Phi_i^{\rho}(i)\downarrow$. We want to define the σ_i so that for every i either $\Phi_i^{\sigma_i}(i)\downarrow$ (and so $i \in G'$) or there is no $\rho \supseteq \sigma_i$ in C_i (and so $\sigma_i \Vdash \Phi_i^G(i)\uparrow$ and $i \notin G'$). Given σ_i we can clearly search computably in $0'$ for a $\tau \supset \sigma_i$ in $\mathcal{F}$ with the properties desired for σ_{i+1}. Assume for a contradiction that there is no such τ. Then there are infinitely many $\rho \supseteq \sigma_i$ in C_{i+1} (as there are infinitely many $\tau \supset \sigma$ in $\mathcal{F}$ and for each such τ there is a $\rho \supseteq \tau$ in C_{i+1}). For each such ρ, we have $\rho(|\rho|) \in \overline{U}$ as otherwise ρ would be a witness for convergence in $\mathcal{F}$, and hence could be taken as σ_{i+1}. As the required property for ρ is computable, we can computably enumerate an infinite increasing (in $<_{\mathbb{N}}$) sequence $\langle b_i \rangle$ with all $b_i \in \overline{U} = \{i \mid (\exists s)(\forall j > s)(i \mid_P j)\}$. We can now computably thin out this sequence to get a computable antichain in $\mathcal{P}$, contrary to our original assumption. More precisely, we define the subsequence $\langle b_{i_j} \rangle$ constituting the antichain by induction so that $b_{i_{j+1}}$ is the first b_i with $i > i_j$ and $b_i \mid_P b_{i_k}$ for all $k \leqslant j$. Such an i exists since each b_{i_k} is comparable with only finitely many elements of $\mathcal{P}$. □

Note that a similar argument for linear orders of type $\omega + \omega^*$ in which U is taken to be the ω part provides low solutions for SADS and so an alternate proof of Theorem 2.11. As for SADS, the existence of low solutions for SCAC establishes various nonimplications.

Corollary 3.5. *There is an ω-model of* SCAC *consisting entirely of low sets.*

Proof. Iterate and dovetail as usual. □

Corollary 3.6. SCAC $\nvdash$ SRT$^2_2$; SCAC $\nvdash$ CADS *and so* SCAC $\nvdash$ COH; SCAC $\nvdash$ ADS.

Proof. Neither SRT$^2_2$ nor CADS have ω-models consisting of low sets by Downey, Hirschfeldt, Lempp and Solomon [2001] and Proposition 2.15, respectively. The second pair of nonimplications then follows by Propositions 2.9 and 2.10. □

We now turn to the question of what is needed on the side of the cohesiveness principle to join SCAC up to CAC. An obvious first choice is the following:

(CCAC) Cohesive CAC: Every partial order has a stable suborder.

This principle, however, is equivalent to ADS.

Proposition 3.7. $\mathrm{RCA}_0 \vdash \mathrm{CCAC} \leftrightarrow \mathrm{ADS}$.

Proof. If $\mathcal{L}$ is a linear order and we view it as a partial order then by definition any stable suborder is of type ω or ω^* (depending on whether it satisfies (1) or (2) in the definition of stability). For the other direction assume ADS and consider a partial order $\mathcal{P}$. Let $\mathcal{L}$ be a linearization of $\mathcal{P}$ (as mentioned in the proof of Proposition 2.15, $\mathcal{L}$ can be obtained computably, and it is easy to see that the proof of its existence works in RCA_0). Now apply ADS to get an ascending or descending sequence $S = \langle s_i \rangle$ in $\mathcal{L}$. Assume that S is ascending in $\mathcal{L}$. Thus $(\forall i)(\forall j > i)(s_i <_P s_j \vee s_i \mid_P s_j)$. Now apply COH (which follows from ADS by Proposition 2.10) to S and the sequence $\langle R_i \rangle$ where $R_i = \{s_j \mid s_i <_P s_j\}$. Any $\vec{R}$-cohesive subset of S with the order of $\mathcal{P}$ is a suborder satisfying (1) in the definition of stability. If S is a descending sequence in $\mathcal{L}$ then a similar argument provides a suborder satisfying (2) in the definition of stability. □

Thus we drop the notation CCAC. We can now use our previous results to split CAC into SCAC and various other principles. The strongest version is the splitting into SCAC and CADS, but COH itself works as well.

Proposition 3.8. $\mathrm{RCA}_0 \vdash \mathrm{CAC} \leftrightarrow \mathrm{SCAC} + \mathrm{ADS} \leftrightarrow \mathrm{SCAC} + \mathrm{COH} \leftrightarrow \mathrm{SCAC} + \mathrm{CADS}$.

Proof. CAC trivially implies SCAC. It implies ADS by Proposition 3.1. ADS implies COH by Proposition 2.10 and COH implies CADS by Proposition 2.9. SCAC implies SADS by Proposition 3.3 and this plus CADS implies ADS by Propositions 2.7. Finally, SCAC plus ADS implies CAC by Proposition 3.7. □

Our final result of this section is the analog of Proposition 2.26 and its corollaries leading up to the result that even CAC does not imply DNR and so, by the results of Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta], it implies neither SRT^2_2 nor RT^2_2.

Proposition 3.9. *Let $\mathcal{P}$ be a stable partial order computable in a set X such that there is no diagonally noncomputable function computable in X. There is then a chain or antichain G in $\mathcal{P}$ such that no diagonally noncomputable function is computable in $G \oplus X$.*

Proof. As usual we assume X is computable. If there is a computable antichain in $\mathcal{P}$ we are done, so assume not. Let $\mathcal{F}$ be the forcing relation defined in the proof of Theorem 3.4 from $\mathcal{P}$ (again assuming $\mathcal{P}$ satisfies (1) in the definition of stability). Let G be any 2-generic for $\mathcal{F}$. Then G is obviously an ascending chain in $\mathcal{P}$ so suppose for the sake of a contradiction that some Φ^G_e is a diagonally noncomputable function. Now argue as in Proposition 2.26 that we can computably produce infinitely many elements $\langle b_i \rangle$ of $\overline{U} = \{i \mid (\exists s)(\forall j > s)(i \mid_P j)\}$. We can now thin out this set as in the proof of Theorem 3.4 to get a computable antichain in $\mathcal{P}$ as required. □

Corollary 3.10. SCAC $\nvdash$ DNR *and hence* SCAC $\nvdash$ WKL_0.

