Defining the Turing Jump

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1 Introduction

The primary notion of effective computability is that provided by Turing machines (or equivalently any of the other common models of computation). We denote the partial function computed by the eth Turing machine in some standard list by φ_e. When these machines are equipped with an "oracle" for a subset A of the natural numbers ω, i.e. an external procedure that answers questions of the form "is n in A", they define the basic notion of relative computability or Turing reducibility (from Turing (1939)). We say that A is computable from (or recursive in) B if there is a Turing machine which, when equipped with an oracle for B, computes (the characteristic function of) A, i.e. for some e, $\varphi_e^B = A$. We denote this relation by $A \leq_T B$ which we read as A is (Turing) reducible to B or A is recursive (computable) in B. This relation is transitive and reflexive and so induces an equivalence relation $\equiv_T$ ($A \equiv_T B \Leftrightarrow A \leq_T B \wedge B \leq_T A$) and a partial order also denoted by $\leq_T$ on the equivalence classes. These equivalence classes are called (Turing) degrees and the equivalence class of a set $A \subseteq \omega$ is called its degree. It is typically denoted by $\boldsymbol{a}$ or $\deg(A)$.

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The structure $\mathcal{D}$ of these degrees has been the object of extensive study over the past fifty or sixty years. (A survey from the early 80s is Shore (1985). Current ones can be found in Griffor (1999).) A central concern in this research over the past twenty years has been the issue of definability. The general question is which (interesting, apparently external) relations on $\mathcal{D}$ are actually definable in terms of relative computability alone. One important line of research has produced a sequence of results of the form that all relations on $\mathcal{D}$ which could possibly be definable, i.e. they are definable in arithmetic with quantification over both numbers and sets, are definable if restricted to "sufficiently" large degrees where sufficiently large has undergone a series of successive weakenings. (See, for example, Shore (1985) and Slaman (1991).) The other major line of investigation into definability in $\mathcal{D}$ has centered on proving that specific important natural but apparently external degrees or relations on $\mathcal{D}$ are definable in $\mathcal{D}$. The first example of what might be considered a natural definition of such a relation which also does not appeal to the general theorems appears in Jockusch and Shore (1984). They define $\mathcal{A}$, the degrees of the arithmetic sets, i.e. those sets definable in arithmetic with quantification only over numbers, in order-theoretic terms within $\mathcal{D}$.

The overarching goal of these investigations has been the definition of the (Turing) jump operator. The (Turing) jump A' of $A \subseteq \omega$ is the halting problem for machines with an oracle for A: $A' = \{e|$ the eth machine with oracle A halts on input $e\}$. So, in particular, $\mathbf{0}'$ is the degree of the halting problem. (We use 0 for the empty set and so its degree $\mathbf{0}$ is the degree of the computable sets.) The undecidability of the halting problem says, when relativized to A, that $A <_T A'$ for every A. As the jump is easily seen to be well defined on degrees, it induces a strictly increasing operation on $\mathcal{D}$ taking $\boldsymbol{a} = \deg(A)$ to $\boldsymbol{a}' = \deg(A')$. By classical results of Kleene and Post, this operator corresponds to definability in arithmetic extended by a predicate for membership in A by formulas with only one quantifier. Its nth iterate $A^{(n)}$ corresponds to definability by such formulas with n quantifiers. Thus, for example, $\mathcal{A} = \{\boldsymbol{x}|\exists n \in \omega(\boldsymbol{x} \leq_T \mathbf{0}^{(n)})\}$. This operator has played a major role in much of the work on $\mathcal{D}$ over the years and the issue of whether it is actually intrinsic to, or definable in, $\mathcal{D}$ was raised already in the fundamental paper of Kleene and Post (1954). This question essentially asks if quantification in arithmetic can be expressed level by level solely in terms of relative computability.

Cooper (1990) argued for the definability of the jump along the lines of the definition of $\mathcal{A}$ provided by Jockusch and Shore (1984). We outline the plan of the proof of the definability of $\mathcal{A}$ so as to be able to both describe

Cooper's proposal and present our own proof, as all of them follow the same general plan. Jockusch and Shore (1984) argued for the definability of $\mathcal{A}$ in $\mathcal{D}$ by combining a completeness and join theorem for certain operators with known structural results for $\mathcal{D}$ and the r.e. degrees. (The recursively enumerable or r.e. sets are those that can be enumerated by a partial recursive function φ_e or, equivalently those that are the domain of such a function. In the relativized case we denote the eth set r.e. in A by W_e^A. It is the domain of φ_e^A. The r.e. in A degrees are those of the sets W_e^A.) We begin with the natural order-theoretic properties crucial to this definition of $\mathcal{A}$ and the definition itself.

Definition 1.1 A degree $\boldsymbol{a}$ is a minimal cover if there is a $\boldsymbol{b}$ such that $\boldsymbol{a} > \boldsymbol{b}$ and there is no $\boldsymbol{c}$ strictly between $\boldsymbol{a}$ and $\boldsymbol{b}$. If $\boldsymbol{b}$ is such a degree for $\boldsymbol{a}$ we say that $\boldsymbol{a}$ is a minimal cover of $\boldsymbol{b}$.

Definition 1.2 $\mathcal{C}_\omega = \{\boldsymbol{c} | \forall \boldsymbol{z}(\boldsymbol{z} \vee \boldsymbol{c}$ is not a minimal cover of $\boldsymbol{z}\}$.
$\overline{\mathcal{C}}_\omega = \{\boldsymbol{d} | \exists \boldsymbol{c} \in \mathcal{C}_\omega (\boldsymbol{d} \leq \boldsymbol{c})\}$.

