INTERPRETING ARITHMETIC IN THE R.E. DEGREES UNDER Σ_4-INDUCTION

C. T. CHONG, RICHARD A. SHORE AND YUE YANG

Abstract. We study the problem of the interpretability of arithmetic in the r.e. degrees in models of fragments of Peano arithmetic. The main result states that there is an interpretation $\varphi \mapsto \varphi^*$ such that every formula φ of Peano arithmetic corresponds to a formula φ^* in the language of the partial ordering of r.e. degrees such that for every model $\mathcal{N}$ of Σ_4-induction, $\mathcal{N} \models \varphi$ if and only if $\mathcal{R}_{\mathcal{N}} \models \varphi^*$, where $\mathcal{R}_{\mathcal{N}}$ is the structure whose universe is the collection of r.e. degrees in $\mathcal{N}$. This supplies, for example, statements φ_m about the r.e. degrees which are equivalent (over $I\Sigma_4$) to $I\Sigma_m$ for every $m > 4$.

§1. Introduction. A basic goal of reverse mathematics is to determine the axiom systems needed to prove particular theorems of mathematics by showing that they are equivalent (over a given base theory) to some specific axiom system. (See [12] for a general introduction to reverse mathematics in the setting of second order arithmetic.) In reverse recursion theory our setting is first order arithmetic and the axiom systems considered are those that include, in addition to the axioms PA^- of Peano arithmetic (PA) without induction, the various syntactic levels of induction and bounding axioms, $I\Sigma_n$ and $B\Sigma_n$, which assert the induction scheme and bounding principle, respectively, for Σ_n formulas. (We refer the reader to [3] or [4] for the basic notions of recursion theory in fragments of arithmetic, and to a summary of recent results in the theory.)

The typical goal of reverse recursion theory then is to find recursion theoretic theorems, usually about the r.e. degrees, that are equivalent to $I\Sigma_n$ over some weaker base theory. Examples include the equivalence

1991 *Mathematics Subject Classification.* Primary 03B30, 03D25, 03F30; Secondary 03B25, 03D25, 03D75.

Key words and phrases. reverse mathematics, fragments of arithmetic, recursively enumerable degrees, interpretability of arithmetic.

Richard A. Shore was partially supported by NSF Grant DMS-9802843 and a Visiting Distinguished Professorship at the National University of Singapore.

Meeting

of the Sacks Splitting Theorem for r.e. degrees with $I\Sigma_1$ (over $B\Sigma_1$) [7] and of a minimal pair of r.e. degrees with $I\Sigma_2$ (over $B\Sigma_2$) [2]. Recent work in [14] supplies a similar example for $I\Sigma_3$. In this paper, we show that over the base theory $I\Sigma_4$ there are sentences φ_m about the r.e. degrees which are equivalent to $I\Sigma_m$ for each $m > 4$. (Of course, these sentences are thus all theorems of PA.) Our proof proceeds by analyzing and extending some of the results of [9] about the definability of (a copy of) the standard model of arithmetic in the r.e. degrees, so as to be able to carry out the necessary arguments in $I\Sigma_4$ to provide an interpretation in the r.e. degrees not of the standard model itself, but only its theory. (The interpretation of the model itself seems to require more induction.) Of course, this suffices for our equivalence results and more. We begin with some background and an outline of our approach.

Let $\mathcal{R}$ denote the collection of recursively enumerable (r.e.) degrees of the set of natural numbers with the partial ordering induced by Turing reducibility. In [9], Nies, Shore and Slaman show that there is an interpretation (in the sense of [5]) of the *standard* model of arithmetic in $\mathcal{R}$. As a consequence, true arithmetic can be interpreted in $\mathrm{Th}(\mathcal{R})$. Since there is an obvious interpretation of $\mathcal{R}$ in the standard model, it follows that $\mathrm{Th}(\mathcal{R})$ is precisely as complicated as true arithmetic. They also use this interpretation to establish the definability of the classes of high$_n$ and low$_{n+1}$ degrees (for $n \geq 1$) as well as a weak form of the bi-interpretability conjecture ([9], [8]), that there is a map f definable over $\mathcal{R}$ with the property that for each $\mathbf{a} \in \mathcal{R}$, $f(\mathbf{a})$ is the index (in the code of the standard model in $\mathcal{R}$) of an r.e. set W_e such that W_e'' has Turing degree $\mathbf{a}''$.

This interpretation of the standard model in $\mathcal{R}$ is achieved through a series of *coding schemes* each of which defines a class of structures in $\mathcal{R}$. Briefly, a coding scheme $S(\vec{\mathbf{p}})$ is prescribed via first order formulas (in the language of $\mathcal{R}$) $\varphi_0(x, \vec{\mathbf{p}})$, $\varphi_1(\vec{x}, \vec{\mathbf{p}})$, $\varphi_2(\vec{x}, \vec{\mathbf{p}})$, ..., $\varphi_k(\vec{x}, \vec{\mathbf{p}})$ and $\psi(\vec{\mathbf{p}})$ where $\vec{\mathbf{p}}$ denotes a finite sequence of elements of $\mathcal{R}$, and φ_0 specifies the domain of the structure being defined. The formulas φ_i, $0 < i \leq k$, correspond to functions or relations such as addition and multiplication or order which are defined on the domain specified by φ_0 to make it into the desired type of structure. The formula ψ is a *correctness condition* which ensures that structures coded by the scheme satisfy certain properties (for example, that the appropriate formulas define functions when so required or that, in addition, the structures defined by such a scheme are models of some finite fragment of Peano arithmetic or even isomorphic to the standard model). For a given coding scheme S, we refer to the structures defined by instances of the parameters $\vec{\mathbf{p}}$ that satisfy the correctness condition ψ as the *models coded* via S. For more details and relevant examples see [5] and [9].

We show in this paper that if $\mathcal{N}$ is a model of a fragment T of Peano arithmetic including Σ_4-induction and $\mathcal{R}_\mathcal{N}$ is the structure of r.e. degrees of $\mathcal{N}$, then there exists an interpretation of $\mathrm{Th}(\mathcal{N})$ in $\mathcal{R}_\mathcal{N}$, i.e. a recursive map $\varphi \mapsto \varphi^*$ taking formulas φ of Peano arithmetic to ones φ^* in the language of the partial ordering of r.e. degrees, such that $\mathcal{N} \models \varphi$ if and only if $\mathcal{R}_\mathcal{N} \models \varphi^*$. A consequence of this is that for $n > m \geq 4$, if $\mathcal{N}_n$ is a model of Σ_n-induction, and $\mathcal{N}_m$ is a model of Σ_m-induction but not of Σ_n-induction, then the structures $\mathcal{R}_{\mathcal{N}_n}$ and $\mathcal{R}_{\mathcal{N}_m}$ are not elementarily equivalent. More precisely, since we can express the scheme of $I\Sigma_m$ induction by a single formula ψ_m, there is a formula ψ_m^* in the language of $\mathcal{R}$ such that $\mathcal{N}$ satisfies $I\Sigma_m$ induction if and only if $\mathcal{R}_\mathcal{N} \models \psi_m^*$. Combining these results with recent ones in [2] and [14], the only case of the general question of whether the amount of induction true in $\mathcal{N}$ is faithfully reflected in $\mathrm{Th}(\mathcal{R}_\mathcal{N})$ is when $n \geq 4$ and $m = 3$. We discuss this further in the next section.

There are essentially two key steps to establishing the interpretability of true arithmetic in $\mathrm{Th}(\mathcal{R})$. The first involves coding the standard model in $\mathcal{R}$. A basic technique for achieving this is provided by Slaman and Woodin in the construction of what are called SW (Slaman-Woodin) sets. An SW-set ([9]) is defined from a finite set of parameters: Given r.e. degrees $\mathbf{b}, \mathbf{p}, \mathbf{q}$ and $\mathbf{r}$, the Slaman-Woodin set coded by $\mathbf{b}, \mathbf{p}, \mathbf{q}, \mathbf{r}$ is the set of minimal elements of $\{\mathbf{x} | \mathbf{b} \leq_T \mathbf{x} \leq_T \mathbf{r} \;\&\; \mathbf{q} \leq_T \mathbf{x} \vee \mathbf{p}\}$. If the parameter $\mathbf{b}$ is omitted, we take it to be $\mathbf{0}$. As shown by Slaman and Woodin, given a recursive partial ordering $\leq_\mathcal{P}$ one may construct a uniformly r.e. sequence (an SW-set with $\mathbf{b} = \mathbf{0}$) $\{\mathbf{g}_i | i \in \omega\}$ of r.e. degrees and an $\mathbf{l} \in \mathcal{R}$ such that $i \leq_\mathcal{P} j$ if and only if $\mathbf{g}_i \leq_T \mathbf{g}_j \vee \mathbf{l}$. Thus we can define a coding scheme and parameters that determine a recursive coding of the standard model of arithmetic in $\mathcal{R}$ by combining the Slaman-Woodin coding of partial orderings with the standard interpretation of an arbitrary theory (here arithmetic) in that of partial orderings.

The second step is to obtain a first order condition on parameters that picks out a subclass of models coded in $\mathcal{R}$ which are standard. This is achieved via a Comparison Theorem which identifies, through a coding scheme, a class of models isomorphic to the standard model. Of course, identifying a definable class of models isomorphic to the standard one gives an interpretation of true arithmetic in the theory of $\mathcal{R}$.

In this paper we are generally guided by this approach in dealing with the problem at hand. There are, however, some obvious issues and difficulties which have to be resolved. First, the term 'standard model' now carries a different meaning. Since one is considering an arbitrary model of a first-order theory (a fragment of Peano arithmetic), which more than likely is itself nonstandard, 'standard' has to be understood in the relative sense, using the given model $\mathcal{N}$ as reference. In such a situation, it is necessary to exercise control on the order type of the models to be considered so that, in order to apply the Comparison Theorem, only those which are end-extensions of $\mathcal{N}$ matter. Second, working with limited mathematical induction always poses the challenge of controlling the complexity of the constructions involved, to ensure that the coding scheme to be defined stays within manageable level, i.e. the constraint of Σ_4-induction.

In the next section, we present the coding mechanism and the technical theorems and lemmas needed to make it work. Assuming the technical recursion theoretic results, we prove (in $I\Sigma_4$) that the coding procedures provide an interpretation of the theory of arithmetic in the r.e. degrees and draw some conclusions about reverse recursion theory and the relations between the theory of the r.e. degrees in a model $\mathcal{N}$ and the amount of induction that holds in $\mathcal{N}$. The technical theorems and lemmas are proven in §3.

§2. Coding Schemes Under $I\Sigma_4$. Let $\mathcal{N}$ be a model of $I\Sigma_4$. We work towards finding a coding scheme S and a formula Ψ in the language of the partial ordering of r.e. degrees such that for appropriate degrees $\mathbf{c}$, any finite set of parameters $\vec{\mathbf{p}}$ below $\mathbf{c}$ which satisfy Ψ code a model $\mathbf{M}$ of arithmetic lying below $\mathbf{c}$ which is also a copy of $\mathcal{N}$ (known as models 'good for $\mathbf{c}$'). The formula Ψ plays the role of a correctness condition for the coding scheme to be used for interpreting the theory of $\mathcal{N}$ in $\mathcal{R}_\mathcal{N}$ by picking out a class of coded models which are isomorphic to $\mathcal{N}$ when $\mathbf{c}$ is low. (We will then get our desired interpretation of $\mathrm{Th}(\mathcal{N})$ by quantifying out the parameter $\mathbf{c}$ in an appropriate way.) The development of S and Ψ proceeds in several steps.

We begin with the initial coding scheme from [9]. As described above, this takes as its parameters degrees $\mathbf{b}, \mathbf{p}, \mathbf{q}, \mathbf{r}$ and $\mathbf{l}$ such that $\mathbf{b}, \mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ define a Slaman-Woodin set $\mathbf{G}$, and $\mathbf{l}$ defines an ordering $\leq_\mathcal{P}$ on $\mathbf{G}$ which codes a model of a finite fragment of PA sufficient for the simple facts about arithmetic that we will need (one implying $I\Sigma_1$ is more than sufficient) on a subset $\mathbf{M}$ of $\mathbf{G}$, using an interpretation of structures for arithmetic in partial orderings. Much of what we do to adapt the codings of [9] will be done to reduce the complexity of various statements about the coded models so as to reduce the amount of induction needed in our verifications.

