BEYOND THE ARITHMETIC

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2005

BEYOND THE ARITHMETIC

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Cornell University 2005

Various results in different areas of Computability Theory are proved.

First we work with the Turing degree structure, proving some embeddablity and decidability results. To cite a few: we show that every countable upper semilattice containing a jump operation can be embedded into the Turing degrees, of course, preserving jump and join; we show that every finite partial ordering labeled with the classes in the generalized high/low hierarchy can be embedded into the Turing degrees; we show that every generalized high degree has the complementation property; and we show that if a Turing degree $\mathbf{a}$ is either 1-generic and Δ^0_1, 2-generic and arithmetic, n-REA, or arithmetically generic, then the theory of the partial ordering of the Turing degrees below $\mathbf{a}$ is recursively isomorphic to true first order arithmetic.

Second, we work with equimorphism types of linear orderings from the viewpoints of Computable Mathematics and Reverse Mathematics. (Two linear orderings are equimorphic if they can be embedded in each other.) Spector proved in 1955 that every hyperarithmetic ordinal is isomorphic to a recursive one. We extend his result and prove that every hyperarithmetic linear ordering is equimorphic to a recursive one. From the viewpoint of Reverse Mathematics, we look at the strength of Fraïssé's conjecture. From our results, we deduce that Fraïssé's conjecture is sufficient and necessary to develop a reasonable theory of equimor-

phism types of linear orderings. Other topics we include in this thesis are the following: we look at structures for which Arithmetic Transfinite Recursion is the natural system to study them; we study theories of hyperarithmetic analysis and present a new natural example; we look at the complexity of the elementary equivalence problem for Boolean algebras; and we prove that there is a minimal pair of Kolmogorov-degrees.

BIOGRAPHICAL SKETCH

I was born and raised in Montevideo, Uruguay. I started to take mathematics seriously when I join the math Olympiads team of Uruguay, in 1994. I spent three years with the math Olympiads solving fun and elementary math problems. With them I got to travel to competitions and meet many people from all around Latin America. In 1997, I entered the undergraduate program in Mathematics at the *Universidad de la República* in Montevideo, Uruguay. There I had Paula Severi and Walter Ferrer as my advisers. They are the ones who introduced me to the subject of logic. In 2000, I got my degree of *Licenciado en Matemáticas* and came to Cornell.

ACKNOWLEDGEMENTS

I am very grateful to my thesis adviser Richard A. Shore for all his help during my time here at Cornell. I could not have had a better adviser. He gave me many interesting problems for me to look at, and he would listen to all my ideas and read all my drafts carefully and comment on them. He advised me not only about mathematics, but also about any other aspect of the academic life.

I also want to thank my co-authors Noam Greenberg, Barbara F. Csima and Bjørn Kjos-Hanssen for letting me include the work we did together in this thesis. It has been fun working with them.

I will not be able to list all the people that supported me in some way or another during the last five years. I will just list a few of them. On the one hand are the people that I met here in Ithaca, without whom I would not have enjoyed my time here as much. Among these people are all friends, my closer ones being Alison, Cristina, Everilis, Fabio, Fernando, Gabor, Johnny, Jorge, Mercedes, Pedro, Rafael and Yannet. I also want to mention Andrés, Carmen, Marta, Marcelo and Ricardo, who helped me out in my first few days in Ithaca. On the other hand are the people at home that, even though where many thousands of kilometers away, where always with me. These include my friends Alejandro and Aldo, and of course my family, my mother, my father and my sister that were always there to support me.

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Chapter 1
General Introduction

The notion of what it means for a set, say of natural numbers, to be *computable* was known from even before computers existed; this is the principal notion in Computability Theory, also known as Recursion Theory. Perhaps a more important notion in Computability Theory is the one of "computable from". A set A is said to be *computable from* (or *recursive in*) a set B, or *Turing reducible to* B, if there is a computable procedure that can tell whether an element is in A or not using B as an *oracle*, that is, we let the procedure use the information of which elements are in B. We can use this notion to measure the amount of information content that a set has; A has *more information content* than B if B can be computed from A. In this case we also say that A is more *complex* than B. We say that A and B are *Turing equivalent* if they can be computed from each other. Other notions of complexity are also relevant to Computability Theory. For example, a set is said to be *arithmetic* if it can be defined by a formula of first order arithmetic. The simplest set which can uniformly compute all the arithmetic set is $0^{(\omega)}$. It is Turing equivalent to the set of all true sentences of first order arithmetic. Another important class of sets, greater than the class of arithmetic set and which contains $0^{(\omega)}$, is the class of hyperarithmetic sets. A set is *hyperarithmetic* if it can be defined by a computable infinitary formula of arithmetic, or equivalently, if it is Δ^1_1.

This thesis contains all the research I have done during my time in Cornell. The main theme is computability theory. Sets which are beyond the arithmetic are considered in almost every chapter, as for example the ωth Turing Jump of $\emptyset$, or other sufficiently complex hyperarithmetic sets.

The thesis is divided in four parts. The first three parts contain my work in the areas of Turing Degree Theory, Reverse Mathematics, and Computable Mathematics. The fourth part contains what did not fit into any of the previous parts. Chapters 2, 3, 6, 7, 9, and 12 contain research that I have done myself. The other chapters (except the introductory one, of course) are joint work: Richard A. Shore is a co-author of Chapters 4 and 10; Noam Greenberg of 4, 5 and 8; Barbara F. Csima of 10, and 11; and Bjørn Kjos-Hanssen of 11. Since most of the chapters have been submitted for publication, each one is written as an individual paper. Hence, each chapter has its own introduction. The idea of this global introduction is to explain what the general ideas behind each of the three areas mentioned above are, and to summarize all the results included in this thesis.

1.1 Turing Degree Theory

The Turing degree structure is a very natural object first studied by Kleene and Post in [KP54]. It is defined as follows. Consider $\mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$

(the set of natural numbers). Given $A, B \in \mathcal{P}(\mathbb{N})$, let $A \leqslant_T B$ if A is computable from B. The relation $\leqslant_T$ is a quasi-ordering on $\mathcal{P}(\mathbb{N})$. As usual, this quasiordering induces an equivalence relation ($A \equiv_T B \Leftrightarrow A \leqslant_T B \;\&\; B \leqslant_T A$) and a partial ordering on the equivalence classes. The equivalence classes are called *Turing degrees*. We use $(\mathbf{D}, \leqslant_T)$ to denote this partial ordering. One of the main goals of Computability Theory is to understand the structure of $(\mathbf{D}, \leqslant_T)$.

We note that we chose to work with subsets of $\mathbb{N}$ because every finite object can be coded by a single number (using, for instance, the binary representation of the number). For example, strings, graphs, trees, simplicial complexes, group presentations, etc., if they are finite, they can be coded effectively by a natural number. Any other set where we can effectively code finite structures will work too.

The first observation about the Turing degrees is that there is a least one, that we denote by $\mathbf{0}$. It is the Turing degree of the computable sets. The Turing degrees form an *upper semilattice*; that is, every pair of elements has a least upper bound. We denote the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \vee \mathbf{b}$. Intuitively, $\mathbf{a} \vee \mathbf{b}$ contains all the information that $\mathbf{a}$ and $\mathbf{b}$ have. In the Turing degrees there is another naturally defined operation called the *Turing jump* (or just *jump*). The jump of a degree $\mathbf{a}$, denoted $\mathbf{a}'$, is given by the degree of the *Halting Problem* relativized to some set in $\mathbf{a}$. (Given $A \subseteq \mathcal{P}(\mathbb{N})$, the *Halting Problem relative to* A, denoted by A', is the set of codes for computer programs, that, when run with oracle A, halt. Note that a computer program is a finite sequence of characters and hence can be encoded by a natural number.) It can be shown that the jump operation is strictly increasing (i.e., $\forall \mathbf{a}(\mathbf{a} <_T \mathbf{a}')$) and monotonic ($\forall \mathbf{a}, \mathbf{b}(\mathbf{a} \leqslant_T \mathbf{b} \Rightarrow \mathbf{a}' \leqslant_T \mathbf{b}')$). A *jump upper semilattice* is an upper semilattice together with a strictly increasing, monotonic function. (*Jump partial orderings* are defined analogously.) So, we have that $(\mathbf{D}, \leqslant_T, \vee, ')$ is a jump upper semilattice.

The study of which structures can be embedded in $(\mathbf{D}, \leqslant_T)$ is part of an ongoing program to understand the shape of the Turing degrees. It follows from the results of Kleene and Post [KP54] that every countable upper semilattice can be embedded into $(\mathbf{D}, \leqslant_T)$. Since then, various other embeddablity results have been proved. Sacks proved in [Sac61] that every partial ordering of size at most $\aleph_1$ with the countable predecessor property can be embedded into $(\mathbf{D}, \leqslant_T)$. (The countable predecessor property says that every element has at most countably many elements below it. Note that since a set can compute at most countably many different sets, $(\mathbf{D}, \leqslant_T)$ has the countable predecessor property.) Abraham and Shore extended this result to upper semilattices of size $\aleph_1$ and of course the countable predecessor property in [AS86]. Moreover, they prove that the embedding can be constructed to be onto an initial segment of $(\mathbf{D}, \leqslant_T)$. Hinman and Slaman proved, in [HS91], that every countable jump partial ordering is embeddable in $(\mathbf{D}, \leqslant_T, ')$. In Chapter 2 we prove the following:

Theorem 2.4.17: *Every countable jump upper semilattice can be embedded into the Turing Degrees (of course, preserving jump and join).*

The proof of this theorem uses a variety of tools from Computability Theory and some ideas from [HS91]. First, via a forcing construction, we prove that every countable jump upper semilattice satisfying certain condition can be embed in $(\mathbf{D}, \leqslant_T, \vee, ')$. This is enough to deduce that the quantifier free formulas that are true in $(\mathbf{D}, \leqslant_T, \vee, ')$ are the exactly the ones that do not contradict the definition of jump upper semilattice. It then follows that the existential theory of $(\mathbf{D}, \leqslant_T, \vee, ')$ is decidable. Then, using hyperarithmetic theory, Harrison linear orderings, Fraïssé limits, and well-quasiorderings, we show that every countable jump upper semilattice can be embedded into one having the condition mentioned above, and that hence can be embedded into the Turing degrees.

The rest of Chapter 2 is dedicated to analyze extensions of this result. We prove that Theorem 2.4.17 is not true about jump upper semilattices with 0, by proving that not every quantifier free 1-type of jump upper semilattice with 0 is realized in $(\mathbf{D}, \leqslant_T, \vee, ', \mathbf{0})$. (Note that embedding a jump upper semilattice with 0 and one other generator can be expressed in terms of realizing quantifier free 1-types.) On the other hand, we show that every quantifier free 1-type of jump partial ordering with 0 is realized in $(\mathbf{D}, \leqslant_T, ', \mathbf{0})$. Moreover, we show that if every quantifier free type, $p(x_1, ..., x_n)$, of jump partial ordering with 0, which contains the formula $x_1 \leqslant 0^{(m)}$ & ... & $x_n \leqslant 0^{(m)}$ for some m, is realized in $(\mathbf{D}, \leqslant_T, ', \mathbf{0})$, then every every quantifier free type of jump partial ordering with 0 is realized in $(\mathbf{D}, \leqslant_T, ', \mathbf{0})$.

We also study the question of whether every jump upper semilattice with the countable predecessor property and size $\kappa \leqslant 2^{\aleph_0}$ is embeddable in $(\mathbf{D}, \leqslant_T, \vee, ')$. We show that for $\kappa = 2^{\aleph_0}$ the answer is no. For cardinals κ between $\aleph_0$ and $2^{\aleph_0}$ we show that, if MA(κ) holds, then the answer is yes. (MA(κ), Martin's axiom for κ, is defined in 2.6.12.) The reason being essentially that Martin's axiom allows us to do the forcing construction we needed to get the embedding in theorem 2.4.17. These last two results imply that whether every jump partial ordering (or jump upper semilattice) of size $\aleph_1$ and with the countable predecessor property is embeddable in $(\mathbf{D}, \leqslant_T, \vee, ')$ or not is independent of ZFC.

In [JP78], Jockusch and Posner defined the *generalized high/low hierarchy* with the intention of classifying the Turing degrees depending on how close a degree is to being computable, and on how close it is to computing the Halting Problem. This classification extended the already known classification of the Δ^0_2 degrees (i.e., the degrees $\leqslant_T \mathbf{0}'$) via the high/low hierarchy. For $n \geqslant 1$ we say that a degree $\mathbf{x}$ is *generalized low$_n$*, or GL$_n$, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee 0')^{(n-1)}$. We say that a degree $\mathbf{x}$ is a *generalized high$_n$ degree*, or GH$_n$, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee 0')^{(n)}$, and it is *generalized intermediate*, or GI, if $\forall n\big((\mathbf{x} \vee 0')^{(n-1)} <_T \mathbf{x}^{(n)} <_T (\mathbf{x} \vee 0')^{(n)}\big)$. (Note that $(\mathbf{x} \vee 0')^{(n-1)}$ is the lowest and $(\mathbf{x} \vee 0')^{(n)}$ is the highest $\mathbf{x}^{(n)}$ could be.) From these classes, taking differences in the obvious way, we can define the proper classes $\mathrm{GL}^*_1, \mathrm{GL}^*_2, ..., \mathrm{GI}^*, ..., \mathrm{GH}^*_2, \mathrm{GH}^*_1$ which partition the Turing degrees. (For instance $\mathrm{GL}^*_2 = \mathrm{GL}_2 \setminus \mathrm{GL}_1$, $\mathrm{GH}^*_2 = \mathrm{GH}_2 \setminus \mathrm{GH}_1$, etc..) When one first sees the definition of this classes, one would think that generalized high degrees should be above generalized low degrees, or at least not below. This intuition is correct for the high/low hierarchy, but we show that it is as wrong at it could be in the generalized case. In Chapter 3, we prove the following.

Theorem 3.2.3: *Every finite partial ordering labeled with elements of the set* $\{\mathrm{GL}_1^*, \mathrm{GL}_2^*, ..., \mathrm{GI}^*, ..., \mathrm{GH}_2^*, \mathrm{GH}_1^*\}$ *can be embedded in the Turing degrees preserving the labels.*

Note that no condition at all is imposed on the labels. This result helps to understand how the degrees in the various classes of the generalized high/low hierarchy are located inside the structure of the Turing degrees. It follows from it that the existential theory of the Turing degrees, in the language with Turing reduction, 0, and unary relations for the classes in the generalized high/low hierarchy, is decidable, answering an open question posed by Lerman [Ler85].

We should mention that these result follow from the decidability of $(\mathbf{D}, \leqslant_T, \vee, ', \mathbf{0})$. Lerman has a proof of this result, that he has not finish writing yet. Our method is very different than Lerman's and definitely simpler. Our construction uses two $\mathbf{0}''$-priority constructions, one in top of the other. Lerman's uses the very interesting, but rather complicated, framework of Iterated Trees of Strategies, which is general method for $\mathbf{0}^{(n)}$-priority constructions for arbitrary n. (See [LL96] for more information about this method.)

By definition, generalized high degrees are the ones that have jumps which are as high as they could be. This is what makes them to be, in some sense, close to computing $\mathbf{0}'$. One then wants to know what other properties of $\mathbf{0}'$, are shared by the generalized high degrees too. For instance, it is known that: every countable partial order can be embedded below $\mathbf{0}'$ (Kleene and Post [KP54]); there are minimal degrees below $\mathbf{0}'$ (Sacks [Sac61]); $\mathbf{0}'$ cups to every degree above it (Friedberg [Fri57]); every degree below $\mathbf{0}'$ joins up to $\mathbf{0}'$ (Robinson [Rob72], Posner and Robinson [PR81]). This properties also hold for generalized high degrees (Jockusch and Posner [JP78]; (Cooper [Coo73]) for $\mathbf{H}_1$ and Jockusch [Joc77] for $\mathbf{GH}_1$; Jockusch and Posner [JP78]; Posner [Pos77].) The property that we consider in Chapter 4 is complementation. A Turing degree $\mathbf{a}$ has the *complementation property* if the partial ordering of degrees below it is *complemented*, i.e., if for every $\mathbf{b} <_T \mathbf{a}$, there exists $\mathbf{c} <_T \mathbf{a}$ such that $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$ and $\mathbf{b} \wedge \mathbf{c}$ (the greatest lower bound of $\mathbf{b}$ and $\mathbf{c}$) exists and is equal to $\mathbf{0}$. Robinson, Epstein and Posner [Rob72, Eps75, Pos77, Pos81] proved that $\mathbf{0}'$ has the complementation property. Posner asked in [Pos81] if every generalized high degree had the complementation property. Noam Greenberg, Richard A. Shore and I answered this question affirmatively.

Theorem 4.1.1: *Every degree* $\mathbf{d} \in \mathbf{GH}_1$ *has the complementation property.*

As usual when dealing with generalized high degrees, rates of growth and domination properties play prominent roles in our constructions.

Another way of analyzing the structure of $(\mathbf{D}, \leqslant_T)$ is by studying the complexity of its theory and the theory of its initial segments. For example, the theory of $(\mathbf{D}, \leqslant_T)$, not only it is known to be undecidable (Lachlan [Lac68]), but also it is know that it is recursively isomorphic to true second order arithmetic (Simpson [Sim77]), or in other words, 1-1-equivalent to $0^{(\omega)}$. For the case of the local theories, it was proved by Shore [Sho81] that the theory of $(\mathbf{D}(\leqslant \mathbf{0}'), \leqslant_T)$ is recursively isomorphic to true first order arithmetic, where $\mathbf{D}(\leqslant \mathbf{0}')$ is the set of degrees which are $\leqslant \mathbf{0}'$. Moreover, he proved that if $\mathbf{a}$ is high or if it bounds an r.e. degree, then the theory of $(\mathbf{D}(\leqslant \mathbf{a}), \leqslant_T)$ interprets true first order arithmetic. Together with Noam Greenberg, we extend this result even further in Chapter 5. We show that, if $\mathbf{g}$ is a degree such that the 1-generic degrees are downwards dense below it, then the theory of $(\mathbf{D}(\leqslant \mathbf{g}), \leqslant_T)$ interprets true first order arithmetic. (See Subsection 5.1.2 for a definition of 1-generic.) In particular, this holds if $\mathbf{g}$ is 2-generic or 1-generic below $\mathbf{0}'$. It follows that for almost every (in the sense of category) degree $\mathbf{g}$, $(\mathbf{D}(\leqslant \mathbf{g}), \leqslant_T)$ interprets true first order arithmetic. It also follows that if $\mathbf{a}$ is n-REA, or if it is arithmetically generic, then the theory of $(\mathbf{D}(\leqslant \mathbf{g}), \leqslant_T)$ is recursively isomorphic to true first order arithmetic.

To prove our result we show that 1-genericity is sufficient to find the parameters needed to code a set of degrees using Slaman and Woodin's method [SW86]. We also prove in Chapter 5 that any recursive lattice can be embedded below a 1-generic degree preserving top and bottom.

1.2 Reverse Mathematics

The questions of which axioms are necessary to do mathematics is of great importance in the Foundations of Mathematics and is the main question behind Friedman and Simpson's program of Reverse Mathematics. To analyze this question formally it is necessary to fix a logical system. Reverse Mathematics deals with the system of second-order arithmetic. Second-order arithmetic, though much weaker than set theory, is rich enough to be able to express an important fragment of classical mathematics. This fragment includes number theory, calculus, countable algebra, real and complex analysis, differential equations, separable metric spaces and countable combinatorics among others. Almost all of mathematics that can be modeled with, or coded by, countable objects can be done in second-order arithmetic.