Theorem 1.3 (Jockusch and Shore (1984)) $\mathcal{A} = \overline{\mathcal{C}}_\omega$ *and the relation* $\boldsymbol{a}$ *is arithmetic in* $\boldsymbol{b}$ *is definable in* $\mathcal{D}$ *(by the relativization of the definition of* $\mathcal{A}$ *to* $\boldsymbol{b}$*).*

We next define the operators on sets needed for the analysis. We restrict ourselves here to the case that $\alpha \leq \omega$ but both of the following definitions have been usefully generalized into the transfinite and the corresponding theorems proven.

Definition 1.4 The $1 - REA$ *operators* J (from 2^ω to 2^ω) are those of the form $J(A) = J_e(A) = A \oplus W_e^A$. The $n - REA$ *operators* J are those of the form $J_{\langle e_1,\ldots,e_n\rangle} = J_{e_n} \circ J_{e_{n-1}} \circ \ldots \circ J_{e_1}$. The $\omega - REA$ operators J are those of the form $J(A) = \bigoplus\{J_{f\upharpoonright n}(A) | n \in \omega\}$ for some recursive f.

Definition 1.5 The $n - r.e.$ *operators* J are those of the form $J(A)(x) = \lim \varphi_e^A(x,s)$ for a (total recursive in A) function φ_e^A which has $\varphi_e^A(x,0) = 0$ for all x and for which there are at most n many s such that $\varphi_e^A(x,s) \neq \varphi_e^A(x,s+1)$. The $\omega - r.e.$ *operators* J are those of the form $J(A)(x) = \lim_{s\to\infty} \varphi_e^A(x,s)$ for a (total recursive in A) function φ_e^A which has $\varphi_e^A(x,0) = 0$ for all x and for which there are at most $f(x)$ many s such that $\varphi_e^A(x,s) \neq \varphi_e^A(x,s+1)$ for some recursive function f.

Now for the completeness theorem needed.

Theorem 1.6 (Jockusch and Shore (1984)) *For any $\alpha-REA$ operator J and any $C \geq_T \mathbf{0}^{(\alpha)}$ there is an A such that $J(A) \equiv_T C$.*

An improvement to the completeness theorem that includes a join operation provides an approach to the first natural definability result for $\mathcal{D}$.

Theorem 1.7 (Jockusch and Shore (1984)) *For any $\omega-r.e.$ operator J and any nonarithmetic D there is an A such that $J(A) \equiv_T D \vee \mathbf{0}^{(\omega)} \equiv_T D \vee A$.*

It is now the minimal degree construction of Sacks (1963)) that provides the operator relevant to the definition of $\mathcal{A}$.

Theorem 1.8 (Sacks (1963)) *There is an $\omega - r.e.$ operator J such that* $\deg J(A)$ *is a minimal cover of* $\deg A$ *for every A.*

Thus for every nonarithmetic degree $\boldsymbol{x}$ there is a $\boldsymbol{z}$ such that $\boldsymbol{x} \vee \boldsymbol{z}$ is a minimal cover of $\boldsymbol{z}$. On the other hand, for all n, $\mathbf{0}^{(n)} \vee \boldsymbol{z}$ is not a minimal cover of $\boldsymbol{z}$ for any $\boldsymbol{z}$ by Jockusch and Soare (1970). This establishes Theorem 1.3.

Cooper (1990) suggested a similar approach to the problem of defining the jump operator. His plan was to use a version of Theorem 1.7 for $2 - r.e.$ operators to define $\mathbf{0}'$ by finding a suitable $2 - r.e.$ operator that would produce a degree with an order-theoretic property that no r.e. degree could have (again even relative to any degree below it). He defined the following notions and classes.

Definition 1.9 $\boldsymbol{d}$ is *splittable over* $\boldsymbol{a}$ *avoiding* $\boldsymbol{b}$ if either $\boldsymbol{a}, \boldsymbol{b} \nleq \boldsymbol{d}$ or $\boldsymbol{b} \leq \boldsymbol{a}$ or there are $\boldsymbol{d}_0, \boldsymbol{d}_1$ such that $\boldsymbol{a} <_T \boldsymbol{d}_0, \boldsymbol{d}_1 <_T \boldsymbol{d}$, $\boldsymbol{d}_0 \vee \boldsymbol{d}_1 = \boldsymbol{d}$ and $\boldsymbol{b} \nleq_T \boldsymbol{d}_0, \boldsymbol{d}_1$. $\mathcal{C}_1 = \{\boldsymbol{c} | \forall \boldsymbol{a}, \boldsymbol{b} (\boldsymbol{a} \vee \boldsymbol{c}$ is splittable over $\boldsymbol{a}$ avoiding $\boldsymbol{b}\}$. $\overline{\mathcal{C}}_1 = \{\boldsymbol{d} | \exists \boldsymbol{c} \in \mathcal{C}_1 (\boldsymbol{d} \leq_T \boldsymbol{c})\}$.

Now, one of the needed results was already well known.

Theorem 1.10 (Sacks (1963)) *Every r.e. degree $\boldsymbol{d}$ is in $\mathcal{C}_1$.*

For the other direction Cooper (1990) claimed as his Main Theorem that there is a suitable $2 - r.e.$ set and so a $2 - r.e.$ operator J such that for every C there are $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{d} \equiv_T \deg(J(C))$ is not splittable over $\boldsymbol{a}$ avoiding $\boldsymbol{b}$. This would have sufficed to define $\mathbf{0}'$ and so, by relativization, the jump operator itself. Cooper (1990) proposed to argue for the existence of such a $2 - r.e.$ set with a $0'''$ priority construction similar to that proving

Lachlan's (1975) Nonsplitting Theorem for the recursively enumerable degrees. However, the following theorem shows that there is no such set and so Cooper's (1990) proposed property does not define the jump.