We begin by assuming that the partial ordering $\mathcal{P}$ used to code arithmetic has specific elements s, s_0 and s_1 such that the elements of the model of arithmetic coded in $\mathcal{P}$ are precisely those below s and the even and odd ones (in the sense of the model of arithmetic) are those below s_0 and s_1 respectively. (In addition, we note that the coding presented in [9] makes the minimal elements of $\mathcal{P}$ be the domain of the coded model of arithmetic.) Thus the Slaman-Woodin set $\mathbf{G}$ has specific elements $\mathbf{s}, \mathbf{s}_0$ and $\mathbf{s}_1$ such that the elements of $\mathbf{M}$ are precisely those $\mathbf{x} \in \mathbf{G}$ satisfying $\mathbf{x} \leq_T \mathbf{s} \vee \mathbf{l}$ and the even (odd) numbers of $\mathbf{M}$ are precisely those $\mathbf{x} \in \mathbf{G}$ satisfying $\mathbf{x} \leq_T \mathbf{s}_0 \vee \mathbf{l}$ ($\mathbf{x} \leq_T \mathbf{s}_1 \vee \mathbf{l}$). We include these elements $\mathbf{s}, \mathbf{s}_0$ and $\mathbf{s}_1$ among the parameters of our coding scheme, and the assertions about their picking out the appropriate sets of numbers in $\mathbf{M}$ as part of our correctness condition.

The next new aspect of our coding scheme that we want to make explicit is the introduction of another parameter $\mathbf{l}'$ which codes a second partial order on $\mathbf{G}$ that coincides with the ordering of $\mathbf{M}$. To be precise, we add the parameter $\mathbf{l}'$ and the correctness condition asserting that for $\mathbf{x}, \mathbf{y} \in \mathbf{M}$, $\mathbf{x} \leq_T \mathbf{y} \vee \mathbf{l}'$ if and only if $\mathbf{x} \leq_{\mathbf{M}} \mathbf{y}$.

We start our search then with a coding scheme S_0 and associated correctness condition ψ_0 with all these properties. All coding schemes considered from now on are assumed to be extensions of S_0. If $\mathbf{M}$ is a model coded by some scheme S, $i \in \mathcal{N}$ and there is (in the sense of $\mathcal{N}$) an ith element in the ordering $\leq_{\mathbf{M}}$, then we denote this element by $i^{\mathbf{M}}$.

Now, given a sufficient amount of induction, the Slaman-Woodin construction as given in [9] may be used to show that there are parameters which, via S_0, code a recursive model $\mathbf{M}$ in $\mathcal{R}_{\mathcal{N}}$ that is isomorphic to $\mathcal{N}$. However, definably picking out copies of $\mathcal{N}$ is considerably more complicated. We continue with a definition that reflects our goal of determining a subclass of the models coded by a scheme S which are isomorphic to $\mathcal{N}$.

DEFINITION 2.1. *Let $\mathcal{C}$ be the class of models of arithmetic coded in $\mathcal{R}_{\mathcal{N}}$ by some coding scheme S. Then $\mathbf{N} \in \mathcal{C}$ is* $\mathcal{N}$-standard *for $\mathcal{C}$ if $\mathbf{N}$ is isomorphic to $\mathcal{N}$ and the domain of $\mathbf{N}$ is isomorphic to an initial segment of the domain of every $\mathbf{M} \in \mathcal{C}$.*

Our goal is to definably (at least in an appropriate parameter) determine a set of models which are $\mathcal{N}$-standard for some (coded) class $\mathcal{C}$. The need to restrict the complexity of various definitions, so as to reduce the amount of induction needed for verifications, drives us to a more complicated mechanism than that used in [9] to define a class of standard models. We use models with 'effectively generated' successor functions, first studied by Shore [10] and used in [9] for their more delicate results on definability in $\mathcal{R}$.

DEFINITION 2.2. *If* $\mathbf{M}$ *is a model of* $I\Sigma_4$ *defined by parameters* $\mathbf{b}, \mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{l}$ *and* $\mathbf{l}'$ *via a coding scheme* S *and* $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{f}_0$, $\mathbf{f}_1$ *are additional parameters from* $\mathcal{R}_{\mathcal{N}}$, *we say that* $\mathbf{M}$ *is* effectively generated *by these parameters if the even and odd integers of* $\mathbf{M}$ *are generated by* $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{f}_0$, $\mathbf{f}_1$. *More precisely if,* $0^{\mathbf{M}} \leq_T \mathbf{f}_0$ *and*

$$\mathbf{x} \leq_T (\mathbf{s}_0 \vee \mathbf{l}) \to (\mathbf{x} \vee \mathbf{e}_1) \wedge \mathbf{f}_1 = (\mathbf{x} +_{\mathbf{M}} 1)^{\mathbf{M}} \textit{ and}$$
$$\mathbf{x} \leq_T (\mathbf{s}_1 \vee \mathbf{l}) \to (\mathbf{x} \vee \mathbf{e}_0) \wedge \mathbf{f}_0 = (\mathbf{x} +_{\mathbf{M}} 1)^{\mathbf{M}}.$$

One concludes from this definition and trivial facts of arithmetic that the relation $\mathbf{g}^+ = \mathbf{h}$ saying $\mathbf{h}$ is the successor of $\mathbf{g}$ is definable in any effectively generated model as follows:

$$\mathbf{g}^+ = \mathbf{h} \Leftrightarrow (\exists i \in \{0,1\})[\mathbf{g} \not\leq_T \mathbf{f}_i \;\&\; \mathbf{h} = (\mathbf{g} \vee \mathbf{e_i}) \wedge \mathbf{f_i}].$$

This fact will be used implicitly in the sequel.

We will show that $\mathbf{N}$ is isomorphic to an initial segment of every model defined by a coding scheme S_1 with extra parameters $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{f}_0$, $\mathbf{f}_1$ and a correctness condition asserting that the models are effectively generated. The following theorem which is similar to Theorem 6.1 of [9] will follow from Theorem 3.8 below. It says that there is a recursively coded effectively generated model.

THEOREM 2.3. *There exist r.e. degrees* $\mathbf{b}, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{l}, \mathbf{l}', \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0$ *and* $\mathbf{f}_1$ *which effectively generate a coded model* $\mathbf{N}$ *with domain* $\{\mathbf{g}_i | i \in \mathcal{N}\}$ *which is isomorphic to* $\mathcal{N}$ *and recursive in the sense that the indices of the* $\mathbf{g}_i$ *and the structures defined on them are uniformly recursive.*

Given an r.e. degree $\mathbf{c} > \mathbf{0}$, we say that a coded model $\mathbf{M}$ lies below $\mathbf{c}$ if the parameters which code $\mathbf{M}$ are all recursive in $\mathbf{c}$. Recall that a degree is promptly simple if it is not half of a minimal pair. (See Definition 3.1 and Theorem 3.2.) By a prompt permitting argument, one may generalize the above theorem as follows:

THEOREM 2.4. *If* $\mathbf{c}$ *is promptly simple, then there exists a recursive, effectively generated model* $\mathbf{N_c}$ *which is recursively isomorphic to* $\mathcal{N}$ *and lies below* $\mathbf{c}$.

This result is also similar to Theorem 6.1 of [9] and will follow from Theorem 3.8. Let $\mathcal{C}_{\mathbf{c}}$ be the class of effectively generated coded models below $\mathbf{c}$ by our scheme S_1. We will now prove that any $\mathbf{N_c}$ as in Theorem 2.4 is $\mathcal{N}$-standard for $\mathcal{C}_{\mathbf{c}}$. We begin by showing that (under suitable hypotheses) $\mathbf{N_c}$ is isomorphic to an initial segment of each effectively generated model coded below $\mathbf{c}$.

LEMMA 2.5. *Let C be an r.e. set of degree $\mathbf{c}$. Assume that $I\Sigma_4^C$ holds in $\mathcal{N}$ and $\mathbf{M}$ is a coded model effectively generated by parameters lying below $\mathbf{c}$ in $\mathcal{R}_{\mathcal{N}}$. Then for any n in $\mathcal{N}$, $n^{\mathbf{M}}$ exists.*

Proof. We prove this by induction using $I\Sigma_4^C$.
Claim 1. For any $n \in \mathcal{N}$, there is an $\mathcal{N}$-finite sequence $\langle \mathbf{x}_i : i \leq n \rangle$ such that

1. $\mathbf{x}_0 = 0^{\mathbf{M}}$;
2. For all $i \leq n$, $\mathbf{x} \leq_T \mathbf{r}$ and $(\mathbf{x}_i \vee \mathbf{p}) \geq_T \mathbf{q}$;
3. For $2i+1 \leq n$, $\mathbf{x}_{2i+1} \leq_T (\mathbf{x}_{2i} \vee \mathbf{e}_1)$ and $\mathbf{x}_{2i+1} \leq_T \mathbf{f}_1$;
4. For $2i+2 \leq n$, $\mathbf{x}_{2i+2} \leq_T (\mathbf{x}_{2i+1} \vee \mathbf{e}_0)$ and $\mathbf{x}_{2i+2} \leq_T \mathbf{f}_0$.

Denote by $\psi(n)$ the conjunction of the four conditions above. Then the complexity of $\psi(n)$, upon replacing the Turing reducibility relation $\leq_T$ by the reduction relation between (indices of) r.e. sets, is Σ_3^C. Suppose that ψ holds for $n = 2k$ (the argument for n odd is similar). We first show that the $\mathbf{x}_i$ are in $\mathbf{M}$. To see this, note that by (2) and (4) $\mathbf{x}_i \notin \mathbf{M}$ is equivalent to

$$\exists \mathbf{y}(\mathbf{y} <_T \mathbf{x}_i \text{ and } \mathbf{y} \vee \mathbf{p} \geq_T \mathbf{q})$$

which is Σ_4^C. Suppose that not all members of the sequence are in $\mathbf{M}$. By $I\Sigma_4^C$, which is equivalent to $L\Pi_4^C$, there is a least counterexample i. Then $\mathbf{x}_i \notin \mathbf{M}$ but $\mathbf{x}_{i-1} \in \mathbf{M}$. Without loss of generality, assume that $\mathbf{x}_{i-1}$ is an even integer in $\mathbf{M}$. Then its successor $\mathbf{h}$ satisfies

$$\mathbf{h} = (\mathbf{x}_{i-1} \vee \mathbf{e}_1) \wedge \mathbf{f}_1 \geq_T \mathbf{x}_i$$

where the last inequality follows from the fact that $\mathbf{x}_i$ is below both $\mathbf{x}_{i-1} \vee \mathbf{e}_1$ and $\mathbf{f}_1$. On the other hand $\mathbf{x}_i \vee \mathbf{p} \geq \mathbf{q}$, thus $\mathbf{x}_i = \mathbf{h}$ since $\mathbf{h}$ is a minimal solution of the inequality and so $\mathbf{x}_i$ is in $\mathbf{M}$. Thus by $I\Sigma_4^C$-induction $\mathbf{x}_i \in \mathbf{M}$ for all $i \leq n$. We can now verify Claim 1 for $n+1$ by taking $\mathbf{x}_{n+1}$ to be the successor of $\mathbf{x}_n$ in $\mathbf{M}$. This proves Claim 1.

Claim 2. For any $n \in \mathcal{N}$, the sequence $\langle \mathbf{x}_i : i \leq n \rangle$ defined above is an initial segment of $\mathbf{M}$. Indeed $\mathbf{x}_i = i^{\mathbf{M}}$. We first use induction on $k \leq n$ to establish the following Π_4^C statement:

$$\forall \mathbf{x}[(\mathbf{x} \leq_T \mathbf{r}, (\mathbf{x}_k \vee \mathbf{l}) \text{ and } (\mathbf{x} \vee \mathbf{p}) \geq_T \mathbf{q}) \rightarrow (\exists i \leq k)(\mathbf{x}_i \leq_T \mathbf{x})]. \qquad (*)$$

Suppose $(*)$ is true for k. By Claim 1, $\mathbf{x}_k$ is in $\mathbf{M}$ and as shown in the proof of that Claim, $\mathbf{x}_{k+1}$ is the successor of $\mathbf{x}_k$ in $\mathbf{M}$. Let $\mathbf{x} \leq_{\mathbf{T}} \mathbf{r}$ be such that $\mathbf{x} \vee \mathbf{p} \geq_T \mathbf{q}$ and $\mathbf{x} \leq_T \mathbf{x}_{k+1} \vee \mathbf{l}$. Then $\mathbf{x}$ is above some $\mathbf{h} \in \mathbf{M}$ and $\mathbf{h} \leq_{\mathbf{M}} \mathbf{x}_{k+1}$. Thus $\mathbf{h} \leq_{\mathbf{M}} \mathbf{x}_k$ or $\mathbf{h} = \mathbf{x}_{k+1}$, which is the successor of $\mathbf{x}_k$. Hence $(*)$ holds.