Proof. By the usual iteration and dovetail argument we can use Proposition 3.9 to build an ω-model of SADS in which there is no diagonally noncomputable function, and which is therefore not a model of DNR. $\square$

Corollary 3.11. CAC $\nvdash$ DNR *and hence* CAC $\nvdash$ WKL_0.

Proof. We can combine the forcing of Proposition 3.9 with those used to prove Corollary 2.28 to get an ω-model in which both SCAC and ADS hold but in which there is no diagonally noncomputable function, and which is therefore not a model of DNR. By Proposition 3.8 this model is also a model of CAC. $\square$

Corollary 3.12. CAC $\nvdash$ SRT^2_2 *and so* CAC $\nvdash$ RT^2_2.

Proof. Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta] show that $\mathrm{SRT}^2_2 \vdash$ DNR and so neither SRT^2_2 nor RT^2_2 can be a consequence of CAC. $\square$

This result answers Question 3.18 of CJS.

We can now add to Diagram 2 as follows:

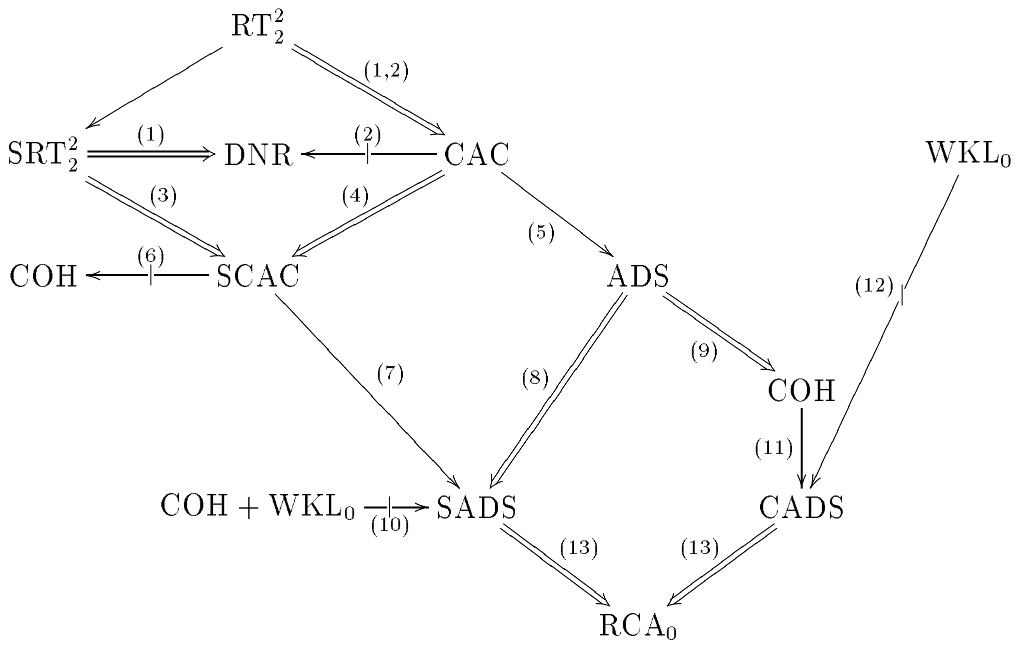

Diagram 3

The references for these results are as follows: (1) Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta]; (2) Corollary 3.11; (3) Proposition 3.3 and Corollary 3.6; (4) Proposition 3.8 and Corollary 3.6; (5) Proposition 3.1; (6) Corollary 3.6; (7) Proposition 3.3; (8) Proposition 2.7 and Corollary 2.16; (9) Proposition 2.10 and Corollary 2.22; (10) Corollary 2.25; (11) Proposition 2.9; (12) Corollary 2.16; (13) Corollary 2.6.

4 $\mathbf{B\Sigma_2}$

As we mentioned in §1, CJS show that $\mathrm{SRT}^2_2 \vdash \mathrm{B}\Sigma_2$ and so SRT^2_2 is not conservative over RCA_0 (even for Σ^0_3 sentences). This shows that COH does not imply SRT^2_2 as COH is conservative over RCA_0 for even more than Π^1_1 sentences (CJS and Corollary 2.21). In this section we show that both SCAC and ADS imply $\mathrm{B}\Sigma_2$ and that there are alternate versions StCOH and StCADS of COH and CADS, respectively, which imply $\mathrm{B}\Sigma_2$ and are equivalent to each other. Each is also equivalent to the original version plus $\mathrm{B}\Sigma_2$. We also show that SADS is not Π^1_1-conservative over RCA_0. It is convenient to work with the following Ramsey type principle, which by Hirst [1987, Theorem 6.4] (or see CJS Theorem 2.10) is equivalent to $\mathrm{B}\Sigma_2$ over RCA_0.

($\mathbf{RT}^1_{<\infty}$) For every $n \in \mathbb{N}$ and every map $f : \mathbb{N} \to n$ there is an infinite set H such that $|f``H| = 1$.

Proposition 4.1. $\mathrm{RCA}_0 \vdash \mathrm{SCAC} \to \mathrm{B}\Sigma_2$.

Proof. We prove $\mathrm{RT}^1_{<\infty}$ using SCAC. Suppose $f : \mathbb{N} \to n$. Define a partial order $\mathcal{P}$ by $x <_P y$ if and only if $x < y$ and $f(x) = f(y)$. If there is an i such that $f^{-1}``\{i\}$ is infinite, we are done. If each such set is finite, then for every x almost every y is $\mathcal{P}$-incomparable with x, as for any $x, y \in \mathbb{N}$, if $f(x) \neq f(y)$ then $x \mid_P y$. Thus $\mathcal{P}$ is stable. A chain in $\mathcal{P}$ would be a subset of a single $f^{-1}``\{i\}$ and so homogeneous for f. An antichain A would be an infinite set every element of which has a different color, and so restricting f to A would yield a one-to-one map from an infinite set into n, which is not possible in RCA_0. ◻

We now introduce alternative versions of COH and CADS that might have been equally plausible principles to consider instead of the original versions.

Definition 4.2. If $\vec{R} = \langle R_i \mid i \in \mathbb{N}\rangle$ is a sequence of sets, an infinite set S is *strongly $\vec{R}$-cohesive* if $(\forall n)(\exists s)(\forall i \leqslant n)[(\forall j > s)(j \in S \to j \in R_i) \vee (\forall j > s)(j \in S \to j \notin R_i)]$.