Theorem 1.11 (Shore and Slaman (n.d.)) *Suppose that $\boldsymbol{d}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are Turing degrees such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are recursive in $\boldsymbol{d}$, $\boldsymbol{b}$ is not recursive in $\boldsymbol{a}$, and $\boldsymbol{d}$ is n-REA relative to $\boldsymbol{a}$. Then there are $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x} \vee \boldsymbol{y} = \boldsymbol{d}$, $\boldsymbol{x} \geq_T \boldsymbol{a}$ and $\boldsymbol{y} \geq_T \boldsymbol{a}$, and $\boldsymbol{x} \not\geq_T \boldsymbol{b}$ and $\boldsymbol{y} \not\geq_T \boldsymbol{b}$.*

As Lachlan has proved that every $2-r.e.$ set is $2-REA$ (see Jockusch and Shore (1984) for a proof that every $\alpha - r.e.$ set is $\alpha - REA$), Theorem 1.11 proves that there is no $2 - r.e.$ degree which fulfills Cooper's requirements.

In our approach, we first prove a stronger version of Theorem 1.7 for all $n - REA$ operators and so answer a question of Jockusch and Shore (1984).

Question 1.12 (Jockusch and Shore (1984)) *Suppose that J is an n-REA operator and A is a subset of the natural numbers which is not Δ^0_n, i.e. not recursive in $0^{(n-1)}$. Does there exist a $G \subseteq \omega$ such that $A \oplus G \equiv_T J(G) \equiv_T A \oplus 0^{(n)}$?*

The case of this question for $\omega - REA$ operators (i.e. replace $\omega - r.e.$ by $\omega - REA$ in Theorem 1.7) was answered positively by Kumabe and Slaman. In Section 2, we provide an affirmative answer to Question 1.12.

Theorem 2.3 *Suppose that $n \geq 1$, J is an n-REA operator, and $A \subseteq \omega$ is not Δ^0_n. Then, there is a $G \subseteq \omega$ such that $A \oplus G \equiv_T J(G) \equiv_T A \oplus 0^{(n)}$.*

The other half of our proof of the definability of the jump, however, comes from a quite different direction.

Theorem 1.13 (Slaman and Woodin (n.d.)) *The double-jump which maps $\boldsymbol{x}$ to $\boldsymbol{x}''$ is definable in $\mathcal{D}$.*

Theorem 2.3 applies to all n-*REA* operators, and the double-jump is the canonical 2-*REA* operator. And so, we define $\mathbf{0}'$ in $\mathcal{D}$.

Theorem 1.14 $\mathbf{0}'$ *is defined within $\mathcal{D}$ as the greatest degree $\boldsymbol{z}$ such that there is no $\boldsymbol{g}$ such that $\boldsymbol{z} \vee \boldsymbol{g}$ is equal to $\boldsymbol{g}''$.*

Proof: Clearly no degree less than or equal to $\mathbf{0}'$ can join any $\boldsymbol{g}$ to $\boldsymbol{g}''$. Theorem 2.3 states that any degree not below $\mathbf{0}'$ does join some $\boldsymbol{g}$ up to $\boldsymbol{g}''$. Consequently, $\mathbf{0}'$ is definable in $\mathcal{D}$ in terms of join and the double-jump. The join is clearly definable in $\mathcal{D}$ and, by Theorem 1.13, so is the double-jump. ■

We can relativize the proofs of Theorem 2.3 and hence of Theorem 1.4. We then obtain the following definition of the function mapping $\boldsymbol{x}$ to $\boldsymbol{x}'$ in $\mathcal{D}$.

Theorem 1.15 *For any degree $\boldsymbol{x}$, $\boldsymbol{x}'$ is definable from $\boldsymbol{x}$ within $\mathcal{D}$ as the greatest degree $\boldsymbol{z}$ such that there is no $\boldsymbol{g}$ greater than or equal to $\boldsymbol{x}$ such that $\boldsymbol{z} \vee \boldsymbol{g}$ is equal to $\boldsymbol{g}''$.*

We end with some comments about the methods by which we prove these theorems and some comments about the problems which remain open.

Theorem 1.13 is proved by applying the Slaman and Woodin analysis of automorphisms of the Turing degrees. This analysis involves a fair amount of metamathematics, but not much technical recursion theory.

Note that it is not crucial that we have the definability specifically of the double jump as our starting point. Any definition of the $(n+1)$st jump for any $n \in \omega$ would suffice. We can define the nth jump from the $(n+1)$st using Theorem 2.3 in essentially the same way we defined the jump from the double jump in Theorem 1.15.

Theorem 2.3 is first proved in the special case of the n-fold iteration of the Turing jump in Theorem 2.1 using a sharp analysis of Kumabe and Slaman forcing. We give this analysis in Section 2. The general case follows from Theorem 2.2, the Jockusch and Shore (1984) Inversion Theorem for n-*REA* operators.

Remarkably, one obtains the definitions of $\mathbf{0}'$ and of the jump presented in Theorems 1.14 and 1.15 without using even one priority construction.

This situation is an advantage in some contexts. Slaman and Woodin showed that the arithmetic jump is definable within the partial order of the arithmetic degrees. In that context, the priority method has limited utility.

However, there are disadvantages as well. For example, the Slaman and Woodin methodology depends on global properties of the degrees. It is open whether $\mathbf{0}'$ is definable within every ideal of $\mathcal{D}$ to which it belongs.

