On isolating r.e. and isolated d-r.e. degrees

Marat Arslanov*
Kazan University, Kazan, Russia

Steffen Lempp†
University of Wisconsin, Madison WI 53706-1388 USA

Richard A. Shore§
Cornell University, Ithaca NY 14853 USA

1. Introduction

The notion of a *recursively enumerable (r.e.) set*, i.e. a set of integers whose members can be effectively listed, is a fundamental one. Another way of approaching this definition is via an approximating function $\{A_s\}_{s\in\omega}$ to the set A in the following sense: We begin by guessing $x \notin A$ at stage 0 (i.e. $A_0(x) = 0$); when x later enters A at a stage $s+1$, we change our approximation from $A_s(x) = 0$ to $A_{s+1}(x) = 1$. Note that this approximation (for fixed) x may change at most once as s increases, namely when x enters A. An obvious variation on this definition is to allow more than one change: A set A is *2-r.e.* (or *d-r.e.*) if for each x, $A_s(x)$ change at most twice as s increases. This is equivalent to requiring the set A to be the difference of two r.e. sets $A_1 - A_2$. (Similarly, one can define n-r.e. sets by allowing at most n changes for each x.)

The notion of d-r.e. and n-r.e. sets goes back to Putnam [1965] and Gold [1965] and was investigated (and generalized) by Ershov [1968a, b, 1970]. Cooper showed that even in the Turing degrees, the notions of r.e. and d-r.e. differ:

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Theorem 1.1. (Cooper [1971]) *There is* a properly d-r.e. *degree, i.e. a Turing degree containing a d-r.e. but no r.e. set.*

In the eighties, various structural differences between the r.e. and the d-r.e. degrees were exhibited by Arslanov [1985], Downey [1989], and others. The most striking difference is probably the following result which stands in contrast with the well-known Sacks Density Theorem for the r.e. degrees:

Theorem 1.2. (Cooper, Harrington, Lachlan, Lempp, Soare [1991]) *There is a maximal incomplete d-r.e. degree below* $\mathbf{0}'$*; thus the d-r.e. degrees are not densely ordered.*

The distribution of r.e. degrees within the structure of the d-r.e. degrees has also been investigated, starting with Lachlan's observation (unpublished) that any noncomputable d-r.e. degree bounds a noncomputable r.e. degree.

Cooper and Yi [1995] defined the notion of an *isolated d-r.e. degree* $\mathbf{d}$ as a Turing degree such that the r.e. degrees strictly below $\mathbf{d}$ contain a greatest r.e. degree $\mathbf{a}$, say. ($\mathbf{a}$ is then said to *isolate* $\mathbf{d}$.) They established the following results about this notion:

Theorem 1.3. (Cooper, Yi [1995]) *(i) There exists an isolated d-r.e. degree.*

(ii) There exists a non-isolated properly d-r.e. degree.

(iii) Given any r.e. degree $\mathbf{a}$ *and d-r.e. degree* $\mathbf{d} > \mathbf{a}$*, there is a d-r.e. degree* $\mathbf{e}$ *between* $\mathbf{a}$ *and* $\mathbf{d}$.

They raise the question of whether the phenomena in (i) and (ii) above occur densely relative to the r.e. degrees (i.e. whether we can find such degrees between any two comparable r.e. degrees), and whether every noncomputable incomplete r.e. degree isolates some d-r.e. degree. LaForte answered the first of these questions positively:

Theorem 1.4. (LaForte [1995]) *Given any two comparable r.e. degrees* $\mathbf{v} < \mathbf{u}$*, there exists an isolated d-r.e. degree* $\mathbf{d}$ *between them.*

(Ding and Qian [1995] independently obtained a partial answer to the above by showing that there is an isolated d-r.e. degree below any noncomputable r.e. degree.)

We answer the other two questions in the present paper:

Theorem 2.1. *Given any two comparable r.e. degrees* $\mathbf{v} < \mathbf{u}$*, there exists a non-isolated d-r.e. degree* $\mathbf{d}$ *between them.*

Before stating the answer to the last question, we state the following proposition, connecting d-r.e. and REA in $\mathbf{a}$ degrees:

Proposition 3.1. *If* $\mathbf{d} > \mathbf{a}$ *is d-r.e. (or* n*-r.e. for any* $n \in \omega$*) then there is a* $\mathbf{c} \leq \mathbf{d}$ *which is r.e. in* $\mathbf{a}$ *and strictly above* $\mathbf{a}$*. So, in particular, if* $\mathbf{a}$ *isolates* $\mathbf{d}$ *then* $\mathbf{a}$ *isolates* $\mathbf{c}$*.*

The following is then a negative answer to the last question of Cooper and Yi mentioned above:

Theorem 3.2. *There is a noncomputable r.e. degree* $\mathbf{a}$ *which isolates no degree REA in it.*

We extend this result by showing that the non-isolating degrees are downward dense in the r.e. degrees and that they occur in any jump class:

Theorem 3.7. *For every noncomputable r.e. degree* $\mathbf{c}$*, there is a noncomputable r.e. degree* $\mathbf{a} \leq \mathbf{c}$ *which isolates no degree REA in it.*

Theorem 3.8. *If* $\mathbf{c}$ *is REA in* $\mathbf{0}'$ *then there is a noncomputable r.e. degree* $\mathbf{a}$ *with* $\mathbf{a}' = \mathbf{c}$ *which isolates no degree REA in it.*

We close with another result relating the d-r.e. degrees to the notion of relative enumerability.

Theorem 4.2. *Given r.e. degrees* $\mathbf{v} < \mathbf{u}$*, there is a d-r.e. degree* $\mathbf{d}$ *between them which is not r.e. in* $\mathbf{v}$*.*

We generally follow the notation of Soare [1987]. Familiarity with the proof of the weak density result of Cooper, Lempp, Watson [1989] is frequently assumed throughout the paper.

2. Non-isolated d-r.e. degrees

In this section we show that between any two r.e. degrees there is a properly d-r.e. degree which is not isolated by any r.e. degree. The proof of this theorem uses an infinite injury argument and is essentially the same as in Cooper, Lempp, Watson [1989] where, given r.e. sets $U >_T V$, a d-r.e. set C of properly d-r.e. degree such that $U >_T C >_T V$ is constructed.