(StCOH) For every sequence $\vec{R}$ of sets there is a strongly $\vec{R}$-cohesive set.

Definition 4.3. An infinite linear order $\mathcal{L}$ with first and last elements (0 and 1, respectively) in which all nonfirst elements have immediate predecessors and all nonlast ones have immediate successors is *strongly of type $\omega+\omega^*$* if, for every finite ascending sequence $0 = x_0 <_L x_1 <_L \cdots <_L x_n = 1$, there is exactly one infinite subinterval $[x_i, x_{i+1})$, and both $[x_0, x_i]$ and $[x_i, x_n]$ are finite.

(StCADS) Every infinite linear order has a suborder that is either of type ω or ω^*, or strongly of type $\omega + \omega^*$.

Each of these strong principles clearly implies the usual version. In the presence of $\mathrm{B}\Sigma_2$ the notions of strongly cohesive and strongly of type $\omega + \omega^*$ coincide with those of cohesive and of type $\omega + \omega^*$, respectively, so the converses also hold in this context. Furthermore, using the strong versions of our principles makes them actually imply $\mathrm{B}\Sigma_2$.

Proposition 4.4. $\mathrm{RCA}_0 \vdash \mathrm{COH} + \mathrm{B}\Sigma_2 \leftrightarrow \mathrm{CADS} + \mathrm{B}\Sigma_2 \leftrightarrow \mathrm{StCADS} \leftrightarrow \mathrm{StCOH}$.

Proof. COH + $\mathrm{B}\Sigma_2$ implies CADS + $\mathrm{B}\Sigma_2$ by Proposition 2.9.

To see that CADS + $\mathrm{B}\Sigma_2$ implies StCADS we just have to show that the two notions of being of type $\omega + \omega^*$ are equivalent given $\mathrm{B}\Sigma_2$. Let $\mathcal{L}$ be of type $\omega + \omega^*$, with first element 0 and last element 1. If $0 = x_0 <_L x_1 <_L \cdots <_L x_n = 1$ is a finite sequence then define $f : \mathbb{N} \to n$ by $f(x) = i$ where i is least such that $x \leqslant_L x_i$. (Such an i exists for every x by $\mathrm{I}\Sigma_1$ and f then exists by Δ^0_1-CA.) By $\mathrm{B}\Sigma_2$ in the form of $\mathrm{RT}^1_{<\infty}$, there is an i such that $f(x) = i$ for infinitely many x. Thus $[x_{i-1}, x_i)$ is infinite. Now x_i has infinitely many predecessors and so only finitely many successors, while x_{i-1} has infinitely many successors and so only finitely many predecessors. Thus $\mathcal{L}$ is strongly of type $\omega + \omega^*$.

Next we assume StCADS and prove StCOH. Consider the proof in Proposition 2.10 that ADS $\to$ COH. Given a sequence $\vec{R}$ of sets, it defines a linear order $\mathcal{L}$ and actually shows that any ascending or descending sequence in $\mathcal{L}$ is a strongly $\vec{R}$-cohesive set. All that remains is to show that we can also get such a set from a suborder S of $\mathcal{L}$ that is strongly of order type $\omega + \omega^*$. Given $n \in \mathbb{N}$, we define an ascending sequence in S with at most $2^{n+1} + 1$ many elements. We first get the set F of $\sigma \in 2^{n+1}$ such that there is an $x \in S$ with $\sigma = \langle R_i(x) \mid i \leqslant n \rangle$. For each $\sigma \in F$ we let x_σ be the $\mathbb{N}$-least $x \in S$ witnessing that $\sigma \in F$. Our ascending sequence in S is given by listing $\{x_\sigma \mid \sigma \in F\}$ in increasing $\mathcal{L}$-order (i.e. increasing lexicographic order on the $\sigma \in F$) adding on if necessary the last element of S. Thus, by the assumption that S is strongly of type $\omega + \omega^*$, there is a unique infinite interval $[x_\sigma, x_\tau)$ and there are only finitely many elements of S that are $\mathcal{L}$-below x_σ or $\mathcal{L}$-above x_τ. Thus $\langle R_i(x) \mid i \leqslant n \rangle = \sigma$ for all but finitely many $x \in S$, and so S is strongly $\vec{R}$-cohesive.

Finally, we assume StCOH and prove $\mathrm{RT}^1_{<\infty}$. (StCOH obviously implies COH.) Let $f : \mathbb{N} \to n$ and define a sequence of sets R_i for $i < n$ by $R_i = \{x \mid f(x) = i\}$. Let S be strongly cohesive for this sequence. Thus there is an s such that $(\forall i < n)[(\forall j > s)(j \in S \to j \in R_i) \vee (\forall j > s)(j \in S \to j \notin R_i)]$. As f is total, there is no j for which the second disjunct can hold for every $i < n$. Thus there must be an $i < n$ for which the first disjunct holds. This R_i is cofinite in S and so infinite as required. □

Proposition 4.5. $\mathrm{RCA}_0 \vdash \mathrm{ADS} \to \mathrm{StCOH}$ *and so* $\mathrm{RCA}_0 \vdash \mathrm{ADS} \to \mathrm{B}\Sigma_2$.

Proof. The proof of Proposition 2.10 that (in RCA_0) ADS $\to$ COH actually shows that ADS $\to$ StCOH. □

Finally, although we do not know whether SADS implies $B\Sigma_2$, we do know that it is not Π^1_1-conservative over RCA_0. To prove this, we introduce the following principle, which follows from $B\Sigma_2$ by the proof of Proposition 4.4.

(PART) Every linear order of type $\omega + \omega^*$ is strongly of type $\omega + \omega^*$.

Proposition 4.6. $RCA_0 \vdash$ SADS $\rightarrow$ PART.

Proof. Let $\mathcal{L}$ be a linear order of type $\omega + \omega^*$ with first element 0 and last element 1. Let $0 = x_0 <_L x_1 <_L \cdots <_L x_n = 1$ for some $n \in \mathbb{N}$ be a finite sequence. By SADS there is an ascending or descending sequence S in $\mathcal{L}$. Suppose it is ascending. By $I\Pi_1$ (which is equivalent to $I\Sigma_1$) there is a least $i \leqslant n$ such that all elements of S are below x_i. The interval $[x_{i-1}, x_i)$ then contains infinitely many elements (a final segment, actually) of S. As in the proof of Proposition 4.4, we see that there are only finitely many elements below x_{i-1} or above x_i as required in the definition of $\mathcal{L}$ being strongly of type $\omega + \omega^*$. The argument for a descending sequence S is symmetric. □

Corollary 4.7. $RCA_0 \nvdash$ PART *and so* SADS *is not* Π^1_1*-conservative over* RCA_0.