It has been common wisdom that, because of the intrinsic complexity of the relation of Turing reducibility itself, coding methods within the Turing degrees cannot resolve relations which are not invariant under the double jump. Though our definition of the jump is a counter-example to this belief, obtaining even finer resolutions is an open problem. For example, it is open whether the relation "$\boldsymbol{x}$ is *REA* in $\boldsymbol{y}$" is definable in $\mathcal{D}$.

Another shortcoming of our proof is that Slaman and Woodin's definition of the double jump involves an explicit translation of isomorphism facts to definability facts via a coding of (second order) arithmetic. Thus, the definition provided is not based on a naturally order-theoretic property of $\mathbf{0}'$. One can hope that the above open questions could also be resolved, if one could find a *natural definition* of the Turing jump. Here natural is meant in the sense that the defining property for $\mathcal{A}$ in Jockusch and Shore (1984) and the one proposed in Cooper (1990) for $\mathbf{0}'$ are natural order-theoretic ones.

2 n-REA Operators and Kumabe–Slaman Forcing

Theorem 2.1 *Suppose that $n \geq 1$ and $A \subseteq \omega$ is not Δ^0_n. Then there is a $G \subseteq \omega$ such that $A \oplus G, A \oplus 0^{(n)} \geq_T G^{(n)}$.*

Before we give the proof of Theorem 2.1, we combine it with Jockusch and Shore's (1984) inversion theorem for REA-operators (relativized) to provide a complete solution to Question 1.12.

Theorem 2.2 (Jockusch and Shore (1984)) *Suppose that $n \geq 1$, J is an n-REA operator, and $K \geq_T H^{(n)}$. Then there is a G such that $G \geq_T H$ and $J(G) \equiv_T K$.*

Theorem 2.3 *Suppose that $n \geq 1$, J is an n-REA operator, and $A \subseteq \omega$ is not Δ^0_n. Then, there is a $G \subseteq \omega$ such that $A \oplus G \equiv_T J(G) \equiv_T A \oplus 0^{(n)}$.*

Proof: By Theorem 2.1, choose H so that $A \oplus H, A \oplus 0^{(n)} \geq_T H^{(n)}$. By Theorem 2.2, choose $G \geq_T H$ so that $J(G) \equiv_T A \oplus H$. But then $A \oplus G \geq_T A \oplus H \geq_T J(G)$. Conversely, since $J(G) \equiv_T A \oplus H$, $J(G) \geq_T A$; since J is an n-REA operator, $J(G) \geq G$; and so $J(G) \geq A \oplus G$. Thus, $A \oplus G \equiv_T J(G)$, as required for the first equivalence. For the second, note that $A \oplus H \equiv_T A \oplus 0^{(n)}$ by our choice of H. ■

The remainder of this section is devoted to the proof of Theorem 2.1.

Definition 2.4 1. A *Turing functional* Φ is a set of sequences (x, y, σ) such that x is a natural number, y is either 0 or 1, and σ is a finite binary sequence. Further, for all x, for all y_1 and y_2, and for all compatible σ_1 and σ_2, if $(x, y_1, \sigma_1) \in \Phi$ and $(x, y_2, \sigma_2) \in \Phi$, then $y_1 = y_2$ and $\sigma_1 = \sigma_2$.

2. Φ is *use-monotone* if the following conditions hold.

   (a) For all (x_1, y_1, σ_1) and (x_2, y_2, σ_2) in Φ, if σ_1 is a proper initial segment of σ_2, then x_1 is less than x_2.

   (b) For all x_1 and x_2, y_2 and σ_2, if $x_2 > x_1$ and $(x_2, y_2, \sigma_2) \in \Phi$, then there are y_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(x_1, y_1, \sigma_1) \in \Phi$.

3. We write $\Phi(x, \sigma) = y$ to indicate that there is a τ such that τ is an initial segment of σ, possibly equal to σ, and $(x, y, \tau) \in \Phi$. If $X \subseteq \omega$, we write $\Phi(x, X) = y$ to indicate that there is an ℓ such that $\Phi(x, X \upharpoonright \ell) = y$, and write $\Phi(X)$ for the function evaluated in this way.

Note that in Definition 2.4, we do not require that Φ be recursively enumerable. Consequently, if Φ is a Turing functional and $X \subseteq \omega$, then $\Phi(X)$ is recursive only in the join of Φ and X.

The following notion of forcing is due to Kumabe and Slaman, who used it to prove a version of Theorem 2.1 in which the nth jump is replaced by the ωth jump.

Definition 2.5 Let P be the following partial order.

1. The elements p of P are pairs $(\Phi_p, \boldsymbol{X}_p)$ in which Φ_p is a finite use-monotone Turing functional and $\boldsymbol{X}_p$ is a finite collection of subsets of ω.

2. If p and q are elements of P, then $p \geq q$ if and only if

   (a) i. $\Phi_p \subseteq \Phi_q$ and

   ii. for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,

   (b) $\boldsymbol{X}_p \subseteq \boldsymbol{X}_q$,

   (c) for every x, y, and $X \in \boldsymbol{X}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

In short, a stronger condition than p can add computations to Φ_p, provided that they are longer than any computation in Φ_p and that they do not apply to any element of $\boldsymbol{X}_p$.

Definition 2.6 If Φ_0 and Φ_1 are finite use-monotone Turing functionals, then $\Phi_0 \geq_0 \Phi_1$ if and only if $(\Phi_0, \emptyset) \geq (\Phi_1, \emptyset)$ in P.

If $G \subseteq P$ is a (sufficiently, or indeed, even slightly) P-generic filter, then G is naturally associated with the functional $\Phi_G = \bigcup\{\Phi_p : p \in G\}$. To prove Theorem 2.1, we will construct a G that is sufficiently P-generic so that every Σ^0_n statement about Φ_G is correctly decided by a condition in

P that belongs to G. We will also show that it possible meet the relevant dense subsets of P and still arrange that $\Phi_G(A)$ is equal to the characteristic function of the complete Σ^0_n set relative to Φ_G. The total effect will be to ensure that $\Phi_G^{(n)}$ is recursive in the join of Φ_G and A

We will treat Φ_G as if it were a subset of ω and suppress the recursive apparatus needed to represent Φ_G in this way.