Furthermore $(*)$ implies that $\langle \mathbf{x}_i : i \leq n \rangle$ is an initial segment of elements in $\mathbf{M}$. To see this, suppose that $\mathbf{h} \in \mathbf{M}$ and $\mathbf{h} \leq_{\mathbf{M}} \mathbf{x}_k$. Then if we substitute $\mathbf{h}$ for $\mathbf{x}$ in $(*)$, we get an $\mathbf{x}_i$ such that $\mathbf{x}_i \leq_T \mathbf{h}$. Since the members of $\mathbf{M}$ form an antichain (with respect to Turing reducibility), we have $\mathbf{h} = \mathbf{x}_i$. Thus if $\mathbf{h} \in \mathbf{M}$ satisfies $\mathbf{h} \leq_{\mathbf{M}} \mathbf{x}_k$, then $\mathbf{h} = \mathbf{x}_i$ for some $i \leq k$. $\square$

Lemma 2.5 allows one to effectively embed $\mathbf{N_c}$ into every other effectively generated model $\mathbf{M}$ below $\mathbf{c}$. The next step is to provide a coding scheme which allows us to compare (via a coding scheme for maps between coded models) initial segments of effectively generated models coded below $\mathbf{c}$. An $\mathcal{N}$-standard model for $\mathbf{c}$ will then be one whose initial segments can be embedded into every other effectively generated model coded below $\mathbf{c}$ via the maps defined by the coding scheme for comparisons. The intuitive argument goes roughly as follows: Suppose $\mathbf{M}$ is effectively generated and embeddable into every other coded model in $\mathcal{C}_{\mathbf{c}}$ as an initial segment. Let $f : \mathbf{M} \to \mathbf{N_c}$ be an embedding that preserves the successor function. Let $g : \mathbf{N_c} \to \mathbf{M}$ be the natural embedding described in Lemma 2.5. If $g(\mathbf{N_c})$ is a proper initial segment of $\mathbf{M}$ (bounded, say, by $\mathbf{m}$), then $fg : \mathbf{N_c} \to \mathbf{N_c}$ maps an initial segment $[0^{\mathbf{N_c}}, n^{\mathbf{N_c}}]$ of $\mathbf{N_c}$ properly into itself (where $n^{\mathbf{N_c}} = f(\mathbf{m})$). Now if $fg \upharpoonright [0^{\mathbf{N_c}}, n^{\mathbf{N_c}}]$ is $\mathcal{N}$-finite, then the pigeon hole principle is violated, and we have a contradiction. All this works if the instance of the pigeon hole principle that we need is sufficiently simple to follow from the amount of induction that we have in $\mathcal{N}$. Thus we need to produce a map f as above that is as simple as possible. (Note that $g \upharpoonright n^{\mathbf{N_c}}$ is $\mathcal{N}$-finite by Lemma 2.5 as long as we have $I\Sigma_4^C$.) We begin by describing a variant of the scheme for comparison maps used in [9] and the technical lemma needed to prove the existence of parameters as required to make the scheme work. We will then consider some extensions of the scheme that allow us to define the map fg as an $I\Sigma_4^C$ relation. Finally, we argue that we can restrict our attention to low degrees $\mathbf{c}$ by an appropriate quantification. This will allow us to get by with $I\Sigma_4$. First the technical lemma which is similar to Theorem 5.1 of [9] and will follow from Theorem 3.10.

LEMMA 2.6. *For any nonzero low degrees* $\mathbf{q}_0, \dots, \mathbf{q}_{m-1}, \mathbf{r}_0$ *and* $\mathbf{r}_1$, *there is a recursive model* $\mathbf{M}$ *coded below a low degree such that*

(i) $\mathbf{M}$ *is recursively isomorphic to* $\mathcal{N}$*;*
(ii) *For each* $i < m$, $(i)^{\mathbf{M}} \leq_T \mathbf{q}_i$*;*
(iii) *For* $i, j < m$, *if* $\mathbf{q}_i \not\leq_T \mathbf{q}_j$ *then* $(i)^{\mathbf{M}} \not\leq_T \mathbf{q}_j$*; and*
(iv) *For all* $\mathbf{x} \geq_{\mathbf{M}} m^{\mathbf{M}}$, $\mathbf{x} \not\leq_T \mathbf{r}_0, \mathbf{r}_1$.

Notice that we do not require $\mathbf{M}$ to be an effectively generated model. (Indeed, there seems to be an incompatibility with $\mathbf{M}$ being effectively generated and the permitting required in this lemma.)

By the $\mathcal{N}$-part of a coded model $\mathbf{M}$ we refer to the union of its initial segments of the type $[0^{\mathbf{M}}, n^{\mathbf{M}}]$ where $n \in \mathcal{N}$. Given two coded low models $\mathbf{M}_0$ and $\mathbf{M}_1$, we want a scheme that (as we vary its parameters) defines comparison maps giving isomorphisms between their $\mathcal{N}$-parts piece by piece. This is done for each n via a third model $\mathbf{M}$ which is obtained by applying Lemma 2.6 with $m = 2n$ and $\mathbf{q}_{2i} = i^{\mathbf{M}_0}$ and $\mathbf{q}_{2i+1} = i^{\mathbf{M}_1}$ for $i < n$.

More precisely, for $\mathcal{N}$-finite segments $[0^{\mathbf{M}_k}, n^{\mathbf{M}_k}]$ in $\mathbf{M}_k$ $(k = 0, 1)$ and a coded model $\mathbf{M}$ with element $\mathbf{n} = n^{\mathbf{M}}$, we define a comparison map $f_{\mathbf{M}, n^M}$ from $\mathbf{M}_0$ to $\mathbf{M}_1$ as follows: $f_{\mathbf{M}, n^{\mathbf{M}}} : \mathbf{x} \mapsto \mathbf{y}$ if and only if $\mathbf{x} \leq_{\mathbf{M}_0} \mathbf{n}_0 = n^{\mathbf{M}_0}$, $\mathbf{y} \leq_{\mathbf{M}_1} \mathbf{n}_1 = n^{\mathbf{M}_1}$ and there exists an $\mathbf{a} \in \mathbf{M}$ such that

(a) $\mathbf{a} <_{\mathbf{M}} \mathbf{n}$ and $\mathbf{a} \leq_T \mathbf{x}$;
(b) $\mathbf{a}$ is even in $\mathbf{M}$;
(c) $\mathbf{a}^+ \leq_T \mathbf{y}$, where $\mathbf{a}^+$ is the successor of $\mathbf{a}$ in $\mathbf{M}$.

Given that $\mathbf{q_{2i}} = i^{\mathbf{M_0}}$ and $\mathbf{q_{2i+1}} = i^{\mathbf{M_1}}$, we apply Lemma 2.6 and see that there is an $\mathbf{M}$ such that $j^{\mathbf{M}} <_T \mathbf{q_j}$. Then if $\mathbf{x} = \mathbf{q_{2i}}$ and $\mathbf{a} = (2i)^{\mathbf{M}}$, we have $\mathbf{a}^+ = (2i+1)^{\mathbf{M}} <_T \mathbf{q_{2i+1}}$ which we can denote by $\mathbf{y}$. Now by Lemma 2.6 (iii) and (iv) this $\mathbf{y}$ is uniquely defined, showing that $f_{\mathbf{M}, n^{\mathbf{M}}}$ provides a one-one correspondence between the integers less than n in $\mathbf{M_0}$ and $\mathbf{M_1}$. We have thus described a formula $\hat{\theta}(\mathbf{x}, \mathbf{y}, \vec{\mathbf{p}}_0, \mathbf{n}_0, \vec{\mathbf{p}}_1, \mathbf{n}_1, \vec{\mathbf{p}}, \mathbf{n})$ of degree theory which provides a scheme for isomorphisms between the $\mathcal{N}$-parts of any two low coded models. In order to guarantee that the map defined is an isomorphism between the specified initial segments of $\mathbf{M}_0$ and $\mathbf{M}_1$ we include a correctness condition in this scheme which simply says that the relation defined between the initial segments of $\mathbf{M}_i$ below $\mathbf{n}_i$ is an order preserving isomorphism. The maps defined by our scheme $\hat{\theta}$ (with this correctness condition) are now candidates for the function f used in our informal argument above for defining $\mathcal{N}$-standard models for $\mathcal{C}_{\mathbf{c}}$.

In particular, if we consider $\mathbf{N}_c$ for $\mathbf{c}$ low and promptly simple then for any effectively generated $\mathbf{M}_0$ coded below $\mathbf{c}$, the scheme given by $\hat{\theta}$ provides isomorphisms from every initial segment of $\mathbf{N}_c$ onto ones of $\mathbf{M}_0$. We would like to claim that any effectively generated $\mathbf{M}_0$ coded below a low promptly simple $\mathbf{c}$, with the property that $\hat{\theta}$ provides isomorphisms between each of its initial segments and ones of every effectively generated $\mathbf{M}_1$ coded below $\mathbf{c}$, is isomorphic to $\mathbf{N}$ (and so $\mathcal{N}$-standard for $\mathcal{C}_{\mathbf{c}}$). As we explained before, the problem is that the maps as defined by $\hat{\theta}$ provide too complicated an instance of the pigeon hole principle to contradict $I\Sigma_4$. The problematic part of the definition of $\hat{\theta}$ is membership in $\mathbf{M}$ (which is Π_4^C) and the definition of the successor function in $\mathbf{M}$. We solve this problem by adding more parameters to the scheme S_0 defining $\mathbf{M}$ as well as new correctness conditions that allow us to argue that there are $I\Sigma_4^C$ definitions of the functions $f_{\mathbf{M},n^M}$ given by $\hat{\theta}$. We add parameters $\mathbf{t},\mathbf{u}$ and $\mathbf{v}$ and the conditions (v)-(vii) of the following lemma to the scheme S_0 to get a new scheme $S_{2,\mathbf{c}}$ defining models of arithmetic which has $\mathbf{c}$ as a parameter as well.

Lemma 2.7. *Let $\mathbf{c}$ be low and promptly simple, and let $\mathbf{M}_i$ ($i=0,1$) be effectively generated models coded below $\mathbf{c}$ and $n\in\mathcal{N}$. There is a model $\mathbf{M}$ coded below $\mathbf{c}$ and degrees $\mathbf{t},\mathbf{u}$ and $\mathbf{v}$ (also below $\mathbf{c}$) which are uniformly defined in terms of the parameters $\vec{\mathbf{p}}_i$ defining $\mathbf{M}_i$ and the sequences $\langle j^{\mathbf{M}_0}|j<n\rangle$ and $\langle j^{\mathbf{M}_1}|j<n\rangle$ such that $\mathbf{M}$ satisfies the conditions (i)-(iv) of Lemma 2.6 substituting n for m, j^{M_0} for $\mathbf{q}_{2j}$ and $j^{\mathbf{M}_1}$ for $\mathbf{q}_{2j+1}$, as well as the following conditions:*

(v) *For any even number $\mathbf{h}$ in $\mathbf{M}$, $\mathbf{h}\vee\mathbf{t}$ is above $\mathbf{h}^+$, but not above any other integer in $\mathbf{M}$ (except, of course, $\mathbf{h}$). (In particular, there is a unique odd (resp. even) number $\mathbf{k}\in\mathbf{M}$ below $\mathbf{h}\vee\mathbf{t}$.)*

(vi) *The join of any nontrivial distinct triple of numbers in $\mathbf{M}$ with $\mathbf{v}$ computes $\mathbf{u}$, but no join of any pair with $\mathbf{v}$ computes $\mathbf{u}$.*

(vii) *For any even number $\mathbf{h}\in\mathbf{M}$, $\mathbf{h}\vee\mathbf{t}\vee\mathbf{v}\not\geq_T\mathbf{u}$.*

This Lemma will follow from Theorem 3.10. We now show that these conditions suffice to make the maps defined by $\hat{\theta}$ be $I\Sigma_4^C$.