Proof. PART is a Π^1_1 consequence of SADS by Proposition 4.6. If it were a theorem of RCA_0 then in RCA_0 we would have that CADS implies StCADS, which implies $B\Sigma_2$ by Proposition 4.4. This would contradict the fact that, as it is a consequence of COH, CADS is Π^1_1-conservative over RCA_0. □

We conclude this section by considering the principle CRT^2_2, introduced in §1, which states that every 2-coloring of pairs has a set on which it is stable, and the following stronger version analogous to those considered above.

($\mathbf{StCRT^2_2}$) Strongly Cohesive Ramsey's Theorem for pairs: For every coloring f of $[\mathbb{N}]^2$ there is an infinite set S on which f is *strongly stable*, i.e. such that $(\forall n)(\exists y)(\forall x \in S)(x < n \rightarrow (\forall z \in S)[z > y \rightarrow f(x,y) = f(x,z)])$.

Proposition 4.8. $RCA_0 \vdash StCRT^2_2 \leftrightarrow CRT^2_2 + B\Sigma_2 \leftrightarrow$ StCOH.

Proof. To see that $StCRT^2_2 \rightarrow B\Sigma_2$ first note that $StCRT^2_2$ clearly implies $StCRT^2_3$ (the same principle for 3-colorings) by the usual argument considering first the 2-coloring gotten by taking as the first color the union of the first two of the given three colors and then if necessary applying $StCRT^2_2$ again to a set homogeneous to this union. Next, consider any counterexample $f : \mathbb{N} \rightarrow n$ to $RT^1_{<\infty}$. As usual, define a coloring by letting $C(x,y)$ be 0 if $f(x) = f(y)$, letting it be 1 if $f(x) < f(y)$, and letting it be 2 if $f(x) > f(y)$. Let S be a set on which C is strongly stable. No x can be stable within S to 0 as each $f^{-1}[i]$ is finite. Let $F = \{i < n \mid (\exists s \in S)(f(s) = i)\}$. This set is finite by $I\Sigma_1$. Let $A = \{a_i \mid i \in F$ and a_i is the least $a \in S$ such that $f(a) = i\}$. Again A is finite and so by assumption there is a $y \in S$ such that $(\forall z \in S)[z > y \rightarrow C(a_i, y) = C(a_i, z)]$ for every $i \in F$. If $C(a_i, y) = 1$ or 2 then $f(y) = j \neq i$ for some $j \in F$ and so $C(a_j, y) = 0$ for the desired contradiction.

Next we assume $\mathrm{CRT}^2_2 + \mathrm{B}\Sigma_2$ and consider any sequence of sets $R_0, R_1, \ldots$ (with the aim of constructing a cohesive set for the sequence). We appeal to the proof that $\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \to \mathrm{COH}$ in Mileti [2004]. Let $d(a, b)$ be the least i such that $R_i(a) \neq R_i(b)$. Since we can always add more sets to the R_i, without loss of generality the function d is total, and we define a coloring C by $C(a, b) = R_{d(a,b)}(a)$. Mileti [2004] shows in RCA_0 that any set homogenous for C is cohesive for the R_i. Let S be such that C is stable on S. If S is not cohesive for the R_i then there is a k for which there are infinitely many $s \in S$ such that $s \in R_k$ and infinitely many $t \in S$ such that $t \notin R_k$. By $\mathrm{B}\Sigma_2$ there are finite strings ρ_0 and ρ_1 of length $k+1$ with $\rho_i(k) = i$ for $i = 0, 1$ such that there are infinitely many $t \in S$ for which $t \in R_j \Leftrightarrow \rho_0(j) = 1$ for $j \leqslant k$ and infinitely many $s \in S$ for which $s \in R_j \Leftrightarrow \rho_1(j) = 1$ for $j \leqslant k$. Let $l \leqslant k$ be least such that $\rho_0(j) \neq \rho_1(j)$. Consider now the sets $A = \{t \in S \mid (\forall j \leqslant k)(t \in R_j \Leftrightarrow \rho_0(j) = 1)\}$ and $B = \{s \in S \mid (\forall j \leqslant k)(s \in R_j \Leftrightarrow \rho_1(j) = 1)\}$. Both sets are infinite and if $a \in A$ and $b \in B$ with $a < b$ then $C(a, b) = R_l(a) = \rho_0(l)$. Thus every $a \in A$ is stable within S to $\rho_0(l)$. The usual construction of a homogeneous set H for C from a set whose elements are all stable to the same color can now be carried out using $\mathrm{B}\Sigma_2$. (Define a sequence a_i by recursion where a_{i+1} is the least element a of A such that $C(a_j, a) = \rho_0(l)$ for $j < i$. There is such an a by $\mathrm{B}\Sigma_2$ and the recursion is then Δ^0_1.) As mentioned above, H is cohesive for the R_i.

Finally, note that $\mathrm{StCOH} \to \mathrm{CRT}^2_2 + \mathrm{B}\Sigma_2$ by Propositions 1.4 and 4.4, while these two together clearly imply StCRT^2_2. □

We summarize our results in the following diagram. We list here only those references not given after Diagram 3: (1) Proposition 4.1; (2) Proposition 4.5; (3) proof of Proposition 4.4; (4) Proposition 4.6; (5) Proposition 4.4; (6) Corollary 4.7; (7) Proposition 4.4; (8) Proposition 4.4 and Corollary 2.21; (9) Corollary 2.21; (10) Corollary 2.6 (since the proof of that corollary shows that COMP is not a model of CADS); (11) Proposition 4.8; (12) Proposition 1.4; (13) Proposition 2.9.