The idea of Reverse Mathematics is as follows. We start by fixing a basic system of axioms. The most commonly used basic system is called RCA_0, which is closely related to Computable Mathematics. (RCA_0 contains the axioms of semirings, Σ^0_1-induction and the axiom scheme of Δ^0_1-comprehension which essentially says that a set exists if it can be computed from sets that we already know exist.) Now, given a theorem of "ordinary" mathematics, the question we ask is what axioms do we need to add to the basic system to prove this theorem. It is often the case in

Reverse Mathematics that we can prove that a certain set of axioms is needed to prove a theorem by proving that the axioms follow from the theorem within some basic system. Because of this idea, this program is called Reverse Mathematics. Many different systems of axioms have been defined and studied. But a very interesting fact is that most of the theorems, whose proof-theoretic strength has been analyzed, have been proved equivalent over RCA_0 to one of five systems: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0 ordered from weakest to strongest. (See Section 6.1 for more information on Reverse Mathematics.)

I started working in Reverse Mathematics trying to find the proof-theoretic strength of Jullien's Theorem which is a classification of the countable extendible linear orderings [Jul69]. This question was posed by Downey and Remmel in [DR00, Question 4.1]. A linear ordering is *extendible* if every countable partial ordering in which it is not embeddable it has a linearization in which it is not embeddable either. The proof theoretic strength of the extendibility of $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$, was analyzed in [DHLS03]. To know the proof-theoretic strength of Jullien's Theorem is also interesting because its proof seems to require more complex axioms than most of the theorems in classical mathematics. The answer that we found is the following.

Theorem 6.5.15: *Jullien's Theorem is equivalent to Fraïssé's conjecture over* $\mathit{RCA}_0 + \Sigma^1_1$-$\mathit{IND}$.

Fraïssé's conjecture (also known as Laver's Theorem [Lav71]) is the statement that says that the countable linear orderings form a well quasiordering with respect to embeddablity. (A *well-quasi-ordering* is a quasi-ordering without infinite descending sequences or infinite antichains.) Fraïssé's conjecture has interested logicians for many years also because of the difficulty of its proof in terms of reverse mathematics.

While studying the proof theoretic strength of Jullien's theorem, we prove the extendibility of many linear orderings, including $\mathbb{N}^2$ and $\mathbb{Q}$, using just ATR_0+Σ^1_1-IND. Moreover, for all these linear orderings, $\mathcal{L}$, we prove that any partial ordering, $\mathcal{P}$, in which $\mathcal{L}$ is not embeddable has a linearization, hyperarithmetic in $\mathcal{P} \oplus \mathcal{L}$, in which $\mathcal{L}$ is not embeddable either.

We also prove that Fraissé's conjecture is equivalent, over RCA_0, to two other interesting statements. One that says that every scattered linear ordering is equimorphic to a finite sum of indecomposable linear orderings. (A linear ordering is *scattered* if $\mathbb{Q}$ cannot be embedded into it. A linear ordering $\mathcal{L}$ is indecomposable if whenever $\mathcal{L}$ embeds into a sum of linear ordering $\mathcal{A} + \mathcal{B}$, we have that $\mathcal{L}$ embeds either into $\mathcal{A}$ or into $\mathcal{B}$. Two linear orderings are *equimorphic* if they can be embedded in each other.) The other statement says that the class of well founded labeled trees, with labels from $\{+, -\}$, with very a natural order relation, is well quasiordered. This trees are called *signed trees*. (See Section 6.2.1.) Signed trees are very useful in this context because they are easier to deal with than linear orderings, from an effective point of view. The notion of signed trees and the operation lin(·) (which assigns linear orderings to sign trees) are essentially new; although they have a similar flavor with the trees $T(\Psi)$ used by Laver [Lav71, pag.104].

In the last section of Chapter 6 we look at a partition theorem about linear orderings that we believe is also equivalent to FRA. This theorem is due to Laver [Lav73] and, when restricted to countable linear orderings, says the following. For every countable linear ordering $\mathcal{L}$ there exists a natural number $n_{\mathcal{L}}$ such that for every coloring of $\mathcal{L}$ with finitely many colors, there exists a subset of $\mathcal{L}$ which is equimorphic to $\mathcal{L}$ and is colored with at most $n_{\mathcal{L}}$ many colors. We call this statement LAV. We show that RCA_0+LAV implies FRA. We do not know whether the reverse implication holds or not. (See the end of Section 6.7 for a short discussion about this reversal.)

The main question that is left open is what is the exact proof-theoretic strength of Fraïssé's conjecture (FRA). It is known that Laver's original proof of FRA can be carried out in Π^1_2-CA_0, and that since FRA is a true Π^1_2 statement, it cannot imply Π^1_1-CA_0. Shore [Sho93] proved that the fact that the class of well orderings is well quasiordered under embeddablity implies ATR_0 over RCA_0, getting as a corollary that FRA implies ATR_0. But we still do not know whether FRA could be proved using just ATR_0 (not even Π^1_1-CA_0), as has been conjectured by Peter Clote [Clo90], Stephen Simpson [Sim99, Remark X.3.31] and Alberto Marcone [Mar].

Chapter 7 is about hyperarithmetic second order arithmetic, also know as hyperarithmetic analysis.

Definition 1.2.1. A *statement of hyperarithmetic analysis* is a sentence of second order arithmetic S such that for every $Y \subseteq \omega$, the minimum ω-model of RCA_0+S containing Y is HYP(Y), the ω-model consisting of the sets hyperarithmetic in Y.

We provide an example of a mathematical theorem which is a statement of hyperarithmetic analysis. This statement, that we call INDEC, is due to Jullien [Jul69]. It says that every scattered indecomposable linear ordering is indecomposable either to the right or to the left. (A linear ordering $\mathcal{L}$ is *indecomposable to the left (right)* if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have that $\mathcal{L}$ embeds in $\mathcal{A}$ (in $\mathcal{B}$).) Theories of hyperarithmetic analysis have been studied in the seventies (see [Fri75], [Van77] and [Ste78]), but (to the author's knowledge) this one is the first natural (already published and purely mathematical) example of a statement of hyperarithmetic analysis. We also prove that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and implies ACA_0, but of course, neither ACA_0, nor $\mathsf{ACA}_0{}^+$ imply it. INDEC is also unusual because is not equivalent to any of the five systems mentioned at the beginning of this section.

Another interesting fact about INDEC is that is incomparable over ACA_0 to other natural statements of mathematics. This is probably the first example of previously published purely mathematical statements which are incomparable and are between ACA_0 and ATR_0. The statements we have in mind are the following: The existence of elementary equivalence invariants for Boolean Algebras, and

Ramsey Theorem. The former statement was studied by Shore [Sho04]. The latter statement, Ramsey's Theorem, has been extensively studied in the context of reverse mathematics (see [Sim99, III.7], [CJS01], or [Mil04, Chapter 7]).

We introduce five other statements of hyperarithmetic analysis and study the relations among them. Four of them are related to finitely-terminating games. The fifth one, related to iterations of the Turing jump, is strictly weaker than all the other statements that we study in this chapter, as we prove using Steel's method of forcing with tagged trees.

ATR_0 is the natural subsystem of second order arithmetic in which one can develop a decent theory of ordinals ([Sim99]). Together with Noam Greenberg, in Chapter 8, we investigate classes of structures which are in a sense the "well-founded part" of a larger, simpler class, for example, superatomic Boolean algebras (within the class of all Boolean algebras). The other classes we study are: well-founded trees, reduced Abelian p-groups, and countable, compact topological spaces. The structures in all of these classes code ordinals as invariants. Using computable reductions between these classes, we show that Arithmetic Transfinite Recursion is the natural system for working with them: natural statements (such as comparability of structures in the class) are equivalent to ATR_0. The reductions themselves are also objects of interest.

1.3 Computable Mathematics

Computable Mathematics deals with the computable aspects of mathematical theorems and objects.

The question "given a mathematical structure, which is the simplest way to represent it?" is of great importance in this area. Part of our work in Computable Mathematics is related to this question. In [Spe55], Clifford Spector proved that every hyperarithmetic well ordering is isomorphic to a computable one. In less technical terms this says that if an ordinal has a representation of a certain complexity (hyperarithmetic, which is quite high) then it has a very simple (computable) representation. We prove a generalization of this result to all countable linear orderings:

Theorem 9.1.2: *Every hyperarithmetic linear ordering is equimorphic with a recursive one.*

Spector's theorem is a special case of Theorem 9.1.2 because if a linear ordering is equimorphic to an ordinal, it is actually isomorphic to it. The proof of Theorem 9.1.2 requires a deep analysis of the structure of the countable linear orderings modulo equimorphisms. This analysis is done by studying the structure of signed trees defined in Section 6.2.1. It is often the case that proofs in Computable Mathematics (and also in Reverse Mathematics) give us a deeper understanding of objects from classical mathematics. This is definitely the case here.

On the way to Theorem 9.1.2 we prove that a linear ordering has Hausdorff rank less than ω_1^{CK} if and only if it is equimorphic to a recursive one. As a corollary of the proof we prove that given a recursive ordinal α, the partial ordering of equimorphism types of linear orderings of Hausdorff rank at most α ordered by embeddablity, is recursively presentable.

Using a result of Ask and Knight [AK00], it is not hard to see that Theorem 9.1.2 is true for the class of Boolean Algebras. With Noam Greenberg, we show in Section 8.7 that Theorem 9.1.2 can also be extended to the class of p-groups.

The other chapter in Computable Mathematics has to do with Boolean algebras.

Tarski [Tar49] defined a way of assigning to each boolean algebra, B, an invariant $\mathrm{inv}(B) \in \mathtt{In}$, where $\mathtt{In}$ is a set of triples from $\mathbb{N}$, such that two boolean algebras have the same invariant if and only if they are elementarily equivalent. Moreover, given the invariant of a boolean algebra, there is a computable procedure that decides its elementary theory. If we restrict our attention to dense Boolean algebras, these invariants determine the algebra up to isomorphism. In Chapter 10, with Barbara F. Csima and Richard A. Shore we analyze the complexity of the question "Does B have invariant x?". For each $x \in \mathtt{In}$ we define a complexity class Γ_x, that could be either Σ_n, Π_n, $\Sigma_n \wedge \Pi_n$, or $\Pi_{\omega+1}$ depending on x, and prove that the set of indices for computable boolean algebras with invariant x is complete for the class Γ_x. Analogs of many of these results for computably enumerable Boolean algebras were proved in [Sel90] and [Sel91]. According to [Sel03] similar methods can be used to obtain the results for computable ones. Our methods are quite different and give new results as well.

As the algebras we construct to witness hardness are all dense, we establish new analogous results for the complexity of various isomorphism problems for dense Boolean algebras.

1.4 Effective Randomness

Within the area of Effective Randomness we are interested in the notion of K-reducibility. K-reducibility is defined with the intention of measuring the relative randomness of infinite binary strings. This reducibility was defined using a function, K, that assigns to each finite binary string the length of its shortest description, in a sense that we will specify (Section 11.1.1). The idea being that if a string is random, there should not be any short way of describing it.

Together with Barbara F. Csima, in Chapter 11, we construct a minimal pair of K-degrees, answering a question of Downey and Hirschfeldt. We do this by showing the existence of an unbounded nondecreasing function f which forces K-triviality in the sense that $\gamma \in 2^\omega$ is K-trivial if and only if for all n, $K(\gamma \upharpoonright n) \leqslant K(n) + f(n) + \mathcal{O}(1)$.

Recently, Downey, J. Miller and Nies have started to study the complexity and other aspects of these kind of functions [DYM].

Another notion that we are interested in is the one of non-continuously-random reals. This class of reals was introduced by Reimann and Slaman [RS]. In March 2005, Slaman gave a talk at the Annual ASL meeting in Stanford presenting their paper. Back then, they knew that there is a real of Turing degree $\mathbf{0}'$ which is not continuously random, that there is one which is not Δ^0_2, and that every real which is not continuously random belongs to Gödel's constructible universe $\mathbf{L}$. With Bjørn Kjos-Hanssen, we show, in Section 11.2, that there are reals which are not continuously random all the way up the hypearithmetic hierarchy. Reimann and Slaman have recently proved that every real which are not continuously random is hypearithmetic.

1.5 Scattered linear orderings

The last chapter is about the structure of the equimorphism types of scattered linear orderings and not really about Computability Theory. The ideas of this section are originated by the work on Chapter 9 were we prove that every hyperarithmetic linear ordering is equimorphic to a computable one.

In this last chapter, Chapter 12, we define invariants for scattered linear orderings of arbitrary cardinality which classify them up to equimorphism. More precisely, we assign to each scattered linear ordering $\mathcal{L}$ a a finite sequence $\texttt{Inv}(\mathcal{L})$ of finite trees labeled by ordinals and signs in $\{-,+\}$. This assignment is an equimorphism invariant, that is, two scattered linear orderings $\mathcal{A}$ and $\mathcal{B}$ are equimorphic if and only if $\texttt{Inv}(\mathcal{A}) = \texttt{Inv}(\mathcal{B})$. We show that the definition of the embeddablity relation on the invariants is relatively simple, and that we can easily characterize the finite sequences of finite trees that correspond to invariants.

Also, for each ordinal α, we explicitly describe the finite set of minimal scattered equimorphism types of Hausdorff rank α. We compute the invariants of each of these minimal types.

Part I

Turing Degree structure

Chapter 2
Embedding Jump upper semilattices into the Turing Degrees.

Published in the Journal of Symbolic Logic, volume 68 (3), year 2003, pages 989-1014.

2.1 Introduction.

We deal with the following kind of structures.

Definition 2.1.1. A *partial jump upper semilattice* (pjusl) is a structure

$$\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \cup, \mathfrak{j} \rangle$$

where $\langle J, \leqslant_{\mathcal{J}} \rangle$ is a partial ordering, $\cup$ is a partial binary operation and $\mathfrak{j}$ are partial unary operation such, that for all $x, y \in J$,

- if $x \cup y$ is defined, it is the least upper bound of x and y, and
- if $\mathfrak{j}(x)$ is defined then $x <_{\mathcal{J}} \mathfrak{j}(x)$; and if $\mathfrak{j}(y)$ is also defined and $x \leqslant_{\mathcal{J}} y$, then $\mathfrak{j}(x) \leqslant_{\mathcal{J}} \mathfrak{j}(y)$.

By partial operation we mean that it does not need to be defined everywhere. A *jump upper semilattice* (jusl) is a pjusl where $\mathfrak{j}$ and $\cup$ are total operations. A *jump partial ordering* (jpo) is a pjusl where $\mathfrak{j}$ is total but $\cup$ is undefined.

Given pjusls, $\mathcal{J}_1$ and $\mathcal{J}_2$, an *embedding of* $\mathcal{J}_1$ *into* $\mathcal{J}_2$ is an injective map $f\colon J_1 \to \mathcal{J}_2$ such that for all $x, y \in J_1$:

- $x \leqslant_{\mathcal{J}_1} y$ if and only if $f(x) \leqslant_{\mathcal{J}_2} f(y)$;
- if $\mathfrak{j}(x)$ is defined, then $f(\mathfrak{j}(x)) = \mathfrak{j}(f(x))$; and
- if $x \cup y$ is defined, then $f(x \cup y) = f(x) \cup f(y)$.

Observe that, $\mathcal{D} = \langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$, the set of Turing degrees together with the Turing reduction, the join operation and the Turing Jump is a jusl.

We address the question of which pjusls can be embedded into $\mathcal{D}$. The first embeddablity result about $\mathcal{D}$ was proved by Kleene and Post in [KP54]. One of the things they proved there is that every finite upper semilattice can be embedded into $\mathcal{D}$. Various others results have been proved. Sacks proved in [Sac61] that every partial ordering of size at most $\aleph_1$ with the c.p.p. can be embedded into $\mathcal{D}$. (Recall that we say that a partial order has the c.p.p. or *countable predecessor property* if every element has at most countably many predecessors.) Abraham and Shore extended this result to upper semilattices in [AS86]. (They even embedded the upper semilattices as initial segments of $\mathcal{D}$.) Hinman and Slaman, proved in

[HS91], that every countable jpo is embeddable in $\mathcal{D}$. We prove here that every countable jusl is embeddable in the Turing degrees. We also construct a jpo of size continuum with the c.p.p. which cannot be embedded in $\mathcal{D}$. For cardinals κ between $\aleph_0$ and $2^{\aleph_0}$, we show that, if MA(κ) holds, then every jusl with the c.p.p. and size κ can be embedded in $\mathcal{D}$. (MA(κ) is defined in 2.6.12.) These two last results imply that whether every jpo (or jusl) of size $\aleph_1$ is embeddable in $\mathcal{D}$ is independent of ZFC.

These kinds of results are always related to decidability results. We know that the elementary theory of $\langle \mathbf{D}, \leqslant_T \rangle$ is undecidable, as was shown by Lachlan in [Lac68]. However, it is still of interest to know which segments of the theory of $\mathcal{D}$ are decidable. For example, form the results of Kleene and Post in [KP54], we get that the $\exists$-theory of $\langle \mathbf{D}, \leqslant_T \rangle$ is decidable. Then Jockusch and Slaman, [JS93], showed that the $\forall\exists$-theory of $\langle \mathbf{D}, \leqslant_T, \vee \rangle$ is decidable. Their result is optimal in the sense that the $\forall\exists\forall$-theory of the same structure is undecidable. This follows from the undecidability of the $\forall\exists\forall$-theory of $\langle \mathbf{D}, \leqslant_T \rangle$, proved by Schmerl (see [Ler83, Corollary VII.4.6]). Another interesting result, proved by Jockusch and Soare is that the whole elementary theory of $\langle \mathbf{D}, ' \rangle$ is decidable (see [Ler83, Exercise III.4.21]). Here, as a corollary of our main result, we get that the existential theory of $\langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$ is decidable. This result is optimal too, since the $\forall\exists$-theory was recently proved undecidable by Shore and Slaman, in [SS].

About $\langle \mathbf{D}, \leqslant_T, ' \rangle$, we know that the $\forall$-theory is decidable and that the $\forall\exists\forall$-theory is undecidable. But, we do not know much about the $\forall\exists$-theory. A sub case of this question, that remains open, is whether the existential theory of $\langle \mathbf{D}, \leqslant_T, ', 0 \rangle$ is decidable. The best approximation to this question is a result due to Lempp and Lerman [LL96]. They proved that every quantifier free formula, $\varphi(x_1, ..., x_n)$, in the language of $\langle \mathbf{D}, \leqslant_T, ', 0 \rangle$, that is consistent with the axioms of jpo with 0 (see 2.5.1 for a definition of jpo with 0) and with the formula $x_1 \leqslant_T 0'$ & ... & $x_n \leqslant_T 0'$, is realized by a n-tuple of r.e. degrees. We call a type, $p(x_1, ..., x_n)$ of jpo with 0 *archimedean* if, for some $m \in \omega$, it contains the formula $x_1 \leqslant_T 0^{(m)}$ & ... & $x_n \leqslant_T 0^{(m)}$. We prove that if every quantifier free (q.f.) archimedean type of jpo with 0 is realized in $\mathcal{D}$, then every q.f. type of jpo with 0 is realized in $\mathcal{D}$. It seems likely that the hypothesis of every q.f. archimedean type being realized in $\mathcal{D}$ can be proved using iterated trees of strategies, which is a method created by Lempp and Lerman (see, for example, [LL96]). Hinman and Slaman proved in [HS91] and [Hin99] that every q.f. archimedean 1-type of jpo with 0 is realized in $\mathcal{D}$. (Actually they proved something equivalent to this. See the proof of Corollary 2.5.9 for an explanation of the equivalence.) We extend their result and prove here that every q.f. 1-type of jpo with 0 is realized in $\mathcal{D}$. We also show that this result cannot be extended to jusl with 0. More precisely, we prove that not every quantifier free 1-type of jusl with 0 is realized in $\mathcal{D}$. This also implies that not every countable jusl with 0 can be embedded in $\mathcal{D}$.

Outline.