Lemma 2.7 *Let $p = (\Phi_p, \boldsymbol{X}_p)$ be an element of P.*

1. *$p \Vdash a \in \Phi_G$ if and only if $a \in \Phi_p$.*
2. *$p \Vdash a \notin \Phi_G$ if and only if*
   (a) *either a is not a suitable triple,*
   (b) *or a is equal to (x, y, σ), $a \notin \Phi_p$, and either*
      i. *there is a $(x_0, y_0, \sigma_0) \in \Phi_p$ such that the length of σ_0 is greater than the length of σ, or x_0 is greater than or equal to x and σ_0 is compatible with σ.*
      ii. *or σ is an initial segment of one of the elements of $\boldsymbol{X}_p$.*

Proof: For the first claim, if $a \in \Phi_p$ then $a \in \Phi_G$ whenever $p \in G$. Consequently, if $a \in \Phi_p$ then $p \Vdash a \in \Phi_G$. Conversely, if $a \notin \Phi_p$ then let σ be a sequence such that σ has length greater than the length of any sequence mentioned in p or a and such that σ is incompatible with all of the elements of $\boldsymbol{X}_p$. Let x be the least number such that $\Phi_p(x, \sigma)$ is not defined. Then $q = (\Phi_p \cup \{(x, 0, \sigma)\}, \boldsymbol{X}_p)$ extends p in P, $a \notin \Phi_p \cup \{(x, 0, \sigma)\}$ and no extension r of q can have $a \in \Phi_r$. Consequently, $q \Vdash a \notin \Phi_G$ and so $p \nVdash a \in \Phi_G$. The proof of the second claim is similar. One observes that if conditions 2(a) and (b) do not hold, then it is possible to extend Φ_p to some Φ_q so that $p \geq (\Phi_q, \boldsymbol{X}_p)$ and $a \in \Phi_q$. ∎

Definition 2.8 Let Φ_p be a finite use-monotone Turing functional. Let $\psi(\Phi_G) = (\forall m)\theta(m, \Phi_G)$ be a Π^0_n sentence about Φ_G in which $\theta(m, \Phi_G)$ is Σ^0_{n-1}. For $\boldsymbol{\tau} = (\tau_1, \dots, \tau_k)$ a sequence of elements of $2^{<\omega}$ all of the same length, we say that $\boldsymbol{\tau}$ is *essential to* $\neg\psi(\Phi_G)$ *over* Φ_p when the following condition holds. For all q and all m, if q is a condition such that $(\Phi_p, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$, then $\Phi_q \setminus \Phi_p$ includes a triple (x, y, σ) such that σ is compatible with at least one component of $\boldsymbol{\tau}$.

Definition 2.9 For Φ_0 a finite use-monotone Turing functional, and k in ω, let $T(\Phi_0, \psi, k)$ be the set of length k vectors $\boldsymbol{\tau}$ which are essential to $\neg\psi(\Phi_G)$ over Φ_0.

We order $T(\Phi_0, \psi, k)$ by extension on all coordinates. That is, $\boldsymbol{\sigma}$ extends $\boldsymbol{\tau}$ if and only if for all i less than or equal to k, the ith coordinate of $\boldsymbol{\sigma}$ extends the ith coordinate of $\boldsymbol{\tau}$. It is immediate that if $\boldsymbol{\sigma}$ extends $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0, then $\boldsymbol{\tau}$ is also essential to $\neg\psi(\Phi_G)$ over Φ_0. Consequently, $T(\Phi_0, \psi, k)$ is a subtree of the tree of length k vectors of binary sequences of equal length ordered as above. That is, $T(\Phi_0, \psi, k)$ is a subtree of a recursively bounded recursive tree. Here, a recursive tree T is recursively bounded if there is a recursive function t such that for all n, $t(n)$ is a finite set which contains the nth splitting level of T.

Lemma 2.10 *Suppose that Φ_0 is a finite use monotone functional, $\psi(\Phi_G)$ is a Π^0_n sentence with $n \geq 1$, and k is a natural number.*

1. *If there is a size k set $\boldsymbol{X}$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$, then $T(\Phi_0, \psi, k)$ is infinite.*
2. *If $T(\Phi_0, \psi, k)$ is infinite, then it has an infinite path Y. Further, each such Y is naturally identified with a size k set $\boldsymbol{X}(Y)$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}(Y)) \Vdash \psi(\Phi_G)$.*

Proof: Say that $\psi(\Phi_G)$ is equal to $(\forall m)\theta(m, \Phi_G)$ where $\theta(m, \Phi_G)$ is Σ^0_{n-1}. For the first claim, suppose there is a size k set $\boldsymbol{X} = (X_1, \dots, X_k)$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$. Fix such an $\boldsymbol{X}$ and consider the set of sequences $\boldsymbol{\tau}_\ell = (X_1 \restriction \ell, \dots, X_k \restriction \ell)$, as ℓ ranges over ω. For all q extending $(\Phi_0, \emptyset)$ and all m, if $q \Vdash \neg\theta(m, \Phi_G)$, then q is incompatible with $(\Phi_0, \boldsymbol{X})$. In particular, $(\Phi_q, \boldsymbol{X}_q \cup \boldsymbol{X})$ does not extend $(\Phi_0, \boldsymbol{X})$ in P. But then, there must be an i such that $\Phi_q \setminus \Phi_0$ contains an element (x, y, σ) such that X_i extends σ. This σ is compatible with the ith component of each $\boldsymbol{\tau}_\ell$. Consequently, each $\boldsymbol{\tau}_\ell$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 and hence $T(\Phi_0, \psi, k)$ is infinite. This verifies the first claim.