Lemma 2.8. *With the notation of Lemma 2.7 and C an r.e. set of degree $\mathbf{c}$, the following Σ_4^C formula θ defines the same map as $f_{\mathbf{M},n^{\mathbf{M}_0}}$ (assuming, of course, the correctness conditions of $\hat{\theta}$ hold):*

$\theta(\mathbf{x},\mathbf{y},\vec{\mathbf{p}}_0,\mathbf{n}_0,\vec{\mathbf{p}}_1,\mathbf{n}_1,\vec{\mathbf{p}},\mathbf{n})$ *if and only if* $\mathbf{x}\leq_{\mathbf{M}_0}\mathbf{n}_0,\mathbf{y}\leq_{\mathbf{M}_1}\mathbf{n}_1$ *and there exist* $\mathbf{w},\mathbf{z}$ *such that*

(d) $\mathbf{w}\leq_T\mathbf{s}_0$ *and* $\mathbf{z}\leq_T\mathbf{s}_1$*;*

(e) $\mathbf{w} \leq_T \mathbf{x}$ *and* $\mathbf{z} \leq_T \mathbf{y}$*;*
(f) $\mathbf{w} \vee \mathbf{p} \geq_T \mathbf{q}$ *and* $\mathbf{z} \vee \mathbf{p} \geq_T \mathbf{q}$*;*
(g) $\mathbf{z} \leq_T \mathbf{w} \vee \mathbf{t}$ *and* $\mathbf{w} \vee \mathbf{t} \vee \mathbf{v} \not\geq_T \mathbf{u}$.

Proof. First we verify that $\theta(\mathbf{x}, \mathbf{y}, \vec{\mathbf{p}}_0, \mathbf{n}_0, \vec{\mathbf{p}}_1, \mathbf{n}_1, \vec{\mathbf{p}}, \mathbf{n})$ holds whenever $\mathbf{y} = f_{\mathbf{M}, n^{\mathbf{M}_0}}(\mathbf{x})$. Let $\mathbf{a}$ be the witness required to show that $f_{\mathbf{M}, n^{\mathbf{M}_0}}(\mathbf{x}) = \mathbf{y}$. We take $\mathbf{w}$ to be $\mathbf{a}$ and $\mathbf{z}$ to be $\mathbf{a}^+$ in the definition of θ. It is easy to see that (d)—(f) hold and (g) follows from the connection between $\mathbf{t}, \mathbf{u}$ and $\mathbf{v}$ specified in (v) and (vii). For the converse, suppose $\theta(\mathbf{x}, \mathbf{y}, \vec{\mathbf{p}}_0, \mathbf{n}_0, \vec{\mathbf{p}}_1, \mathbf{n}_1, \vec{\mathbf{p}}, \mathbf{n})$ holds via witnesses $\mathbf{w}$ and $\mathbf{z}$. By (f), $\mathbf{w}$ is above some $\mathbf{a}$ in $\mathbf{M}$ and by (d) $\mathbf{a}$ is an even number in $\mathbf{M}$. By (e) and (iii) and (iv) of Lemma 2.6, $\mathbf{a}$ is the unique even element in $\mathbf{M}$ below $\mathbf{x}$ as the $j^{\mathbf{M}_0}$'s form an antichain. We show that $\mathbf{a}$ is a witness that $f_{\mathbf{M}, n^{\mathbf{M}_0}}(\mathbf{x}) = \mathbf{y}$. Clearly (a) and (b) hold. We now prove (c) by showing that $\mathbf{a}^+ \leq_T \mathbf{z}$, which is sufficient in view of (e). By (g), which says $\mathbf{w} \vee \mathbf{t} \vee \mathbf{v} \not\geq_T \mathbf{u}$, there are at most two integers in $\mathbf{M}$ below $\mathbf{w} \vee \mathbf{t}$. Now $\mathbf{a} \leq_T \mathbf{w}$ is one and $\mathbf{a}^+ \leq_T \mathbf{w} \vee \mathbf{t} \leq_T \mathbf{a} \vee \mathbf{t}$ is another by (v). Hence only $\mathbf{a}$ and $\mathbf{a}^+$ are below $\mathbf{w} \vee \mathbf{t}$. By (f) and (d), $\mathbf{z}$ is above some odd integer in $\mathbf{M}$. Since $\mathbf{z} \leq_T \mathbf{w} \vee \mathbf{t}$, that odd integer must be $\mathbf{a}^+$. □

Recall that if $\mathbf{c}$ is promptly simple, then $\mathcal{C}_{\mathbf{c}}$ denotes the class of effectively generated coded models below $\mathbf{c}$ via S_1. Heading toward our definition of a class of $\mathcal{N}$-standard models for $\mathcal{C}_{\mathbf{c}}$, we define the relation $\Theta(\vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1)$ to hold if and only if

1. $\vec{\mathbf{p}}_0$ and $\vec{\mathbf{p}}_1$ code models $\mathbf{M}_0$ and $\mathbf{M}_1$, respectively, via S_1 and
2. $(\forall \mathbf{x} \in \mathbf{M}_0)(\exists \mathbf{y} \in \mathbf{M}_1)(\exists \vec{\mathbf{p}}, \mathbf{n})(\vec{\mathbf{p}}$ codes a model $\mathbf{M}$ via $S_{2,\mathbf{c}}$, $\mathbf{n} \in \mathbf{M}$ and
$$\theta(\mathbf{x}, \mathbf{y}, \vec{\mathbf{p}}_0, \mathbf{n}_0, \vec{\mathbf{p}}_1, \mathbf{n}_1, \vec{\mathbf{p}}, \mathbf{n})).$$

That is, every initial segment $[0^{\mathbf{M}_0}, \mathbf{x}]$ of $\mathbf{M}_0$ is isomorphic to one $[0^{\mathbf{M}_0}, \mathbf{y}]$ of $\mathbf{M}_1$ by a map defined by θ through a third model $\mathbf{M}$ coded by $\vec{\mathbf{p}}$ via $S_{2,\mathbf{c}}$ and an $\mathbf{n} \in \mathbf{M}$.

We will now show that $\Psi(\vec{\mathbf{p}}_0, \mathbf{c}) \equiv (\forall \vec{\mathbf{p}}_1)(\vec{\mathbf{p}}_1$ codes a model below $\mathbf{c}$ via $S_1 \rightarrow \Theta(\vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1))$ is the required correctness condition for a coding scheme that guarantees that $\mathbf{M}_0$ is $\mathcal{N}$-standard for $\mathcal{C}_{\mathbf{c}}$.

Lemma 2.9. *Assume that C is promptly simple and that $I\Sigma_4^C$ holds. Let $\mathbf{M}_0$ be an effectively generated model coded by $\vec{\mathbf{p}}_0$ below $\mathbf{c}$. Suppose that $\Psi(\vec{\mathbf{p}}_0, \mathbf{c})$ holds. Then $\mathbf{M}_0$ is isomorphic to $\mathbf{N}$.*

Proof. By Theorem 2.4 there is a coded model $\mathbf{N}_{\mathbf{c}} = \mathbf{M}_1$ below $\mathbf{c}$ which is a recursive copy of $\mathcal{N}$. By Lemma 2.5 there is an order-preserving one-to-one map g from $\mathbf{M}_1$ to $\mathbf{M}_0$ which is closed downwards so that $\mathbf{M}_0$ is an end-extension of $\mathbf{M}_1$. Moreover, g restricted to any initial segment of $\mathbf{M}_1$ is $\mathcal{N}$-finite. We show that this map is onto. Suppose for the sake of a contradiction that there is an $\mathbf{x} \in \mathbf{M}_0$ which is not in the range of g. Then, as guaranteed by Θ, there is an $\mathbf{M}$ and an $\mathbf{n} \in \mathbf{M}$ which provide an order-preserving isomorphism from $[0^{\mathbf{M}_0}, \mathbf{x}]$ to some initial segment $[0^{\mathbf{M}_1}, \mathbf{y}]$ of $\mathbf{M}_1$. By Lemma 2.8 this map f is Σ_4^C. The $\mathcal{N}$-finite function $g \restriction [0^{\mathbf{M}_1}, \mathbf{y}]$ maps this interval one-to-one and onto some $[0^{\mathbf{M}_0}, \mathbf{x}']$ in $\mathbf{M}_0$ and $\mathbf{x}' <_{\mathbf{M}_0} \mathbf{x}$ by our choice of $\mathbf{x}$. Thus $fg \restriction [0^{\mathbf{M}_1}, \mathbf{y}]$ is one-one, Σ_4^C and maps some $[0^{\mathbf{M}_1}, \mathbf{y}]$ which is recursively isomorphic to some interval $[0, n]$ in $\mathcal{N}$ into a proper subset of itself. As this map is Σ_4^C it is actually $\mathcal{N}$-finite by $I\Sigma_4^C$ induction, and so contradicts the pigeon hole principle. $\square$

Our analysis so far shows that the next definition may be formalized in the language of $\mathcal{R}_{\mathcal{N}}$.

Definition 2.10. $\mathbf{M}_0$ *is* good *for* $\mathbf{c}$ *if* $\mathbf{M}_0$ *is an effectively generated model coded below* $\mathbf{c}$ *by the parameters* $\vec{\mathbf{p}}_0$ *and* $\Psi(\vec{\mathbf{p}}_0, \mathbf{c})$ *holds.*

Summarizing the results obtained thus far, and noticing that for C low $I\Sigma_4^C$ is equivalent to $I\Sigma_4$, we have the following.

Theorem 2.11. *If* $\mathcal{N} \models I\Sigma_4$ *and* $\mathbf{c}$ *is promptly simple and low, then*

1. *there is an* $\mathbf{M}_0$ *which is good for* $\mathbf{c}$*, and*
2. *any* $\mathbf{M}_0$ *good for* $\mathbf{c}$ *is isomorphic to* $\mathcal{N}$*.*

We are now ready to provide an interpretation of the theory of $\mathcal{N}$ in $\mathcal{R}_{\mathcal{N}}$. Theorem 2.11 says that the theory of $\mathcal{N}$ may be interpreted below any low, promptly simple degree $\mathbf{c}$ whenever $\mathcal{N}$ satisfies $I\Sigma_4$. We can define the class $\mathbf{PS}$ of promptly simple degrees in $\mathcal{R}_{\mathcal{N}}$ as the ones which are not half of a minimal pair by [1]. (See Theorems 3.2.) The problem is that we do not have a first-order degree-theoretic definition for the low degrees. However, we can get the same effect in terms of interpreting the theory of $\mathcal{N}$ (although not of $\mathcal{N}$ itself) by quantifying over promptly simple degrees in the way used by Slaman and Woodin in their (unpublished) proof that there is an interpretation of the theory of $\mathcal{N}$ in $\mathcal{R}$.

Definition 2.12. *We define a translation* * *taking a sentence* φ *in the language of Peano arithmetic to a* φ^* *in the language of* $\mathcal{R}$*. First let* $\varphi'(\vec{\mathbf{p}})$ *be the corresponding sentence of* $\mathcal{R}_{\mathcal{N}}$ *asserting that* φ *holds in the model coded by the parameters* $\vec{\mathbf{p}}$ *(using the translation provided by the scheme* S_0*). Then let* φ^* *be*

$$(\forall \mathbf{d} \in \mathbf{PS})(\exists \mathbf{c} \in \mathbf{PS})(\exists \mathbf{M}_0)(\mathbf{c} \leq_T \mathbf{d},\ \mathbf{M}_0 \textit{ is good for } \mathbf{c} \textit{ and } \mathbf{M}_0 \models \varphi').$$

THEOREM 2.13. *For every formula φ of arithmetic, if $\mathcal{N} \models I\Sigma_4$, then*

$$\mathcal{N} \models \varphi \Leftrightarrow \mathcal{R}_{\mathcal{N}} \models \varphi^*.$$

Proof. Suppose that $\mathcal{N} \models \varphi$ and $\mathbf{d}$ is promptly simple. By Theorem 3.6 (taken from [6]) we can choose a degree $\mathbf{c} \leq \mathbf{d}$ which is promptly simple and low. Let $\mathbf{M}_0$ be a the recursive copy of $\mathcal{N}$ as given by Lemma 2.4. Clearly $\mathbf{M} \models \varphi'$ as it is isomorphic to $\mathcal{N}$. It remains to check that $\mathbf{M}_0$ is good for $\mathbf{c}$, which can be done as carried out before under $I\Sigma_4^C$, and this is equivalent to $I\Sigma_4$ since $\mathbf{c}$ is low. For the converse, suppose that $\mathcal{R}_{\mathcal{N}} \models \varphi^*$. Choose $\mathbf{d}$ low and promptly simple. Fix $\mathbf{c}$ and $\mathbf{M}_0$ as in the hypothesis. Then by the lowness of $\mathbf{c}$ and Theorem 2.11, $I\Sigma_4$ shows that $\mathbf{M}_0$ is isomorphic to $\mathcal{N}$. Since $\mathbf{M}_0 \models \varphi'$, we get $\mathcal{N} \models \varphi$. $\square$

We now draw some consequences of our work which illustrate the extent to which the theory of $\mathcal{R}_{\mathcal{N}}$ depends on, and reflects, the amount of induction that holds in $\mathcal{N}$. First, in terms of reverse mathematics we see that we have equivalents for $I\Sigma_n$ for each $n > 4$ over the base theory $I\Sigma_4$.