We start by proving that any countable pjusl which supports a jump hierarchy is embeddable in $\mathcal{D}$. (We define jump hierarchies in 2.2.1.) We do this via a forcing construction that uses some ideas from the one that Hinman and Slaman used in [HS91]. We both simplify the construction in [HS91] and add new features to it. Then, in section 2.3, we show that certain simple pjusls support jump hierarchies and we deduce that the existential theory of $\langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$ is decidable. In section 2.4 we prove our main result: Every countable jusl is embeddable in $\mathcal{D}$. To do this we show that every countable jusl can be embedded into one that supports a jump hierarchy. Part of this proof uses Fraïssé limits which are somewhat similar to the geometric part of the forcing notion used by Hinman and Slaman in [HS91]. In the last two sections we study pjusls with 0 and uncountable pjusls.

2.2 The Main construction

Definition 2.2.1. Given a structure $\mathcal{P} = \langle P, \leqslant_P, ...\rangle$, where $\langle P, \leqslant_P\rangle$ is a partial ordering, a *Jump Hierarchy over* $\mathcal{P}$ is a map $H\colon P \to \omega^\omega$ such that, for all $x, y \in P$,

- $\mathcal{P} \leqslant_T H(x)$;
- $\bigoplus_{x \leqslant_P y} H(x) \leqslant_T H(y)$;
- if $x <_P y$ then $H(x)' \leqslant_T H(y)$.

When such an H exists, we say that $\mathcal{P}$ *supports* a jump hierarchy.

This section is devoted to proving the following theorem.

Theorem 2.2.2. *Every countable partial jump upper semilattice which supports a jump hierarchy can be embedded in* $\mathcal{D}$.

We shall use a forcing construction (see [SWa]). We shall also use different kinds of codings. Here is a description of them.

Definition 2.2.3. For any X, Y, $Z \in \omega^\omega$, and any $n \in \omega$:

1. X *codes* Y *(directly) in the nth column* if $X^{[n]} = Y$. (Where $X^{[n]}(m) = X(\langle n, m\rangle)$.)
2. X *jump codes* Y *in the nth column* if for all m,

$$Y(m) = \lim_z X(\langle n, m, z\rangle);$$

   that is, for some function S and all m and $z \geqslant S(m)$, $Y(m) = X(\langle n, m, z\rangle)$. S is called a *Skolem function* for the coding.

3. *X codes Y lazily in the nth column* if for all m and z, either $X(\langle n,m,z\rangle) = 0$ or $X(\langle n,m,z\rangle) = Y(m)+1$, and for each m there is at least one z such that $Y(m)+1 = X(\langle n,m,z\rangle)$.

4. *X and Y code Z lazily in the nth column* if for all k, l and $m \in \omega$,

$$X(\langle n,m,l\rangle) = Y(\langle n,m,l\rangle) = k \neq 0 \Rightarrow Z(m) = k-1,$$

and for each m there is at least one l such that $X(\langle n,m,l\rangle) = Y(\langle n,m,l\rangle) = Z(m)+1$.

Observation 2.2.4. For X, Y and $Z \in \omega^\omega$,

- If X codes Y directly or lazily in some column, then $Y \leqslant_T X$.
- If X jump codes Y in some column, then $Y \leqslant_T X'$.
- If X and Y code Z lazily, then $Z \leqslant_T X \oplus Y$.

Fix $\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \cup, \mathrm{j}\rangle$, a countable partial jump upper semilattice. Assume that J, the universe of $\mathcal{J}$, is a recursive subset of ω. Let $H : J \to \omega^\omega$ be a jump hierarchy over $\mathcal{J}$.

In a first reading of this proof, the reader can assume that $\cup$ and j are total: there are no essential changes in the proof when we allow $\cup$ and j to be partial.

We shall define a function $R_G\colon J \to \omega^\omega$ via a forcing construction. The map $x \mapsto degree(R_G(x))\colon \mathcal{J} \to \mathcal{D}$ is going to be the desired embedding. For each $x \in J$, $R_G(x)$ consists of:

- A direct code of $H(x)$ in the 0th column.
- A jump coding of $R_G(\mathrm{j}(x))$ in the 2nd column if $\mathrm{j}(x)\downarrow$. This jump coding has $Sk_G(x)$ as a Skolem function.
- A lazy coding of $R_G(y)$ in the $(3y)$th column for all $y <_{\mathcal{J}} x$.
- A lazy code of the Skolem function $Sk_G(y)$ in the $(3y+1)$st column for all y such that $\mathrm{j}(y) = x$.
- In the $(3\lfloor x,z\rfloor + 2)$nd column, $R_G(x)$ and $R_G(z)$ code $R_G(x \cup z)$ lazily for each $z \mid_{\mathcal{J}} x$ such that $x \cup y$ is defined, where $\lfloor x,z\rfloor = min(\langle x,z\rangle, \langle z,x\rangle)$ (it is a code for the unordered pair $\{x,z\}$), and $x \mid_{\mathcal{J}} z$ stands for $x \nleqslant_{\mathcal{J}} z$ & $z \nleqslant_{\mathcal{J}} x$.

2.2.1 The forcing notion.

Now we define a partial ordering $\mathbb{P}$. Then we consider a generic filter G over $\mathbb{P}$, and from it define $R_G\colon J \to \omega^\omega$.

Construction of $\mathbb{P}$ and R_G: Let $\bar{\mathbb{P}}$ be the set of pairs $p = \langle R_p, Sk_p \rangle$, where R_p and Sk_p are finite partial functions $J \times \omega \rightharpoonup \omega$. We order $\bar{\mathbb{P}}$ by reverse inclusion in both coordinates. (i.e: $\langle r, s \rangle \leqslant \langle r', s' \rangle \Leftrightarrow r \supseteq r'$ & $s \supseteq s'$.) For $x \in J$, we write $R_p(x)$ for the partial function $\omega \rightharpoonup \omega$ defined by $R_p(x)(n) = R_p(x, n)$. The same for $Sk_p(x) \colon \omega \rightharpoonup \omega$. Let $\mathbb{P}$ be the set of $p \in \bar{\mathbb{P}}$ such that, for all $x, y, z \in J$, all the following conditions are satisfied:

1. $R_p(x)$ can be consistently extended to code $H(x)$ in the 0th column. i.e:
$$R_p(x)^{[0]} \subset H(x)$$
as partial functions.

2. If $\mathfrak{j}(x)\downarrow$, $Sk_p(x)$ specifies part of a Skolem function for a jump coding of $R_p(\mathfrak{j}(x))$ in the 2nd column of $R_p(x)$. More specifically, if $n \in \omega$ and $Sk_p(x)(n)\downarrow = k$, then $R_p(\mathfrak{j}(x))(n)\downarrow$ and for all $m \geqslant k$,
$$R_p(x)(\langle 2, n, m \rangle)\downarrow \Rightarrow R_p(x)(\langle 2, n, m \rangle) = R_p(\mathfrak{j}(x))(n).$$

3. If $y <_{\mathcal{J}} x$, $R_p(x)$ is compatible with coding $R_p(y)$ lazily in the $3y$th column. i.e:
$$\exists m \big(R_p(x)(\langle 3y, n, m \rangle)\downarrow = k \neq 0\big) \Rightarrow R_p(y)(n)\downarrow = k - 1.$$

4. If $\mathfrak{j}(y) = x$, $R_p(x)$ is compatible with lazy coding $Sk(y)$ in the $(3y + 1)$st column. i.e:
$$\exists m \big(R_p(x)(\langle 3y + 1, n, m \rangle)\downarrow = k \neq 0\big) \Rightarrow Sk_p(y)(n)\downarrow = k - 1.$$

5. If $z \mid_{\mathcal{J}} x$, $y = x \cup z$ and for some $n, m \in \omega$ we have that
$$R_p(x)(\langle 3\lfloor x, z \rfloor + 2, n, m \rangle)\downarrow = R_p(z)(\langle 3\lfloor x, z \rfloor + 2, n, m \rangle) = k \neq 0$$
then $R_p(y)(n)\downarrow = k - 1$.

Let G be an arithmetically (in $\mathbb{P}$) $\mathbb{P}$-generic filter. Let $R_G(x) = \bigcup\{R_p(x) : p \in G\}$ and $Sk_G(x) = \bigcup\{Sk_p(x) : p \in G\}$. $\Diamond$

Lemma 2.2.5. *(The conditions on $\mathbb{P}$ are not contradictory.) For each $x \in \mathcal{J}$ and $n \in \omega$ the sets $\{q \in \mathbb{P} : R_q(x, n)\downarrow\}$ and $\{q \in \mathbb{P} : Sk_q(x, n)\downarrow\}$ are dense in $\mathbb{P}$. Hence $R_G(x), Sk_G(x) \in \omega^\omega$.*

Moreover, given $p \in \mathbb{P}$ and $n = \langle k, m \rangle \in \omega$, there exists $t \in \omega$ such that if we define q just by extending p so that $R_q(x, n)\downarrow = t$ (i.e. $R_q = R_p \cup \{\langle \langle x, n \rangle, t \rangle\}$ and $Sk_q = Sk_p$), then $q \in \mathbb{P}$. t can be obtained as follows.

1. *If n is in the 0th column, i.e. $k = 0$, let $R_q(x, n) = H(x)(m)$.*

2. *If* $k = 2$ *and* $m = \langle m_1, m_2 \rangle$ *then, if* $\mathrm{j}(x)\downarrow$, $Sk_p(x, m_1)\downarrow = s \neq 0$ *and* $s \leqslant m_2$, *define* $R_q(x, n) = R_p(\mathrm{j}(x), m_1)$. *(Observe that if* $Sk_p(x, m_1)\downarrow$, *then* $R_p(\mathrm{j}(x), m_1)\downarrow$.*) Otherwise, define* $R_q(x, n)$ *arbitrarily.*

3. *Now, suppose that* $k = 3y$, $y <_{\mathcal{J}} x$ *and* $m = \langle m_1, m_2 \rangle$. *We can always set* $R_q(x, n) = 0$. *But, if we also know that* $R_p(y, m_1)\downarrow = l$, *then we could set* $R_q(x, n) = l + 1$.

4. *Now suppose that* $k = 3y + 1$ *with* $\mathrm{j}(y) = x$, *and* $m = \langle m_1, m_2 \rangle$. *Then, if* $Sk_p(y, m_1)\downarrow = m_3$ *set* $R_q(x, n) = 0$ *or* $= m_3 + 1$, *otherwise set* $R_q(x, n) = 0$.

5. *If* $k = 3\lfloor x, z \rfloor + 2$ *with* $z \mid_{\mathcal{J}} x$ *and* $x \cup z$ *defined, then we can always set* $R_q(x, n) = 0$. *Actually, we can set* $R_q(x, n)$ *to be anything we want as long as* $R_q(x, n) \neq R_q(z, n)$ *or* $R_q(x, n) = R_q(x \cup z, m_1) + 1$ *where* $n = \langle m_1, m_2 \rangle$.

6. *In any other case we can set* $R_q(x, n)$ *arbitrarily.*

Sketch of the Proof: We have to show that $q \in \mathbb{P}$. To do this we have to check all the conditions in the definition of $\mathbb{P}$. For example, suppose that $n = \langle 0, m \rangle$ and that we have set $R_q(x, n) = H(x)(m)$. Since $R_p(x)^{[0]} \subset H(x)$, we also have that $R_q(x)^{[0]} \subset H(x)$. Hence q satisfies condition 1 in the definition of $\mathbb{P}$. Conditions 2-5 are trivially satisfied. We leave the other cases to the reader.

Since this is true for all p, it implies that $\{q \in \mathbb{P} : R_q(x, n)\downarrow\}$ is dense in $\mathbb{P}$.

Now we have to show that $\{q \in \mathbb{P} : Sk_q(x, n)\downarrow\}$ is dense in $\mathbb{P}$. Consider $p \in \mathbb{P}$ and suppose that $Sk_p(x, n)\uparrow$. We want to show that there is an extension p_2 of p such that $Sk_{p_2}(x, n)\downarrow$. Let p_1 be an extension of p such that $R_{p_1}(\mathrm{j}(x), n)\downarrow$. Let m be such that for all $i \geqslant m$ $R_{p_1}(z, \langle 3x + 1, n, i \rangle)\uparrow$ and let p_2 be such that $R_{p_2} = R_{p_1}$ and $Sk_{p_2} = Sk_{p_1} \cup \{\langle \langle x, n \rangle, m \rangle\}$. It is easy to verify that $p_2 \in \mathbb{P}$. □

Lemma 2.2.6. *For all* $x, y, z \in J$*:*

1. $H(x)$ *is directly coded in the 0th column of* $R_G(x)$.

2. *if* $\mathrm{j}(x)\downarrow$, *then* $R_G(\mathrm{j}(x))$ *is jump coded in the 2nd column of* $R_G(x)$ *with Skolem function* $Sk_G(x)$.

3. *If* $y <_{\mathcal{J}} x$, *then* $R_G(y)$ *is lazily coded in the* $(3y)$*th column of* $R_G(x)$.

4. *If* $\mathrm{j}(y) = x$, *then,* $R_G(x)$ *codes* $Sk_G(y)$ *lazily in the* $(3y + 1)$*st column.*

5. *If* $x \mid_{\mathcal{J}} z$ *and* $x \cup z\downarrow$, *then* $R_G(x)$ *and* $R_G(z)$ *code* $R_G(x \cup z)$ *lazily in the* $(3\lfloor x, z \rfloor + 2)$*nd column.*

Proof: For example, for the third part use Lemma 2.2.5 and observe that, once $R_p(y, n)\downarrow$, the set

$$\{q : \exists i\big(R_q(x, \langle 3y, n, i \rangle) = R_q(y, n) + 1\big)\}$$

is dense below p. The other parts are similar. □

Corollary 2.2.7. *For all x and y in J,*

1. *$H(x) \leqslant_T R_G(x)$;*
2. *if $\mathfrak{j}(x)\downarrow$, then $R_G(\mathfrak{j}(x)) \leqslant_T R_G(x)'$;*
3. *if $y \leqslant_{\mathcal{J}} x$, then $R_G(y) \leqslant_T R_G(x)$;*
4. *if $\mathfrak{j}(y)\downarrow \leqslant_{\mathcal{J}} x$, then $Sk_G(y) \leqslant_T R_G(x)$;*
5. *if $x \mid_{\mathcal{J}} y$ and $x \cup y\downarrow$, then $R_G(x \cup y) \equiv_T R_G(x) \oplus R_G(y)$.*

Moreover, all these Turing reductions are uniform in x and y.

PROOF: All the proofs are immediate from the previous lemma and observation 2.2.4. For (4) observe that $Sk_G(y) \leqslant_T R_G(\mathfrak{j}(y)) \leqslant_T R_G(x)$. □

2.2.2 Preservation of nonorder.

We have already proved that $x \leqslant_{\mathcal{J}} y$ implies that $R_G(x) \leqslant_T R_G(y)$. In this subsection we prove that if $x \not\leqslant_{\mathcal{J}} y$, then $R_G(x) \not\leqslant_T R_G(y)$. To do this we need to analyze $\mathbb{P}$ a little bit more. We shall prove a combinatorial lemma about $\mathbb{P}$ that is going to be useful in the next subsection too.

Definition 2.2.8. For $x \in J$, define

1. $J_x = \{y \in J : y \leqslant_{\mathcal{J}} x\}$ and $J_x^{\mathfrak{j}} = \{y \in J : \mathfrak{j}(y)\downarrow \leqslant_{\mathcal{J}} x\}$;
2. $\bar{\mathbb{P}}_x = \{\langle r, s\rangle : r$ and s are finite partial functions, $r : J_x \times \omega \rightharpoonup \omega$ and $s : J_x^{\mathfrak{j}} \times \omega \rightharpoonup \omega\}$;
3. $p \restriction x = \langle R_p \restriction J_x, Sk_p \restriction J_x^{\mathfrak{j}}\rangle \in \bar{\mathbb{P}}_x$.

Definition 2.2.9. Say that $p \in \mathbb{P}$ is *nice at $x \in J$* if for all y, z with $y \mid_{\mathcal{J}} z$ and $y \leqslant_{\mathcal{J}} x \not\geqslant_{\mathcal{J}} z$, and for all $i \in \omega$

$$R_p(z)^{[3\lfloor y,z\rfloor+2]} \subseteq R_p(y)^{[3\lfloor y,z\rfloor+2]}$$

as partial functions.

Observation 2.2.10. For every x, the set of p which are nice at x is dense.

PROOF: Use Lemma 2.2.5. Given y, z with $y \mid_{\mathcal{J}} z$ and $y \leqslant_{\mathcal{J}} x \not\geqslant_{\mathcal{J}} z$, extend $R_p(y)$ by adding 0's to its $(3\lfloor y, z\rfloor + 2)$nd column at the same places where $R_p(z)^{[3\lfloor y,z\rfloor+2]}$ is defined. □

In the next lemma we need to consider $p \restriction \mathfrak{j}(x)$ even if $\mathfrak{j}(x)$ is undefined. In that case define $p \restriction \mathfrak{j}(x) = p \restriction x \cup \bigcup\{p \restriction \mathfrak{j}(y) : y \leqslant_{\mathcal{J}} x \,\&\, \mathfrak{j}(y)\downarrow\}$. Where $\cup$ is the union of compatible partial functions.

Lemma 2.2.11. *For all $p, q \in \mathbb{P}$ and $x \in J$ such that p is nice at x, we have that*

$$q \leqslant p \restriction \mathrm{j}(x) \Rightarrow q \restriction x \cup p \in \mathbb{P}.$$

PROOF: Let $r = q \restriction x \cup p$. We have to check that all the conditions in the definition of $\mathbb{P}$ are satisfied by r.

Condition 1: $R_r(y)^{[0]} \subseteq R_p(y)^{[0]} \cup R_q(y)^{[0]} \subset H(y)$ as partial functions.

Condition 2: Consider $y \in J$ such that $\mathrm{j}(y)\downarrow$. We want to show that $Sk_r(y)$ is part of a Skolem function for a jump coding of $R_r(\mathrm{j}(y))$ in the 2nd column of $R_r(y)$. There are three possible cases: $\mathrm{j}(y) \leqslant_{\mathcal{J}} x$; $y \leqslant_{\mathcal{J}} x$ but $\mathrm{j}(y) \nleqslant_{\mathcal{J}} x$; and $y \nleqslant_{\mathcal{J}} x$. If $\mathrm{j}(y) \leqslant_{\mathcal{J}} x$, then, since $r \restriction x = q \restriction x$, the condition holds because it does at q. If $y \nleqslant_{\mathcal{J}} x$, then the condition holds because it does at p. So, suppose that $y \leqslant_{\mathcal{J}} x$ but $\mathrm{j}(y) \nleqslant_{\mathcal{J}} x$. We have that $Sk_r(y) = Sk_p(y)$. So, whenever $Sk_r(y,n)\downarrow = k$, $Sk_p(y,n)\downarrow = Sk_q(y,n) = k$, because $q \leqslant p \restriction \mathrm{j}(x)$. Then, $R_p(\mathrm{j}(y),n)\downarrow = R_q(\mathrm{j}(y),n) = R_r(\mathrm{j}(y),n)$. Now, if $R_r(y,\langle 2,n,m\rangle)\downarrow = l$ for some $m \geqslant k$, then $l = R_r(\mathrm{j}(y),n)$.

Condition 3: Suppose that $y <_{\mathcal{J}} z$ and we want to check that $R_r(z)$ is compatible with lazy coding of $R_r(y)$ in the $(3y)$th column. If $z \leqslant_{\mathcal{J}} x$, then everything works fine, because it does at q. Otherwise, $R_r(z) = R_p(z)$. So, if for some n, i and k, $R_r(z,\langle 3y,n,i\rangle)\downarrow = k \neq 0$, then $R_p(z,\langle 3y,n,i\rangle)\downarrow = k \neq 0$; and therefore, $R_r(y,n) = R_p(y,n)\downarrow = k-1$.