For the second claim of the lemma, suppose that $T(\Phi_0, \psi, k)$ is infinite. By König's Lemma, since $T(\Phi_0, \psi, k)$ is a finitely branching tree, it has at least one infinite path. Now suppose that Y is such an infinite path. Let $\boldsymbol{X}(Y)$ be the size k set $\{X_1, \dots, X_k\}$ in which each X_i is the limit of the ith coordinates of the elements of Y. For every extension q of $(\Phi_0, \emptyset)$ and every m, if $q \Vdash \neg\theta(m, \Phi_G)$ then $\Phi_q \setminus \Phi_0$ includes an element (x, y, σ) such that σ is compatible with at least one component of each element of

Y. But then, for all sufficiently large elements of Y, σ is extended by a coordinate of Y, and so σ is extended by at least one of the elements of $\boldsymbol{X}(Y)$. Thus, for all m, no extension of $(\Phi_0, \boldsymbol{X}(Y))$ can force $\neg\theta(m, \Phi_G)$. Therefore, $(\Phi_0, \boldsymbol{X}(Y)) \Vdash \psi(\Phi_G)$, as required to verify the second claim. ■

Lemma 2.11 *For each finite use monotone functional Φ_0, each Π^0_n sentence $\psi(\Phi_G)$ with $n \geq 1$, and each number k, $T(\Phi_0, \psi, k)$ is Π^0_n, uniformly in Φ_0, ψ, and k.*

Proof: First consider the forcing relation for sentences in which all of the quantifiers are bounded. Suppose $\neg\theta(\Phi_G)$ is a bounded sentence about Φ_G. Applying Lemma 2.7, whether $(\Phi_0, \boldsymbol{X}) \Vdash \neg\theta(\Phi_G)$ is a bounded property, given uniformly in terms of Φ_0, $\neg\theta(\Phi_G)$, and $\boldsymbol{X}$. Fix a bound m on the quantifiers in the formula which defines this property. Again, by referring to Lemma 2.7, if $\boldsymbol{X}_0$ is a subset of $\boldsymbol{X}$ such that for all $X \in \boldsymbol{X}$, there is an $X_0 \in \boldsymbol{X}_0$ such that X and X_0 agree on the numbers less than m, then $(\Phi_0, \boldsymbol{X}) \Vdash \neg\theta(\Phi_G)$ if and only if $(\Phi_0, \boldsymbol{X}_0) \Vdash \neg\theta(\Phi_G)$. Since there are only finitely many incompatible binary sequences of length m, we can capture the possible behaviors of sets $\boldsymbol{X}$ by quantifying over the possible behaviors of subsets of the set of length m binary sequences. Consequently, uniformly in $\neg\theta$, whether there is a finite set $\boldsymbol{X}$ such that $(\Phi_0, \boldsymbol{X}) \Vdash \neg\theta(\Phi_G)$ is a bounded property given uniformly in terms of Φ_0 and $\neg\theta(\Phi_G)$.

We now prove Lemma 2.11 by induction on n. First consider the base case when n is equal to 1. That is ψ is of the form $(\forall x)\theta(x, \Phi_G)$ and $\theta(x, \Phi_G)$ is bounded. Let k be fixed and suppose that $\boldsymbol{\tau}$ is a length k sequence of elements of $2^{<\omega}$ all of the same length. By Definition 2.8, $\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 if and only if for all $q \in P$ and all $m \in \omega$, if $(\Phi_0, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$, then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one component of $\boldsymbol{\tau}$. By the analysis of the forcing relation for bounded sentences, for each finite use-monotone functional Φ_q, whether there is a finite set $\boldsymbol{X}_q$ such that $(\Phi_0, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$ is a bounded property of Φ_q and m. Thus, the quantifier over q in P can be replaced by a quantifier over finite use-monotone functionals Φ_q with $\Phi_0 \geq_0 \Phi_q$. (See Definition 2.6.) Consequently, $\boldsymbol{\tau}$'s being essential to $\neg\psi(\Phi_G)$ over Φ_0 is a Π^0_1 property of $\boldsymbol{\tau}$, and so $T(\Phi_0, \psi, k)$ is a Π^0_1 tree, verifying the lemma for $n = 1$. Note that the Π^0_1 definition of $T(\Phi_0, \psi, k)$ was obtained uniformly in terms of Φ_0, ψ, and k.