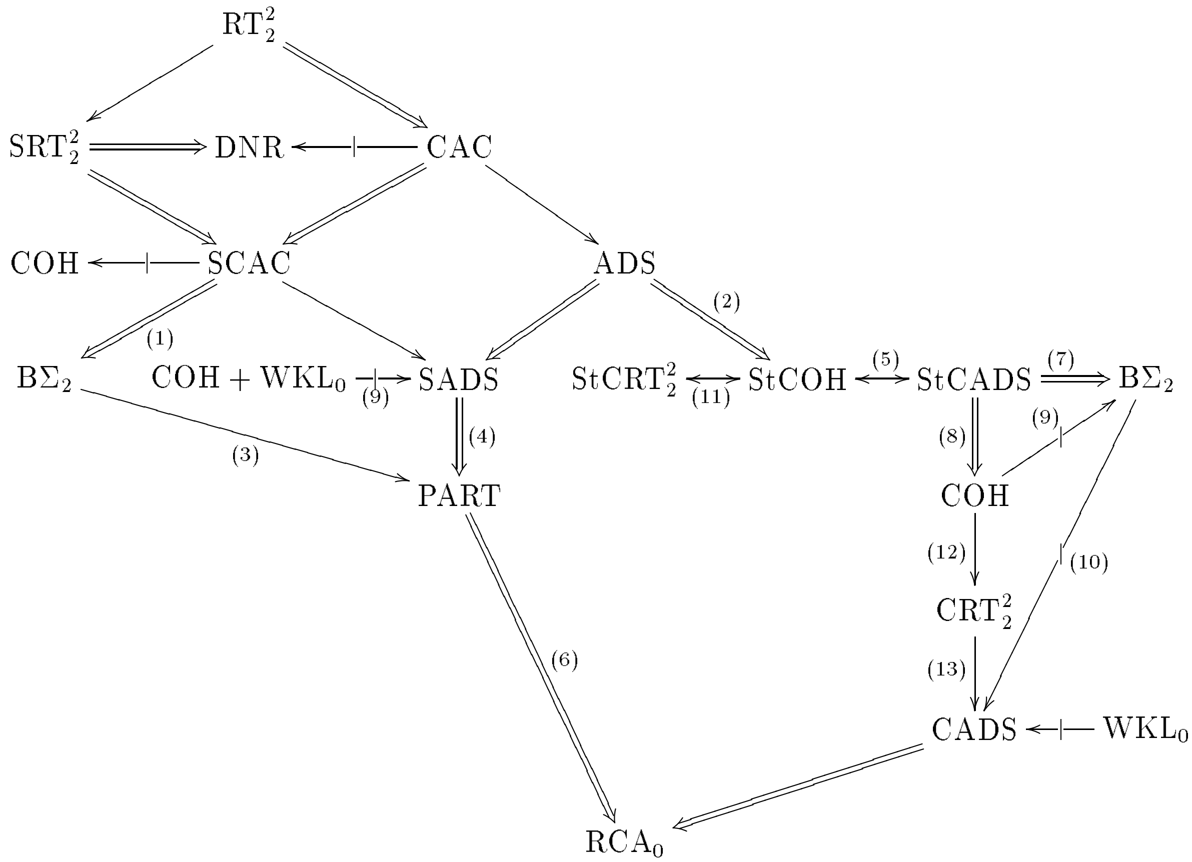

Diagram 4

5 Transitive and Semitransitive Colorings

In this section we will try to isolate the properties of colorings that allow us to prove that ADS or CAC is weak in some way that RT^2_2 is not or is not known to be. In particular, each of ADS and CAC can be split into a stable version that has low solutions and the cohesive principle COH that does not, but is conservative over RCA_0. Moreover, neither implies WKL_0 nor even DNR. If we examine the proofs of these results the primary issue is seen to be one of some transitivity of the coloring. We isolate the required properties in the definitions of transitive and semitransitive colorings below. The existence of homogeneous sets for such colorings clearly implies ADS and CAC, respectively. We will prove that the implications can be reversed and so provide classes of colorings corresponding to ADS and CAC. We also examine the relation between the stable versions of the coloring principles and SCAC and SADS.

Definition 5.1. An *n-coloring* of $[\mathbb{N}]^2$ (as in Definition 1.1) is *transitive* if each $f^{-1}[i]$ (i.e. the set C_i of pairs in the natural ordering of $\mathbb{N}$ with color i) is a transitive relation, i.e. $C_i(x,y) \wedge C_i(y,z) \to C_i(x,z)$. (With this view we identify the coloring with the sequence $\vec{C}$.) An n-coloring is *semitransitive* if each C_i is a transitive relation, except possibly for one i. Homogeneity and stability for such colorings is as usual.

Thus, for example, if $\mathcal{L}$ is a linear order (on $\mathbb{N}$) then $C_0(x,y) \leftrightarrow x <_{\mathbb{N}} y \wedge x <_L y$ and $C_1(x,y) \leftrightarrow x <_{\mathbb{N}} y \wedge x >_L y$ define a transitive 2-coloring. Moreover, any H homogeneous for this coloring is an ascending or descending sequence in $\mathcal{L}$ (depending on whether H is homogeneous to C_0 or C_1, respectively). Similarly, if $\mathcal{P}$ is a partial order (on $\mathbb{N}$), let $C_0(x,y) \leftrightarrow x <_{\mathbb{N}} y \wedge x \mid_P y$, let $C_1(x,y) \leftrightarrow x <_{\mathbb{N}} y \wedge x <_P y$, and let $C_2(x,y) \leftrightarrow x <_{\mathbb{N}} y \wedge x >_P y$; this defines a semitransitive 3-coloring. If H is homogeneous, then H is an antichain in $\mathcal{P}$ (if homogeneous to C_0), an ascending sequence (if homogeneous to C_1), or a descending sequence (if homogeneous to C_2). Thus the assumptions that there are always homogeneous sets for transitive 2-colorings and for semitransitive 3-colorings imply ADS and CAC, respectively.

Now for semitransitive n-colorings an inductive argument shows that the principle of existence of homogeneous sets for n-colorings is equivalent to that for 2-colorings: Given an n-coloring $\vec{C}$ with $n > 2$ and C_0 not transitive define a semitransitive 2-coloring $\vec{D}$ by setting $D_0 = C_0 \cup \cdots \cup C_{n-2}$ and $D_1 = C_{n-1}$. A homogeneous set for $\vec{D}$ is either homogeneous to $D_1 = C_{n-1}$, and so is a homogeneous set for $\vec{C}$, or to D_0. In the latter case we define a new 2-coloring $\vec{E}$ with $E_0 = C_0 \cup \cdots \cup C_{n-3}$ and $E_1 = C_{n-2}$ and consider a homogeneous set for $\vec{E}$. Iterating at most $n-1$ times we arrive at a set homogeneous for the original n-coloring $\vec{C}$. The same argument works for the principles limited to stable colorings, as the union operation preserves stability. Such considerations lead to the usual situation that, in a context with limited induction, we have a difference between the assertions of the instances of some principle for each standard n and the assertion that the principle holds for every n. We leave these issues aside and turn to the proofs that CAC and ADS imply the existence of homogeneous sets for the appropriate types of colorings.