Condition 4: Suppose that $\mathrm{j}(y) = z$ and we want to check that $R_r(z)$ is compatible with lazy coding of $Sk_r(y)$ in the $(3y+1)$th column. If $z \leqslant_{\mathcal{J}} x$, then everything works fine, because it does at q. Otherwise, $R_r(z) = R_p(z)$. So, if for some n, i and k, $R_r(z,\langle 3y+1,n,i\rangle)\downarrow = k \neq 0$, then $R_p(z,\langle 3y+1,n,i\rangle)\downarrow = k \neq 0$; and therefore, $Sk_r(y,n) = Sk_p(y,n)\downarrow = k-1$.

Condition 5: Suppose that $y \mid_{\mathcal{J}} z$ and that $y \cup z$ is defined. If both y and z are $\leqslant_{\mathcal{J}} x$ or neither of them is, then the condition holds: in the former case because it holds at q, and in the later case because it does at p. So assume that $y \leqslant_{\mathcal{J}} x \ngeqslant_{\mathcal{J}} z$. Also assume that for some m

$$R_r(y,\langle 3\lfloor y,z\rfloor + 2, m\rangle)\downarrow = R_r(z,\langle 3\lfloor y,z\rfloor + 2, m\rangle)\downarrow = k \neq 0$$

then $R_p(z,\langle 3\lfloor y,z\rfloor + 2, m\rangle)\downarrow = k$ too, because $R_r(z) = R_p(z)$. Thus, we also have that $R_p(y)(\langle 3\lfloor y,z\rfloor + 2, m\rangle)\downarrow$, because p is nice at x. Necessarily $R_p(y)(\langle 3\lfloor y,z\rfloor + 2, m\rangle) = k$. Therefore $R_p(z \cup x)(m)\downarrow = k+1$, and then $R_r(z \cup x)(m)\downarrow = k+1$ too. □

Corollary 2.2.12. *For all $p \in \mathbb{P}$ and $x \in J$, $p \restriction x \in \mathbb{P}$.*

PROOF: Observe that the empty condition, $\emptyset$, is nice at x and that $p \leqslant \emptyset \restriction \mathrm{j}(x) = \emptyset$. Therefore $p \restriction x = p \restriction x \cup \emptyset \in \mathbb{P}$. □

Corollary 2.2.13. *If $y \nleqslant_{\mathcal{J}} x$, then $R_G(y) \nleqslant_T R_G(x)$.*

PROOF: Suppose, toward a contradiction, that for some $p \in G$, $p \Vdash \{e\}^{R_G(x)} = R_G(y)$, where $\{e\}$ is the eth Turing functional and $\Vdash$ is the strong forcing relation, $\Vdash^*_{\mathbb{P}}$, as defined in [SWa]. Moreover, by observation 2.2.10, we can assume that p is nice at x. Let n be of the form $\langle 1, m\rangle$ such that $R_p(y,n)\uparrow$. (Remember that the 1st column is the one that is not coding anything.) Let $q \leqslant p$ be such that $q \Vdash \{e\}^{R_G(x)}(n) = i$ for some $i \in \omega$. Let $r = q\restriction x \cup p$. Since $q \leqslant p \leqslant p\restriction \mathrm{j}(x)$, we have that $r \in \mathbb{P}$ by the previous lemma. Then, since $R_r(x) = R_q(x)$, $r \Vdash \{e\}^{R_G(x)}(n) = i$. But $R_r(y,n)$ is undefined. Extend r to r^* by setting $R_{r^*}(y,n) = i+1$. From Lemma 2.2.5 we get that $r^* \in \mathbb{P}$. Then, $r^* \Vdash \{e\}^{R_G(x)}(n) \neq R_G(y)(n)$, contradicting our assumption about p. $\square$

2.2.3 R_G preserves the jump.

Fix $x \in J$ such that $\mathrm{j}(x)$ is defined. We have already proved that $R_G(\mathrm{j}(x)) \leqslant_T R_G(x)'$. Now, we want to prove that $R_G(x)' \leqslant_T R_G(\mathrm{j}(x))$.

We start by studying the complexity of the statement "p decides $\{e\}^{R_G(x)}(e)\downarrow$".

Definition 2.2.14. 1. $\mathbb{P}_x = \bar{\mathbb{P}}_x \cap \mathbb{P}$;

2. $\mathbb{P}_{p,x} = \{q\restriction x : q \in \mathbb{P} \,\&\, q \leqslant p\}$;

3. $G\restriction x = G \cap \mathbb{P}_x$.

Three easy facts about $\mathbb{P}_x$ that we are going to use are:

Observation 2.2.15.
- $\mathbb{P}_{p,x} \subseteq \mathbb{P}_x$.
- $\mathbb{P}_x$ is recursive in $H(x)$.
- Every $p \in \mathbb{P}_x$ is nice at x.

We are going to show later that some $p \in G\restriction \mathrm{j}(x)$ decides $\{e\}^{R_G(x)}(e)\downarrow$. First we study the complexity of $G\restriction \mathrm{j}(x)$.

Lemma 2.2.16. *$G\restriction x \equiv_T R_G(x)$.*

PROOF: Clearly $R_G(x) \leqslant_T G\restriction x$. Now we prove that $G\restriction x \leqslant_T R_G(x)$. Observe that $\bar{q} \in \mathbb{P}_x$ is in $G\restriction x$ iff for all y, $R_{\bar{q}}(y) \subset R_G(y)$ and $Sk_{\bar{q}}(y) \subset Sk_G(y)$. We observed in 2.2.15 that $\mathbb{P}_x \leqslant_T H(x)$, and by corollary 2.2.7, $H(x) \leqslant_T R_G(x)$, $\forall y \in J_x(R_G(y) \leqslant_T R_G(x))$ and $\forall y \in J^{\mathrm{j}}_x(Sk_G(y) \leqslant_T R_G(x))$, uniformly in y. $\square$

Lemma 2.2.17. *For p nice at x,*

1. $\mathbb{P}_{p,x} = \mathbb{P}_{p\restriction \mathrm{j}(x),x} = \{\bar{q} \in \bar{\mathbb{P}}_x : \bar{q} \leqslant p\restriction x \,\&\, \bar{q} \cup p \in \mathbb{P}\}$;

2. *$\mathbb{P}_{p,x}$ is recursive in $H(x)$ uniformly in p.*

Proof: We shall prove that

$$\mathbb{P}_{p,x} \subseteq \mathbb{P}_{p\restriction \mathrm{j}(x),x} \subseteq \{\bar{q} \in \bar{\mathbb{P}}_x : \bar{q} \leqslant p\restriction x \;\&\; \bar{q} \cup p \in \mathbb{P}\} \subseteq \mathbb{P}_{p,x}.$$

Since $p \leqslant p\restriction \mathrm{j}(x)$, we have that $\mathbb{P}_{p,x} \subseteq \mathbb{P}_{p\restriction \mathrm{j}(x),x}$. Now consider $\bar{q} \in \mathbb{P}_{p\restriction \mathrm{j}(x),x}$. There is some $q \leqslant p\restriction \mathrm{j}(x)$ such that $\bar{q} = q\restriction x$. From Lemma 2.2.11, we get that $q\restriction x \cup p \in \mathbb{P}$. So $\bar{q} \in \{\bar{q} \in \bar{\mathbb{P}}_x : \bar{q} \leqslant p\restriction x \;\&\; \bar{q} \cup p \in \mathbb{P}\}$. Now consider $\bar{q} \in \bar{\mathbb{P}}_x$ such that $\bar{q} \leqslant p\restriction x$ and $r = \bar{q} \cup p \in \mathbb{P}$. Clearly $r \leqslant p$ and $r\restriction x = \bar{q}$, so $\bar{q} \in \mathbb{P}_{p,x}$. This proves the first part.

For second part, given $\bar{q} \in \bar{\mathbb{P}}_x$, we want to decide, recursively in $H(x)$, whether $\bar{q} \in \mathbb{P}_{p.x}$. Note that checking if $\bar{q} \leqslant p\restriction x$ is clearly recursive, uniformly in p. To check if $r = \bar{q} \cup p \in \mathbb{P}$ one has to check the conditions in the definition of $\mathbb{P}$. All but the first condition, can be checked recursively in $\mathcal{J}$. For the first condition we already have that, for $y \not\leqslant_{\mathcal{J}} x$, $R_r(y)^{[0]} = R_p(y)^{[0]} \subseteq H(y)$. So we only have to check if $\forall y \leqslant_{\mathcal{J}} x(R_r(y)^{[0]} \subseteq H(y))$, which we can do recursively in $H(x)$. □

Lemma 2.2.18. *For p nice at x, and $e \in \omega$:*

1. *The following are equivalent:*
   (a) $p \Vdash \{e\}^{R_G(x)}(e)\uparrow$;
   (b) $p\restriction \mathrm{j}(x) \Vdash \{e\}^{R_G(x)}(e)\uparrow$;
   (c) $p\restriction x \Vdash_{\mathbb{P}_{p,x}} \{e\}^{R_G(x)}(e)\uparrow$.
2. *Whether p decides $\{e\}^{R_G(x)}(e)\downarrow$ can be decided recursively in $H(x)'$, uniformly in p and e. Moreover, if p decides $\{e\}^{R_G(x)}(e)\downarrow$, we can also tell whether p forces $\{e\}^{R_G(x)}(e)\downarrow$ or its negation.*

Proof: By definition of forcing we have that, $p \Vdash \{e\}^{R_G(x)}(e)\uparrow$ if and only if

$$\forall q \leqslant p \forall s \in \omega(\{e\}_s^{R_q(x)}(e)\uparrow).$$

This is equivalent to

$$\forall \bar{q} \in \mathbb{P}_{p,x} \forall s \in \omega(\{e\}_s^{R_{\bar{q}}(x)}(e)\uparrow), \tag{2.2.1}$$

which, by definition of $\Vdash_{\mathbb{P}_{p,x}}$, is equivalent to $p\restriction x \Vdash_{\mathbb{P}_{p,x}} \{e\}^{R_G(x)}(e)\uparrow$. We have shown that (1a) is equivalent to (1c). We get that (1b) is equivalent to (1c) in the same way because $\mathbb{P}_{p,x} = \mathbb{P}_{p\restriction \mathrm{j}(x),x}$. Whether $p \Vdash \{e\}^{R_G(x)}(e)\downarrow$, can be decided recursively, because we only have to check if $\{e\}^{R_p(x)}(e)\downarrow$. Whether $p \Vdash \{e\}^{R_G(x)}(e)\uparrow$, is a $\Pi_1^{H(x)}$ question as shown in (2.2.1), so $H(x)'$ can answer it. □

Corollary 2.2.19. *if $\mathrm{j}(x)\downarrow$, then $R_G(x)' \equiv_T R_G(\mathrm{j}(x))$.*

Proof: We showed that $R_G(\mathrm{j}(x)) \leqslant_T R_G(x)'$ in corollary 2.2.7. Now we compute $R_G(x)'$ from $R_G(\mathrm{j}(x))$. Consider $e \in \omega$. Find $p \in G\restriction \mathrm{j}(x)$ such that p decides $\{e\}^{R_G(x)}(e)\downarrow$. By Lemma 2.2.16, $G\restriction \mathrm{j}(x) \leqslant_T R_G(\mathrm{j}(x))$, and by Lemma 2.2.18, $H(x)'$ knows whether p decides $\{e\}^{R_G(x)}(e)\downarrow$. Since H is a jump hierarchy, $H(x)' \leqslant_T H(\mathrm{j}(x)) \leqslant_T R_G(\mathrm{j}(x))$. So, we can find such a p recursively in $R_G(\mathrm{j}(x))$. We can also tell whether p forces $\{e\}^{R_G(x)}(e)\downarrow$ or its negation; in the former case we get that $R_G(x)'(e) = 1$ and in the later that $R_G(x)'(e) = 0$. □

This finishes the proof of Theorem 2.2.2.

2.3 Decidability results.

As a corollary of Theorem 2.2.2, we prove that the existential theory of the Turing degrees with $\leqslant_T$, join and jump is decidable.

Proposition 2.3.2 is stronger than what we actually need to prove decidability, but we shall use it again later. To prove it we need the following lemma.

Lemma 2.3.1. *Given a recursive well founded partial ordering $\mathcal{P}$ of rank α, and a recursive presentation, $\mathcal{A}$, of α, the usual rank map, $rk : \mathcal{P} \to \mathcal{A}$, is recursive in $0^{2\alpha+2}$.*

Sketch of the Proof: We claim that there is a recursive function f such that, for $\beta < \alpha$, $f(\beta)$ is a $0^{2\beta+2}$-index for the function $\varphi_\beta(x)$ that answers whether $rk(x) \geqslant \beta$. The definition of f is by transfinite recursion using that $rk(x) \geqslant \beta$ iff for all $\gamma < \beta$, there exists $y \in \mathcal{P}$ such that $y < x$ & $rk(y) \geqslant \gamma$. So, $\{f(\beta)\}^{0^{2\beta+2}}(x) =$ $\mathtt{yes}$ if and only if

$$\forall \gamma < \beta \; \exists y \in \mathcal{P}\big(y < x \;\&\; \{f(\gamma)\}^{0^{2\gamma+2}}(y) = \mathtt{yes}\big).$$

$\square$

Proposition 2.3.2. *Every well founded partial ordering supports a jump hierarchy.*

Proof: Let $\mathcal{P} = \langle P, \leqslant_P\rangle$ be a well founded partial ordering. Assume that $\mathcal{P}$ is recursive. Otherwise relativize the proof to the degree of $\mathcal{P}$.

Let $rk(\mathcal{P})$ be the rank of $\mathcal{P}$ and $\beta = 2rk(\mathcal{P}) + 2$. Let $\{H_a\}_{a\in O}$ be the hyperarithmetic hierarchy, where O is the set of ordinal notations (see [Sac90a]). Fix an initial segment of O of length $\beta + rk(\mathcal{P})$, and think of the ordinals below $\beta + rk(\mathcal{P})$ as elements of that segment of O. For $x \in P$, let $rk(x)$ be the usual rank of x in $\mathcal{P}$. Now, for each $x \in P$ define

$$K(x) = H_{\beta + rk(x)}$$

Clearly $\mathcal{P} \leqslant_T K(x)$ for all $x \in P$. We get that $x <_T y$ implies $K(x)' \leqslant_T K(y)$ because $x <_P y$ implies that $rk(x) < rk(y)$. We get that $\bigoplus_{x \leqslant_P y} K(x) \leqslant_T K(y)$ because given $\langle x, m\rangle$ with $x \leqslant_P y$ we can compute $rk(x)$ recursively in H_β, and then compute $H_{rk(x)}(m)$. Therefore, K is a jump hierarchy over $\mathcal{P}$. $\square$

Remark 2.3.3. Moreover, for every $X \subseteq \omega$, every well founded partial ordering, $\mathcal{P}$, supports a jump hierarchy, K, such that $\forall x \in P(X \leqslant_T K(x))$. The construction is the same as above, but now relativize to $X \oplus \mathcal{P}$.

Corollary 2.3.4. *Every finite pjusl can be embedded into $\mathcal{D}$.*

Proof: Every finite pjusl is well founded, so it supports a jump hierarchy. Therefore, by Theorem 2.2.2, it can be embedded into $\mathcal{D}$. □

Theorem 2.3.5. *The* $\exists$*-theory of* $\mathcal{D} = \langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$ *is decidable.*

Proof: Consider an existential sentence φ in the language of $\mathcal{D}$. It is equivalent to a disjunction of sentences of the form

$$\exists x_1 ... \exists x_n (\psi_1 \ \& \ ... \ \& \ \psi_m), \tag{2.3.1}$$

where each ψ_i has one of the following forms: $x_{j_1} \leqslant_T x_{j_2}$, $x_{j_1} \nleqslant_T x_{j_2}$, $x_{j_1} \neq x_{j_2}$, $x'_{j_1} = x_{j_2}$, or $x_{j_1} \vee x_{j_2} = x_{j_3}$. We have to decide whether one of these disjuncts holds in $\mathcal{D}$. So, suppose that φ is the formula in (2.3.1). We claim that $\mathcal{D} \models \varphi$ if and only if φ holds in some pjusl with at most n elements. If $\mathcal{D} \models \varphi$, then the degrees $\mathbf{x}_1, ..., \mathbf{x}_n$ which witness φ form the desired pjusl. If $\mathcal{J} \models \varphi$, for some pjusl $\mathcal{J}$ with at most n elements, then, since we can embed $\mathcal{J}$ into $\mathcal{D}$, we have that $\mathcal{D} \models \varphi$. Clearly we can recursively check whether φ holds in some pjusl with at most n elements. □

2.4 Jump upper semilattices which support Jump Hierarchies.

Now we show how to embed any countable jusl into one which supports a jump hierarchy.

This section is divided into five subsections. First we show how to define a Harrison Linear Ordering in such a way that we have recursive operations of addition and multiplication. In subsection 2.4.2 we define, for each $\alpha < \omega_1^{CK}$, a pjusl $\mathcal{P}_\alpha$ which supports a jump hierarchy, and we show that any pjusl with a certain property can be embedded in $\mathcal{P}_\alpha$. In subsection 2.4.4 we show that every recursive jusl has that property. But first we need to prove that every finitely generated pjusl is well quasiordered (we define well quasiorderings in 2.4.9); we do this in subsection 2.4.3. In the last subsection we put all the pieces together and prove that every countable jusl embeds into $\mathcal{D}$.

2.4.1 Pseudo-well orderings with Jump Hierarchies.

In [Har68], Harrison proved that there is a recursive linear ordering of type $\omega_1^{CK} \cdot (1 + \eta)$, (i.e: ω_1^{CK} followed by η copies of ω_1^{CK}, where η is the order type of the rational numbers.) which supports a jump hierarchy. Here we show that we can get such a linear ordering also having recursive addition and multiplication. These operations should have the same properties as ordinal addition and multiplication.

Definition 2.4.1. A *chain of structures of length* α is a sequence $\langle A_i : i < \alpha \rangle$ of structures together with a set of embeddings $\{\varphi_{ij} \colon A_i \to A_j\}_{i<j<\alpha}$, such that $\forall i < j < k < \alpha (\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik})$.

A *recursive chain of length* α ($< \omega_1^{CK}$) is a chain where $\langle A_i : i <_O a\rangle$ and $\langle \varphi_{ij} : i <_O j <_O a\rangle$ are uniformly recursive, and a is an ordinal notation for α.

Lemma 2.4.2. *Let $\langle\langle A_i : i <_O a\rangle, \langle\varphi_{ij} : i <_O j <_O a\rangle\rangle$ be a recursive chain. Its direct limit, A_a, and the set of embeddings $\varphi_{ia}\colon A_i \to A_a$, for $i <_O a$, are uniformly recursive. Furthermore, indices for A_a and $\langle\varphi_{ia}\colon A_i \to A_a : i <_O a\rangle$ can be found recursively from an index for $\langle\langle A_i : i <_O a\rangle, \langle\varphi_{ij} : i <_O j <_O a\rangle\rangle$.*

(See [Hod93, page 50] for a general definition of direct limits.)