Theorem 2.1. *Given r.e. sets* $U >_T V$ *there is a d-r.e. set* C *of properly d-r.e. degree such that* $U >_T C >_T V$*, and, for any r.e. set* B*, if* $B <_T C$ *then* $B <_T W <_T C$ *for some r.e. set* W*.*

Proof. We construct r.e. sets $A_1, A_2 \leq_T U$. If $A = A_1 - A_2$ then $C = V \oplus A$ will be the desired set. To ensure that $V \oplus A$ is not of r.e. degree we satisfy for every e the requirement

$$R_e : A \neq \Theta_e^{W_e} \quad \vee \quad W_e \neq \Phi_e^{V \oplus A}.$$

To ensure that the degree of $V \oplus A$ is not isolated we satisfy for every e the requirement

$$S_e : W_e = \Psi_e^{V \oplus A} \Rightarrow (\exists \text{ r.e. } U_e \leq_T V \oplus A)(\forall\, i)(U_e \neq \Omega_i^{W_e}).$$

Here $\{(W_e, \Theta_e, \Phi_e, \Psi_e, \Omega_e)\}_{e \in \omega}$ is some enumeration of all possible five-tuples of r.e. sets W and partial recursive functionals Θ, Φ, Ψ and Ω.

Since we handle the requirements $\{R_e\}_{e \in \omega}$ in the same way as in Cooper, Lempp, Watson [1989] we will consider here only the requirements $\{S_e\}_{e \in \omega}$. In satisfying S_e we shall construct a r.e. set U_e with the intention that if $W_e = \Psi_e^{V \oplus A}$ then $U_e \neq \Omega_i^{W_e}$ for all i and $U_e \leq_T V \oplus A$ through a modified permitting argument. We break S_e up into subrequirements $S_{e,i}$:

$$S_{e,i} : W_e = \Psi_e^{V \oplus A} \Rightarrow U_e \neq \Omega_i^{W_e}.$$

Basic module. Let us first consider requirements $S_{e,i}$ without the claim that $A \leq_T U$ and in the absence of any V-changes. (This is just the proof that there is a non-isolated d-r.e. degree.) The strategy proceeds as follows:

(1) Choose an unused candidate x for $S_{e,i}$ greater than any number mentioned in the construction thus far.

(2) Wait for a stage s such that

$$\Omega_{i,s}^{W_{e,s}}(x) \downarrow = 0,$$

and for some least u such that

$$W_{e,s} \upharpoonright \omega_{i,s}(x) = \Psi_{e,s}^{(V \oplus A)_s \upharpoonright u} \upharpoonright \omega_{i,s}(x).$$

(If this never happens then x is a witness to the success of $S_{e,i}$).

(3) Protect $A \upharpoonright u$ from other strategies from now on.

(4) Put x into U_e and A.

(5) Wait for a stage s' such that

$$\Omega_{i,s'}^{W_{e,s'}}(x) \downarrow = 1,$$

and for some least u' such that

$$W_{e,s'} \upharpoonright \omega_{i,s'}(x) = \Psi_{e,s'}^{(V \oplus A)_{s'} \upharpoonright u'} \upharpoonright \omega_{i,s'}(x).$$

(If this never happens then again x is a witness to the success of $S_{e,i}$. If it does happen then the change in $\Omega_i^{W_e}(x)$ between stages s and s' can only be brought about by a change in $W_e \upharpoonright \omega_{i,s}(x)$, which is irreversible since W_e is a r.e. set.)

(6) Remove x from A and protect $A \upharpoonright u'$ from other strategies from now on.

(Now x is a permanent witness to the success of $S_{e,i}$ because

$$\Psi_e^{V \oplus A} \upharpoonright \omega_{i,s}(x) = \Psi_{e,s}^{(V \oplus A)_s} \upharpoonright \omega_{i,s}(x) = W_{e,s} \upharpoonright \omega_{i,s}(x) \neq W_e \upharpoonright \omega_{i,s}(x).)$$

We see that the $S_{e,i}$-strategy in isolation and without the claims $A \leq_T U$ and $V \leq_T A$ is essentially the same as the R_e-strategy under similar assumptions. (Note that since we have refuted the overall hypothesis of S_e we no longer need to maintain the reduction $U_e \leq_T A$.) It allows us to meet all requirements $\{S_{e,i}\}_{e,i \in \omega}$ and $\{R_e\}_{e \in \omega}$ together in the same way as in the similar theorem from Cooper, Lempp, Watson [1989].

As in Cooper, Lempp, Watson [1989], we handle the condition $V \leq_T A$ by imposing "indirect" restraints to protect V, threatening $U \leq_T V$ via a functional Γ. We make infinitely many attempts to satisfy $S_{e,i}$ as above by an ω-sequence of "cycles", each cycle k proceeding as above with its own witness and with the following step inserted after step 3:

($3\frac{1}{2}$) Set $\Gamma_e^V(k) = U_s(k)$ with *use* $\gamma(k) = u$, start cycle $k+1$ simultaneously, wait for $U(k)$ to change, then stop cycles $k' > k$ and proceed.

Finally, we ensure that $A \leq_T U$ through a permitting argument. So x has to be permitted to enter A by U at step (4) and to leave A at step (6). The former permission is already given by the $U(k)$-change, the latter we build into the strategy as in Cooper, Lempp, Watson [1989].

Now the basic module for the $S_{e,i}$-strategy repeats the module for the R_e-strategy from Cooper, Lempp, Watson [1989]. It consists of an $(\omega \times \omega)$-sequence of cycles (j,k), $j,k \in \omega$. Cycle $(0,0)$ starts first, and each cycle (j,k) can start cycles $(j,k+1)$ or $(j+1,0)$ and stop, or cancel, cycles $(j',k') > (j,k)$ (in the lexicographical ordering). Each cycle (j,k) can define $\Gamma_j^V(k)$ and $\Delta^V(j)$.

A cycle (j,k) now proceeds as follows:

(1) Choose an unused candidate x such that $x-1$ is greater than any number mentioned thus far in the construction.

(2) Wait for a stage s_1 such that

$$\Omega_{i,s_1}^{W_{e,s_1}}(x) \downarrow = 0,$$

and for some least u such that

$$W_{e,s_1} \upharpoonright \omega_{i,s_1}(x) = \Psi_{e,s_1}^{(V\oplus A)_{s_1}\upharpoonright u} \upharpoonright \omega_{i,s_1}(x).$$

(3) Protect $A \upharpoonright u$ from other strategies from now on.

(4) Set $\Gamma_j^V(k) = U_{s_1}(k)$ with *use* $\gamma_j(k) = u$, and start cycle $(j, k+1)$ to run simultaneously.

(5) Wait for $V \upharpoonright u$ or $U(k)$ to change.

If $V \upharpoonright u$ changes first then cancel cycles $(j', k') > (j, k)$, drop the A-protection of cycle (j, k) to 0, and go back to step (2).