Theorem 5.2. *For each $n \in \omega$, we can prove in* RCA_0 *that* CAC *implies the principle that every semitransitive n-coloring has a homogeneous set, and so for $n \geqslant 2$, each such principle is equivalent to* CAC*.*

Proof. Consider first the case that $n = 2$ and assume that C_0 is not necessarily transitive, but C_1 is transitive. Define a relation $\mathcal{P}$ on $\mathbb{N}$ by $x <_P y \leftrightarrow x < y \ \wedge \ C_1(x,y)$. As C_1 is transitive, $\mathcal{P}$ is a partial order. A chain in $\mathcal{P}$ is homogeneous to C_1. An antichain is homogeneous to C_0. We now proceed by induction (on ω) as above to deduce the principle for $n > 2$. $\square$

Theorem 5.3. *In* RCA_0*, we can prove that* ADS *implies that every transitive* 2*-coloring has a homogeneous set, and so this principle is equivalent to* ADS*.*

Proof. Suppose that C_0, C_1 define a transitive 2-coloring of $\mathbb{N}$. We define a linear order $\mathcal{L}$ on $\mathbb{N}$ by recursion. Begin by putting 0 into the domain of $\mathcal{L}$. Suppose at stage n we have ordered the numbers less than n in $\mathcal{L}$. Find the $\mathcal{L}$-largest $k <_{\mathbb{N}} n$ such that $C_0(k, n)$. If there is no such k, declare n to be $\mathcal{L}$-below every number less than n. Otherwise, insert n into the $\mathcal{L}$ order $\mathcal{L}$-above k and $\mathcal{L}$-below the $\mathcal{L}$-immediate successor of k among the numbers less than n. This defines the linear order $\mathcal{L}$ on $\mathbb{N}$. Let H be an ascending or descending sequence in $\mathcal{L}$.

If H is a descending sequence in $\mathcal{L}$, we may take it to also be ascending in natural order as it is infinite. We then claim that H is homogeneous to C_1. If not then there are $s <_{\mathbb{N}} t$ in H such that $C_0(s, t)$. By the definition of $\mathcal{L}$, when t is put into the order, it is put in $\mathcal{L}$-above s, which contradicts the fact that H is a descending sequence.

If H is ascending, we may again take it to be ascending in natural order. We claim H is then homogeneous to C_0. In fact, we claim that if $s <_{\mathbb{N}} t$ and $s <_{\mathcal{L}} t$ then $C_0(s, t)$. If not, let s be $\mathbb{N}$-least such that there is a t greater than s in both natural and $\mathcal{L}$ order such that $C_1(s, t)$ and then let t be the $\mathbb{N}$-least such number for this s. (This pair exists by Σ_1 induction.) Consider the situation when t is added to $\mathcal{L}$. As $s <_{\mathcal{L}} t$, there must be some $t_0 <_{\mathbb{N}} t$ that is greater than or equal to s in the $\mathcal{L}$ order such that $C_0(t_0, t)$. We cannot have $s = t_0$ by our assumption that $C_1(s, t)$. If $t_0 <_{\mathbb{N}} s$ then, by the definition of the order construction at s, we have $C_1(t_0, s)$, as otherwise we would have put s in $\mathcal{L}$-above t_0. As we also have by assumption that $C_1(s, t)$, transitivity of C_1 gives $C_1(t_0, t)$ for a contradiction. Finally we cannot have $s <_{\mathbb{N}} t_0$ by our leastness assumption on t. $\square$

We next consider the stable versions of these principles. Our first result provides the final evidence that SCAC is the correct stable version of CAC. The others deal with SADS and stable transitive partitions.

Proposition 5.4. $\mathrm{RCA}_0 \vdash \mathrm{SCAC} \leftrightarrow$ *every stable semitransitive* 2*-coloring has a homogeneous set.*

Proof. Suppose C is a stable semitransitive 2-coloring. Let $\mathcal{P}$ be as defined in the proof of Theorem 5.2. If x is stable to C_1 then for almost every y, we have $C_1(x, y)$ and so $x <_P y$. On the other hand, if x is stable to C_0 then for almost every y, we have $x \mid_P y$. So $\mathcal{P}$ is stable. Thus the proof of Theorem 5.2 provides our desired homogeneous set. For the other direction, suppose that $\mathcal{P}$ is stable. The standard 3-coloring as defined above to show that this implication works for arbitrary partial orders and colorings is stable and the same proof as in the general case shows that a homogeneous set gives the desired chain or antichain. $\square$

Proposition 5.5. $\mathrm{RCA}_0 \vdash$ *every stable transitive* 2*-coloring has a homogeneous set* $\rightarrow$ SADS.

Proof. The proof of Proposition 2.8 works here as well, as the coloring used there is transitive. $\square$

Proposition 5.6. $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2 \vdash \mathrm{SADS} \rightarrow$ *every stable transitive* 2*-coloring has a homogeneous set.*

Proof. Consider the proof of Theorem 5.3. We only have to show that the linear order $\mathcal{L}$ defined there is of type $\omega + \omega^*$ if the coloring is stable. Consider any n. If there is an $m >_{\mathcal{L}} n$ that is stable to C_0 then by definition $x >_{\mathcal{L}} n$ for almost every x. Otherwise, consider the construction of $\mathcal{L}$ at stage n. There are some numbers $x_1, \ldots, x_k \leqslant_{\mathbb{N}} n$ that are $\mathcal{L}$-above or equal to n. We claim by Σ_1 induction that, for $m >_{\mathbb{N}} n$, we have $m >_{\mathcal{L}} n$ if and only if the following condition holds: there is a finite chain $m = y_0 >_{\mathcal{L}} \cdots >_{\mathcal{L}} y_j = x_i$ for some $i \leqslant k$ such that the y_l are also descending in natural order, and for each $l < j$, we have $C_0(y_{l+1}, y_l)$. To establish this claim, we argue as follows. When m is put into the ordering and is placed above n it must be placed above some $z \geqslant_{\mathcal{L}} n$ with $z <_{\mathbb{N}} m$ and $C_0(z, m)$. The claim now follows by the transitivity of the coloring (and Σ_1 induction). Now, by $\mathrm{B}\Sigma_2$ and our case assumption there is an s after which all the x_i are stable to 1 and so there are only finitely many $m >_{\mathcal{L}} n$ as required. □

Note that we can run through the versions of these principles and their connections with $\mathrm{B}\Sigma_2$ as in §4. As an example, we note that an examination of the preceding two proofs shows that, in RCA_0, SADS for orders strongly of type $\omega + \omega^*$ is equivalent to the principle stating that every strongly stable transitive 2-coloring has a homogeneous set.