Sketch of the Proof: One just has to observe that the usual construction of direct limits is uniformly recursive. What one does is to consider the disjoint union of the A_i:

$$B = \bigcup_{i <_O a} A_i \times \{i\},$$

and define an equivalence relation in B:

$$(x, i) \sim (y, j) \Leftrightarrow y = \varphi_{ij}(x)$$

for $i \leqslant_O j$. If $i >_O j$ say that $(x, i) \sim (y, j) \Leftrightarrow (y, j) \sim (x, i)$. This equivalence relation is clearly recursive. So $B/\!\sim$ is recursive: for each equivalence class take the element with least index as its representative. It is also easy to see that all the operations on $B/\!\sim$ and the embeddings φ_{ia} are recursive too. $\square$

Lemma 2.4.3. *For every $\alpha < \omega_1^{CK}$ there is a recursive well ordering, of order type at least α, in which the operations of addition and multiplications are recursive.*

Sketch of the Proof: For each $a \in O$ we shall define a recursive chain, c_a, of length $|a|$. c_a consists of recursive well orderings with addition and multiplication such that, for all $i <_O a$, the ith well ordering in the chain has order type at least $|i|$. We also want that if $a <_O b$, c_a is included in c_b. We shall use transfinite recursion. For $|a| = 1$, set c_a to be a chain with only one element consisting of ω with its usual addition and multiplication. If $a = 3 \cdot 5^e$ and we are given $c_{\{e\}(n)}$ for all $n \in \omega$, define c_a to be the union of all the $c_{\{e\}(n)}$. Now suppose that $a = 2^b$ and we are given c_b. If $|b|$ is a limit ordinal, extend c_b by adding its direct limit at the end. We can do this uniformly by the previous lemma. The last case is when $a = 2^b$ and $b = 2^d$, for some $d \in O$. Let l_d be the last well ordering in the chain c_b ($l_d = c_b(d)$). We shall construct a well ordering, l_b, with addition and multiplication, extending l_d. Then we define c_a by putting l_b at the end of c_b. Let β be a new symbol. (β represents the order type of l_d.) Define l_b as a set of formal sums as follows:

$$l_b = \{\sum_{i=0}^{n} \beta^i x_i : n < \omega, x_i \in l_d, x_n \neq 0\}.$$

The order relation and the addition operation are defined in the obvious way. Define multiplication as follows:

$$(\sum_{i=0}^{n} \beta^i x_i) \cdot (\sum_{j=0}^{m} \beta^j y_j) = \sum_{j=0}^{m} \beta^{n+j} y_i.$$

Is not hard to prove that l_b is a well ordering and that the multiplication defined this way is the usual ordinal multiplication. It is also clear that l_b is recursive. Embed l_d into l_b by mapping x to $\beta^0 x$. □

Theorem 2.4.4. *There is a structure $\mathcal{L} = \langle L, \leqslant, +, \cdot \rangle$ which supports a jump hierarchy, $\mathcal{H}$, such that: $\langle L, \leqslant \rangle$ is a recursive linear ordering of order type $\omega_1^{CK} \cdot (1+\eta)$; $+$ and $\cdot$ are recursive and satisfy the axioms of ordinals addition and multiplication; and for all $x \in L$, $\mathcal{H}(x)$ computes every hyperarithmetic set.*

Sketch of the Proof: We want to get $\langle \mathcal{L}, \mathcal{H} \rangle$ satisfying:

- $\langle L, \leqslant \rangle$ is a recursive linear ordering;
- for all $a \in O$, there is an x such that the set of predecessors has order type $|a|$;
- for all $a \in O$, there is no infinite descending sequence in $\mathcal{L}$ computable form 0^a;
- $+$ and $\cdot$ are recursive and satisfy the axioms of the inductive definition of addition and multiplication of ordinals;
- $\mathcal{H}$ is a jump hierarchy, and, for all $a \in O$, 0^a is recursive in $\mathcal{H}(x)$ for all $x \in L$.

(We write 0^a for the set corresponding to a in the hyperarithmetical hierarchy. Sometimes we shall write 0^α meaning 0^a for some a, in some fixed path through O, such that $|a| = \alpha$.)

All these axioms can be expressed by a Π^1_1 set, Γ, of computable infinitary formulas as in [AK00]. By the Barwise-Kreisel compactness theorem, as stated in [AK00, Theorem 8.3], if we prove that every Δ^1_1 subset of Γ has a model, then so does Γ. Any Δ^1_1 subset, Λ, of Γ will mention only a Δ^1_1 subset of O, so there has to be a $\beta < \omega_1^{CK}$, which bounds the norm of all of these ordinal notations (see [AK00, Proposition 5.20]). Then, by the previous lemma, we can always get a recursive well ordering with addition and multiplication of length at least β. The hyperarithmetical hierarchy, starting at 0^γ, and going up to $0^{\gamma+\beta}$, would be a jump hierarchy on it, where $\gamma = 2\beta + 2$, as in Proposition 2.3.2. This well ordering satisfies Λ. So we have that Γ has a model. Harrison proved in [Har68], that every recursive linear ordering with no hyperarithmetic descending sequences has order type either β or $\omega_1^{CK} \cdot (1+\eta) + \beta$ for some $\beta < \omega_1^{CK}$. The second set of conditions rules out the first case, and the fact that $\mathcal{L}$ is closed under addition makes $\beta = 0$ the only possibility. □

2.4.2 Partial upper semilattices with level function.

Now, we shall construct, for each $\alpha < \omega_1^{CK}$, a pjusl $\mathcal{P}_\alpha$ which supports a jump hierarchy. To define a jump hierarchy on $\mathcal{P}_\alpha$, we assign to each element of $\mathcal{P}_\alpha$ a member of $\mathcal{L}$, where $\mathcal{L}$ is defined in Theorem 2.4.4.

We work with the following kind of structures.

Definition 2.4.5. A *partial jump upper semilattice with levels in* $\mathcal{L}$ is a pjusl $\mathcal{J}$ together with a map $lev\colon J \to \mathcal{L}$ which preserves strict order. (i.e. $x <_{\mathcal{J}} y \Rightarrow lev(x) < lev(y)$.)

Fix $\alpha < \omega_1^{CK}$. Let $\mathcal{K}_\alpha$ be the set of finitely generated pjusl $\mathcal{J}$ with levels in $\mathcal{L}$ which are arithmetic in 0^α and such that $\forall x \in J(\mathfrak{j}(x)\downarrow)$.

Lemma 2.4.6. *$\mathcal{K}_\alpha$ has the Uniform Amalgamation Property. i.e: Given A, A_1 and $A_2 \in \mathcal{K}_\alpha$ and embeddings $\varphi_1\colon A \to A_1$ and $\varphi_2\colon A \to A_2$, there are a $C \in \mathcal{K}_\alpha$ and embeddings $\psi_1\colon A_1 \to C$ and $\psi_2\colon A_2 \to C$ such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. Moreover, indices for C, ψ_1 and ψ_2 can be found recursively from indices for A, A_1, A_2, φ_1 and φ_2.*

(An index for an embedding only has to code the embedding restricted to the finitely many generators.)

Proof: Let $\bar{A}_1 = A_1 \smallsetminus \varphi_1[A]$ and $\bar{A}_2 = A_2 \smallsetminus \varphi_2[A]$. Define the domain of C to be the disjoint union of A, $\bar{A}_1$ and $\bar{A}_2$. Define the embeddings ψ_1 and ψ_2 in the obvious way. Define the jump, join, level and the order relation in $A \cup \bar{A}_1$ as induced by A_1, and in $A \cup \bar{A}_2$ as induced by A_2. Do not define the join between elements of $\bar{A}_1$ and $\bar{A}_2$. (Here is where it is useful to work with pjusl and not with jusl.) To make $\leqslant$ transitive, define, for $x \in \bar{A}_1$ and $y \in \bar{A}_2$,

$$x \leqslant y \Leftrightarrow \exists z \in A(x \leqslant \varphi_1(z) \;\&\; \varphi_2(z) \leqslant y)$$

and

$$x \geqslant y \Leftrightarrow \exists z \in A(x \geqslant \varphi_1(z) \;\&\; \varphi_2(z) \geqslant y).$$

It is not hard to verify that we obtain a partial ordering. We also have to show that what we get is actually a pjusl. The properties for join and level are easily verified too. Let us verify that if $x \in \bar{A}_1$, $y \in \bar{A}_2$, $\mathfrak{j}(x)$ and $\mathfrak{j}(y)$ are defined and $x \leqslant y$, then $\mathfrak{j}(x) \leqslant \mathfrak{j}(y)$. Since $x \leqslant y$, there exists $z \in A$ such that $x \leqslant \varphi_1(z) \;\&\; \varphi_2(z) \leqslant y$. Therefore $\mathfrak{j}(x) \leqslant \varphi_1(\mathfrak{j}(z))$ and $\varphi_2(\mathfrak{j}(z)) \leqslant \mathfrak{j}(y)$. So $\mathfrak{j}(x) \leqslant \mathfrak{j}(y)$. $\square$

Now we shall consider $\mathcal{P}_\alpha$, the Fraïssé limit of $\mathcal{K}_\alpha$ (see [Hod93]). We construct $\mathcal{P}_\alpha$ is such a way that it is recursive in $0^{\alpha+\omega}$.

Construction of $\mathcal{P}_\alpha$: Enumerate all the tuples $\langle A, A_1, A_2, \varphi_1, \varphi_2\rangle$ such that A, A_1, $A_2 \in \mathcal{K}_\alpha$ and φ_1 and φ_2 are embeddings from A to A_1 and to A_2 respectively. (Actually, enumerate the tuples of indices.) We can get such an enumeration recursively in $0^{\alpha+\omega}$. We shall construct a sequence $\langle D_i : i < \omega\rangle$ together with embeddings $f_{ij}\colon D_i \to D_j$ recursively in $0^{\alpha+\omega}$.

Let $D_0 = \emptyset$. Now, suppose we have defined D_i for all $i < n$. Take the first tuple $\langle A, A_1, A_2, \varphi_1, \varphi_2\rangle$ from the list, not already taken, such that A_1 is equal to some D_i, $i < n$. Using Lemma 2.4.6, as in the diagram below, construct $D_n \in \mathcal{K}_\alpha$, and embeddings $f_{n-1,n}\colon D_{n-1} \to D_n$ and $\psi\colon A_2 \to D_n$ such that $f_{n-1,n} \circ f_{i,n-1} \circ \varphi_1 = \psi \circ \varphi_2$. For $j < n$, let $f_{j,n} = f_{n,n-1} \circ f_{j,n-1}$.

$$\begin{tikzcd}
 & D_i \arrow[rr, bend left, "f_{i,n-1}"] \arrow[rr, phantom, "\cdots"] & & D_{n-1} \arrow[dr, dotted, "f_{n-1,n}"] & \\
A \arrow[ur, "\varphi_1"] \arrow[urrr, "f_{i,n-1}\circ\varphi_1"'] \arrow[dr, "\varphi_2"'] & & & & D_n \\
 & A_2 \arrow[urrr, dotted, "\psi"'] & & &
\end{tikzcd}$$

Let $\mathcal{P}_\alpha$ be the direct limit of the chain constructed. By Lemma 2.4.2 relativized to $0^{\alpha+\omega}$, we can get $\mathcal{P}_\alpha \leqslant_T 0^{\alpha+\omega}$. $\Diamond$

Lemma 2.4.7. *$\mathcal{P}_\alpha$ supports a Jump Hierarchy.*

PROOF: $\mathcal{P}_\alpha$ is a pjusl with a level function to $\mathcal{L}$ and $\mathcal{L}$ supports a jump hierarchy, $\mathcal{H}$. So, for each $x \in \mathcal{P}_\alpha$, we can define

$$R(x) = \mathcal{H}(lev(x)).$$

We claim that R is a jump hierarchy over $\mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha$ is hyperarithmetic, we have that $\mathcal{P}_\alpha \leqslant_T R(x)$, for all $x \in \mathcal{P}_\alpha$. We also have that

$$\bigoplus_{x\leqslant y} R(x) \leqslant_T R(y),$$

because, given $x \leqslant y$, we can compute $lev(x)$ recursively in $R(y)$, and then, compute $\mathcal{H}(lev(y))$. The third thing that needs to be verified is that $x < y$ implies $H(x)' \leqslant_T H(y)$. This is true because $\mathcal{H}$ is a jump hierarchy over $\mathcal{L}$ and lev preserves strict order. $\Box$

Lemma 2.4.8. *Let $\mathcal{J}$ be a pjusl with levels in $\mathcal{L}$ such that there is a sequence*

$$\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \dots . \subseteq \mathcal{J},$$

with $\mathcal{J} = \bigcup_{i<\omega} \mathcal{J}_i$, and for all i, $\mathcal{J}_i \in \mathcal{K}_\alpha$. Then $\mathcal{J}$ can be embedded in $\mathcal{P}_\alpha$.

PROOF: We have constructed $\mathcal{P}_\alpha$ as the direct limit of $\langle D_i : i < \omega\rangle$ with embeddings f_{ij}. We shall get a subsequence $\{D_{i_k}\}_{k<\omega}$ such that for each k there is an embedding $g_k\colon J_k \to D_{i_k}$ such that if $k_1 \leqslant k_2$, then

$$f_{i_{k_1} i_{k_2}} \circ g_{k_1} = g_{k_2} \restriction \mathcal{J}_{k_1}.$$

This would imply that $\mathcal{J}$, the direct limit of $\langle \mathcal{J}_i : i < \omega\rangle$, embeds into $\mathcal{P}_\alpha$.

$$\begin{tikzcd}
D_{i_0} \arrow[r] \arrow[rr, bend left, "f_{i_0 i_1}"] & \cdots \arrow[r] & D_{i_1} \arrow[r] \arrow[rr, bend left, "f_{i_1 i_2}"] & \cdots \arrow[r] & D_{i_2} \arrow[r] & \cdots\cdots \arrow[r] & \mathcal{P}_\alpha \\
 & \mathcal{J}_0 \arrow[ul, "g_0"] \arrow[r, "\mathrm{id}_0"'] & \mathcal{J}_1 \arrow[u, "g_1"] \arrow[r, "\mathrm{id}_1"'] & \mathcal{J}_2 \arrow[ur, "g_2"] \arrow[r, "\mathrm{id}_2"'] & \cdots\cdots \arrow[r] & \mathcal{J} \arrow[ur] &
\end{tikzcd}$$

Let $\mathcal{J}_{-1} = \emptyset$, $i_{-1} = 0$ and $g_{-1} : \emptyset \to D_0$ be the empty map. Now suppose we have defined i_n and $g_n \colon \mathcal{J}_n \to D_{i_n}$. Consider the tuple $\langle \mathcal{J}_n, D_{i_n}, \mathcal{J}_{n+1}, g_n, \mathrm{id}_n \rangle$, where id_n is the inclusion map $\mathcal{J}_n \hookrightarrow \mathcal{J}_{n+1}$. Eventually, say at step i_{n+1}, this tuple is going to be considered in the construction of $\mathcal{P}_\alpha$. So, $D_{i_{n+1}}$ is going to be defined, together with a map $g_{n+1} \colon J_{n+1} \to D_{i_{n+1}}$, so that $f_{i_n i_{n+1}} \circ g_n = g_{n+1} \restriction \mathcal{J}_n$. $\square$

2.4.3 Well quasiorderings.

Now we move into the direction of proving that every recursive pjusl embeds in some $\mathcal{P}_\alpha$.

Definition 2.4.9. A *well quasiordering* is a set Q together a transitive and reflexive relation $\leqslant$ such that for every sequence $\{x_i\}_{i \in \omega}$, there are $i < j$ with $x_i \leqslant x_j$.

Observation 2.4.10.
1. A partial ordering which is well quasiordered, is well founded.
2. The image of a well quasiordering under an order preserving map is well quasiordered too.

Proof: The first observation is trivial. For the second one consider: Q, a well quasiordering; $f \colon Q \to P$, an order preserving map; and a sequence $\{x_i\}_{i<\omega} \subseteq P$. Let $\{y_i\}_{i<\omega} \subseteq Q$ be such that for all i, $f(y_i) = x_i$. There exist $i < j$ with $y_i \leqslant y_j$. Then $x_i \leqslant x_j$. $\square$

Definition 2.4.11.
1. Given a set F of variables, let $\mathcal{T}_F$ be the set of terms over the language with j, $\cup$, and variables from F.
2. For $t \in \mathcal{T}_F$, the *Jump Rank* of t is defined by recursion:

$$jrk(t) = \begin{cases} 0 & \text{if } t \text{ is a variable;} \\ \max(jrk(t_1), jrk(t_2)) & \text{if } t = t_1 \cup t_2; \\ jrk(t_1) + 1 & \text{if } t = \mathsf{j}(t_1). \end{cases}$$

3. The *support* of t, $supp(t)$, is the set of variables that actually occur in t.

4. For t with $supp(t) \subseteq F$, we define the *Jump Rank of t over F* by recursion:

$$jrk_F(t) = \begin{cases} -\infty & \text{if } supp(t) \subset F; \\ 0 & \text{if } t \text{ is a variable } x_i \text{ and } F = \{x_i\}; \\ \max(jrk_F(t_1), jrk_F(t_2), 0) & \text{if } t = t_1 \cup t_2 \text{ and } supp(t) = F; \\ jrk(t_1) + 1 & \text{if } t = \mathfrak{j}(t_1). \end{cases}$$

5. For terms $t_1(\bar{x})$ and $t_2(\bar{x})$, say that $t_1 \leqslant t_2$ if for every jusl U

$$U \vDash \forall \bar{x}\big(t_1(\bar{x}) \leqslant t_2(\bar{x})\big).$$

6. We say that t_1 is equivalent to t_2, and write $t_1 \equiv t_2$, if $t_1 \leqslant t_2$ and $t_2 \leqslant t_1$.

We shall write $\mathfrak{j}^m(b)$ for $\overbrace{\mathfrak{j}(\mathfrak{j}(...\mathfrak{j}}^{m}(x)..))$ and $\bigcup\{b_1, ..., b_n\}$ for $b_1 \cup b_2 \cup ... \cup b_n$.

Lemma 2.4.12. *For every term $t \in \mathcal{T}_F$;*

1. $\bigcup supp(t) \leqslant t$;
2. $t \leqslant \mathfrak{j}^{jrk(t)}(\bigcup F)$;
3. *if* $supp(t) = F$ *then* $\mathfrak{j}^{jrk_F(t)}(\bigcup F) \leqslant t$.

Proof: The first two parts are straightforward by induction on t. The third part can be proved by induction on $jrk_F(t)$ as follows. If $jrk_F(t) = 0$ then $\mathfrak{j}^0(\bigcup F) = (\bigcup F) \leqslant t$ by the first part. Now suppose that $jrk_F(t) > 0$. If $t = \mathfrak{j}(t_1)$, then $\mathfrak{j}^{jrk_F(t)}(\bigcup F) = \mathfrak{j}(\mathfrak{j}^{jrk_F(t_1)}(\bigcup F)) \leqslant \mathfrak{j}(t_1) = t$ by inductive hypothesis. If $t = t_1 \cup t_2$, then either $jrk_F(t_1)$ or $jrk_F(t_2)$ is equal to $jrk_F(t)$. Say the first one. Then $\mathfrak{j}^{jrk_F(t)}(\bigcup F) = \mathfrak{j}^{jrk_F(t_1)}(\bigcup F) \leqslant t_1 \leqslant t$. □

Lemma 2.4.13. *For finite F, $\mathcal{T}_F$ is a well quasiordering.*

Proof: We use induction on $|F|$, so we can assume that $\mathcal{T}_G$ is a well quasiordering for every $G \subset F$. (Note that the empty set is well quasiordered.) Now consider a sequence $\{t_i\}_{i\in\omega} \subseteq \mathcal{T}_F$. We want to show that there are $i < j$, such that $t_i \leqslant t_j$. Let $m_0 = jrk(t_0)$. If for some $i \neq 0$, $jrk_F(t_i) \geqslant m_0$, we are done because, by Lemma 2.4.12,

$$t_0 \leqslant \mathfrak{j}^{m_0}(\bigcup F) \leqslant \mathfrak{j}^{jrk_F(t_i)}(\bigcup F) \leqslant t_i.$$

So, assume that there is some $m \in \omega$ such that for all i, $jrk_F(t_i) < m$. Let $\mathcal{T}_{F,m} = \{t \in \mathcal{T}_F : jrk_F(t) < m\}$. We shall prove, by induction on m, that $\mathcal{T}_{F,m}$ is well quasiordered. This will imply that there are $i < j$ as we want, and hence that $\mathcal{T}_F$ is a well quasiordering.