If $U(k)$ changes first then stop cycles $(j', k') > (j, k)$ and proceed to step (6).

(6) Put x into A and U_e.

(7) Wait for a stage s_2 such that

$$\Omega_{i,s_2}^{W_{e,s_2}}(x) \downarrow = 1,$$

and for some least u' such that

$$W_{e,s_2} \upharpoonright \omega_{i,s_2}(x) = \Psi_{e,s_2}^{(V\oplus A)_{s_2}\upharpoonright u'} \upharpoonright \omega_{i,s_2}(x).$$

(8) Protect $A \upharpoonright u'$ from other strategies from now on.

(9) Set $\Delta^V(j) = U_{s_2}(j)$ with *use* $\delta(j) = u'$ and start cycle $(j+1, 0)$ simultaneously.

(10) Wait for $V \upharpoonright u'$ or $U(j)$ to change.

If $V \upharpoonright u'$ changes first then cancel cycles $(j', k') \geq (j+1, 0)$, drop the A-protection of cycle (j, k) to u, and go back to step (7).

If $U(j)$ changes first then stop cycles $(j', k') \geq (j+1, 0)$ and proceed to step (11).

(11) Remove x from A.

(12) Wait for $V \upharpoonright u \neq V_{s_1} \upharpoonright u$.

(13) Reset $\Gamma_j^V(k) = U(k)$, put $x+1$ into A, cancel cycles $(j', k') > (j, k)$, start cycle $(j, k+1)$, and halt cycle (j, k).

Whenever a cycle (j, k) is started, any previous version of it has been cancelled and its functionals have become undefined through V-changes and, therefore, Γ_j and Δ are defined consistently.

The basic module has four possible outcomes similar to those of the basic module of the R_e-strategy.

(A) There is a stage s after which no cycle acts. Then some cycle (j_0, k_0) eventually waits at step 2, 7 or 12 forever. Thus we win requirement $S_{e,i}$ through the cycle (j_0, k_0).

(B) Some cycle (j_0, k_0) acts infinitely often but no cycle $< (j_0, k_0)$ does so. Then it goes from step 5 to step 2, or from step 10 to step 7, infinitely often. Thus Ψ_e or Ω_e is partial. Notice that the overall restraint of all cycles has finite liminf.

(C) There is a (least) j_0 such that every cycle $(j_0, k), k \in \omega$, eventually waits at step 5 or 13 forever. ("Row j_0 acts infinitely".) This means that $U \leq_T V$ via Γ_{j_0} contrary to hypothesis.

(D) For every j there is a cycle (j, k_j) that eventually waits at step 10 forever. ("Every row acts finitely"). This means that $U \leq_T V$ via Δ contrary to hypothesis.

The verification now proceeds as in Cooper, Lempp, Watson [1989], and we leave the details to the reader, except for the following item: When we remove x from A, we also lose the U_e-permission for x (which must, of course, remain in U_e). But note that the win on S_e is global (and so U_e is no longer needed) unless $V \upharpoonright u$ changes later. In that case, however, $x+1$ is enumerated into A, and so $V \oplus A$ can recognize this. $\Box$

3. Nonisolating r.e. degrees

Following Cooper and Yi [1995] we say that a r.e. degree $\mathbf{a}$ *isolates* the degree $\mathbf{d} > \mathbf{a}$ if, for every r.e. $\mathbf{b} \leq \mathbf{d}$, we have $\mathbf{b} \leq \mathbf{a}$. Cooper and Yi ask (**Q 4.3**) if every r.e. degree $\mathbf{a}$ isolates some d-r.e. degree $\mathbf{d}$. In this section we supply a strong negative answer to this question. Our basic construction shows (Theorem 3.2) that there is a noncomputable r.e. degree $\mathbf{a}$ which does not isolate any $\mathbf{d}$ which is REA in it. The answer to **Q 4.3** then follows immediately from the next proposition.

Proposition 3.1. *If* $\mathbf{d} > \mathbf{a}$ *is d-r.e. (or* n*-r. e. for any* $n \in \omega$*) and* $\mathbf{a}$ *is r.e. then there is a degree* $\mathbf{c} \leq \mathbf{d}$ *which is r.e. in* $\mathbf{a}$ *and strictly above it. So, in particular, if* $\mathbf{a}$ *isolates* $\mathbf{d}$ *then* $\mathbf{a}$ *isolates* $\mathbf{c}$.

Proof. By Jockusch and Shore [1984], $\mathbf{d}$ is 2-REA, i.e. there is a r.e. degree $\mathbf{e}$ such that $\mathbf{d}$ is REA in $\mathbf{e}$. Now if $\mathbf{e} \leq \mathbf{a}$ then $\mathbf{d}$ itself is REA in $\mathbf{a}$ and so the degree $\mathbf{c}$ required in the Proposition. If not, then $\mathbf{a} < \mathbf{e} \vee \mathbf{a} \leq \mathbf{d}$ and so $\mathbf{e} \vee \mathbf{a}$ is the degree $\mathbf{c}$ required by the Proposition. (Essentially the same argument now works for $\mathbf{d}$ n-r.e. by induction on n.) The assertion about $\mathbf{a}$ isolating $\mathbf{c}$ follows by definition.

We also supply two variations on this basic construction that show that the degrees $\mathbf{a}$ not isolating any $\mathbf{d}$ which is REA in $\mathbf{a}$ are widely distributed in the r.e. degrees. Theorem 3.7 shows that such degrees exist below every nonrecursive r.e. $\mathbf{c}$ and Theorem 3.8 shows that they exist in every jump class, i.e. for every $\mathbf{c}$ REA in $\mathbf{0}'$ there is such a r.e. degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{c}$.

We begin with the basic construction.

Theorem 3.2. *There is a nonrecursive r.e. set A such that its degree $\mathbf{a}$ isolates no degree REA in it, i. e. $\forall e(A <_T W_e^A \rightarrow \exists B(B$ is r.e. & $B \leq_T W_e^A \,\&\, B \not\leq_T A))$.*

There are two types of basic requirements:

$P_e : \Phi_e \neq A$ (for each partial recursive function Φ_e).

$N_e : A <_T W_e^A \rightarrow B_e \leq W_e^A \,\&\, B_e \not\leq A$ (for each e we construct an appropriate r.e. set B_e).

The requirements N_e are divided up into subrequirements:

$N_{e,i} : \Phi_i^A \neq B_e$ (for each partial recursive functional Φ_i).