6 Questions

From our point of view, the main combinational/reverse mathematics questions left open in CJS are whether SRT^2_2 implies COH and so RT^2_2, whether CAC implies RT^2_2 and whether RT^2_2 implies WKL_0 (all over RCA_0). We have answered the analogous questions for CAC and ADS negatively (and on the positive side have shown that CAC and ADS do imply COH and $\mathrm{B}\Sigma_2$). In addition, we have shown (in conjunction with Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [ta]) that CAC does not imply RT^2_2. The other questions for RT^2_2 remain open. To these we add the following:

Question 6.1. Does ADS imply CAC?

Question 6.2. Does SADS imply SCAC?

Of course a positive answer to Question 6.2 would imply a positive one to Question 6.1.

One obvious question is raised by our results on (semi)transitive colorings:

Question 6.3. Are there $m > n \geqslant 2$ such that the existence of homogeneous sets for all transitive n-colorings implies the same for m-colorings?

If, for example, there are no such m and n then we would have a whole hierarchy of Ramsey type principles intermediate between ADS and CAC as, by Theorem 5.2, CAC implies that every transitive m-coloring has a homogeneous set for every m.

There are also two obvious combinatorial/reverse mathematics questions raised by our analysis of the relationships between $\mathrm{B}\Sigma_2$, StCADS, and StCOH.

Question 6.4. Does CRT^2_2 or even CADS imply COH?

Question 6.5. Does SADS or even PART imply $\mathrm{B}\Sigma_2$?

Next, we see the primary proof theoretic question left open in CJS as whether SRT^2_2 or even RT^2_2 is conservative over $\mathrm{B}\Sigma_2$. The problem of proving conservativity over $\mathrm{B}\Sigma_2$ seems difficult. External forcing arguments of the sort used in CJS and in our proof of Theorem 2.20 and Corollary 2.21 can at times be used to get conservation results over $\mathrm{I}\Sigma_2$ (or more) when the forcing relation for Σ_2 (or Π_2) sentences is Σ^0_2 (Π^0_2, respectively). This is done in CJS to prove that even RT^2_2 is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2$. The problem at the level of $\mathrm{B}\Sigma_2$ is that what one would want to do is consider each sentence of the form $(\forall i < n)(\exists s)(\forall t)\varphi(i,s,t)$ that is true in the forcing extension and so forced by a condition p. Of course, for each $i < n$ there is an s and a condition $q \Vdash \forall t\, \varphi(i,s,t)$ with $q \leqslant p$. As long as this assertion about i is Σ^0_2 in the ground model one might hope to apply $\mathrm{B}\Sigma_2$ in the ground model to get a single condition forcing everything and so the desired bound in the extension. The problem is that producing such a condition seems to require a recursive construction of n many successive conditions rather than a bounding argument. Unfortunately, the existence of n many recursive iterations of even Π^0_1 functions for each $n \in \mathbb{N}$ implies $\mathrm{I}\Sigma_2$ over RCA_0 (indeed, over just $\mathrm{I}\Delta^0_1$, which follows from $\mathrm{P}_0 + \Delta^0_1$-CA) and so cannot be applied in arbitrary models of $\mathrm{B}\Sigma_2$ to get such a conservation result. We can make this statement more precise by considering the following family of principles:

($\mathbf{PREC}_n$) If $\varphi(x,y) \in \Pi_{n-1}$ defines a total function then $(\forall z)(\forall m)(\exists \sigma)[|\sigma| = m \wedge \sigma(0) = z \wedge (\forall i < m)\varphi(\sigma(i), \sigma(i+1))]$.

In the presence of Δ^0_1-CA (even without $\mathrm{I}\Sigma_1$), this principle for $n = 1$ is easily seen to be equivalent to the following:

(PREC) $\forall z\, \forall f\, \exists g\, [g(0) = z \wedge \forall n\, (g(n+1) = f(g(n))]$ where we use f and g to range over functions (with the understanding that this is an abbreviation for the translation into formulas with variables over sets).

Proposition 6.6. *For each $n \geqslant 1$, we have* $\mathrm{P}_0 + \mathrm{I}\Delta^0_1 \vdash \mathrm{PREC}_n \leftrightarrow \mathrm{I}\Sigma_n$.

Proof. To see that $\mathrm{I}\Sigma_n$ implies PREC_n, fix z and prove the assertion of PREC_n by Σ_n-induction on m.

For the other direction, we proceed by induction on n. We begin with PREC_1. Consider a Σ^0_1 formula $\exists x\, \psi(i,x)$ such that $\exists x\, \psi(0,x) \wedge \forall k\, (\exists x\, \psi(k,x) \rightarrow \exists x\, \psi(k+1,x))$.

We must show that $\forall k\,\exists x\,\psi(k,x)$. Define the formula $\varphi(\langle i,x\rangle,\langle n,z\rangle)$ by

$$[\neg\psi(i,x) \wedge n = z = 0] \vee [\psi(i,x) \wedge n = i+1 \wedge \psi(i+1,z) \wedge (\forall z' < z)\neg\psi(i+1,z')].$$

This formula is Π^0_0 and defines a total function by $\mathrm{I}\Delta^0_1$ and our assumptions on ψ. To prove $\forall k\,\exists x\,\psi(k,x)$, consider any k. By PREC_1 there is a σ of length $k+2$ such that $\sigma(0)$ is a witness that $\exists x\,\psi(0,x)$ and $\varphi(\sigma(i),\sigma(i+1))$ holds for $i<k$. Using $\mathrm{I}\Delta^0_1$, we can now prove that $\psi(i,\sigma(i))$ holds for every $i \leqslant k$ and so $\exists x\,\psi(k,x)$ as required.