For the inductive argument, we assume that the lemma holds for n. We repeat the argument for the base case, with the inductive assumption used to analyze the forcing relation for Π^0_n sentences. Let k be fixed and suppose that $\boldsymbol{\tau}$ is a length k sequence of elements of $2^{<\omega}$ all of the same length. Again, $\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 if and only if, for all $q \in P$ and all $m \in \omega$, if $(\Phi_0, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$, then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one element of $\boldsymbol{\tau}$. This condition is equivalent to "for all Φ_q with $\Phi_0 \geq_0 \Phi_q$, for all k, and all $m \in \omega$, if there is a size k set $\boldsymbol{X}$ such that $(\Phi_q, \boldsymbol{X}) \Vdash \neg\theta(m, \Phi_G)$ then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one element of $\boldsymbol{\tau}$". By Lemma 2.10, "there is a size k set $\boldsymbol{X}$ such that $(\Phi_q, \boldsymbol{X}) \Vdash \neg\theta(m, \Phi_G)$" can be replaced by "$T(\Phi_q, \neg\theta(m), k)$ is infinite". Thus, $\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 if and only if, for all Φ_q such that $\Phi_0 \geq_0 \Phi_q$, for all k, and all $m \in \omega$, if $T(\Phi_q, \neg\theta(m), k)$ is infinite then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one element of $\boldsymbol{\tau}$" Since $\neg\theta(m, \Phi_G)$ is a Π^0_n sentence, we can apply induction to conclude that $T(\Phi_q, \neg\theta(m), k)$ is uniformly Π^0_n in terms of Φ_q, ψ, m, and k. As a fact of pure definability, whether a Π^0_n subtree of a recursively bounded recursive tree is infinite is itself Π^0_n: it is Σ^0_n to state that there is a splitting level in the recursive tree which is disjoint from the Π^0_n subtree. So, "$\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0" is equivalent to a condition of the form "for all Φ_q with $\Phi_0 \geq_0 \Phi_q$, for all k, and all $m \in \omega$, if a Π^0_n condition holds, then so does a bounded one". Thus, "$\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0" is a Π^0_{n+1} property of $\boldsymbol{\tau}$, Φ_0 and ψ. Consequently, for each k and for each Π^0_{n+1} sentence ψ, $T(\Phi_0, \psi, k)$ is Π^0_{n+1}, uniformly in Φ_0, ψ, and k. This completes the verification of the lemma. ■

Corollary 2.12 *Suppose that A is not Δ^0_n. Let Φ_0 be a finite use-monotone functional, $\psi(\Phi_G)$ be a Π^0_n sentence about Φ_G, and k be a positive natural number. If there is a size k set $\boldsymbol{X}$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$, then there is such a set $\boldsymbol{X}$ such that $A \notin \boldsymbol{X}$. Moreover, we can find such an $\boldsymbol{X}$ all of whose members are recursive in $0^{(n)}$ uniformly in ψ, k and $A \oplus 0^{(n)}$.*

Proof: Suppose that there is a size k set $\boldsymbol{X}$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$. By Lemmas 2.10 and 2.11, $T(\Phi_0, \psi, k)$ is a Π^0_n subtree of a recursively bounded recursive tree T which has an infinite path. Thus, the infinite paths through $T(\Phi_0, \psi, k)$ form a nonempty Π^0_n class in the sense of Jockusch and Soare (1972) and so by their Corollary 2.11 (relativized to $0^{(n-1)}$), there is an infinite path Y in $T(\Phi_0, \psi, k)$ in which A is not recursive and so, in particular, $A \notin \boldsymbol{X}(Y)$. By Lemma 2.10, $(\Phi_0, \boldsymbol{X}(Y)) \Vdash \psi(\Phi_G)$ for any such Y. Thus all we need to do is find an infinite path Y in $T(\Phi_0, \psi, k)$ such that $A \notin \boldsymbol{X}(Y)$ recursively in $A \oplus 0^{(n)}$ given that there is one. Since there is such a path, there is one with an initial segment σ such that all of the components of σ are incompatible with A. Moreover, any infinite path through $T(\Phi_0, \psi, k)$ beginning with such a σ has the desired properties. As A can tell which σ have all their components incompatible with A and $0^{(n)}$ can tell which nodes have infinite paths through them and so then actually construct such a path, we have the desired conclusion. ∎

Lemma 2.13 *Suppose that n is greater than 0, A is not Δ^0_n, and $\psi(\Phi_G)$ is a Π^0_n sentence about Φ_G; say $\psi(\Phi_G) = (\forall x)\theta(x, \Phi_G)$ in which θ is Σ^0_{n-1}. For any condition $p = (\Phi_p, \boldsymbol{X}_p)$ with $A \notin \boldsymbol{X}_p$, there is a stronger condition $q = (\Phi_q, \boldsymbol{X}_q)$ which we can find uniformly in Φ_p and $A \oplus 0^{(n)} \oplus \boldsymbol{X}_p$ such that the following conditions hold.*

1. *$A \notin \boldsymbol{X}_q$ and each $X \in \boldsymbol{X}_q - \boldsymbol{X}_p$ is recursive in $0^{(n)}$.*
2. *For all x, if $\Phi_q(x, A)$ is defined, then $\Phi_p(x, A)$ is defined. That is, q does not add any new computations to Φ_G which apply to A.*
3. *Either $q \Vdash \psi(\Phi_G)$ or there is an m such that $q \Vdash \neg\theta(m, \Phi_G)$ (and we can tell which formula we have forced).*