We order the requirements $P_e, N_{e,i}$ in an ω type list $\langle R_n \rangle$. The procedures for satisfying the individual requirements are fairly standard. We will diagonalize against Φ_e by putting some witness x into A at a stage s to satisfy P_e when $\Phi_e(x) = 0[s]$. For $N_{e,i}$ we will wait until some $x \in \omega^{[\langle e,i \rangle]}$ with $\Phi_i(A; x) = 0[s]$ is permitted by W_e^A at s and then put x into B_e. To implement the permitting we first approximate W_e^A in the usual way: $x \in W_e^A[s]$ iff $\Phi_e(A; x) \downarrow [s]$. We then say that x is *permitted by* W_e^A at s if it looks as if some $y < x$ is in W_e^A at s but it does not, at s, look as if it was in at $s-1$ by an A-correct computation, i.e.

$$\exists y < x\{y \in W_e^A[s] \,\&\, (y \notin W_e^A[s-1] \;\vee\; \exists z < \varphi_e(y, s-1)[z \in A_s - A_{s-1}])\}.$$

The restraint necessary to preserve the A-use relevant to this computation will be imposed automatically by our procedure for choosing potential witnesses for the P_e. We now present the formal construction and verifications.

Construction:

At stage s we find the first requirement R_n in our list such that one of the following two cases holds:

1) $R_n = P_e$; there is no z such that $\Phi_e(z) = 0\,[s]$ and $z \in A$; $\Phi_e(x) = 0\,[s]$ for the least $x \in \omega^{[e]}$ which is larger than any stage at which we have acted for any requirement of higher priority than P_e. We call this x the *current potential witness for* P_e.

2) $R_n = N_{e,i}$; there is no z such that $\Phi_i(A; z) = 0\,[s]$ and $z \in B_e$; there is a least $x \in \omega^{[\langle e,i\rangle]}$ larger than any stage at which we have acted for any requirement of higher priority than $N_{e,i}$ such that $\Phi_i(A; x) = 0\,[s]$ and larger than any current potential witness for any higher priority P-requirement; and x is permitted by W_e^A at s.

If there is no such n, we go on to stage $s+1$. Otherwise, we now act for requirement R_n according to which of the above two cases applies:

1) If $R_n = P_e$ then we put x into A.

2) If $R_n = N_{e,i}$ we put x into B_e.

Verifications:

Lemma 3.3. *We act for each requirement only finitely often.*

Proof. We proceed by induction through the priority ordering. Suppose we never act for any R_m with $m < n$ after stage s. If $R_n = P_e$ and we act for this requirement at $t > s$ by putting x into A then it is clear that we never act for it again as $\Phi_e(x) = 0\,[t]$ and $x \in A\,[t+1]$. If $R_n = N_{e,i}$ and we act for this requirement at $t > s$ by putting x into B_e, we never put any number less than t into A at any later stage since no P_j of lower priority can do so by construction and none of higher priority can act by our choice of s. Thus, by the usual conventions that the Φ_i use at t is at most t, no number less than the use $\varphi_i(x,t)$ can ever enter A after t. In particular, $\Phi_i(A; x)\,[t] = \Phi_i(A; x)\,[t'] = 0$ and $x \in B_e$ for every $t' > t$. So we never act again for $N_{e,i}$. $\square$

Lemma 3.4. *Each requirement P_e is satisfied, i. e. $\Phi_e \neq A$.*

Proof. Let s be the last stage at which we act for a requirement of higher priority than P_e and let x be the least element of $\omega^{[e]}$ larger than s. If we ever act for P_e after s we have $\Phi_e(x) = 0$ and we put x into A to satisfy P_e. If we never act for P_e after s, then either there is some other z such that $\Phi_e(z) = 0$ and $z \in A$ or $x \notin A$ and $\neg(\Phi_e(x) = 0)$. In either case $\Phi_e \neq A$ as required. $\square$

Lemma 3.5. *If $A <_T W_e^A$ then we satisfy each requirement $N_{e,i}$, i. e. $\Phi_i^A \neq B_e$ for each i.*

Proof. Consider any requirement $N_{e,i}$ and let s be a stage after which we never act for any requirement of higher priority than $N_{e,i}$. If we ever act for $N_{e,i}$ at a stage $t > s$ by putting some x into B_e then the argument for Lemma 3.3 shows that $\Phi_i(A;x) = 0$ and so $\Phi_i^A \neq B_e$ as required. If we never act for $N_{e,i}$ after stage s then $x \notin B_e$ for each $x \in \omega^{[\langle e,i\rangle]}$ which is larger than some fixed $s' > s$. Unless $\Phi_i(A;x) = 0$ for each such x, we have also shown that $\Phi_i^A \neq B_e$.

If neither of these situations satisfying $N_{e,i}$ occurs, we show that $W_e^A \leq_T A$ for a contradiction. To compute $W_e^A(x)$ for $x > s'$, then find a $z \in \omega^{[e]}$ such that $z > x$ and a stage $t > z$ such that $\Phi_i(A;z) = 0\,[t]$ by an A-correct computation, i. e. $A \upharpoonright \varphi_i(z,t) = A_t \upharpoonright \varphi_i(z,t)$. We claim that $x \notin W_e^A$ unless $x \in W_e^A[t]$ by an A-correct computation, i. e. $\Phi_e(A;x)\!\downarrow[t]$ and $A \upharpoonright \varphi_e(x,t) = A_t \upharpoonright \varphi_e(x,t)$. Of course, if $x \in W_e^A[t]$ by an A-correct computation, then $x \in W_e^A$. On the other hand, if $x \in W_e^A$ but not by an A-correct computation at t, then there must be a $v > t$ (the first stage at which we have the A-correct computation of $\Phi_e(A;x)$) at which W_e^A would permit z and so we would act for $N_{e,i}$ at v by putting z into B_e contrary to our assumption. □

Lemma 3.6. $B_e \leq_T W_e^A \oplus A$.

Proof. To determine if $x \in \omega^{[\langle e,i\rangle]}$ is in B_e, wait until a stage s such that $A \upharpoonright x = A_s \upharpoonright x$ and, for every $y < x$ such that $y \in W_e^A$, $y \in W_e^A[s]$ by an A-correct computation. We claim that if $x \notin B_{e,s}$ then $x \notin B_e$. The only way x can enter B_e at some $t > s$ is by our acting for $N_{e,i}$ at t and so, in particular, by W_e^A permitting x at t. Thus some $y < x$ is in $W_e^{A[}[t]$ that was not previously (and so not at s) in W_e^A by an A-correct computation. By construction, no requirement of lower priority than $N_{e,i}$ can injure the computation of $\Phi_e(A;y)[t]$. On the other hand, the current potential witnesses for P_j of higher priority than $N_{e,i}$ must all be less than x by construction.