COROLLARY 2.14. *For each $n > 4$ there is a theorem ψ_n about the r.e. degrees such that ψ_n is equivalent to $I\Sigma_n$ over the base theory $I\Sigma_4$.*

Proof. For each $n > 1$ there is a sentence φ_n of arithmetic that is equivalent to $I\Sigma_n$ over $I\Sigma_1$. Let ψ_n be φ_n^*. For $n > 4$, our results show that $\mathcal{N} \models \varphi_n \Leftrightarrow \mathcal{R}_{\mathcal{N}} \models \varphi_n^*$ for every model $\mathcal{N}$ of $I\Sigma_4$. Thus φ_n^* is the desired formula ψ_n. $\square$

Another way of looking at our results and other older ones is in terms of the r.e. degree theory of various fragments of arithmetic. If T is a theory of arithmetic, the *r.e. degree theory of T* is the set of all first-order sentences (without parameters) in the language of the r.e. degrees which are true in every model of T. For example, consider the theory $I\Sigma_n^+$ which is $PA^- + I\Sigma_n$ together with a sentence which asserts the failure of Σ_{n+1}-induction. We call this theory *strict* $I\Sigma_n$. Our results here show that the r.e. degree theories of $I\Sigma_n^+$ are not complete for $n \geq 4$ as the theories $I\Sigma_n$ are themselves not complete. On the other hand, results such as those on the existence of minimal pairs [2] and on the density of branching r.e. degrees [14] show that the r.e. degree theories of $I\Sigma_1^+$ and $I\Sigma_2^+$ are incomplete as well.

We now give some other corollaries.

COROLLARY 2.15. *For $n > m \geq 4$, the r.e. degree theories of $I\Sigma_n^+$ and $I\Sigma_m^+$ are different.*

Proof. If $\mathcal{N}_m$ is a model of $I\Sigma_m^+$, it satisfies the sentence φ which states that $I\Sigma_{m+1}$ fails. Hence φ^* holds in $\mathcal{R}_{\mathcal{N}_m}$. Now φ^* is parameter free and true in the r.e. degree theory of $I\Sigma_m^+$. On the other hand, φ is false in every model $\mathcal{N}_n$ of $I\Sigma_n^+$, so that φ^* is false in the r.e. theory of degrees for $I\Sigma_n^+$. $\square$

COROLLARY 2.16. *The r.e. degree theory of* $I\Sigma_1^+$ *is different from the r.e. degree theory of* $I\Sigma_n^+$ *for* $n \geq 2$.

Proof. It is shown in [2] that the existence of a minimal pair of r.e. degrees is equivalent to $I\Sigma_2$ over the base theory $B\Sigma_2$. Since there exist models satisfying $B\Sigma_2$ (hence $I\Sigma_1$) but not $I\Sigma_2$, we have a sentence that differentiates between the r.e. degree theories of $I\Sigma_1^+$ and $I\Sigma_n^+$ for $n \geq 2$. $\square$

Yang [14] has recently shown that over $PA^- + B\Sigma_3$, Slaman's theorem that branching r.e. degrees are dense is equivalent to Σ_3-induction. This gives an example of the equivalence of $I\Sigma_3$ with a $\mathbf{0}'''$ priority argument. It also gives a difference between the r.e. degree theories of $I\Sigma_2^+$ and $I\Sigma_3^+$. We believe that Corollary 2.15 holds for all $n > m \geq 1$. The only case left open is when $n > m = 3$. A recursion-theoretic statement whose proof requires a $\mathbf{0}''''$ priority argument would be a good candidate to establish this.

In terms of reverse mathematics, the natural question along these lines is whether Corollary 2.14 can be improved by weakening the base theory to $I\Sigma_1$ (or any $I\Sigma_k$ for $k < 4$) and establishing equivalences for all $n > 1$ (or $n > k$). Another question to ask is the extent to which Theorem 2.13, whose proof uses $I\Sigma_4$ in an essential way, can be strengthened. More specifically, for $1 \leq k \leq 3$, is there an interpretation $*$ (a recursive map from sentences in the language of Peano arithmetic to sentences in the language of partial ordering for r.e. degrees) such that for all models $\mathcal{M}$ of $I\Sigma_k$ and all sentences φ, $\mathcal{M} \models \varphi$ if and only if $\mathcal{R}_{\mathcal{M}} \models \varphi^*$?

§3. Recursion-theoretic Constructions. Throughout this section, we fix $\mathcal{N}$ to be a model of $I\Sigma_4$. All r.e. degrees are members of $\mathcal{R}_{\mathcal{N}}$. We need to argue that all the facts about the $\mathcal{R}_{\mathcal{N}}$ used in the previous section can be proven in $I\Sigma_4$. An examination of our arguments shows that four Theorems 3.2, 3.6, 3.7 and 3.9 suffice. Some of the arguments merely require reading proofs in the literature and noticing that $I\Sigma_4$ suffices to carry them out. Others are new results about the r.e. degrees which are extensions and elaborations of results in [9]. We assume familiarity with that paper as well as [6] and [1] or the presentation of that material in Ch. XIII of [13] and with tree arguments as in, for example, Ch. XIV of [13]. For our purposes we may take the promptly simple degree theorem of [6] as a definition of **PS**, the class of promptly simple degrees.

DEFINITION 3.1. *An r.e. degree* $\mathbf{a}$ *is promptly simple if there is an* $A \in \mathbf{a}$ *with an enumeration* A_s *and a recursive function* p *such that for all* s, $p(s) \geq s$ *and for all* e

$$W_e \textit{ infinite } \Rightarrow (\exists x)(\exists s)[x \in W_{e,at\, s} \,\&\, A_s \restriction x \neq A_{p(s)} \restriction x].$$

THEOREM 3.2. (Ambos-Spies et al. [1]) *The promptly simple degrees* $\mathbf{c}$ *are precisely those which are not halves of minimal pairs, i.e.* $(\forall \mathbf{x} > \mathbf{0})$ $(\exists \mathbf{y} > \mathbf{0})$ $(\mathbf{y} \leq_T \mathbf{x}, \mathbf{c})$.

The proof that no promptly simple $\mathbf{c}$ is half of a minimal pair is Theorem 1.11 of [6]. In terms of Definition 3.1, it appears as Theorem XIII.2.1 in [13]. Its proof is easily seen to work in far less than $I\Sigma_4$. The other direction is Theorem 1.13 of [1] (and XIII.2.2 of [13]). Unfortunately there is a gap in these write ups of this result. It will be corrected in the new edition of [13] but we take this opportunity to present a different presentation based on a gap-cogap tree argument taken from [11].

We are given a nonrecursive r.e. set B and wish to build an r.e. A and various candidates p_α such that either some p_α is a witness to B's being of promptly simple degree or $A > 0$ and $A \wedge B = 0$. The requirements are as follows:

N_e: $\Phi_e(A) = \Phi_e(B) = h_e \Rightarrow h_e \leq_T 0$ (in fact we will define partial recursive functions $\psi_{e,i}$ such that $h_e = \psi_{e,i}$ for some i).

P_e: $\Phi_e \neq A$

R_e: There is a recursive p_e such that $\forall i[W_i$ infinite $\Rightarrow \exists x, s(x \in W_{i,at\, s} \& B_s \restriction x \neq B_{p(s)} \restriction x)]$.

The requirements N_e, P_e and R_e are assigned to nodes α with length $3e, 3e+2$ and $3e+1$, respectively. Thus our tree construction will, for some α with $|\alpha| = 3e+1$, build a recursive p_α as in R_e or guarantee that N_e and P_e are satisfied for all e.

To understand the module for one requirement R_e or N_e consider first the problem of just trying to build A for any given B following the usual minimal pair construction on a tree. The problem is that once we get a new length of agreement, if we allow A to change later on the use, we cannot preserve the B-computations. (And so, in fact, not every $\mathbf{b}$ is half of a minimal pair.) Our plan then is to allow A to change only at those stages (inside a gap) at which a B change would get us closer to satisfying $R_{e,i}$ for some new i: If we have an $x \in W_{e,at\, s}$ and if $B \restriction x$ changes "promptly", i.e. before $p_e(s)$, we would satisfy $R_{e,i}$. At those stages (inside the cogap) at which we would get no advantage from a B change (after $p_e(s)$) we must preserve the A-computations to keep the minimal pair argument working.

To be more precise if $\Phi_e(A) = \Phi_e(B) = h_e$ we will define p_e and $\psi_{e,i}$ such that if i is least such that p_e does not satisfy $R_{e,i}$ then $\psi_{e,i} = h_e$ so as to meet N_e. At a stage $s+1$ at which the usual length of agreement function $\ell(e,s)$ has reached a new maximum we may open an $R_{e,i}$ gap for the least i such that $R_{e,i}$ is not yet satisfied and for which we have an $x \in W_{i,at\ s}$ and a $y \notin \operatorname{dom} \psi_{e,i,s}$ with $y < \ell(e,s)$ such that $\forall z \leq y[\phi_e(B;z)[s] < x]$. We now set $\psi_{e,i}(z) = \Phi_e(B;z)[s]$ for $z \leq y$, $z \notin \operatorname{dom} \psi_{e,i,s}$. The idea is that we need not restrain A as long as the gap is open since any change in B below the use (at some $y' \leq y$) would give us a win on $R_{e,i}$. For this threat to be credible we must, however, eventually define $p_e(s)$. We wait for the first stage t at which $\ell(e,t) > \ell(e,s)$. We then set $p_e(s) = t$ and close the gap. The point now is that we no longer get any advantage from a B change on the use (for $y' \leq y$) and must now impose restraint on the use from A during the cogap , i.e. until the next gap is opened.

Note that we really need to deal with R_e only if $\ell(e,s) \to \infty$. The positive requirements have, as usual, disjoint, infinite, recursive sets Q_α assigned to them with $x \in Q_\alpha \to x > \alpha$. The interactions among the various strategies (in particular different R_e) is now handled by a tree construction.

The priority tree: We specify the assignment of requirements and their outcomes:

a) $|\alpha| = 3e$ is assigned to N_e and the possible outcomes are i (if $\ell(\alpha,s) \to \infty$) and $n \in \omega$ (the last stage at which $\ell(\alpha,s)$ reached a maximum).

b) $|\alpha| = 3e+1$ is assigned to R_e. If the outcome of N_e is not i at the immediate predecessor of α, then that of R_e at α is 0 (no action is needed for R_e). Otherwise, the outcome can be g_n or $d_n (n > 0$, $n \in \omega)$. The idea here is first that g_n (gap) represents the possibility that $\Phi_e(B)$ is total and n is least such that $R_{e,n}$ fails, i.e. we open an $R_{e,n}$ gap infinitely often at α. In this case we will claim that, if h_e is defined then it equals $\psi_{\alpha,n}$. The outcomes d_n (divergent) represent the possibility that $\Phi_e(B;y_n)\uparrow$ where we define y_n at the last stage we go through g_n (i.e. the last time we open or close an $R_{e,n}$ gap). The outcomes are ordered by $0 <_L g_1 <_L d_1 <_L g_2 <_L d_2 <_L \cdots$.

c) $|\alpha| = 3e+2$ is assigned to P_e. The outcomes here are s if $\exists x \in Q_\alpha[\Phi_e(x) = 0 \,\&\, x \in A]$ and w otherwise. Of course $w <_L s$.

The Construction:

The accessible nodes at stage t: The sequence δ_t of length $\leq t$ of nodes accessible at stage t is defined by induction beginning with $\emptyset$ which is always accessible. Say $\alpha \subseteq \delta_t$, $|\alpha| < t$. We let $u_e(y,t) = \max\{\phi_e(B;z)[t] | z \leq y\}$.

$|\alpha| = 3e$: $\alpha\hat{\ }i \subseteq \delta_t$ if $\ell(\alpha,t) > \ell(\alpha,r)$ for every $r < t$ such that $\alpha \subset \delta_r$. Otherwise $\alpha\hat{\ }r \subseteq \delta_t$ where r is the last stage at which $\alpha\hat{\ }i \subseteq \delta$ or 0 if there is no such stage.