Next, assume PREC_{n-1} for $n>1$, which by induction implies $\mathrm{I}\Sigma_{n-1}$. Consider any Σ^0_n formula $\rho = \exists x\,\forall y\,\theta(k,x,y)$ and assume that $\rho(0) \wedge \forall k(\rho(k) \to \rho(k+1))$. We wish to prove $\forall k\,\rho(k)$. The proof is like that in the $n=1$ case with one added layer of uniformization. Let $\varphi(\langle i,x_0,x_1\rangle,\langle m,z_0,z_1\rangle)$ be

$$\begin{aligned}
&[\neg\theta(i,x_0,z_0) \wedge (\forall w < z_0)\theta(i,x_0,w) \wedge m = z_1 = 0] \vee \\
&\quad [\forall y\,\theta(i,x_0,y) \wedge m = i+1 \wedge \forall y\,\theta(i+1,z_0,y) \wedge z_1 \text{ is a sequence of length } z_0 \wedge \\
&\qquad (\forall w < z_0)\neg\theta(i+1,w,z_1(w)) \wedge (\forall z' < z_1)(\text{if } z' \text{ is a sequence of length } z_0 \text{ then} \\
&\qquad\qquad \neg(\forall w < z_0)\neg\theta(i+1,w,z'(w))\,].
\end{aligned}$$

This formula is Π_{n-1} and, by our assumptions (including $\mathrm{I}\Pi_{n-1}$, which is equivalent to $\mathrm{I}\Sigma_{n-1}$), defines a total function. Iterating it for n many steps starting at $\langle 0,x_0,0\rangle$ for an x_0 such that $\forall y\,\theta(0,x_0,y)$ produces the sequence of witnesses needed to show that $(\forall i \leqslant n)\rho(i)$. Thus PREC_n proves $\forall n\,\rho(n)$ by reducing the induction needed one quantifier level as in the $n=1$ case. □

We note that PREC or PREC_1 give a comprehension axiom that could be adopted in place of $\mathrm{I}\Sigma_1$ in the definition of RCA_0 (given by Simpson [1999]). These axioms assert only the existence of (even just finite) iterations of given functions and make no additional induction assumptions. This is the route followed by Friedman [1976] to define his EFT (elementary theory of functions) as the base theory to which Δ^0_1-CA is added to get RCA_0.

Returning now to $\mathrm{B}\Sigma_2$, we note that the only conservation results at this level of which we are aware are by Hájek [1993] and Avigad [2002] for WKL_0.[2] Hájek uses a priority argument in RCA_0 (and so one that is carried out internally in each model of $\mathrm{B}\Sigma_2$) to produce a definable universal solution G to all instances of WKL_0 computable in a fixed set X such that the verification of $\mathrm{B}\Sigma_2$ for sentences involving G can be reduced to ones for sentences with X but without G. (Computability theoretically, what he proves is that the construction of the Jockusch-Soare Low Basis Theorem can be used to produce a path P such that $P' \leqslant_{\mathrm{tt}} 0'$ and that this suffices.) Following this outline one might hope to prove that SADS, for example, is conservative over $\mathrm{B}\Sigma_2$. The idea would be to use the priority argument in the proof of Theorem 2.11 to produce low solutions for SADS in an arbitrary models of $\mathrm{B}\Sigma_2$. As G is low, $\Sigma^0_2(G) \in \Sigma^0_2$ and so $\mathrm{B}\Sigma_2$ would hold in the

[2] Chong, Slaman, and Yang have recently announced a proof that COH is Π^1_1-conservative over $\mathrm{B}\Sigma_2$.

extension generated by G. One would then iterate as in Hájek [1993] to get a model of SADS + $\mathrm{B}\Sigma_2$ with the same first order part as the original model. (This could then be extended to get Π^1_1-conservativity by relativizing.)

Unfortunately, it does not seem as if the priority argument of Theorem 2.11 can be carried out in $\mathrm{B}\Sigma_2$ as it is a Sacks type argument with unbounded injuries rather than a Friedberg-Muchnik one where the number of injuries is computably bounded. Also, the iteration used in Hájek [1993] is done through a cut (a (typically) proper initial segment of the first order part or the given model). Unfortunately, this means that the proof that all instances of the problem at hand get solutions added on during the iteration relies on the generic G being a universal solution, i.e. on there being a solution computable in G for every computable instance of the problem. (In the case of WKL_0 an instance of the problem is an infinite binary tree and the solution an infinite path in the tree. In the case of SADS an instance of the problem is a linear order of type $\omega + \omega^*$ and the solution a suborder of type ω or ω^*.) Theorem 2.18, however, shows that there is no such universal instance of SADS. (If there were one then it would have a low solution S by Theorem 2.11, but by Theorem 2.18 there would be another instance of SADS that has no solution computable in S.)

On the other path in our diagrams of implications and nonimplications the weakest principle is CADS, which, in the presence of $\mathrm{B}\Sigma_2$, is equivalent to COH. Here we seem to have a sufficiently universal instance of the problem given by the sequence of primitive recursive sets. Our problem for COH is that no internal argument in $\mathrm{B}\Sigma_2$ produces even a low_2 solution. It is true that CJS prove that there always is a low_2 solution (and so, in particular, a low_2 p-cohesive set) but the proof does not work in $\mathrm{B}\Sigma_2$. Indeed, it seems likely that by using and extending the ideas of Mourad [1988, Ch. II] as well as some further computability theoretic information about solutions to COH one could show that it is not possible to produce low_2 solutions for COH in $\mathrm{B}\Sigma_2$ in the desired way. Indeed, it seems that if, provably in $\mathrm{B}\Sigma_2$, there is a formula that for every sequence $\vec{R}$ uniformly defines an $\vec{R}$-cohesive set S such that the complete Σ^0_1 set in S is Δ^0_3 in $\vec{R}$, then $\mathrm{I}\Sigma_2$ holds.[3] Nonetheless, as mentioned above Chong, Slaman, and Yang have recently announced a proof that COH is Π^1_1-conservative over $\mathrm{B}\Sigma_2$. Perhaps their methods can be extended to other principles considered here.

Thus we close with an open-ended question about conservativity over $\mathrm{B}\Sigma_2$.

Question 6.7. Are any of the principles weaker than ACA_0 that we have considered (other than those following from WKL_0 or COH) Π^1_1- or even just arithmetically conservative over $\mathrm{B}\Sigma_2$? [4]

[3] We thank Carl Jockusch for some useful conversations and Chi Tat Chong for some useful correspondence about this point.

[4] As mentioned above, Hirschfeldt, Shore, and Slaman have recently studied another principle, AMT. It asserts that every complete atomic theory has an atomic model. This principle lies strictly between SADS and RCA_0 but is Π^1_1-conservative over $\mathrm{B}\Sigma_2$ as well as RCA_0.

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