Proof: Fix $p = (\Phi_p, \boldsymbol{X}_p)$ in P. Let $X_1, \ldots, X_k$ be an enumeration of the elements of $\boldsymbol{X}_p$. First, use $0^{(n)}$ to determine if $T(\Phi_p, \psi, k+1)$ is infinite. If so, Corollary 2.12 supplies a condition $r = (\Phi_p, \boldsymbol{X})$ forcing $\psi(\Phi_G)$ with $A \notin \boldsymbol{X}$ and every $X \in \boldsymbol{X}$ recursive in $0^{(n)}$. As (2) is trivially satisfied if Φ_p is kept fixed, our desired condition q is $(\Phi_p, \boldsymbol{X}_p \cup \boldsymbol{X})$. If $T(\Phi_p, \psi, k+1)$ is not infinite, then $\boldsymbol{X}_p \cup \{A\}$ does not provide an infinite path through it. Thus, for some ℓ, $\boldsymbol{\tau}(\ell) = (X_1 \restriction \ell, \ldots, X_k \restriction \ell, A \restriction \ell)$ is not essential to $\neg\psi(\Phi_G)$ over Φ_p. Then, there is an number m and a condition $r = (\Phi_r, \boldsymbol{X}_r)$ extending $(\Phi_p, \emptyset)$ such that Φ_r does not add any new computations compatible with any of the components of $\boldsymbol{\tau}(\ell)$ and $r \Vdash \neg\theta(m, \Phi_G)$. In particular, Φ_r does not add any new computations which apply to A or to any element of $\boldsymbol{X}_p$ and, by Lemma 2.10 there is a k such that $T(\Phi_r, \neg\theta(m, \Phi_G), k)$ is infinite. As we can decide which Φ_r extending Φ_p add no new computations which apply to A or to any element of $\boldsymbol{X}_p$ recursively in $A \oplus \boldsymbol{X}_p$ and then whether $T(\Phi_r, \neg\theta(m, \Phi_G), k)$ is infinite recursively in $0^{(n-1)}$ we can find such a Φ_r and k recursively in $A \oplus 0^{(n)} \oplus \boldsymbol{X}_p$. We can now apply Corollary 2.12 again to get an $\boldsymbol{X}$ of size k with $A \notin \boldsymbol{X}$ such that every $X \in \boldsymbol{X}$ is recursive in $0^{(n)}$ and $(\Phi_r, \boldsymbol{X}) \Vdash \neg\theta(m, \Phi_G)$. Our desired condition q is thus $(\Phi_r, \boldsymbol{X}_p \cup \boldsymbol{X})$. ∎

Now we can complete the proof of Theorem 2.1. Suppose that $A \subseteq \omega$ is not Δ^0_n. Let $(\psi_i(\Phi_G) : i \geq 1)$ be a recursive enumeration of the Π_n sentences about Φ_G. We build a sequence of conditions $(p_i : i \in \omega)$ recursively in

$A \oplus 0^{(n)}$ so that $p_0 = (\emptyset, \emptyset)$, $p_i > p_{i+1}$, and for all i, p_i decides $\psi_i(\Phi_G)$, $A \notin \boldsymbol{X}_{p_i}$ and every $X \in \boldsymbol{X}_{p_i}$ is recursive in $0^{(n)}$.

Given p_{i-1}, we obtain p_i in two steps. Suppose that ψ_i is $\forall x \theta_i(x, \Phi_G)$ and $\theta_i(x, \Phi_G)$ is Σ_{i-1}. First, we apply Lemma 2.13 to find a condition $q = (\Phi_q, \boldsymbol{X}_q)$ extending p_{i-1} such that $A \notin \boldsymbol{X}_q$, every $X \in \boldsymbol{X}_q$ is recursive in $0^{(n)}$, $\Phi_q(A)$ is equal to $\Phi_p(A)$, and either $q \Vdash \psi_i(\Phi_G)$ or there is an m in ω such that $q \Vdash \neg\theta_i(m, \Phi_G)$. Let ℓ be so large that it is greater than m, it is greater than the length of any sequence mentioned in Φ_q, and for each X in $\boldsymbol{X}_q$ there is an x less than ℓ with $X(x) \neq A(x)$. We define p_i to be $(\Phi_q \cup \{(n, 0, A \upharpoonright \ell)\}, \boldsymbol{X}_q)$, if $q \Vdash \psi_i(\Phi_G)$, and $(\Phi_q \cup \{(n, 1, A \upharpoonright \ell)\}, \boldsymbol{X}_q)$, if $q \Vdash \neg\psi_i(\Phi_G)$. In other words, we build p_i by first deciding the ith Π_n sentence about Φ_G without extending $\Phi_G(A)$, and then defining $\Phi_G(n, A)$ to record the value decided.

Let Φ_G be the union of the Φ_{p_i}. By induction on the logical complexity of its subformulas, for each Π_n sentence ψ_i about Φ_G, $\psi_i(\Phi_G)$ is true if and only if $p_i \Vdash \psi_i(\Phi_G)$. But then $\Phi_G(A)$ is the characteristic function of a complete Σ_n set relative to Φ_G. So, $\Phi_G \oplus A \geq_T \Phi_G(A) \geq_T \Phi_G^{(n)}$. As the whole construction is recursive in $A \oplus 0^{(n)}$ and it decides the truth of all Π_n sentences about Φ_G, we also have $A \oplus 0^{(n)} \geq_T \Phi_G^{(n)}$ as required. ■

The uniformity of this construction clearly provides a proof of the theorem for the $\omega - REA$ operators as well.

Theorem 2.14 (Kumabe and Slaman) *Suppose that J is an ω-REA operator, and $A \subseteq \omega$ is not Δ_n^0 for any $n \in \omega$. Then, there is a $G \subseteq \omega$ such that $A \oplus G \equiv_T J(G) \equiv_T A \oplus 0^{(\omega)}$.*

We can now continue into the transfinite. Using our results, it is not hard to see that deciding if there is an extension q of a condition p which forces a bounded sentence with a predicate for $\Phi_G^{(\omega)}$ is recursive in $0^{(\omega)}$. One can then employ an inductive analysis similar to that of Lemmas 2.11 and 2.13 to see that such decisions can be made for sentences which are Π_n in $\Phi_G^{(\omega)}$ can be made recursively in $0^{(\omega+n)}$. At limit levels of the transfinite jump hierarchy the uniformity of our constructions carries us through. Thus we can answer Question 1.12 for all the recursive ordinals as asked originally in Jockusch and Shore (1984).

Theorem 2.15 *Suppose that J is an $\alpha - REA$ operator, and $A \subseteq \omega$ is not recursive in $0^{(\beta)}$ for any $\beta < \alpha$. Then, there is a $G \subseteq \omega$ such that $A \oplus G \equiv_T J(G) \equiv_T A \oplus 0^{(\alpha)}$.*

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