For $m = 0$ we have that

$$\mathcal{T}_{F,0} = \bigcup\{\mathcal{T}_G : G \subset F\}.$$

It is not hard to see that a finite union of well quasiordering is a well quasiordering. So, since we are assuming that each $\mathcal{T}_G$ is well quasiordered, $\mathcal{T}_{F,0}$ is well quasiordered. Now assume that $\mathcal{T}_{F,m}$ is well quasiordered and consider $\{t_i\}_{i\in\omega} \subseteq \mathcal{T}_{F,m+1}$. Suppose, toward a contradiction, that for all $i<j$, $t_i \not\leqslant t_j$. There cannot be infinitely many terms in $T_{F,0}$ because of the base case we have just proved. If we eliminate the terms in $T_{F,0}$, we can assume that $\{t_i\}_{i\in\omega}$ is a sequence where all the terms have support F. First observe that every t_i in the sequence can be written, up to equivalence, as

$$\bigcup_{j<r_i} \mathrm{j}(s_{ij}) \cup \bigcup G_i,$$

where $G_i \subseteq F$ and $s_{ij} \in \mathcal{T}_{F,m}$. For each $i>0$, since $t_0 \not\leqslant t_i$ and $\bigcup G_i \leqslant \bigcup F \leqslant t_i$, we have that for some $j<r_0$, $\mathrm{j}(s_{0j}) \not\leqslant t_i$. Therefore

$$\exists j<r_0 \exists^\infty i \in \omega \big(\mathrm{j}(s_{0j}) \not\leqslant t_i\big).$$

Let j_0 be one of those j's. Let $s_0 = s_{0j_0}$, and $I_0 = \{i \in \omega : \mathrm{j}(s_0) \not\leqslant t_i\}$. Now consider i_1, the first element in I_0. For the same reason,

$$\exists j<r_{i_1} \exists^\infty i \in I_0 \big(\mathrm{j}(s_{i_1 j}) \not\leqslant t_i\big).$$

Let j_1 be one of those j's. Let $s_1 = s_{i_1 j_1}$, and $I_1 = \{i \in I_0 : \mathrm{j}(s_1) \not\leqslant t_i\}$. Repeat this procedure to get a sequence $\{s_i\}_{i\in\omega} \subseteq \mathcal{T}_{F,m}$ such that

$$\forall i<j \big(\mathrm{j}(s_i) \not\leqslant \mathrm{j}(s_j)\big).$$

But, by inductive hypothesis, there are $i<j$ such that $s_i \leqslant s_j$. Which implies that $\mathrm{j}(s_i) \leqslant \mathrm{j}(s_j)$. Contradiction. □

Corollary 2.4.14. *Every finitely generated pjusl is well quasiordered.*

Proof: Every finitely generated pjusl, $\mathcal{J}$, is the image of a subset of $\mathcal{T}_F$, for some finite F, under an order preserving map. Therefore, since $\mathcal{T}_F$ is well quasiordered, so is $\mathcal{J}$ by observation 2.4.10. □

2.4.4 The decomposition of $\mathcal{J}$.

Consider $\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \cup, \mathrm{j}\rangle$, a recursive pjusl such that j is a total function. We want to show that we can define a level function to $\mathcal{L}$ on it and a sequence

$$\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq .... \subseteq \mathcal{J},$$

with $\mathcal{J} = \bigcup_{i<\omega} \mathcal{J}_i$, such that for some $\alpha < \omega_1^{CK}$, $\mathcal{J}_i \in \mathcal{K}_\alpha$ for all $i \in \omega$.

Enumerate J as $\{a_0, a_1, ..., a_n, ...\}$. Let $\mathcal{J}_n = \langle a_i : i<n\rangle_{\mathcal{J}}$, the pjusl generated by $a_0, ..., a_{n-1}$. Let J_n be the domain of $\mathcal{J}_n$. Note that for each n, $\mathcal{J}_n \leqslant_T 0'$. Let $\preceq$ be a recursive linear ordering extending the ordering of $\mathcal{J}$. In other words, $\langle J, \preceq\rangle$

is a recursive linear ordering and $\leqslant_{\mathcal{J}} \subseteq \preceq$. A proof of the fact that every recursive partial ordering has a recursive linear extension can be found in [Dow98, Obs. 6.1].

Let $\preceq_n$ be $\preceq$ restricted to J_n. Since $\langle J_n, \leqslant_{\mathcal{J}} \rangle$ is well quasiordered, $\preceq_n$ is well quasiordered too. Since $\preceq_n$ is linear, it is actually a well ordering. Let γ be the supremum of the order types of $\preceq_n$, for $n < \omega$. We know that $\gamma < \omega_1^{CK}$ because $\langle \preceq_n : n < \omega \rangle$ is a an arithmetic sequence of well orderings. Think of γ as an initial segment of $\mathcal{L}$. The rank function of $\langle J_n, \leqslant_{\mathcal{J}} \rangle$, $rk_{\preceq_n} \colon J_n \to \gamma$ is recursive in $0^{2\gamma+2}$ by Lemma 2.3.1. Let $\alpha = 2\gamma + 2$.

Lemma 2.4.15. *There is a level function $lev \colon \mathcal{J} \to \mathcal{L}$ such that for each n, $lev \restriction J_n$ is recursive in 0^α.*

Proof: To simplify the definitions, add to $\preceq$ an element, ∞, on top: Let $\bar{J}_n = J_n \cup \{\infty\}$ and for all $x \in J_n$ set $x \preceq \infty$. Together with lev we define a sequence $\{\sigma_n\}_{n\in\omega} \subseteq \mathcal{L}$ and for each $y \in \ J_n$ an element $b_y^n \in \mathcal{L}$. We require that each $\sigma_n \notin \omega_1^{CK}$ (we identify ω_1^{CK} with the initial segment of $\mathcal{L}$ of order type ω_1^{CK}), that

$$\forall x \in J_n (x < y \Rightarrow lev(x) < b_y^n),$$

and that

$$lev(y) \geqslant b_y^n + \sigma_n.$$

Construction of lev: The construction is done by recursion on n. For $n = 0$, we have that $J_0 = \{\infty\}$. Let $b_\infty^0 = 0$ and let σ_0 be anything in $\mathcal{L} \setminus \omega_1^{CK}$. Define $lev(\infty) = \sigma_0$. Now suppose we have defined $lev(y)$, and b_y^n for all $y \in J_n$, recursively in 0^α and the finite sequence $\langle \sigma_0, ..., \sigma_n \rangle$. Let σ_{n+1} be such that $\sigma_{n+1} \notin \omega_1^{CK}$ and $\sigma_{n+1} \cdot (\alpha + 1) < \sigma_n$. Such a σ_{n+1} exists because the set $\{\beta \in \mathcal{L} : \beta \cdot (\alpha + 1) < \sigma_n\}$ is recursive and contains ω_1^{CK}, but since ω_1^{CK} is not recursive, there is some σ_{n+1} in that set which is not in ω_1^{CK}. For $x \in J_{n+1} \setminus J_n$, define:

$$\begin{aligned} \beta_x &= rk_{\preceq_{n+1}}(x) \ \in \omega_1^{CK} \subset \mathcal{L}; \\ y_x &= \mu y \in J_n (x \prec y); \\ lev(x) &= b_{y_x}^n + \sigma_{n+1} \cdot (\beta_x + 1); \\ b_x^{n+1} &= b_{y_x}^n + \sigma_{n+1} \cdot \beta_x. \end{aligned}$$

For $y \in J_n$, let $b_y^{n+1} = b_y^n + \sigma_{n+1} \cdot \alpha$. ◊

Since $rk_{\preceq_{n+1}}(x)$ can be found recursively in 0^α, we can find β_x recursively in 0^α. Also note that $y = \mu y \in J_n(x \prec y)$ can be found recursively in $0''$, because such a y always exists and $\langle J_n, \preceq \rangle \leqslant_T 0'$. Is easy to verify, by induction on n, that the construction does what we want. □

2.4.5 Putting the pieces together.

Proposition 2.4.16. *Let $\mathcal{J}$ be a countable pjusl such that its jump operation is total. There exists a countable pjusl $\mathcal{P}$ which extends $\mathcal{J}$ and supports a jump hierarchy R.*

Proof: Assume that $\mathcal{J}$ is recursive. Otherwise we can relativize the proof. Let α be as defined in the beginning of subsection 2.4.4 and let $\mathcal{P} = \mathcal{P}_\alpha$. By Lemma 2.4.15, $\mathcal{J}_i \in \mathcal{K}_\alpha$ for all $i \in \omega$, so, from Lemma 2.4.8 we get that $\mathcal{J}$ embeds into $\mathcal{P}_\alpha$. By Lemma 2.4.7, $\mathcal{P}_\alpha$ supports a jump hierarchy. □

Theorem 2.4.17. *Every countable jump upper semilattice can be embedded into $\mathcal{D}$.*

Proof: Immediate from the previous proposition and Theorem 2.2.2. □

2.5 Adding 0 to the Language

In this section we add 0 to the structure and we ask the same kind of questions we asked for jump upper semilattices. We are concerned with the following kind of structures.

Definition 2.5.1. A *partial jump upper semilattice with 0* is a structure $\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \cup, \mathrm{j}, 0\rangle$ such that $\langle J, \leqslant_{\mathcal{J}}, \cup, \mathrm{j}\rangle$ is a pjusl, 0 is the least element of $\langle J, \leqslant_{\mathcal{J}}\rangle$, and for all $n \in \omega$, $\mathrm{j}^n(0)$ is defined. A *jump upper semilattice with 0* is a pjusl with 0 where join and jump are total, and a *jump partial ordering with 0* is one where jump is total but join is undefined.

In this section $\mathcal{D}$ represents $\langle \mathbf{D}, \leqslant_T, \vee, ', 0\rangle$.

2.5.1 A negative answer.

The direct generalization of Theorem 2.4.17 to jusls with 0 is false.

Theorem 2.5.2. *Not every quantifier free 1-type of jusl with 0 is realizable in $\mathcal{D}$.*

Proof: We shall prove that there are continuum many quantifier free 1-types of jusl with 0 which contain a formula of the form $x \leqslant \mathrm{j}^n(0)$. But there are only countably many arithmetic Turing degrees. Therefore, not all of these types can be realized in $\mathcal{D}$.

Given a set $A \subseteq \omega$, we construct, $p_A(x)$, a quantifier free 1-type of jusl with 0. Put in $p_A(x)$ all the formulas

$$\mathrm{j}^n(x) \geqslant_{\mathcal{J}} \mathrm{j}^n(0), \qquad \mathrm{j}^n(x) \mid_{\mathcal{J}} \mathrm{j}^{n+1}(0), \qquad \mathrm{j}^n(x) \leqslant_{\mathcal{J}} \mathrm{j}^{n+2}(0),$$

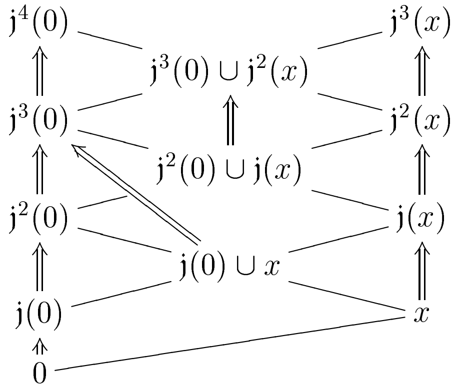
for all $n \in \omega$. Also, for each $n \in A$, add the formula

$$\mathrm{j}\big(\mathrm{j}^n(x) \cup \mathrm{j}^{n+1}(0)\big) = \mathrm{j}^{n+1}(x) \cup \mathrm{j}^{n+2}(0),$$

and for $n \notin A$, the formula

$$\mathrm{j}\big(\mathrm{j}^n(x) \cup \mathrm{j}^{n+1}(0)\big) = \mathrm{j}^{n+3}(0).$$

Of course, we add to $p_A(x)$ all the formulas which can be deduced form the ones already in $p_A(x)$.



In the picture above the reader can see how a realization of $p_A(x)$ would look like, and convince himself that $p_A(x)$ is consistent with the axioms of jusl with 0. (In the picture, the double arrows ($\Rightarrow$) represent the jump operator. In the example drawn, $0 \notin A$ but $1 \in A$.) It is also easy to see that for $A \neq B$, $p_A \neq p_B$. $\square$

Remark 2.5.3. If $p(x)$ is the type of an arithmetic degree $\mathbf{x} \in \mathbf{D}$, then necessarily $p(x) \leqslant_T 0^\omega$. Because given an index for a set in $\mathbf{x}$, all the quantifier free formulas of jusl with 0 can be decided uniformly in 0^ω.

Since realizing quantifier free n-types of jusl is equivalent to embedding jusl with n generators, we get the following corollary.

Corollary 2.5.4. *Not every countable jusl with 0 is embeddable into $\mathcal{D}$.*

2.5.2 A positive answer.

Now we consider jpo with 0. The situation here changes because there are only countably many quantifier free 1-types of jpo with 0 containing a formula of the form $x \leqslant \mathrm{j}^n(0)$. Moreover, all of these types are recursive.

We need a stronger version of Theorem 2.2.2.

Definition 2.5.5. Given a jpo $\mathcal{P}$, we say that $H : P \to \omega^\omega$ is *almost a jump hierarchy over $\mathcal{P}$* if for all $x \in P$

- $\mathcal{P} \restriction \mathrm{j}(x) \leqslant_T H(x)$, where $\mathcal{P} \restriction x$ is the restriction of $\mathcal{P}$ to $\{y \in P : y \leqslant_P x\}$.
- $\bigoplus_{y \in \mathcal{P} \restriction x} H(y) \leqslant_T H(x)$;
- $H(x)' \leqslant_T H(\mathrm{j}(x))$.

Theorem 2.5.6. *Suppose that $\mathcal{J}$ is a countable jpo that supports an almost jump hierarchy H. Then there is an embedding from $\mathcal{J}$ into $\mathcal{D}$ presented by $R : \mathcal{J} \to \omega^\omega$ such that*

$$\forall x, y \in J\big(H(x) \leqslant_T R(y) \Rightarrow H(x) \leqslant_T H(y)\big). \tag{2.5.1}$$

Proof: We construct R in the same way as in Theorem 2.2.2. We have to prove that an almost jump hierarchy is enough to guarantee that R is an embedding, and that we also get (2.5.1). To prove that R represents an embedding, we have to verify that the proof in section 2.2 works in the same way as there. We only used that $\mathcal{J} \leqslant_T H(x)$ for all $x \in \mathcal{J}$ in observation 2.2.15 and Lemma 2.2.17. Observe that in both cases we only needed that $\mathcal{J} \upharpoonright \mathfrak{j}(x) \leqslant_T H(x)$. We used that $x <_{\mathcal{J}} y \Rightarrow H(x)' \leqslant_T H(y)$ in corollary 2.2.19, but we only used that $H(x)' \leqslant_T H(\mathfrak{j}(x))$.

Let us prove now that (2.5.1) holds. Suppose that $H(x) = \{e\}^{R(y)}$. Then, there is some $p \in \mathbb{P}$ such that $p \Vdash \{e\}^{R(y)} = R(x)^{[0]}(= H(x))$. So, for every $q \leqslant p$ and $m \in \omega$ such that $\{e\}^{R_q(y)}(m)\downarrow$, we have that $\{e\}^{R_q(y)}(m) = H(x)(m)$. We also know that for every m there is some $q \leqslant p$ such that $\{e\}^{R_q(y)}(m)\downarrow$. Now, given $m \in \omega$, we can find $\bar{q} \in \mathbb{P}_{p,y}$ such that $\{e\}^{R_{\bar{q}}(y)}(m)\downarrow$, recursively in $H(y)$, because $\mathbb{P}_{p,y} \leqslant_T H(y)$. Then $H(x)(m) = \{e\}^{R_{\bar{q}}(y)}(m)$. This shows that $H(x) \leqslant_T H(y)$. $\square$

Definition 2.5.7. Given a jpo with 0 $\mathcal{J}$, the *archimedean part* of $\mathcal{J}$ is

$$J_a = \{x \in J : \exists n \in \omega \big(x \leqslant_{\mathcal{J}} \mathfrak{j}^n(0)\big)\}.$$

We say that $\mathcal{J}$ is *archimedean* if $J = J_a$. Observe that J_a is closed under jump. So, let $\mathcal{J}_a$ be the restriction to J_a of $\mathcal{J}$ as a jpo.

We say that a type of jpo with 0, $p(x_1, ..., x_n)$ is *archimedean* if for some $m \in \omega$ it contains the formula "$x_1 \leqslant \mathfrak{j}^m(0)$ & ... & $x_n \leqslant \mathfrak{j}^m(0)$".

Theorem 2.5.8. *Let $\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \mathfrak{j}, 0\rangle$ be a finitely generated jpo with 0 such that every pair $x, y \in J_a$ has a least upper bound. Then, any embedding of $\mathcal{J}_a$ into $\mathcal{D}$ extends to an embedding of $\mathcal{J}$ into $\mathcal{D}$ (not necessarily preserving join but preserving 0).*

Proof: Suppose that we have an embedding of $\mathcal{J}_a$ presented by $R\colon J_a \to \omega^\omega$. We start by defining a particular almost jump hierarchy, K, over $\mathcal{J}$. We need to begin with a couple of observations. First observe that, by corollary 2.4.14, since $\mathcal{J}$ is finitely generated, it is well founded. So, by Remark 2.3.3, there is a jump hierarchy, H, over $\mathcal{J}$ such that for all $x \in J$, $H(x) \geqslant_T R \oplus (\mathcal{J})'$. Second, say that $x \gg 0$ if $\forall n (x \geqslant_{\mathcal{J}} \mathfrak{j}^n(0))$. Now observe that for every $x \in J$ either $x \gg 0$ or there is a $x_a \in J_a$ such that

$$\forall y \in J_a (y \leqslant_{\mathcal{J}} x \Leftrightarrow y \leqslant_{\mathcal{J}} x_a).$$

This is because $\mathcal{J}$ is finitely generated: Let $\bar{a} = \{a_1, ..., a_{n-1}\}$ be a set of generators of $\mathcal{J}$, let $F = \{x_1, ..., x_{n_1}\}$, and suppose that $x \not\gg 0$. Then, there is some m such that $\mathfrak{j}^m(0) \nleqslant_T x$. So, each $y \leqslant_{\mathcal{J}} x$ has to be of the form $\mathfrak{j}^k(a_i)$ for some $i < n$ and $k < m$. Therefore, there are only finitely many $y \in J_a$ with $y \leqslant_{\mathcal{J}} x$. Let x_a be the least upper bound of $\{y \in J_a : y \leqslant_{\mathcal{J}} x\}$, which exists by hypothesis.

Now we define $K\colon \mathcal{J} \to \omega^\omega$ as follows

$$K(x) = \begin{cases} R(x_a) & \text{if } x \not\gg 0, \\ H(x) & \text{if } x \gg 0. \end{cases}$$

We claim that K is almost a jump hierarchy over $\mathcal{J}$. For each $x \in J$ we have to check the conditions in Definition 2.5.5. For $x \not\gg 0$ we have that $\mathcal{J} \restriction \mathrm{j}(x) \leqslant_T K(x)$ because $\mathcal{J} \restriction \mathrm{j}(x)$ is finite; we have that $\bigoplus_{y \in \mathcal{J} \restriction x} K(y) \leqslant_T K(x)$ because $\mathcal{J} \restriction x$ is finite and for all $y \leqslant_{\mathcal{J}} x$, $y_a \leqslant_{\mathcal{J}} x_a$; and we have that $K(x)' \leqslant_T K(\mathrm{j}(x))$ because $\mathrm{j}(x_a) \leqslant_{\mathcal{J}} (\mathrm{j}(x))_a$. For $x \gg 0$, we have that $\mathcal{J} \leqslant_T K(x)$ and that $K(x)' \leqslant_T K(\mathrm{j}(x))$ because H is a jump hierarchy over $\mathcal{J}$. To prove that $K(y) \leqslant_T K(x)$ uniformly in y observe that using $\mathcal{J}'$ we can decide whether $y \gg 0$, and if $y \not\gg 0$ we can find y_a. Then since R and $\bigoplus_{y \leqslant_{\mathcal{J}} x} H(y)$ are recursive $H(x) = K(x)$, we get that $K(y) \leqslant_T K(x)$ uniformly in y.