Thus none of them can enter A by our choice of s. If any of these potential witnesses changes at a later stage $v > t$ (because of some action by a yet higher priority $N_{k,l}$ requirement), it must change to a number grater than $v > t > \varphi_e(y,t)$ and so also cannot injure the computation putting y into W_e^A. Thus $y \in W_e^A$ but is not in W_e^A by an A-correct computation at s for the desired contradiction. □

We may combine this last construction with r.e. permitting to construct the desired A below any given nonrecursive r.e. set C.

Theorem 3.7. *For every nonrecursive r.e. set C there is a nonrecursive r.e. set $A <_T C$ such that $\forall e(A <_T W_e^A \rightarrow \exists B(B$ is r.e. & $B \leq_T W_e^A$ & $B \not\leq_T A))$.*

Proof. We adjust the previous construction by possibly appointing many current potential witnesses for each requirement P_e. More specifically, if there is no z such that $\Phi_e(z) = 0\,[s]$ and $z \in A$ and $\Phi_e(y) = 0\,[s]$ for every current potential witness for P_e at s, then we act for P_e by appointing as a potential witness the least $x \in \omega^{[e]}$ which is larger than any stage at which we have acted for any requirement of higher priority than P_e and larger than every current potential witness. We cancel this potential witness at any later stage at which we act for some requirement of higher priority than P_e. If there is now a potential witness x with $\Phi_e(x) = 0$ which is permitted by C (i.e. some $y < x$ enters C at s) then we act for P_e by putting x into A. Otherwise, the construction is the same as before. The verifications now follow the usual pattern of a permitting argument. Assuming we never act for any requirement of higher priority than P_e after stage s, we use the nonrecursiveness of C to show that we act only finitely often for P_e and eventually satisfy it. (If we act infinitely often without putting a number into A (necessarily by appointing more and more potential witnesses) then we calculate C by noting that once $\Phi_e(x) = 0$ at a stage $t > s$ for some potential witness x, we can never later have a number $y < x$ enumerated in C.) The other verifications now proceed as before. □

Finally we show that there is a nonrecursive r.e. set A in every jump class which does not isolate any set D which is REA in A and so (by Proposition 3.1) not any d-r.e. degree above it either.

Theorem 3.8. *If C is REA in $\emptyset'$ then there is a nonrecursive r.e. set A such that $A' \equiv_T C$ and $\forall e(A <_T W_e^A \to \exists B(B$ is r.e. & $B \leq_T W_e^A$ & $B \not\leq_T A))$.*

We follow the usual proof of the Sacks jump theorem by starting with an r.e. D such that $x \in C$ implies $D^{[x]} = \{y|y < n\}$ for some n and $x \notin C$ implies $D^{[x]} = \omega$. Moreover, for technical convenience we assume that if a number z is enumerated in $D^{[e]}$ at stage s then $z < s$ and every $x < s$ which is in $\omega^{[e]}$ and not already in $D^{[e]}$ is enumerated in $D^{[e]}$ at s. We will make A a thick subset of D, i. e. for every x, $A^{[x]} \subseteq D^{[x]}$ and $D^{[x]} - A^{[x]}$ is finite. As usual this guarantees that $C \leq_T A'$. We now use a typical tree construction to satisfy the following requirements:

$P_e : A^{[e]} \subseteq D^{[e]}$ and $D^{[e]} - A^{[e]}$ is finite.

$N_{e,i} : \Phi_i^A \neq B_e$ (for each e we construct a r.e. set B_e satisfying $N_{e,i}$ for each partial recursive functional Φ_i if $A <_T W_e^A$).

Our priority tree is constructed as usual given that we assign nodes on level $2e$ to P_e and their possible outcomes are, in left to right order, $i < 0 < 1 < \ldots < n < \ldots$ while ones on level $2\langle e, i\rangle + 1$ area assigned to $N_{e,i}$ and their possible outcomes are $w < 0 < 1 < \ldots < s < \ldots$. (The intended meaning of the outcomes for P_e are $i : D^{[e]} = \omega$ and $n : n$ is the last stage at which a number is enumerated in $D^{[e]}$ (and so the first number not in $D^{[e]}$). The intended meaning of the outcomes for $N_{e,i}$ are w : we are waiting for a chance to diagonalize and $s \in \omega$: we succeed in diagonalizing by putting some x into B_e for which $\Phi_i(A; x) = 0$ at stage s. The nodes α assigned to requirements $N_{e,i}$ may impose *restraint* $r(\alpha, s)$ at stage s. We define $R(\alpha, s)$ the *restraint* imposed at s on a requirement α assigned to a requirement P_e as $\max\{r(\beta, s) | \beta < \alpha\}$.

Construction:

At each stage s we define a sequence of length s of *accessible* nodes and act accordingly. We begin with $\emptyset$, the root of our priority tree, as the first accessible node at each stage s. Suppose a node α of length less than s has just been declared accessible. If P_e is assigned to α then we see if there has been a number enumerated in $D^{[e]}$ since the last stage at which α was accessible (since stage 0 if this is the first stage at which α is accessible). If so, then the outcome of P_e is i; we declare $\alpha\hat{}i$ to be accessible; and we put every $x > R(\alpha, s)$ which is in $D^{[e]}_s$ into A. If not, we declare $\alpha\hat{}n$ to be accessible where n is the last stage at which some number was enumerated in $D^{[e]}$. If α is assigned to $N_{e,i}$ and we have acted for α at some previous stage t, then the outcome of α is t and we declare $\alpha\hat{}t$ to be accessible. Otherwise, the outcome of α is w and $\alpha\hat{}w$ is accessible. When we reach a node β of length s we see if there is any node α for which we have not yet acted which has previously been accessible (but is not necessarily accessible now) and is assigned to a requirement $N_{e.i}$ and any $x \in \omega^{[\alpha]}$ satisfying the following conditions:

(1) x is larger than the first stage $u' > u$ at which α was accessible where u is the last stage at which any $\beta <_L \alpha$ has been accessible.

(2) x is smaller than the last stage at which α was accessible.

(3) x is permitted by W_e^A at s.

(4) $\Phi_i(A; x) = 0\,[s]$ via an α-believable computation. (We say that the computation $\Phi_i(A; x) = 0$ is *α-believable* at s if $\forall z \forall k(z \in \omega^{[k]} \,\&\, \alpha(k) = i \,\&\, z > R(\alpha^- k, s) \,\&\, z < \varphi_i(x, s) \to z \in A_s)$.)