$|\alpha| = 3e+1$: If $\delta_t(3e) \neq i$, $\delta_t(3e+1) = 0$. Otherwise we go through the following steps to define $\delta_t(3e+1)$:

1. Let $r < t$ be the last stage at which $\alpha \subseteq \delta_r$. (If there is no such r, $\delta_t(3e+1) = 0$.) If we opened an $R_{e,n}$ gap for α at r we now close it, set $\delta_t(3e+1) = g_n$ and terminate the definition of δ_t. Otherwise,
2. We find the least n for which
   (a) we can *open an $R_{e,n}$ gap (for α) at t*, i.e.
      (i) $R_{e,n}$ is not satisfied for $p = p_\alpha$ which is given by $p_\alpha(r) = (\mu s > r)(\alpha \subseteq \delta_s$ and we closed an $R_{e,n}$ gap at $s)$ and
      (ii) $\exists x \in W_{n,t} - W_{n,r}$ such that $u_e(y_n,t) < x$; or
   (b) $\operatorname{dom}\psi_{\alpha,n,r} = \operatorname{dom}\ \psi_{\alpha,n,t} = y_n$ and $u_e(y_n,t) > u_e(y_n,v)$ for all $v < t$ with $\alpha \subseteq \delta_v$.

   Now, if we can open a gap for $R_{e,n}$ (2a), then we set $\delta_t(3e+1) = g_n$, and $\psi_{\alpha,n}(y_n) = \Phi_{e,s}(B_s;y_n)$. (Note that we will automatically have $y_n < \ell(\alpha,t)$ as we only extend $\operatorname{dom}\psi_{\alpha,n}$ at stages at which $\ell(\alpha,s)$ is above the value it had when we last extended $\operatorname{dom}\psi_{\alpha,n}$). If we cannot open a gap but satisfy (2b) then we set $\delta_t(3e+1) = d_n$ and continue.
3. If there is no n as required in (2) we terminate the definition of δ_t.

$|\alpha| = 3e+2$: $\alpha\hat{\ }s \subseteq \delta_t$ if $\exists x \in Q_\alpha[\ \Phi_{e,t}(x) = 0 \,\&\, x \in A_t]$. Otherwise $\alpha\hat{\ }w \subseteq \delta_t$.

The action at stage t: Find the shortest $\alpha \subseteq \delta_t$ such that $\delta_t(|\alpha|) = w$ and $\exists x \in Q_\alpha[\Phi_{e,t}(x) = 0\ \&\ x > r$ for every $r < t$ at which $\delta_r <_L \alpha]$ and put the least such x into A. If there is no such α go on to the next stage. This finishes the construction.

The Verifications:

The basic situation here is a bit more complicated than in many usual tree arguments. As we might go through g_n (and d_n) successively there needn't be a leftmost infinite path through the tree that is accessible infinitely often. In this case, however, we show that $B \in \mathbf{PS}$.

LEMMA 3.3. *If $B \notin \mathbf{PS}$ then there is an infinite leftmost path f. (We take this assertion as shorthand for the existence, for each n, of an α of length n which is accessible infinitely often such that no β to the left of α is accessible infinitely often. We denote this property by $\alpha \subseteq f$.)*

Proof. We proceed by induction on $n = |\alpha|$. If $|\alpha| = 3e$ or $3e+2$ then if $\alpha \subset \delta_t$ for infinitely many t then $\alpha\hat{\ }x \subset \delta_t$ for some outcome x and so for a leftmost one as there are only finitely many outcomes. If $|\alpha| = 3e+1$ and some immediate successor x is accessible infinitely often then there is again a leftmost one as there are only finitely many outcomes to the left of x. If no successor node is accessible infinitely often, we claim that p_α is a witness to B being of promptly simple degree. If not let W_n be the least counterexample. Note first that $\alpha(3e) = i$ for otherwise we would eventually have $\alpha\hat{\ }0 \subseteq \delta_t$ for all t with $\alpha \subset \delta_t$. Thus $\ell(\alpha, t) \to \infty$. As no successor g_m or d_m is traversed infinitely often we may choose t_0 such that $\forall t \geq t_0(\alpha\hat{\ }g_m \nsubseteq \delta_t \,\&\, \alpha\hat{\ }d_m \nsubseteq \delta_t)$ for $m \leq n$. As we never again go through $\alpha\hat{\ }g_n$, $\psi_{a,n,t}$ is fixed, say as $\psi_{\alpha,n}$, for $t \geq t_0$ and we can set $y_n = \operatorname{dom}\psi_{\alpha,n}$. As we never again go through $\alpha\hat{\ }d_n$, $u_e(y_n, t)$ is bounded for t with $\alpha \subseteq \delta_t$ say by u. As W_n is infinite we eventually get a $t > t_0$ with an $x \in W_{n,at\ t}$ and $x > u$. Now at the next stage $s > t$ at which $\alpha \subset \delta_s$ we would, by the definition of being able to open an $R_{e,n}$ gap, have $\alpha\hat{\ }g_n \subseteq \delta_t$ unless $R_{e,n}$ were already satisfied for our desired contradiction. □

We now assume that f is the leftmost infinite path.

LEMMA 3.4. *Each P_e is satisfied.*

Proof. Say $|\alpha| = 3e+2$ and $\alpha \subset f$. If $\alpha\hat{\ }s \subset f$ we are done so suppose $\alpha\hat{\ }w \subset f$ but $\Phi_e = A$. Thus $Q_\alpha \cap A = \emptyset$ and there is an $x \in Q_\alpha$ with $x > t$ for every t with $\delta_t <_L \alpha$ and an s with $\Phi_{e,s}(x) = 0$ for which $\alpha \subset \delta_s$. We would now, by our action at s, put such an x into A unless P_e were already satisfied and so $\alpha\hat{\ }s \subset f$ after all. □

Note now that if $\alpha \subset f$ and $\alpha\hat{\ }d_n \subset f$ then $\Phi_e(B)$ is not total and so to finish the proof of the theorem it suffices to prove the following:

LEMMA 3.5. *If $\Phi_e(A) = \Phi_e(B) = h_e$, $|\alpha| = 3e+2$ and $\alpha\hat{\ }g_n \subset f$ then $\psi_{\alpha,n} = h_e$.*

Proof. Let t_0 be such that $\forall t > t_0(\delta_t \not<_L \alpha\hat{\ }g_n)$. We argue by induction from one stage $t > t_0$ at which $\alpha\hat{\ }g_n \subseteq \delta_t$ and at which we open an $R_{e,n}$ gap to the next such t' that if $\psi_{\alpha,n,t}(y)\downarrow$ then for every r, $t \leq r \leq t'$

$$\psi_{\alpha,n,r}(y) = \ \Phi_{e,r}(A_r; y) \text{ or } \ \Phi_{e,r}(B_r; y).$$

Now at the first stage t at which $\psi_{\alpha,n,t}(y)\downarrow$ its value is $\Phi_{e,t}(B_t;y)$ by definition. As we never satisfy $R_{e,n}$, B cannot change on $u_e(y,t)$ until the next stage r at which $\alpha\hat{\ }g_n \subseteq \delta_r$ at which we close the $R_{e,n}$ gap. At this stage $\Phi_{e,r}(A_r;y) = \Phi_{e,r}(B_r;y) = \Phi_{e,t}(B_t;y)$ and no element enters A. (The part to notice here is that $\delta_r = \alpha\hat{\ }g_n$ as its definition is terminated at this level. Thus no $P_{e'}$ for $e' \geq e$ can put an element into A. On the other hand no $P_{e'}$ for $e' < e$ can put one in since that would cause a shift in outcome for P'_e from w to s contradicting $\alpha \subset f$. From now until t', however, we are never to the left of $\alpha\hat{\ }g_n$ as $t > t_0$ and so no elements can be put into A below $r > u_e(y,t)$ until t'. Thus $\Phi_{e,t'}(A_{t'};y) = \Phi_{e,r}(A_r;y) = \Phi_{e,t}(A_t;y)$. We now simply note that if we define $\psi_{\alpha,n}(y')$ at t' then $y' > y$, $u_e(y') > u_e(y)$ and so we can continue the induction by seeing that again B cannot change on $u_e(y',t') > u_e(y,t)$ until we next close the $R_{e,n}$ gap. $\square$

This completes the proof of Theorem 3.2 (in ordinary mathematics). As an aside we point out that this proof shows that the definition of promptly simple degree is independent of both the r.e. member of the degree considered and its enumeration. All we have to do now is check that we did not need anything more than $I\Sigma_4$. The most complicated use of induction is the proof of Lemma 3.3. The statement being proved by induction on n is that there is an α of length n which is accessible infinitely often and such that there is a bound on the stages at which any $(\alpha \restriction i)\hat{\ }x$ is accessible for $i < n$ and $x <_L \alpha(i)$. This sentence is Σ_3 and so our induction can be carried out in the model $\mathcal{N}$ of $I\Sigma_4$. Everything else used in the proof is easily seen to be simpler yet.

THEOREM 3.6. (Maass, Shore and Stob [6]) *For every promptly simple* $\mathbf{d}$ *there is a low promptly simple* $\mathbf{c} \leq_{\mathbf{T}} \mathbf{d}$.

Proof. This is simply the special case of Theorem 1.5 of [6] with $A = B$. The proof is routine (once one knows about prompt permitting) and clearly needs far less induction than Σ_4. $\square$

Our next theorem was originally stated as Theorem 2.4.

THEOREM 3.7. *For every promptly simple* $\mathbf{c}$ *there is an effectively generated recursive model* $\mathbf{M}$ *isomorphic to* $\mathcal{N}$ *with all its parameters below* $\mathbf{c}$.

For this Theorem and Theorem 3.9, we use the notation of [9], assume the reader is familiar with its constructions and verifications, and describe only the changes needed to get the construction satisfying Theorems 3.7 and 3.9. Our starting point for Theorem 3.7 is Theorem 6.1 of [9] to which we must add requirements (8 and 9 below) to take care of the only new aspect of our theorem, namely the addition of the second partial order coded by $\mathbf{1}'$ that coincides with $\leq_{\mathbf{M}}$ on $\mathbf{M}$. The following Theorem is then the technical result needed to verify Theorem 3.7.

THEOREM 3.8. *Given any $0 <_T A \leq_T U$ with U promptly simple, and given recursive partial ordering $\mathcal{P} = \langle \omega, \preceq \rangle$ with a specified infinite recursive set $H = \{h_i | i \in \omega\}$ of minimal elements, there are r.e. sets $E_0, E_1, F_0, F_1, B, P, Q, R$ and G_i (for $i \in \omega$) with $R = \oplus G_i, F_0 = \oplus G_{h_{2i}}$, $F_1 = \oplus G_{h_{2i+1}}$ and $B = G_i^{[0]}$ (for each $i \in \omega$) such that $P, Q \leq_T U$, all the other sets constructed are recursive in A and*

1. $(T) : G_i \oplus P \geq_T Q$.
2. $(D) : G_i \not\geq_T G_j$ *for* $i \neq j$.
3. (M) : *If B is recursive in an r.e. W which is recursive in R and $W \oplus P \geq_T Q$, then there is a $j \in \omega$ such that $G_j \leq_T W$.*
4. (K) : $R \oplus P$ *is low.*
5. $(O) : i \succeq j \Rightarrow G_i \oplus L \geq_T G_j$.
6. $(N) : i \not\succeq j \Rightarrow G_i \oplus L \not\geq_T G_j$.
7. (Y) : *For each $i \in \omega$,* $\deg\left(G_{h_{2i}} \oplus E_1\right) \wedge \deg(F_1) = \deg(G_{h_{2i+1}})$ *and*
   $\deg\left(G_{h_{2i+1}} \oplus E_0\right) \wedge \deg(F_0) = \deg(G_{h_{2i+2}})$.
8. $(O') : i \geq j \Rightarrow G_{h_i} \oplus L' \geq_T G_{h_j}$.
9. $(N') : i \not\geq j \Rightarrow G_{h_i} \oplus L' \not\geq_T G_{h_j}$.

For the purposes of Theorem 3.7, we can take $A = U \in \mathbf{c}$. Also note that our coding of a model $\mathbf{M}$ of arithmetic in the partial order $\mathcal{P}$ makes the domain of $\mathbf{M}$ the minimal elements of $\mathcal{P}$ which we take to be H listed in the order of $\mathbf{M}$.