Now, by Theorem 2.5.6, there is an embedding $\mathcal{J} \to \mathcal{D}$ presented by some $R_1 \colon J \to \omega^\omega$ such that

$$\forall x\, y \in J\big(K(x) \leqslant_T R_1(y) \Rightarrow K(x) \leqslant_T K(y)\big). \tag{2.5.2}$$

Extend R to $\mathcal{J}$ by defining $R(x) = R_1(x)$ for all $x \in J \smallsetminus J_a$. R preserves the jump because it does it for $x \in J_a$ and it does it for $x \in J \smallsetminus J_a$. All we have to prove, to show that R represents an embedding of $\mathcal{J}$ into $\mathcal{D}$, is that for all $x \in J_a$ and $y \in J \smallsetminus J_a$ we have that

$$x \leqslant_{\mathcal{J}} y \Leftrightarrow R(x) \leqslant_T R(y)$$

If $y \gg 0$ then $x \leqslant_{\mathcal{J}} y$ and $R(x) \leqslant_T R(y)$. So, suppose that $y \not\gg 0$. First assume that $x \leqslant_{\mathcal{J}} y$. Then $x_a = x \leqslant_{\mathcal{J}} y_a$. Therefore

$$R(x) = R(y_a) \leqslant_T K(y_a) \leqslant_T R_1(y) = R(y).$$

Now suppose that $R(x) \leqslant_T R(y)$. Since $x \in J_a$, $R(x) = K(x)$, and since $y \notin J_a$, $R(y) = R_1(y)$. So, $K(x) \leqslant_T R_1(y)$. Then, by (2.5.2), $K(x) \leqslant_T K(y)$. Hence, we have that $R(x) \leqslant_T R(y_a)$. But we know that R restricted to $\mathcal{J}_a$ is an embedding, so $x \leqslant_{\mathcal{J}} y_a \leqslant_{\mathcal{J}} y$. $\square$

Corollary 2.5.9. *Every quantifier free 1-type of jpo with 0 is realized in $\mathcal{D}$.*

PROOF: We start by defining the notion of jump trace introduced in [HS91]. A *consistent jump trace* is a pair of sequences $(h_0, h_1, h_2, ...; ..., l_2, l_1, l_0)$ such that for all $k \in \omega$ $h_k \leqslant h_{k+1} \leqslant l_{k+1} \leqslant l_k \leqslant l_{k+1} + 1$. The *jump trace* of an arithmetic degree $\mathbf{x}$ is $(h_0, h_1, ...; ...l_1, l_0)$ where h_i is the greatest h such that $\mathbf{x}^{(i)} \geqslant_T 0^{(i+h)}$, and l_i is the least l such that $\mathbf{x}^{(i)} \leqslant_T 0^{(i+l)}$ is in $p(x)$. Given $p(x)$, an archimedean type of jpo with 0 we can associate to it the jump trace $(h_0, h_1, ...; ...l_1, l_0)$ where h_i is the greatest h such that "$\mathrm{j}^i(x) \geqslant \mathrm{j}^{i+h}(0)$" is in $p(x)$, and l_i is the least l such that "$\mathrm{j}^i(x) \leqslant \mathrm{j}^{i+l}(0)$" is in $p(x)$. It is easy to see that an arithmetic degree $\mathbf{x}$ realizes $p(x)$ if and only if $\mathbf{x}$ and $p(x)$ have the same jump trace. Hinman proved in [Hin99], finishing the cases left by Hinman and Slaman in [HS91], that every consistent jump trace is realizable in $\mathcal{D}$. Hence every archimedean quantifier free 1-type of jpo with 0 is realizable in $\mathcal{D}$.

Now let $p(x)$ be a quantifier free 1-type of jpo with 0 and suppose that no formula of the form "$x \leqslant \mathrm{j}^m(0)$" is in $p(x)$. Consider a jpo with 0, $\mathcal{J}$, with one generator a, such that $\mathcal{J} \models p(a)$. By our assumption on $p(x)$, $a \notin J_a$, and hence $J_a = \{0, \mathrm{j}(0), \mathrm{j}^2(0), ...\}$. Obviously, $\mathcal{J}_a$ embeds into $\mathcal{D}$, and every pair of elements in $\mathcal{J}_a$ has a least upper bound. So, by Theorem 2.5.8, the embedding of $\mathcal{J}_a$ into $\mathcal{D}$ extends to $\mathcal{J}$. Therefore, $p(x)$ is realizable in $\mathcal{D}$. □

Lemma 2.5.10. *Every finitely generated archimedean jpo with 0, $\mathcal{P} = \langle P, \leqslant_{\mathcal{P}}, \mathrm{j}, 0\rangle$, can be embedded into a finitely generated archimedean jpo with 0, $\mathcal{J}$, such that every pair of elements has a least upper bound.*

Proof: The idea is to consider the usl with 0 generated by $\mathcal{P}$ and define the jump operator on it by imposing that $\mathrm{j}(x \cup y) = \mathrm{j}(x) \cup \mathrm{j}(y)$. Let $J' = \{F \subset P : F \text{ finite } \& \ F \neq \emptyset\}$ and define an order on J' as follows:

$$F \leqslant' G \qquad \Leftrightarrow \qquad \forall x \in F \ \ \exists y \in G(x \leqslant_{\mathcal{P}} y)$$

Observe that $\leqslant'$ is transitive and reflexive. Say that F is equivalent to G, $F \equiv G$, if $F \leqslant' G$ & $G \leqslant' F$, and write $[F]$ for the equivalence class of F. Let $J = J'/\equiv$, define $[F] \leqslant [G] \Leftrightarrow F \leqslant' G$, and $[F] \vee [G] = [F \cup G]$. It is easy to show that $\mathcal{J}$ is an usl with 0 and that the map that sends $x \in P$ into $[\{x\}]$ is an embedding of $\mathcal{P}$ into $\mathcal{J}$. Define a jump operation on $\mathcal{J}$ as follows:

$$\mathrm{j}([F]) = [\{\mathrm{j}(x) : x \in F\} \cup \{0\}]$$

One can easily check that j is well defined, that it is monotone and strictly increasing and that $\mathcal{J}$ is archimedean.

Now we need to prove that $\mathcal{J}$ is finitely generated as a jpo with 0. Let $\{a_1, ..., a_n\}$ be a set of generators of $\mathcal{P}$. Let m be such that all the generators of $\mathcal{P}$ are below $\mathrm{j}^m(0)$. We claim that the set

$$A = \{[F] : F \subseteq \{\mathrm{j}^i(a_j) : i = 0, ..., m-1; \ \ j = 1, ..., n\}\}$$

generates $\mathcal{J}$. Take any $[G] \in \mathcal{J}$. G is equivalent to some $G_1 = \{x_1, ..., x_k\}$ such that $\forall i \neq j (x_i \not\leqslant_{\mathcal{P}} x_j)$. Each x_i is of the form $\mathrm{j}^{r_i}(a_{s_i})$ for some r_i and for some generator a_{s_i}. Let $r = \min\{r_1, ..., r_k\}$, suppose, without lost of generality, that $r = r_1$. Then

$$[G] = \mathrm{j}^r([F]) \qquad \text{where } F = \{\mathrm{j}^{r_i - r}(a_{s_i}) : i = 1, ..., k\}.$$

We have to that for all $i = 1, ..., k$, $r_i - r < m$. Suppose that $r_i - r \geqslant m$, then $\mathrm{j}^{r_i - r}(a_{s_i}) \geqslant_P \mathrm{j}^m(0) \geqslant_{\mathcal{P}} a_{s_1}$. Therefore $x_i = \mathrm{j}^{r_i}(a_{s_i}) \geqslant \mathrm{j}^r(a_{s_1}) = x_1$, contradicting our assumption on $G_1 = \{x_1, ..., x_k\}$. □

Corollary 2.5.11. *If every finitely generated archimedean jpo with 0 can be embedded into $\mathcal{D}$, then every finitely generated jpo with 0 can be embedded into $\mathcal{D}$. Equivalently: If every archimedean quantifier free type of jpo with 0 is realizable in $\mathcal{D}$, then every quantifier free type of jpo with 0 is realizable in $\mathcal{D}$.*

Proof: Let $\mathcal{P}$ be a finitely generated jpo with 0. Let $\bar{\mathcal{J}}$ be an extension of $\mathcal{P}_a$ as in the previous lemma. Let $\mathcal{J}$ be the jpo with 0 obtained by amalgamating $\mathcal{P}$ and $\bar{\mathcal{J}}$ as in Lemma 2.4.6. Note that $\mathcal{J}$ is still finitely generated, and that its archimedean part is $\bar{\mathcal{J}}$, in which every pair of elements has a least upper bound. By hypothesis $\mathcal{J}_a = \bar{\mathcal{J}}$ can be embedded into $\mathcal{D}$. Then, by Theorem 2.5.6, $\mathcal{J}$ can be embedded into $\mathcal{D}$. Hence $\mathcal{P}$ can be embedded too. □

2.6 Uncountable jump upper semilattices

So far we have studied countable pjusls. Now, given κ, with $\aleph_0 \leqslant \kappa \leqslant 2^{\aleph_0}$, we address the following question: Is every jusl with the size κ and the c.p.p. embeddable in $\mathcal{D}$? In the first subsection we answer this question negatively for $\kappa = 2^{\aleph_0}$. In the second subsection we answer this question positively for κ such that MA(κ) holds.

2.6.1 A negative answer.

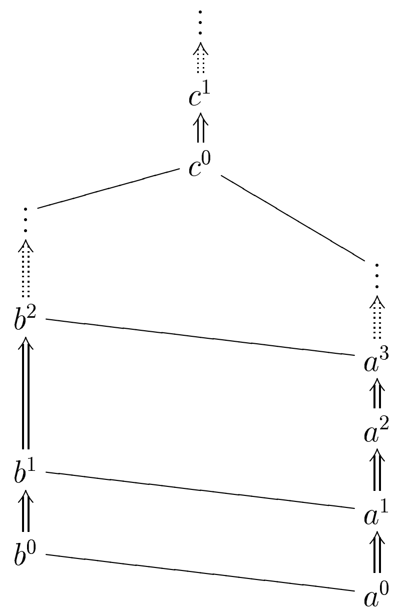
We construct a jpo of size $2^{\aleph_0}$ which cannot be embedded into the degrees.

Definition 2.6.1. Given a strictly increasing function $f\colon \omega \to \omega$, we define a jpo $\mathcal{P}_f = \langle P_f, \leqslant, \mathrm{j}\rangle$ as follows:

- $P_f = \{a^i : i \in \omega\} \cup \{b^i : i \in \omega\} \cup \{c^i : i \in \omega\}$.
- $\mathrm{j}(a^i) = a^{i+1}$, $\mathrm{j}(b^i) = b^{i+1}$ and $\mathrm{j}(c^i) = c^{i+1}$ for all $i \in \omega$.
- $a^i < a^j$ iff $i < j$, $b^i < b^j$ iff $i < j$, and $c^i < c^j$ iff $i < j$.
- $a^i < c^j$ and $b^i < c^j$ for all i, j.
- $a^i \leqslant b^j$ iff $f(j) \geqslant i$.
- for all $i, j \in \omega$, $b^i \not\leqslant a^j$, $c^i \not\leqslant a^j$ and $c^i \not\leqslant b^j$.

In the figure below we draw an example where $f(0) = 0$, $f(1) = 1$, $f(2) = 3$,...

(The double arrows ($\Rightarrow$) represent the jump operator.)



It is easy to see that, for every strictly increasing f, $\mathcal{P}_f$ is a jpo.

Lemma 2.6.2. *Let $f\colon \omega \to \omega$ be strictly increasing, and let ψ be an embedding of $\mathcal{P}_f$ into $\mathcal{D}$. Then $\psi(c^3) \geqslant_T f$.*

PROOF: Let A be a member of $\psi(a^0)$, B be a member of $\psi(b^0)$ and C be a member of $\psi(c^0)$. Since for all $i \in \omega$, $A^{(i)}$ and $B^{(i)}$ are recursive in C, there are functions g and h, recursive in $C^{(3)}$ (actually recursive in $C^{(2)}$ too), such that

$$A^{(i)} = \{g(i)\}^C \qquad \text{and } B^{(i)} = \{h(i)\}^C.$$

Therefore, we can decide whether $B^{(j)} \geqslant_T A^{(i)}$ recursively in $C^{(3)}$, uniformly in i and j. So, we can compute f from $C^{(3)}$. $\square$

Definition 2.6.3. Let d be a new symbol and $\mathcal{J}$ be the jpo with generator d (i.e. $J = \{d, \mathrm{j}(d), \mathrm{j}^2(d), ...\}$), and let $\mathcal{F}$ be the set of all strictly increasing functions from ω into itself. Define

$$\mathcal{P} = \mathcal{J} \oplus \bigoplus_{f \in \mathcal{F}} \mathcal{P}_f.$$

In other words: the domain of $\mathcal{P}$ is the disjoint union of $\mathcal{J}$ and all the $\mathcal{P}_f$ with $f \in \mathcal{F}$; the jump operation is defined in the obvious way; and the $\leqslant$ relation in $\mathcal{P}$ is the disjoint union of the $\leqslant$ relations of each jpo.

Proposition 2.6.4. *$\mathcal{P}$ cannot be embedded into $\mathcal{D}$.*

PROOF: Suppose that there is an embedding $\psi\colon \mathcal{P} \to \mathcal{D}$. In the degree $\mathbf{d} = \psi(d)$ there is some $f \in \mathcal{F}$. Let $\mathbf{c}^3_f$ be the image under ψ of the element c^3 of $\mathcal{P}_f$ (call that element c^3_f). Then, by the previous lemma, $\mathbf{d} = \deg(f) \leqslant \mathbf{c}^3_f$. This contradicts the fact that ψ is an embedding since d and c^3_f are incomparable. $\square$

2.6.2 A positive answer

Now, we prove that if MA(κ) holds, then every jusl with the c.p.p. and size κ can be embedded into $\mathcal{D}$. The idea is the if we have a pjusl of size κ, with the c.p.p. and supporting an almost jump hierarchy, we can carry out the forcing construction of Section 2.2 as long as we can get generic enough filters. MA(κ) give us the existence of such generic filters.

The hard part is to prove that every pjusl with the c.p.p. extends to another one, also with the c.p.p., which supports an almost jump hierarchy (ajh) and has the same cardinality. We start by proving some facts we will use about end extensions and amalgamations of pjusls. (We say that a partial order $\mathcal{P}$ is an *end extension* of $\mathcal{Q}$ if $\mathcal{Q} \subseteq \mathcal{P}$ and $\mathcal{Q}$ is closed downward in $\mathcal{P}$.)

In this section, the jump operation of every pjusl is total.

Definition 2.6.5. Given pjusls A, A_1 and A_2, and embeddings $\varphi_1 \colon A \to A_1$ and $\varphi_2 \colon A \to A_2$, let $A_1 \oplus_{A,\varphi_1,\varphi_2} A_2$ be the structure defined in Lemma 2.4.6. We write $A_1 \oplus_A A_2$ if φ_1 and φ_2 are clear from the context, and we write $A_1 \oplus A_2$ when $A = \emptyset$.

In Lemma 2.4.6 we also constructed two embeddings, $\psi_1 \colon A_1 \to A_1 \oplus_A A_2$ and $\psi_1 \colon A_1 \to A_1 \oplus_A A_2$, such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. Observe that if φ_1 and φ_2 are inclusions, we can think of ψ_1 and ψ_2 as inclusions too.

Lemma 2.6.6. *Let A, A_1, A_2, φ_1 and φ_2 be as in the definition above. Then:*

1. *Given a pjusl B and two homomorphisms (of pjusl) $\chi_1 \colon A_1 \to B$ and $\chi_2 \colon A_2 \to B$ such that $\chi_1 \circ \varphi_1 = \chi_2 \circ \varphi_2$, there is a unique homomorphism $\chi : A_1 \oplus_A A_2 \to B$ such that the following diagram commutes.*

$$\begin{tikzcd}
& A_1 \arrow[dr, "\psi_1"] \arrow[drr, "\chi_1"] & & \\
A \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & A_1 \oplus_A A_2 \arrow[r, dotted, "\chi"] & B \\
& A_2 \arrow[ur, "\psi_2"] \arrow[urr, "\chi_2"'] & &
\end{tikzcd}$$

2. *If A_1 is an end extension of A, then $A_1 \oplus_A A_2$ is an end extension of A_2.*

3. *If A_1 is an end extension of $A_1 \setminus A$, then $A_1 \oplus_A A_2$ is an end extension of $A_1 \setminus A$.*

4. *If A_1 and A_2, both have the c.p.p., then so does $A_1 \oplus_A A_2$.*

Lemma 2.6.7. *If $\langle A_\xi : \xi < \alpha \rangle$ is a chain of pjusl with the c.p.p. such that for all $\beta < \gamma < \alpha$, A_γ is an end extension of A_β, then $A = \bigcup_{\xi<\alpha} A_\xi$ is a pjusl which is an end extension of each A_ξ and has the c.p.p.*

The proofs of these lemmas are straightforward.

Lemma 2.6.8. *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two countable pjusls, such that $\mathcal{J}_2$ is an end extension of $\mathcal{J}_1$. Let H and K be ajhs over $\mathcal{J}_1$ and $\mathcal{J}_2 \smallsetminus \mathcal{J}_1$ respectively, such that $\forall x \in J_2 \smallsetminus J_1(\mathcal{J}_1 \oplus H \leqslant_T K(x))$. Define $R\colon \mathcal{J}_2 \to \omega^\omega$ by $R(x) = H(x)$ if $x \in J_1$ and $R(x) = K(x)$ if $x \in J_2 \smallsetminus \mathcal{J}_1$. Then R is an ajh over $\mathcal{J}_2$.*

PROOF: Just check the conditions of Definition 2.5.5. $\square$

Lemma 2.6.9. *Let $\mathcal{Q}$ and $\mathcal{J}$ be two countable pjusls, such that $\mathcal{J}$ is an end extension of $\mathcal{Q}$. Let H be an almost Jump Hierarchy over $\mathcal{Q}$. Then, there is a pjusl $\mathcal{P}$ extending $\mathcal{J}$ which is an end extension of $\mathcal{Q}$ and supports an almost jump hierarchy extending H.*

PROOF: Let $\bar{\mathcal{P}}$ be a pjusl extending $\mathcal{J}$ and supporting an almost jump hierarchy K such that $\forall x \in \bar{\mathcal{P}}(K(x) \geqslant_T H \oplus \mathcal{J})$. (A relativized version of Proposition 2.4.16 would give us such a $\bar{\mathcal{P}}$.) Let $\mathcal{P} = \bar{\mathcal{P}} \oplus_{\mathcal{J} \smallsetminus \mathcal{Q}} J$. Note that $\mathcal{J} \smallsetminus \mathcal{Q}$ is a pjusl and is closed under jump because $\mathcal{J}$ is an end extension of $\mathcal{Q}$. Also observe that, since $\mathcal{J}$ is an end extension of $\mathcal{Q} = \mathcal{J} \smallsetminus (\mathcal{J} \smallsetminus \mathcal{Q})$, $\mathcal{P}$ is an end extension of $\mathcal{Q}$. Now define $R\colon \mathcal{P} \to \omega^\omega$ by $R(x) = H(x)$ if $x \in Q$ and $R(x) = K(x)$ if $x \in \bar{\mathcal{P}}$. By lemma 2.6.8, R is an ajh over $\mathcal{P}$ extending H. $\square$

Lemma 2.6.10. *Let $\mathcal{Q}$ and $\mathcal{J}$ be two pjusls, such that $\mathcal{J}$ is an end extension of $\mathcal{Q}$ and let $\kappa = |J|$. Let H be an almost Jump Hierarchy over $\mathcal{Q}$. Then, there is a pjusl $\mathcal{P}$ extending $\mathcal{J}$ which is an end extension of $\mathcal{Q}$, has size κ, and supports an ajh extending H.*

PROOF: We use induction on κ. When $\kappa = \aleph_0$, the result is given by the previous lemma. Now suppose that $\kappa > \aleph_0$ and that the lemma is true for all cardinals smaller that κ. Let $\{a_\xi : \xi < \kappa\}$ be a well ordering of the elements of $\mathcal{J}$, let $\mathcal{J}_\gamma$ be the downward closure of the pjusl generated by $\{a_\xi : \xi < \gamma\}$, and let $\mathcal{Q}_\gamma = \mathcal{Q} \cap \mathcal{J}_\gamma$. Note that $|\mathcal{J}_\gamma| = |\gamma| + \aleph_0$. Now, we construct a sequence $\{\mathcal{P}_\gamma\}_{\gamma \leqslant \kappa}$ as in the figure below (where $A \hookrightarrow B$ indicates that B is an end extension of A). We do it by induction on γ, and we want the sequence to have the following properties.