If there is such an α, we act for the highest priority one by putting the smallest such x into B_e and set $r(\alpha, s) = s$. If not, we go on to stage $s + 1$.

Verifications:

As each node that is accessible infinitely often clearly has a leftmost immediate successor which is accessible infinitely often, there is a path TP in the priority tree consisting of the leftmost nodes which are accessible infinitely often.

Lemma 3.9. *If $\alpha \in TP$ and α is first accessible at s then no $\beta <_L \alpha$ is ever accessible at $t > s$. Moreover, there is a stage $t \geq s$ after which we never act for any $\beta < \alpha$ assigned to a requirement $N_{e,i}$. Thus if α is assigned to some P_e then $R(\alpha, v)$ is constant for $v \geq t$.*

Proof. Proceeding by induction along TP the first claim is obvious from the definition of when the various outcomes of each node are accessible. As we can act at most once for each β assigned to a requirement $N_{e,i}$, the other assertions are also immediate. □

Lemma 3.10. *Suppose $\alpha \in TP$ is assigned to requirement P_e and s is the first stage at which α is accessible and $t \geq s$ is the first stage after which we never act for any $\beta < \alpha$ assigned to a requirement $N_{e,i}$ (such a stage exists by Lemma 3.9). If $D^{[e]} = \omega$ then $\alpha\hat{\ }i \in TP$ and for all $x \in \omega^{[e]}$, $x \in A^{[e]}$ iff $x > R(\alpha, t) \vee x \in D_t^{[e]}$. Otherwise, $D^{[e]}$ is finite and if n is the last stage at which some number is enumerated in $D^{[e]}$ then $\alpha\hat{\ }n \in TP$ and no number is put into $A^{[e]}$ after the first stage at which $\alpha\hat{\ }n$ is accessible.*

Proof. Suppose $D^{[e]} = \omega$. It is immediate from the definition of the accessible successor of α that $\alpha\hat{\ }i \in TP$. Now, $R(\alpha, t) = R(\alpha, v)$ for every $v > t$ by our choice of s and Lemma 3.9. Thus if $x < R(\alpha, t)$ and $x \notin D_t^{[e]}$ then $x \notin A^{[e]}$ by construction. On the other hand, if $x > R(\alpha, t)$ then there is a stage $v > t$ after x has entered $D^{[e]}$ at which $\alpha\hat{\ }i$ is accessible. By construction, we put x into $A^{[e]}$ at v.

If $D^{[e]}$ is finite and n is the last stage at which a number is enumerated in $D^{[e]}$ then it is clear from the definition of the accessible successor of α that $\alpha\hat{\ }n \in TP$ and from the construction and Lemma 3.9 that no number is put into $A^{[e]}$ after the first stage at which $\alpha\hat{\ }n$ is accessible. □

Lemma 3.11. *If $\alpha \in TP$ is assigned to $N_{e,i}$ and we act for α at s by putting x into B_e, then $\Phi_i(A; x) = 0$.*

Proof. By construction, α has been accessible before stage s or we could not act for it. Thus by Lemma 3.9 no node to the left of α can ever be accessible again. In particular, no action for a node $\beta <_L \alpha$ can put any number into A after stage s. No node of lower priority can put any number less than $\varphi_i(x, s) < s$ into A after stage s as we set $r(\alpha, s) = s$ and never change it. Finally, we claim no node $\beta \subset \alpha$ will ever put a number less than $\varphi_i(x, s)$ into A after stage s. If $\beta \subset \alpha$ and $\beta\hat{\ }n \subseteq \alpha$ for some n, then β puts no numbers at all into A after s by Lemma 3.10. On the other hand, if $\beta\hat{\ }i \subseteq \alpha$, then note that $R(\beta, t)$ is nondecreasing in t and so we will not put in any number less than $R(\beta, s)$ for β after s while all others that it might ever put into A less than $\varphi_i(x, s)$ are already in A by the definition of $\Phi_i(A; x)[s]$ being α-believable. □

Lemma 3.12. *We satisfy N_e, i.e. if $A <_T W_e^A$ then $\Phi_i^A \neq B_e$ for each i.*

Proof. Suppose $A <_T W_e^A$ and consider the node $\alpha \in TP$ assigned to $N_{e,i}$. If we ever act for α by putting some $x \in \omega^{[\alpha]}$ into B_e then, by Lemma 3.11, $\Phi_i(A;x) = 0 \neq B_e(x)$ as required. If we never act for α then $B_e \cap \omega^{[\alpha]} = \emptyset$ by construction. In this case, if, contrary to the conclusion of our Lemma, $\Phi_i^A = B_e$ then $\Phi_i(A;x) = 0$ for every sufficiently large $x \in \omega^{[\alpha]}$. Let s be the first stage at which α is accessible. We now argue exactly as in Lemma 3.5 with α replacing e that $W_e^A \leq_T A$ for the desired contradiction. □

Lemma 3.13. *For every e, $B_e \leq_T W_e^A \oplus A$.*

Proof. To determine if $x \in \omega^{[\alpha]}$ with α assigned to $N_{e,i}$ is in B_e assume we have already calculated $B_e {}^{\frown} x$ and all numbers in $B_e {}^{\frown} x$ have already been enumerated in B_e by stage $u > x$. Now choose a w such that for every e in the domain of α there is a $z \in \omega^{[e]}$ with $w > z > u$ and wait until a stage $s > w$ such that $A {}^{\frown} w = A_s {}^{\frown} w$ and, for every $y < x$ such that $y \in W_e^A$, $y \in W_e^A[s]$ by an A-correct computation. We claim that if $x \notin B_{e,s}$ then $x \notin B_e$. First, note that if any node $\beta <_L \alpha$ has been accessible since stage x then x cannot later enter B_e by condition (1) on our choice of x. Moreover, until such a β becomes accessible, no number greater than x can be put into any B_j for any $\beta <_L \alpha$ by condition (2) on our choice of x. Thus, by our choice of u, the restraints imposed by such β remain constant after stage u and are less than u until some such β becomes accessible. Now, the only way x can enter B_e at some $t > s$ is by our acting for α at t and so, in particular, by W_e^A permitting x at t. Thus some $y < x$ is in $W_e^{A[}[t]$ that was not previously (and so not at s) in W_e^A by an A-correct computation. By construction no requirement of lower priority than α can injure the computation of $\Phi_e(A;y)[t]$ after t. On the other hand, no action for a node $\beta \subset \alpha$ or $\beta <_L \alpha$ can injure the computation without our moving to the left of α or already having first moved to its left. Suppose then that we move to the left of α at some $v > t$. This can happen only when some $\alpha {}^{\frown} e$ is accessible at v and some has been enumerated in $D^{[e]}$ since stage n where $\alpha(e) = n \in \omega$. When this happens, we must enumerate all numbers in $\omega^{[e]}$ which are less than v into $A^{[e]}$ unless they are below $R(\alpha {}^{\frown} e, v)$. Our previous remarks, however, show that $R(\alpha {}^{\frown} e, v) < u$ and so some number $z \in \omega^{[e]}$ with $w > z > u$ is enumerated into A at v contradicting our choice of s. Thus $\varphi_e(y,t)$ would never be injured contradicting our choice of s once again. □

Lemma 3.14. $A' \equiv_T C$.