As a first approximation, requirements O' and N' should be treated exactly as were O and N. The plan for O' then would be that whenever a number x enters some G_{h_j} it must also be put either into L' or into every G_{h_k} with $k > j$. Similarly, requirements N' would be treated exactly as are the ones for N. If we were constructing a coded model via S_0 (and so without the lattice structure required by S_1 and effective generation), we could do this without any difficulty. One only needs to check that no new conflicts are introduced. For O and O', we consider any x targeted for some G. It is a follower of some requirement $D_{i,j}$, $N_{i,j,e}$ or $N'_{i,j,e}$ and so we must preserve the corresponding computation from G_i, $G_i \oplus L$ or $G_{h_i} \oplus L'$, respectively. In the first case, when we put x into G_j we can put x into L and L' as well without injuring the computation from G_i. In the second case, when we put x into G_j we can put x into L' and every G_k with $k \succeq j$ without injuring the computation from $G_i \oplus L$ since $i \not\succeq j$. In the third case, when we put x into G_{h_j}

we can put it into L and every G_{h_k} for $k > j$ without injuring the computation from $G_{h_i} \oplus L'$ as $i \not\succeq j$. Thus no essential changes to the basic constructions in [9] through §4.4 would be necessary to construct, for example, a recursive coded model isomorphic to $\mathcal{N}$ via our scheme S_0 by omitting requirements Y in Theorem 3.8 but including O' and N'.

The new problem in proving Theorem 3.8 comes from the need to satisfy the infima requirements Y in the associated pinball construction. This construction does not allow us to put numbers simultaneously into distinct G_{h_j} and G_{h_k} (as they appear as components of both sides of the computations at single gates). (There is no problem putting numbers simultaneously into any G_j and all G_i with $i \succ j$ as the only G_k involved in the infima requirements are ones minimal with respect to $\succeq$.) To solve this problem, we first change the nature of the coding procedure for $O'_{k,j}$ to compute G_{h_j} from $G_{h_k} \oplus L'$. To decide if $x \in G_{h_j}$ we ask if $x \in G_{h_k}$ or $x \in L'$. If not, $x \notin G_{h_j}$. If so, we find the stage s at which it entered. If x entered at s we will set up a sequence of coding markers targeted for L' in the standard fashion. We appoint a new large one $t > s$ at s and may change it to a larger number when it or some smaller number enters L'. Eventually, either x enters G_{h_j} (perhaps already at s) or we appoint a marker targeted for L' which never enters L'. The full solution to our problem also involves weakening even this coding procedure for requirements O' so that they work for all but finitely many x's. We also have to add an appropriate sequence of balls to the entourage for any x which is a follower of some $N'_{i,j,e}$.

As in the basic construction without the infima requirements, D and N requirements only need to preserve computations from G_i and L. When one of their followers x enters G_{h_j} we can simply put x into L' to maintain the codings for all O' requirements. (We can also put it into G_k for all $k \succ h_j$ as well to maintain the O codings.) Similarly, any ball x appointed as a trace in the pinball construction is too large to injure the computation realizing the follower to whose entourage it belongs and so when it enters its target G_{h_k} we can put it into both L and L' to satisfy all the codings requirements for O and O'. Thus our only worries are for followers x of some $N'_{i,j,e}$ requirement that we wish to target for G_{h_j}. We place x in the hole for this requirement which we assume is the nth requirement on our master list of all the requirements. Our plan is to also put in the hole balls labelled $\langle x, i, j, e, k\rangle$, say, targeted for G_{h_k} for $j < k \leq n$. When the ball $\langle x, i, j, e, k\rangle$ leaves the permitting bin of the pinball machine (i.e. it first reaches the bin and then x is permitted by A) we put x into G_{h_k}. Of course, each such ball is provided with an appropriate sequence of traces at all stages before it leaves the machine. The balls $\langle x,i,j,e,k\rangle$ (with their traces) are arranged in the hole in order (of k) so that the first one to roll out and stop at a gate is $\langle x,i,j,e,n\rangle$ (preceded, as usual, by its sequence of traces).

The appointment of traces and movement of balls down the pinball machine is now as in [9]. When the first of these new balls in x's entourage, $\langle x,i,j,e,n\rangle$, leaves the permitting bin, say at stage s_n, we put x into G_{h_n} but not into any other G_{h_k}. We also appoint $t_n > s_n$ as the coding marker for the reduction associated with $O'_{n,k}$ at x for $k<n$. This marker is targeted for L'. This action does not injure the computation that realized x for $N'_{i,j,e}$ as $i<j\leq n$. It does, however, violate our coding requirements $O'_{k,n}$ for $k>n$ but such violations will happen only finitely often. When the next ball $\langle x,i,j,e,n-1\rangle$ leaves the permitting bin, say at stage s_{n-1}, we put x into $G_{h_{n-1}}$ and so violate the coding requirements $O'_{k,n-1}$ for $k>n$ but for $k=n$ we can put t_n into L' (and so make the reduction described above for computing $G_{h_{n-1}}$ from G_{h_n} correct at x) without injuring the computation that realized x for $N'_{i,j,e}$. Of course, we now redefine the coding markers for x targeted for L' for $O'_{n,k}$ and $O'_{n-1,k}$ for $k<n-1$ to be $t_{n-1}>s_{n-1}$. We continue on with this procedure putting x into each G_{h_k} in turn until we get to put x into G_{h_j} as desired. At this point we injure $O'_{k,j}$ for $k>n$ but can put the large marker t_{j+1} for each $O'_{k,j}$ for $j<k\leq n$ into L' and so make these reductions correct. Thus we see that the reduction described above for computing an arbitrary G_{h_j} from $G_{h_m}\oplus L'$ with $m>j$ is violated only when we act for a requirement N' which appears on our list of requirements before position m. As these requirements will act only finitely often, we will satisfy the requirement $O'_{m,j}$.

The other changes needed in the construction are minor and straightforward given our plans for satisfying the O' requirements. For example, we place $N'_{i,j,e}$ immediately after $N_{i,j,e}$ in our master list of requirements. Suppose it is in position n. The function giving the number of permissions need to get a particular follower x into G_{h_j} is then $(n-j+1)$ times that for $N_{i,j,e}$ $(g(i,j,e))$ as we must get all the balls $\langle x,i,j,e,k\rangle$ for $j<k\leq n$ as well as x through the permitting bin. The restraint imposed to preserve the suitability of the follower x of $N'_{i,j,e}$ in the Slaman-Woodin part of the construction is now calculated so as to also preserve the suitability of x entering G_{h_k} for $j\leq k\leq n$. This then concludes our description of the (changes in the) construction (from that for Theorem 6.1 of [9]).

We must now say a word or two about the amount of induction needed for the verifications. The global structure of the Slaman-Woodin construction (requirements T, D, M and K) has finitary action by all the requirements except M and so the induction statement of the outcomes for these requirements is Σ_2. The M requirements are like the minimal pair requirements in Ch. IX of [13]. The essential point here is that for each M_i there are either finitely many expansionary stages or infinitely many. In the first case, the action by M is finitary. In the second, we argue that the restraint is eventually constant on the expansionary stages. In either case, the positive action is finitary. Thus the statements needed here are clearly no worse than Σ_3. We also need to know that the subsets of any initial segment of M requirements that fall into each case is $\mathcal{N}$-finite and then that the total restraint is bounded. Again $I\Sigma_3$ is sufficient. Except for the interactions with the pinball machine requirements N and N' are like D. The coding requirements can be ignored at this stage as they impose no restraint and act whenever they want to. So all that remains at the global level is the pinball construction. Here the situation is again like the minimal pair argument. We must know the subset of gates within any initial segment that open infinitely often. Then we must argue that there is a bound on the stages at which those (in any initial segment) that are permanently occupied have been occupied by their permanent residents and that after that stage every ball that reaches the beginning of this sequence of gates gets through to the end. Again $I\Sigma_3$ clearly suffices. The inductive assumptions for the diagonalization and preservation requirements are again just that the outcomes are finitary. This brings us to the consideration of the inductive steps of the argument. For the basic construction, all except the M requirements are essentially immediate. The arguments for M involve a few Lemmas (4.2–4.6 of [9]) which at worst are of the form that if something happens infinitely often (e.g. some functional $\Delta_{i,j}$ being constructed gives the wrong answer at infinitely many x) then there are infinitely many x with some additional property. These are proved by induction on n by showing that under the assumed hypotheses there are at least n many x's as required. As the required properties are Π^0_1, these inductions are Σ_2. The other type of inductions used are on j as it is shown that each $\Delta_{i,j}$ $(j \leq n_i)$ are total. This looks like a Π_3 induction as the functionals $\Delta_{i,j}$ are computed relative to arbitrary r.e. sets W_e. Although this would not be a problem in $I\Sigma_4$, we note that as all the sets W_e considered are below the set R that we are constructing to be low, we can actually convert this into a Π_2 induction.

With the pinball construction added in, the difficult part of the proof is actually showing that the action for the D, N and N' requirements is finitary. The inductive assumptions about the actions of the gates described above, however, are sufficient along with $I\Sigma_3$. The argument for the success of infima requirements is now the standard pinball construction argument that, recursively in the infimum, finds a certain type of stage at which we have a computation at a given x and then argues by induction that the computation remains correct (on one side or the other of the pair being considered) at every later stage. Once again the amount of induction needed is small. The arguments for the success of the finitary requirements are straightforward given the inductive assumptions about the actions of the requirements and involve just standard (and prompt) permitting arguments that show that if we get infinitely many candidates we eventually put one in (at an expansionary stage). The arguments for the success of the reductions constructed is a simple induction showing that they are correct at every stage (with at most finitely many exceptions for each one which are bounded by the actions of higher priority requirements). Thus the verifications for Theorem 3.8 can be carried out in $I\Sigma_3$.

The following theorem combines Lemmas 2.6 and 2.7.

THEOREM 3.9. *For every low promptly simple degree* $\mathbf{c}$ *and degrees* $\mathbf{q}_0, \ldots, \mathbf{q}_{m-1}, \mathbf{r}_0, \mathbf{r}_1 \leq_T \mathbf{c}$ *there is a recursive model* $\mathbf{M}$ *coded below* $\mathbf{c}$ *and degrees* $\mathbf{t}, \mathbf{u}$ *and* $\mathbf{v}$ *also below* $\mathbf{c}$ *such that*

(i) $\mathbf{M}$ *is recursively isomorphic to* $\mathcal{N}$*;*
(ii) *For each* $i < m$, $(i)^{\mathbf{M}} \leq_T \mathbf{q}_i$*;*
(iii) *For* $i, j < m$, *if* $\mathbf{q}_i \not\leq_T \mathbf{q}_j$ *then* $(i)^{\mathbf{M}} \not\leq_T \mathbf{q}_j$*; and*
(iv) *For all* $\mathbf{x} \geq_{\mathbf{M}} m^{\mathbf{M}}$, $\mathbf{x} \not\leq \mathbf{r}_0, \mathbf{r}_1$.
(v) *For any even number* $\mathbf{h}$ *in* $\mathbf{M}$, $\mathbf{h} \vee \mathbf{t}$ *is above* $\mathbf{h}^+$, *but not above any other integer in* $\mathbf{M}$ *(except, of course,* $\mathbf{h}$*). (In particular, there is a unique odd (respectively, even) number from* $\mathbf{M}$ *below* $\mathbf{h} \vee \mathbf{t}$*.)*
(vi) *The join of any nontrivial distinct triple of numbers in* $\mathbf{M}$ *with* $\mathbf{v}$ *computes* $\mathbf{u}$, *but no join of any pair with* $\mathbf{v}$ *computes* $\mathbf{u}$.
(vii) *For any even number* $\mathbf{h} \in \mathbf{M}$, $\mathbf{h} \vee \mathbf{t} \vee \mathbf{v} \not\geq \mathbf{u}$.

Our starting point here is Theorem 5.1 of [9] to which we add requirements to take care of the new aspects of our theorem: (9) and (10) for $\mathbf{l}'$ to code the second partial order that coincides with $\leq_{\mathbf{M}}$ on $\mathbf{M}$, and (11)–(15) for $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ as described above. The following Theorem is then the technical result needed to verify Theorem 3.9. Note that we still have a coding of a model $\mathbf{M}$ of arithmetic in the partial order $\mathcal{P}$

that makes the domain of $\mathbf{M}$ the minimal elements of $\mathcal{P}$ which we take below to be H listed in the order of $\mathbf{M}$. We may assume that all the sets U_i and $V_{i,j}$ of Theorem 5.1 of [9] are uniformly below $C \in \mathbf{c}$ (and so uniformly low as in [9]) but we must also add in ordinary and prompt permitting by C to get all the sets constructed recursive in C.

