Minimal degrees which are Σ_2^0 but not Δ_2^0

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Abstract

We give a short proof of the existence of minimal Turing degrees which are Σ_2^0 but not Δ_2^0.

Manaster [1971] proved that there are minimal Turing degrees which are Δ_{n+2} but not Σ_{n+1} for every n. A natural question then was whether there are ones which are Σ_{n+2} but not Δ_{n+2}. For $n > 0$ we noticed long ago (as did Carl Jockusch) that there is a simple solution using 1-trees from Lachlan [1971]. Take such a function $f : \omega \to 3$ recursive in $0''$ with $D = \{n| \ f(n) = 2\}$ infinite such that every function gotten by replacing each of the values 2 by either 0 or 1 (independently) gives a function of minimal degree. If D is listed in increasing order as d_i and $C \in \Sigma_{n+2} - \Delta_{n+2}$ then the function g which equals f on $\bar{D}$ and $C(i)$ on d_i is of the desired degree when $n > 2$. The problem for $n = 2$ remained open. We present here a similarly simple solution for $n = 2$. After we found this proof and inquired after the status of the problem, S. B. Cooper informed us that he and Yue Yang had already recently found a direct (recursive in $0'$) construction of such a degree. Our proof should be noticeably simpler since it basically consists pointing out how to combine two known types of tree constructions from different settings.

There are two common notions of trees on $2^{<\omega}$, the set of binary sequences of finite length ordered by inclusion. One (common in, for example, the study or Π_1^0 classes as in Jockusch and Soare [1972] and [1972a]), defines a *tree* T as a subset of $2^{<\omega}$ closed downward (i.e. under initial segments). The other (common in minimal degree constructions as in Lerman [1987]) defines a "tree" F as a function from $2^{<\omega}$ to $2^{<\omega}$ that preserves both order ($\subseteq$) and nonorder ($\nsubseteq$). We will take the former definition as our basic one here. Typically the objects of interest are the paths (infinite by definition) on the trees: $[T] = \{f : \omega \to 2 | \forall x (f \restriction x \in T)\}$ and $[F] = \{f : \omega \to 2 | (\exists g : \omega \to 2)(f = \cup\{n \in \omega | F(g \restriction n)\})\}$. We will augment the latter definition of "tree" by the requirement that

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the function preserve lexicographic order as well. This is innocuous in terms of the paths in the tree. We refer to such a function as an *f-tree*. If $\mathbf{x}$ is a Turing degree, we say an (f-)tree is an $\mathbf{x}$-(f-)tree if it is computable in $\mathbf{x}$ as a (function) set.

We note that for each $\mathbf{x}$-f-tree F there is an $\mathbf{x}$-tree T_F (indeed one uniformly recursive in F) with the same paths: $T_F = \{\sigma | \exists\tau(|\tau| = |\sigma| \,\&\, \sigma \subseteq F(\tau)\}$. (The relation in the other direction is more complicated since, for example, $[F]$ is always a nonempty perfect set while T may have terminal nodes and so isolated paths.) We also remark that the fact (Jockusch and Soare [1972a]) that the leftmost path in a recursive tree T is of r.e. degree relativizes in a strong way. If f is the leftmost path in an $\mathbf{x}$-tree T then f (and not just $\mathbf{f} \oplus \mathbf{x}$) is of r.e. degree in $\mathbf{x}$. The proof is essentially the same as in the unrelativized case as long as one realizes that the $\sigma \in 2^{<\omega}$ not in T and lexicographically to the left of f can be enumerated recursively in $\mathbf{x}$ along with the ones which are nodes in T: We enumerate σ once we verify recursively in $\mathbf{x}$ that we have found a level n of the full binary tree $2^{<\omega}$ at which no node above or to the left of σ has a successor on T. As f is clearly of the same degree as the set of all $\sigma \in 2^{<\omega}$ which are lexicographically to its left, the degree of f is r.e. in $\mathbf{x}$.

Consider now an $\mathbf{x}$-tree S and an $\mathbf{x}$-f-tree F. We define the *S-subtree* of F as the tree $S(F) = \{\sigma \in 2^{<\omega} | (\exists\tau)(|\tau| \leq |\sigma| \,\&\, \sigma \subseteq F(\tau) \,\&\, \tau \in S)\}$. For example, if S is $2^{<\omega}$ then $S(F) = T_F$. Note that $S(F)$ is also an $\mathbf{x}$-tree and $[S(F)] \subseteq [F]$ (hence the terminology subtree). Every path $g \in [S]$ corresponds to one $\cup F(g \restriction n)$ in $[S(F)]$ and vice versa. Moreover, if $f \in [S(F)]$ then the $g \in 2^{<\omega}$ such that $f = \cup F(g \restriction n)$ is a path in S which is recursive in $\mathbf{f} \vee \mathbf{x}$. It is obviously a path in S by the definition of $S(F)$. To compute $g(n)$ assume we have $g \restriction n = \tau$ and $F(\tau) = f \restriction m$. Exactly one of $F(\tau\hat{\ }0)$, $F(\tau\hat{\ }1)$ is an initial segment of f and so provides the next value of g.

To prove our main result we now need two trees. We first let S be a $0'$-tree with infinite paths but none recursive in $0'$. The existence of a tree S as required is well known. For example, the tree S of $f \in 2^{<\omega}$ which are DNR$^{0'}$, i.e. diagonally nonrecursive relative to $0'$: $S = \{\sigma | (\forall e \leq |\sigma|)(\neg(\varphi^{0'}_{e,|\sigma|}(e) \downarrow = \sigma(e))\}$. Next we need a $\mathbf{0}'$-f-tree F all of whose paths are of minimal degree. The existence of such an f-tree is perhaps not well known but one is easily defined by using the Sacks construction of a minimal degree below $0'$. The construction appears in Sasso [1970]. He asserts only that every path in his tree is either of minimal degree or recursive. An application of Posner's trick as in Exercise 5.8 of Soare [1987] would show that none of them are actually recursive. Alternatively, his claim for F suffices for our application. These two trees now give us the desired minimal degree.

Theorem 1 *There is a minimal degree* $\mathbf{f}$ *which is* Σ^0_2 *but not* Δ^0_2.

Proof. The (lexicographically) leftmost path f in $S(F)$ is a function of minimal degree (or possibly recursive if relying only on Sasso's stated theorem) since it is a path in F. As it is leftmost in the $\mathbf{0}'$-tree $S(F)$, it is of degree r.e. in $0'$, i.e. of Σ^0_2 degree, by our remarks above. Finally, the g such that $f = \cup F(g \restriction n)$ is a path in S and so not recursive in $0'$. It is however recursive in $f \vee 0'$ and so f is not recursive in $0'$, i.e. f is not Δ^0_2 (and so of minimal degree). $\square$

We have also used the idea of combining trees in this way in the setting of ω-branching trees to show that (with a quite liberal definition) there is no hyperarithmetic ω-branching f-tree of minimal degrees and to solve some problems of Jockusch [1975] and Friedman [1975], [1976] about the Turing degrees of paths in $\mathcal{O}$. These results will appear in Goncharov, Knight, Harizanov and Shore [2004].

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