Proof. By Lemma 3.10, A is a thick subset of D and so $C \leq_T A'$. We claim that $TP \leq_T C$ and that $A' \leq_T TP$. We first recursively calculate TP from

C. Suppose we have $\alpha \in TP$ and want to find the immediate successor of α on TP. If α is assigned to some $N_{e,i}$ then $\alpha\hat{\ }w \in TP$ unless there is a stage s at which we act for α. In this case, $\alpha\hat{\ }s \in TP$. Of course, $\emptyset'$ can tell if there is such a stage and $\emptyset' \leq_T C$. If α is assigned to some P_e then $\alpha\hat{\ }i \in TP$ if $D^{[e]}$ is infinite and otherwise $\alpha\hat{\ }n \in TP$ where n is the last stage at which a number is enumerated in $D^{[e]}$. As $D^{[e]}$ is infinite if and only if $e \notin C$, C can tell which case applies and so (using $\emptyset'$ again in the second case) find the correct immediate successor of α on TP.

Now, we calculate A' from TP. We begin with a fixed e such that $A <_T W_e^A$. To determine if $j \in A'$ find an i such that, for every z, $\Phi_i(A; z) = 0$ iff $\Phi_j(A; j)\downarrow$. It is now clear from the proof of Lemma 3.12 that $j \in A'$ iff $\alpha\hat{\ }w \notin TP$ for the node $\alpha \in TP$ assigned to $N_{e,i}$. □

Proof (of Theorem 3.8). If $C \equiv_T \emptyset'$ then Theorem 3.7 provides the required A. Otherwise, our last construction supplies the desired set by Lemmas 3.12, 3.13 and 3.14. □

4. D-r.e. degrees and REA degrees

Theorem 4.1. *Let* $\mathbf{v}$ *be a r.e. degree such that* $\mathbf{0}' > \mathbf{v}$. *Then there is a d-r.e. degree* $\mathbf{d} > \mathbf{v}$ *which is not r.e. in* $\mathbf{v}$.

Proof. Let $K \in \mathbf{0}'$ and $V \in \mathbf{v}$ be fixed r.e. sets. We will construct a d-r.e. set A so that $D = V \oplus A$ does not have degree r.e. in $\mathbf{v}$.

To satisfy the last property we meet the following requirements for all e,

$$R_e : A \neq \Phi_e^{W_e^V} \vee W_e^V \neq \Psi_e^{V \oplus A},$$

where $\{(W_e^V, \Phi_e, \Psi_e)\}_{e \in \omega}$ is some enumeration of all possible triples consisting of sets W^V r.e. in V and partial recursive functionals Φ and Ψ.

We use a common convention (see, for example, Rogers [1967]) that W_e^V enumerates an element x by listing in W_e a quadruple $\langle x, 1, u, v\rangle$ with $D_u \subseteq V$ and $D_v \subseteq \bar{V}$.

Obviously, if for some finite set $X \subset \omega$, $X \subseteq W_e^V$ then there is a stage s such that for all $t \geq s$ we have $X \subseteq W_{e,t}^{V_t}$. (Note here that we denote by $X \subseteq W_e^V$ that X is a *subset* and *not* necessarily a substring.) Besides, if for some s and a (least) number θ, $X \subseteq W_{e,s}^{V_s}$, and

$$\forall x(x \in X \to \langle x, 1, u, v\rangle \in W_{e,s} \wedge (D_u \subset V_s \restriction \theta \wedge D_v \subset \bar{V}_s \restriction \theta)),$$

then $X \not\subseteq W_e^V$ implies $V \restriction \theta \neq V_s \restriction \theta$. We call θ the X-use for $W_{e,s}^{V_s}$.

In satisfying R_e we shall construct functionals $\Gamma_j (j \in \omega)$ and Δ with the intention that if R_e fails then $K \leq_T V$ via some Γ_j, or Δ, contrary to our hypothesis.

Basic module. As usual, we will choose a sequence of candidates (one for each "cycle" of the strategy), one of which will witness the failure of one or both of the statements:

1. $A = \Phi_e^{W_e^V}$,
2. $W_e^V = \Psi_e^{V \oplus A.}$

This will be sufficient for R_e to succeed.

We make infinitely many attempts to satisfy R_e by an $\omega \times \omega$-sequence of "cycles", where each cycle (j, k) proceeds as follows:

(1) Choose an unused candidate $x_{j,k}$ greater than any number mentioned thus far in the construction.

(2) Wait for a stage s at which for some n, $\Psi_{e,s}^{V_s \oplus A_s} \upharpoonright n$ is defined, and for $X \subseteq \{0, \ldots, n\}$, $X \upharpoonright n = \Psi_{e,s}^{V_s \oplus A_s} \upharpoonright n$,

$$X \subseteq W_{e,s}^{V_s}$$

(with X-use θ), and

$$A(x_{j,k}) = \Phi_e^{X \upharpoonright n}(x_{j,k}).$$

(It is easy to see that if this never happens then $x_{j,k}$ is a witness to the success of R_e.)

(3) Protect $A \upharpoonright \psi_{e,s}\varphi_{e,s}(x_{j,k})$ from other strategies from now on.

(4) Set $\Gamma_j^V(k) = K_s(k)$ with *use* $\gamma_j(k) = \max\{\theta, \psi_{e,s}\varphi_{e,s}(x_{j,k})\}$, and start cycle $(j, k+1)$ to run simultaneously with cycle (j, k).

(5) Wait for $K(k)$ to change (at a stage s', say).

(If there is a $V \upharpoonright \psi_e\varphi_e(x_{j,k})$-change between stages s and s', we kill the cycles $(j', k') > (j, k)$, drop the A-protection of this cycle (j, k) to 0, and go back to step (2). If there is a $V \upharpoonright \theta$-change between stages s and s', but there is no $V \upharpoonright \psi_e\varphi_e(x_{j,k})$-change, we kill the cycles $(j', k') > (j, k)$, and go back to step (2). In both cases, the parts of the functionals Γ_j, Δ defined by cycles $(j', k') > (j, k)$ become undefined by the V-change.