THEOREM 3.10. *Suppose $C \in \mathbf{c}$ is low and promptly simple; $\mathcal{P} = \langle \omega, \preceq \rangle$ is a recursive partial order; H is a recursive set of minimal elements of $\mathcal{P}$; $\langle U_i, V_{i,j} \rangle_{i \in H}$ is a uniformly r.e. array of pairs of sets such that the U_i and $V_{i,j}$ are uniformly recursive in C (and so we can recursively in i,j calculate an index for computing $V'_{i,j}$ from $\emptyset'$); and, for every $i \in H$, $U_i \not\leq_T V_{i,j}$ (and so in particular $U_i > 0$ for $i \in H$). Then there are r.e. sets L, L', P, Q, R, T, U, V and G_i (for $i \in \omega$) with $R = \oplus G_i$ all uniformly recursive in C such that*

1. $(T) : G_i \oplus P \geq_T Q$.
2. $(D) : G_i \not\geq_T G_j$ *for* $i \neq j$.
3. (M) : *If W is r.e., recursive in R and $W \oplus P \geq_T Q$, then there is a $j \in \omega$ such that $G_j \leq_T W$.*
4. $(K) : R \oplus P$ *is low.*
5. $(O) : i \succeq j \Rightarrow G_i \oplus L \geq_T G_j$.
6. $(N) : i \not\succeq j \Rightarrow G_i \oplus L \not\geq_T G_j$.
7. $(Z) : G_i \not\leq_T V_{i,j}$ *for* $i \in H$.
8. $(Q) : G_i \leq_T U_i$ *for* $i \in H$.
9. $(O^1) : i \geq j \Rightarrow G_{h_i} \oplus L' \geq_T G_{h_j}$.
10. $(N^1) : i \not\geq j \Rightarrow G_{h_i} \oplus L'_T \not\geq_T G_{h_j}$.
11. $(O^2) : G_{h_{2j}} \oplus T \geq_T G_{h_{2j+1}}$.
12. $(N^2) : G_{h_{2i}} \oplus T \not\geq_T G_{h_j}$ *for* $j \neq 2i+1$.
13. $(O^3) : G_{h_i} \oplus G_{h_j} \oplus G_{h_k} \oplus V \geq_T U$ *for i, j, k all distinct.*
14. $(N^3) : G_{h_i} \oplus G_{h_j} \oplus V \not\geq_T U$.
15. $(N^4) : G_{h_{2i}} \oplus T \oplus V \not\geq_T U$.

Once again we have the problem of not being able to put numbers simultaneously into G_{h_i} and G_{h_j} as they have to be permitted by U_{h_i} and U_{h_j}, respectively. Thus we need generalized reduction procedures for the O^1, O^2 and O^3 type requirements like those used for O' in Theorem 3.8. We include the rules for the O requirements in our list as well because of their interactions with other requirements. Again these rules will be satisfied except for finitely many numbers x. Note that the only numbers put into any of the sets G_i or U are followers of various requirements. Thus we restrict our attention in computing reductions to such numbers (which form an infinite coinfinite recursive set by convention).

- (O) : As in [9], for $i \succ j$, to decide if $x \in G_j$, see if $x \in L$ or $x \in G_i$. If not, $x \notin G_j$. If so, let s be the stage at which x L or G_i. At s we appointed a new large number t_s targeted for L and we proceed until either x enters G_j or the sequence of markers that we appointed beginning with t_s comes to a marker that is not in L. (A new marker t' can be appointed in this sequence only when the previous one t enters L).
- (O^1) : As in Theorem 3.8, for $i > j$, to decide if $x \in G_{h_j}$, see if $x \in L'$ or $x \in G_{h_i}$. If not, $x \notin G_{h_j}$. If so, let s be the stage at which x entered. At s we appointed a new large number t_s targeted for L' and we proceed until either x enters G_{h_j} or the sequence of markers that we appointed beginning with t_s comes to a marker that is not in L'.
- (O^2) : For any $j \in \omega$, to decide if $x \in G_{h_{2j+1}}$, see if $x \in T$ or $x \in G_{h_{2j}}$. If not, $x \notin G_{h_{2j+1}}$. If so, let s be the stage at which x entered. At s we appointed a new large number t_s targeted for T and we proceed until either x enters $G_{h_{2j+1}}$ or the sequence of markers that we appointed beginning with t_s comes to a marker that is not in T.
- (O^3) : For any distinct i, j and k, to decide if $x \in U$, see if $x \in V$ or $x \in G_{h_m}$ for $m \in \{i, j, k\}$. If not $x \notin U$. If so, let s be the stage at which x entered. At s we appointed a new large number t_s targeted for V and we proceed until either x enters U or the sequence of markers that we appointed beginning with t_s comes to a marker that is not in T.

Now we list the diagonalization requirements and explain how they can be satisfied by putting numbers into the target sets (but at most one G_{h_i} at a time) while preserving the sets needed in the computations producing the desired diagonalizations and still obeying all the rules for the above reductions.

- (D) : We can put x into G_j (when permitted by the associated U_j if $j \in H$ and by C otherwise) and preserve G_i (for $i \neq j$) and satisfy all the O type requirements by putting x into L, L' and T.
- (Z) : We can put x into G_i (when permitted by the associated U_i if $i \in H$ and by C otherwise) and also put it into L, L' and T.
- (N) : When we want to put x into G_j (when permitted by the associated U_j if $j \in H$ and by C otherwise) and preserve $G_i \oplus L$ with $i \not\succeq j$, we put x into G_k for $k \succ j$, L' and T. (No permissions

are needed for G_k with $k \notin H$ and if $k \succ j$ then $k \notin H$ as the elements of H are all minimal elements with respect to $\preceq$.)

- (N^1) : When we want to put x into G_{h_j} and preserve $G_{h_i} \oplus L'$ with $i \not\geq j$, we first try to put x into each G_{h_k} for each k with $j < k \leq n$ in reverse order where n is the index of the requirements we are considering on our master list. So our first step is to put x into G_{h_n}, L and T when permitted by U_{h_n} at s_n and choose a new large marker t_n targeted for L', L and T. We then proceed in turn to try to put x into each G_{h_k} for $j < k < n$ when permitted by U_{h_k}. When we get to do so at stage s_k we put t_{k+1} into L', L and T and choose a new large marker t_k. Finally, when we have put x into $G_{h_{j+1}}$ we wait for permission from U_j to put x into G_{h_j}.When we get this permission we put x into G_{h_j} and t_{j+1} into L', L and T.
- (N^2) : When we want to put x into G_{h_j} and preserve $G_{h_i} \oplus T$ with $j \neq 2i+1$, we first try to put x into $G_{h_{2k}}$ for $j-1 \leq 2k \leq n$ (again n is the number of the requirement we are considering) in reverse order and appoint markers targeted for T, L' and L. We begin with the largest such k. When permitted by $U_{h_{2k}}$ we put x into $G_{h_{2k}}$, L and L' and appoint a marker t_{2k} targeted for T, L and L'. When we are permitted to put x into $G_{h_{2k+1}}$ by $U_{h_{2k+1}}$ we put it in and t_{2k} into L, L' and T and appoint t_{2k-1}. We then turn to $2(k-1)$ and repeat the process getting (if permitted) x first into $G_{h_{2(k-1)}}$ and t_{2k-1} into L, L' and T while appointing $t_{2(k-1)}$ and then getting x into $G_{h_{2k-1}}$, $t_{2(k-1)}$ into L, L' and T while appointing $t_{2(k-2)}$. Finally, if we get all the required permissions, we will put x into G_{h_j} and the last marker appointed in this sequence into T, L and L'.
- (N^3) : When we want to put x into U and preserve G_{h_i}, G_{h_j} and V, we first try to put x into each G_{h_k} for each $k \neq i, j$ with $k \leq n$ in reverse order (n as before). We proceed as for N^1 but first putting x into G_{h_n} and L, L' and T and appointing a marker t_n targeted for V as well as L, L' and T. At each successive step we put x into G_{h_k} for $k \neq i, j$ in turn in reverse order when permitted by $U_{h_k}$and the last marker appointed into L, L', T and V and appoint a new marker until we get x into the least $k \leq n$ not equal to i or j at which point we also put it into U (and the final marker into L, L', T and V).
- (N^4) : When we want to put x into U and preserve $G_{h_{2i}}$, T and V note that we also need to preserve $G_{h_{2i+1}}$ (for O^2). Our plan is to put x into all G_{h_k} for $k \leq n$ and $k \neq 2i, 2i+1$ and appropriate

markers into various sets. Our order is like that for N^2. We begin with the largest k such that $2k \leq n$ and $k \neq i$ (without loss of generality $n \geq 2$). When permitted by U_{2k} we put x into $G_{h_{2k}}$, L and L' and appoint a marker t_{2k}. When x is permitted by U_{2k+1} we put x into $G_{h_{2k+1}}$, t_{2k} into L, L', T and V and appoint a new marker t_{2k+1}. We then move on to $2(k-1)$. Note that we skip $2i$ and $2i+1$ moving instead to $2i-2$ and $2i-1$ when we reach that point. At the end we put x into U and the last marker into L, L', T and V.

It is clear that these procedures preserve the required sets for each diagonalization requirement. It is slightly complicated but reasonably straightforward to check that they also make all the desired reductions correct. For example, consider numbers put into U by action for some requirement of type N^3 and the reduction procedure for $O^3_{i,j,k}$. Once all action by requirements of priority higher than (i.e. occurring on the master list before) $\max\{i,j,k\}$ have finished their positive actions, any number x put into U by an N^3 requirement for which it is a follower must first enter one of G_{h_i}, G_{h_j} or G_{h_k} since it enters every G_{h_m} for $m \leq \max\{i,j,k\}$ other than the two being preserved for the computation realizing the N^3 requirement. Once it enters one G_{h_n} for $n \in \{i,j,k\}$, we have a sequence of markers appointed targeted for V by construction. As required by the reduction procedure, x enters U only if all of these markers enter V. Similarly, if x is a follower of some N^4 requirement of priority less than $\max\{i,j,k\}$, it must enter every G_{h_m} for $m \leq \max\{i,j,k\}$ except for the two being preserved for the computation needed by N^4 (i.e. $2n$ and $2n+1$ if the requirement we are considering is N^4_n). As another example consider O^2_j and an x which is a follower of some diagonalization requirement of priority less than $2j+1$. We argue by cases according to the type of requirement for which x is a follower. If x follows a requirement of type D, Z or N the argument is trivial since if x enters any set it enters its target for the requirement and at the same stage enters T. For the other requirements x may enter sets not mentioned in the requirement it follows but for requirements of type N^1 and N^3 once it enters any set, it enters T and a sequence of followers is set up as required. For N^2 and N^4 type requirements each time x enters a G_{h_n} we appoint a marker targeted for T and following the markers until either x enters the set associated with the N^2 or N^4 requirement completely determines which sets x enters. The other cases are similar.

Thus no further essential changes to the basic constructions in [9] for Theorem 5.1 are necessary. (As before the P restraint associated with any diagonalization requirement witness (targeted for G_j) is the restraint needed to maintain suitability for the witness to enter all of the sets G_i it might yet have to enter as the above procedure is carried out. The prompt permitting works for the M requirements as in Theorem 3.8 to get $P, Q \leq_T C$. Otherwise, we see that every number going into any other set is associated with some particular diagonalization requirements as a follower or marker. We can recursively tell which requirement any given number is associated with by the usual conventions of assigning witnesses and markers for the action of each requirement form some fixed infinite recursive set. Once the requirement is identified all the action for it putting numbers into any set is controlled by permitting the follower by members of some fixed finite set of U_i's and C and so can be entirely determined recursively in C. Other aspects of the construction and verification are either like those for Theorem 3.8 or standard such as multiple permitting of a single number x by a finite sequence of nonrecursive sets U_i.

Finally, we conclude with a word about the verifications from the viewpoint of the amount of induction needed. The global structure of the verification is like that of Theorem 3.8. The diagonalization requirements are finitary and the others are like those for the minimal pair argument. Thus the global inductions are at worst $I\Sigma_3$. The arguments for the inductive steps for each requirement are simpler. One new item is the use of guessing to approximate some computation from a low oracle using the recursion theorem as well. Another, mentioned above, is the requirements needing successive permissions form a sequence of finite nonrecursive sets to succeed. Neither of these present any difficulties. Thus, once again $I\Sigma_3$ would be sufficient to verify that the constructions succeed.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NATIONAL UNIVERSITY OF SINGAPORE, LOWER KENT RIDGE ROAD, SINGAPORE 119260

E-mail: scicct@leonis.nus.edu.sg

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA NY 14853, U.S.A.

E-mail: shore@math.cornell.edu

URL: http://www.math.cornell.edu/~shore/

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NATIONAL UNIVERSITY OF SINGAPORE, LOWER KENT RIDGE ROAD, SINGAPORE 119260

E-mail: matyangy@leonis.nus.edu.sg