(6) Stop cycles $(j', k') > (j, k)$ and put $x_{j,k}$ into A.

(7) Wait for a stage s'' at which, for some n', $\Psi_{e,s''}^{(V \oplus A)_{s''}} \upharpoonright n'$ is defined, and for $X' \subseteq \{0, \ldots, n'\}$, $X' \upharpoonright n' = \Psi_{e,s''}^{(V \oplus A)_{s''}} \upharpoonright n'$,

$$a) X \subsetneq X',$$

$$b) X' \subseteq W_{e,s''}^{V_{s''}}$$

with X'-use θ', and

$$c) A(x_{j,k}) = \Phi_e^{X' \upharpoonright n'}(x_{j,k}).$$

(Note that if this never happens then $x_{j,k}$ is again a witness to the success of R_e. Indeed, if b) and c) never happen then obviously either $A(x_{j,k}) \neq \Phi_e^{W_e^V}(x_{j,k})$ or $W_e^V \neq \Psi_e^{V \oplus A}$. If b) and c) do happen with some X', but $X \not\subseteq X'$, then while enumerating V we must have seen some $V \upharpoonright \psi_{e,s''}\varphi_{e,s}(x_{j,k})$-change or a $V \upharpoonright \theta$-change and would go back to step 2, otherwise we would win R_e by $x_{j,k}$: we have $X \upharpoonright n' \subsetneqq W_e^V \upharpoonright n'$ and $\Psi_e^{V \oplus A} \upharpoonright n' = X' \upharpoonright n'$, therefore $\Psi_e^{V \oplus A} \upharpoonright n' \neq W_e^V \upharpoonright n'$).
Notice also that if we now remove $x_{j,k}$ from A, we would have (in the absence of a $V \upharpoonright \theta'$-change or $V \upharpoonright \psi_e\phi_e(x_{j,k})$-change)

$$\Psi_e^{V \oplus A} \upharpoonright \varphi_e(x_{j,k}) = \Psi_{e,s}^{(V \oplus A)_s} \upharpoonright \varphi_e(x_{j,k}) = X \upharpoonright \varphi_e(x_{j,k}) \subsetneqq X' \upharpoonright \varphi_e(x_{j,k}).$$

So if $X' \upharpoonright \varphi_e(x_{j,k}) \subseteq W_e^V \upharpoonright \varphi_e(x_{j,k})$ then this is enough for the success of R_e. But, unfortunately, W_e^V is reversible through a $V \upharpoonright \theta'$-change (even if $V \upharpoonright \theta$ does not change) and we may again have $X \upharpoonright \varphi_e(x_{j,k}) = W_e^V \upharpoonright \varphi_e(x_{j,k})$.

To avoid this difficulty we will use these changes of $V \upharpoonright \theta'$ to threaten $K \leq_T V$ via a new functional Δ.

(8) Set

$$\Delta^V(j) = K(j)$$

with *use* $\delta(j) = \max\{\theta', \psi_{e,s''}(\varphi_{e,s''}(x_{j,k}))\}$, and start cycle $(j+1, 0)$ to run simultaneously.

(9) Wait for $K(j)$ to change (at stage s^*, say).

(10) Stop all cycles $(j', k') \geq (j+1, 0)$, remove the number $x_{j,k}$ from A, and preserve $A \upharpoonright \psi_{e,s^*}\varphi_{e,s^*}(x_{j,k})$.

(11) Wait for a $V \upharpoonright \delta(j)$-change.

(12) Drop the A-protection of this cycle to 0, set

$$K(j) = \Delta^V(j)$$

with a new use $\delta(j)$, stop cycle (j, k), cancel all cycles $> (j, k)$, and start cycle $(j+1, 0)$.

Whenever some cycle sees a $V \upharpoonright \delta(j)$-change between stages s'' and s^*, it will kill the cycles $(j',k') > (j,k)$, make their functionals (including Δ^V) undefined, and go back to step 7.

If some cycle sees a $V \upharpoonright \psi_{e,s}\varphi_{e,s}(x_{j,k})$-change between stages s and s^*, it will again kill the cycles $(j',k') > (j,k)$, make their functionals and $\Delta^V(j)$ undefined, and go back to step 2.

Note that if a cycle (j,k) sees a $V \upharpoonright \delta(j)$-change between stages s'' and s^* but there is no $V \upharpoonright \psi_{e,s}\varphi_{e,s}(x_{j,k})$-change after stage s then it goes back to step 7 and proceeds. If later the cycle again comes to step 8 it redefines $\Delta^V(j)$ (with the same j) with a new *use* $\delta(j)$. So in this case (when there is no $V \upharpoonright \psi_{e,s}\varphi_{e,s}(x_{j,k})$-change), other cycles $(j',k') \neq (j,k)$ cannot define $\Delta^V(j)$.

The module has the following possible outcomes:

(A) There is a stage s after which no cycle acts. Then some cycle (j_0,k_0) eventually waits at step (2), (7) or (11) forever. It means that we were successful in satisfying R_e through the cycle (j_0,k_0).

(B) Some (least) cycle (j_0,k_0) acts infinitely often. Then it goes from step (5) to step (2), or from step (9) to step (7) or (2) infinitely often. Thus Φ_e or Ψ_e is partial. Notice that the overall restraint of all cycles has finite liminf.

(C) Every cycle acts only finitely often but there are infinitely many cycles (j_0,k) (for some least j_0) which collectively act infinitely often. Then $\Gamma^V_{j_0} = K$, contrary to hypothesis.

(D) Otherwise. Then, for each j, the last time some cycle (j,k) acts, it defines $\Delta^V(j)$ permanently and correctly, so $\Delta^V = K$, contrary to hypothesis.

The explicit construction and the remaining parts of the proof are now essentially the same as in Cooper, Lempp and Watson [1989] with only obvious changes. So we will not give them here.

Moreover, adding to the construction a permitting argument in exactly the same way as in Cooper, Lempp, Watson [1989], we can prove the following theorem.

Theorem 4.2. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be r.e. degrees such that* $\mathbf{v} < \mathbf{u}$. *Then there is a d-r.e. degree* $\mathbf{d}$ *such that* $\mathbf{v} < \mathbf{d} < \mathbf{u}$ *and* $\mathbf{d}$ *is not r.e. in* $\mathbf{v}$.

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