- $\mathcal{P}_\gamma$ supports an ajh, K_γ.
- if $\beta < \gamma$ then $\mathcal{P}_\gamma$ is an end extension of $\mathcal{P}_\beta$, and $K_\beta \subseteq K_\gamma$.
- $\mathcal{P}_\gamma$ extends $\mathcal{J}_\gamma$.
- $\mathcal{P}_\gamma$ is an end extension of Q_γ and K_γ extends $H \restriction \mathcal{Q}_\gamma$.
- $|\mathcal{P}_\gamma| \leqslant |\gamma|$.

$$
\begin{tikzcd}
& \mathcal{P}_0 \arrow[r, hook] & \mathcal{P}_1 \arrow[r, hook] & \cdots\cdots & \cdots\mathcal{P}_\gamma \arrow[r, hook] & \mathcal{P}_{\gamma+1} \arrow[r, hook] & \cdots\cdots & \mathcal{P} \\
\mathcal{J}_0 \arrow[r, hook] \arrow[ur] & \mathcal{J}_1 \arrow[r, hook] \arrow[ur] & \cdots\cdots & \cdots\mathcal{J}_\gamma \arrow[r, hook] \arrow[ur] & \mathcal{J}_{\gamma+1} \arrow[r, hook] \arrow[ur] & \cdots\cdots & \mathcal{J} \arrow[ur] & \\
\mathcal{Q}_0 \arrow[r, hook] \arrow[u, hook] \arrow[uur] & \mathcal{Q}_1 \arrow[r, hook] \arrow[u, hook] \arrow[uur] & \cdots\cdots & \cdots\mathcal{Q}_\gamma \arrow[r, hook] \arrow[u, hook] \arrow[uur] & \mathcal{Q}_{\gamma+1} \arrow[r, hook] \arrow[u, hook] \arrow[uur] & \cdots\cdots & \mathcal{Q} \arrow[u, hook] \arrow[uur] &
\end{tikzcd}
$$

Suppose that $\mathcal{J}_0 = \mathcal{Q}_0 = \emptyset$ and let $\mathcal{P}_0$ be empty too. Now assume we have defined $\mathcal{P}_\gamma$ and we want to define $\mathcal{P}_{\gamma+1}$. (We do it as in the diagram below.) Let $H_\gamma = \mathcal{K}_\gamma \cup H \restriction Q_{\gamma+1}$; it is an ajh over $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$. Let $\mathcal{P}_{\gamma+1}$ be an extension of $\mathcal{P}_\gamma \oplus_{\mathcal{J}_\gamma} \mathcal{J}_{\gamma+1}$, such that $\mathcal{P}_{\gamma+1}$ is an end extension of $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$ and supports an ajh, $\mathcal{K}_\gamma$, extending H_γ. We know such a $\mathcal{P}_{\gamma+1}$ exists because $|\mathcal{P}_\gamma \oplus_{\mathcal{J}_\gamma} \mathcal{J}_{\gamma+1}| \leqslant |\gamma| < \kappa$. Moreover we can get $\mathcal{P}_{\gamma+1}$ of size $\leqslant |\gamma|$. Note that since both $\mathcal{P}_\gamma$ and $\mathcal{Q}_{\gamma+1}$ are end extensions of $\mathcal{Q}_\gamma$, $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$ is an end extension of both $\mathcal{P}_\gamma$ and $\mathcal{Q}_{\gamma+1}$. Now, since $\mathcal{P}_{\gamma+1}$ is an end extension of $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$, it is also an end extension of both $\mathcal{P}_\gamma$ and $\mathcal{Q}_{\gamma+1}$.

$$
\begin{tikzcd}
\mathcal{P}_\gamma \arrow[rr, hook] \arrow[ddrr, dashed] & & \mathcal{P}_\gamma \oplus_{\mathcal{J}_\gamma} \mathcal{J}_{\gamma+1} \arrow[r, dotted] & \mathcal{P}_{\gamma+1} \\
\mathcal{J}_\gamma \arrow[u] \arrow[r, hook] & \mathcal{J}_{\gamma+1} \arrow[ur] \arrow[dr, dashed] & & \\
& & \mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1} \arrow[uu, hook] \arrow[uur, dotted, hook] & \\
\mathcal{Q}_\gamma \arrow[uu, hook] \arrow[uuu] \arrow[r, hook] & \mathcal{Q}_{\gamma+1} \arrow[uu, hook] \arrow[ur, hook] & &
\end{tikzcd}
$$

When γ is a limit ordinal let $\mathcal{P}_\gamma = \bigcup_{\xi<\gamma} \mathcal{P}_\xi$ and $K_\gamma = \bigcup_{\xi<\gamma} K_\xi$. It is easy to check that $\{\mathcal{P}_\gamma\}_{\gamma \leqslant \kappa}$ has the properties mentioned above and that $\mathcal{P} = \mathcal{P}_\kappa$ is as wanted. $\square$

Proposition 2.6.11. *Every pjusl with the c.p.p. can be extended to one of the same cardinality which also has the c.p.p. and supports an ajh.*

Proof: Apply the previous lemma with $\mathcal{Q} = \emptyset$. $\square$

Now we use Martin's Axiom to prove that some uncountable jusls can be embedded into $\mathcal{D}$.

Definition 2.6.12. MA(κ) is the statement: Whenever $\langle \mathbb{P}, \leqslant \rangle$ is a non-empty c.c.c. partial order, and $\mathcal{F}$ is a family of $\leqslant \kappa$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathcal{F}(G \cap D \neq \emptyset)$. We say that a p.o. has the *countable chain condition* (c.c.c.) if every antichain is at most countable.

It is consistent with ZFC that $2^{\aleph_0} > \aleph_1$ and MA(λ) for all $\lambda < 2^{\aleph_0}$ (see [Jec03, Theorem 16.13]).

Proposition 2.6.13. *If MA(κ) holds, then every jusl with the c.p.p. and of size $\leqslant \kappa$ can be embedded into $\mathcal{D}$.*

PROOF: Consider a jusl $\mathcal{Q}$ with the c.p.p. and of size $\leqslant \kappa$. By Proposition 2.6.11, there is a pjusl $\mathcal{J}$, extending $\mathcal{Q}$, which supports an ajh H, has cardinality $\leqslant \kappa$ and has the c.p.p. We claim that the construction done in Section 2.2 works for $\mathcal{J}$ too. Therefore, we would get that $\mathcal{J}$, and hence $\mathcal{Q}$, can be embedded into $\mathcal{D}$. Note that we do not have a jump hierarchy here, but an almost jump hierarchy. As mentioned in Theorem 2.5.6, this is not a problem.

Let $\mathbb{P}$ be the partial order defined in Subsection 2.2.1. $\mathbb{P}$ is a set of pairs of finite partial functions from $J \times \omega$ to ω. Actually we can view $\mathbb{P}$ as a set of finite partial functions from $J \times \omega \times 2$ to ω. Such a partial ordering always has the c.c.c. (see [Kun80, Lemma VII.5.4]). Now consider the set of all first order formulas in the language with signature $\{0, S, +, \cdot, <\} \cup \{R_p(x)(\cdot), Sk_p(x)(\cdot) : p \in \mathbb{P}, x \in \mathcal{J}\}$. For every such a formula we can define what it means that $p \in \mathbb{P}$ forces it (see [SWa]). We know that given a formula φ, $\{p : p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ is dense in $\mathbb{P}$, and, since $\mathbb{P}$ and $\mathcal{J}$ have cardinality $\leqslant \kappa$, there are at most κ such formulas. Hence, because of MA(κ), there is a filter G in $\mathbb{P}$ such that every such formula is decided by some element of G. This generic filter G satisfies all the properties that we needed in Section 2.2. Therefore, in particular, it gives us an embedding from $\mathcal{J}$ into $\mathcal{D}$. □

Corollary 2.6.14. *Whether every jpo (or jusl) with the c.p.p. and size $\aleph_1$ is embeddable into $\mathcal{D}$ is independent of ZFC.*

PROOF: On the one hand, we get from 2.6.4 that, if CH holds, not every jpo of size $\aleph_1 = 2^{\aleph_0}$ with the c.p.p. is embeddable in $\mathcal{D}$. On the other hand, if MA($\aleph_1$) holds, we just proved that every jusl with the c.p.p. and size $\aleph_1$ is embeddable into $\mathcal{D}$. □

Chapter 3
There is no order in the Generalized High/Low Hierarchy.

This chapter will appear in the Archive of Mathematical Logic.

3.1 Introduction

The high/low hierarchy was introduced by Soare in [Soa74] and independently by Cooper in a preprint of [Coo74] with the idea of classifying the Turing degrees below $\mathbf{0}'$ depending on how close they are to being recursive and how close they are to being complete. This classification has been very helpful in the study of the structure of the Δ_2^0 Turing degrees. A generalization of this classification to all the Turing degrees is the generalized high/low hierarchy introduced by Jockusch and Posner in [JP78]. Many properties have been proved about members of certain classes in this hierarchy. To cite a few: every 1-generic set is GL_1 (see [Ler83, IV.2]); every minimal degree is GL_2 [JP78]; every non-GL_2 cups to every degree above it [JP78]; every GH_1 degree bounds a minimal degree [Joc77] but not every GH_2 [Ler86]; every GH_1 degree has the complementation property [GMS] (Chapter 4 of this thesis).

In [Ler85], Lerman proved that the $\exists$-theory of the Δ_2^0 Turing degrees in the language $\mathcal{L}_H$, which has a relation for the Turing reduction, constants for $\mathbf{0}$ and $\mathbf{0}'$, and one unary relation for each class in the high/low hierarchy, is decidable. In that paper he leaves as an open question the decidability of the $\exists$-theory of the Turing Degrees in the language with predicates for the classes in the generalized high/low hierarchy. We prove here that $\exists$-theory of the Turing Degrees in the language $\mathcal{L}_{\mathrm{GH}}$, which has relations for the classes in the generalized high/low hierarchies instead of the high/low hierarchy and does not have a constant for $\mathbf{0}'$ is decidable. The language $\mathcal{L}_{\mathrm{GH}}$ does not contain a relation symbol for GH_0 ($\mathbf{x} \in \mathrm{GH}_0 \Leftrightarrow \mathbf{x} \geqslant \mathbf{0}'$), and whether the $\exists$-theory of the Turing Degrees in the language $\mathcal{L}_{\mathrm{GH}_0}$ with a symbol for GH_0 is decidable or not is unknown. A proof of this decidability result would probably use different techniques than ours.

The result we are proving, like Lerman's, is also interesting because it helps to understand how the degrees from the various classes of the generalized high/low hierarchy are located in the poset of the Turing Degrees. To prove it we show that every finite GH-poset can be embedded in the Turing Degrees. A GH-poset is a poset labeled with elements of $\mathcal{G}^*$, satisfying certain trivial conditions. $\mathcal{G}^*$ is the partition of $\mathcal{D}$ induced by the generalized high/low hierarchy (see Definition 3.1.1). The GH-posets generalize Lerman's H-posets, where the labels are elements of $\mathcal{C}^*$. $\mathcal{C}^*$ is the partition of the Δ_2^0 degrees induced by the high/low hierarchy. An important condition that H-poset have to satisfy is that if $x \leqslant y$ are elements of the H-poset, then the label of x is less than or equal the label of y with respect to the ordering of $\mathcal{C}*$. (The ordering of $\mathcal{C}^*$ is the obvious one. We define it below.)

No analogous of this condition has to be satisfied by GH-posets. This is why we say that there is no ordering on the classes in the generalized high/low hierarchy.

The proof is divided into two parts. In section 3.2 we analyze the problem and reduce it to a technical proposition which is an extension of Harrington's ZBC Lemma. One of the main tools in simplifying the problem is Lerman's Bounding Lemma [Ler85, 2.8]. We prove our technical result in section 3.3.

We have to note that the decidability of the $\exists$-theory of the Turing degrees in the language $\mathcal{L}_{GH}$, without a symbol for $\mathrm{GI}(\cdot)$, would follow from the decidability of the $\exists$-theory of $\langle \mathbf{D}, \leqslant, \vee, ', \mathbf{0}\rangle$. But this problem is still open. Another observation is that the $\exists$-theory of the Turing degrees in the language which has a relation for the Turing reduction and one binary relation for each class in the Relativized generalized high/low hierarchy (but does not have a constant for $\mathbf{0}$), is decidable. If we remove the symbol $\mathrm{GI}(\cdot,\cdot)$ from the language, this follows from the decidability of the $\exists$-theory of $\langle \mathbf{D}, \leqslant, \vee, ' \rangle$, which was proved in Chapter 2. Otherwise we have to use that every countable jump upper semilattice can be embedded in the Turing degrees, which was also proved in Chapter 2.

Basic Notions

Define

$$\mathcal{C} = \{\mathrm{L}_1, \mathrm{L}_2, ...\} \cup \{I\} \cup \{\mathrm{H}_1, \mathrm{H}_2, ...\},$$

where L_n is the class of low$_n$ degrees, I the class of intermediate degrees, and H_n the class of high$_n$ degrees. A degree $\mathbf{x} \leqslant \mathbf{0}'$ is *low$_n$* if $\mathbf{x}^{(n)} = \mathbf{0}^{(n)}$, is *high$_n$* if $\mathbf{x}^{(n)} = \mathbf{0}^{(n+1)}$, and is *intermediate* if $\forall n\big(\mathbf{0}^{(n)} <_T \mathbf{x}^{(n)} <_T \mathbf{0}^{(n+1)}\big)$. Note that for all n, $\mathrm{L}_n \subseteq \mathrm{L}_{n+1}$, $\mathrm{H}_n \subseteq \mathrm{H}_{n+1}$, and L_n, H_n and I are disjoint. These classes induce a partition, $\mathcal{C}^*$, of the degrees $\leqslant \mathbf{0}'$.

$$\mathcal{C}^* = \{\mathrm{L}_1^*, \mathrm{L}_2^*, ...\} \cup \{I^*\} \cup \{\mathrm{H}_1^*, \mathrm{H}_2^*, ...\},$$

where $\mathrm{L}_1^* = \mathrm{L}_1$, $\mathrm{H}_1^* = \mathrm{H}_1$, $I^* = I$ and for $n > 1$, $\mathrm{L}_n^* = \mathrm{L}_n \smallsetminus \mathrm{L}_{n-1}$, and $\mathrm{H}_n^* = \mathrm{H}_n \smallsetminus \mathrm{H}_{n-1}$. We define an ordering, $\prec$, on $\mathcal{C}^*$ as follows:

$$\mathrm{L}_1^* \prec \mathrm{L}_2^* \prec \cdots \prec I^* \prec \cdots \prec \mathrm{H}_2^* \prec \mathrm{H}_1^*.$$

Observe that if $\mathbf{x} \leqslant \mathbf{y}$, $\mathbf{x} \in X \in \mathcal{C}^*$ and $\mathbf{y} \in Y \in \mathcal{C}^*$, then $X \preceq Y$. Let $\mathcal{L}_H$ be the first order language with a binary relation $\leqslant$, two constant symbols $\mathbf{0}$ and $\mathbf{0}'$, and an unary relation for each class in $\mathcal{C}$. Lerman proved that every existential formula of $\mathcal{L}_H$ which is consistent with the observation above, and consistent with the axioms of partial orderings with bottom and top elements, $\mathbf{0}$ and $\mathbf{0}'$, is true about the degrees below $\mathbf{0}'$, and also about the r.e. degrees.

As a generalization of these notions to all the Turing degrees we get the *generalized high/low hierarchy.*

Definition 3.1.1. For $n \geqslant 1$ we say that a degree $\mathbf{x}$ is *generalized low$_n$*, or GL_n, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee \mathbf{0}')^{(n-1)}$. We say that a degree $\mathbf{x}$ is a *generalized high$_n$ degree*, or GH_n, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee \mathbf{0}')^{(n)}$, and it is *generalized intermediate*, or GI, if $\forall n\big((\mathbf{x} \vee \mathbf{0}')^{(n-1)} <_T \mathbf{x}^{(n)} <_T (\mathbf{x} \vee \mathbf{0}')^{(n)}\big)$. Let

$$\mathcal{G} = \{\mathrm{GL}_1, \mathrm{GL}_2, ...\} \cup \{\mathrm{GI}\} \cup \{\mathrm{GH}_1, \mathrm{GH}_2, ...\}.$$

and

$$\mathcal{G}^* = \{\mathrm{GL}_1^*, \mathrm{GL}_2^*, ...\} \cup \{\mathrm{GI}^*\} \cup \{\mathrm{GH}_1^*, \mathrm{GH}_2^*, ...\},$$

where $\mathrm{GL}_1^* = \mathrm{GL}_1$, $\mathrm{GH}_1^* = \mathrm{GH}_1$, $\mathrm{GI}^* = \mathrm{GI}$ and for $n > 1$, $\mathrm{GL}_n^* = \mathrm{GL}_n \setminus \mathrm{GL}_{n-1}$, and $\mathrm{GH}_n^* = \mathrm{GH}_n \setminus \mathrm{GH}_{n-1}$. Let $\mathcal{L}_{\mathrm{GH}}$ be the first order language with a binary relation $\leqslant$, a constant symbol $\mathbf{0}$, and a unary relation for each class in $\mathcal{G}$.

We make two observations. The first one is that, as in the high/low hierarchy, for all n, $\mathrm{GL}_n \subseteq \mathrm{GL}_{n+1}$, $\mathrm{GH}_n \subseteq \mathrm{GH}_{n+1}$, and GL_n, GH_n and GI are disjoint. The second one is that $\mathbf{0}$ is GL_1. We will prove that every existential formula of $\mathcal{L}_{\mathrm{GH}}$ which is consistent with the observations above, and consistent with the axioms of partial orderings with bottom element $\mathbf{0}$, is true about the Turing degrees. We mention earlier that there is an ordering $\preceq$ on $\mathcal{C}^*$ that in some sense preserves Turing reduction. No analogous of this property holds for the generalized high/low hierarchy. For example, it is known that there are Turing degrees $\mathbf{a} \leqslant_T \mathbf{b}$ such that $\mathbf{a}$ is GH_1 and $\mathbf{b}$ is GL_1.

By relativizing these notions we get the *Relativized generalized high/low hierarchy*. We say that $\mathbf{a}$ is GH_n *relative to* $\mathbf{b}$, and we write $\mathbf{a} \in \mathrm{GH}_n(\mathbf{b})$ (or $\mathrm{GH}_n(\mathbf{a}, \mathbf{b})$) if $\mathbf{a} \geqslant_T \mathbf{b}$ and $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{b}')^{(n)}$. Analogously we can define $\mathrm{GL}_n(\mathbf{b})$ and $\mathrm{GI}(\mathbf{b})$.

3.2 GH-posets.

In this section we show how to use our technical result, which extends Harrington's ZBC lemma, to prove our main result. (Harrington's ZBC lemma will be stated, and its extension will be proved, in the next section.) First we define GH-posets as the generalized version of Lerman's H-posets (see [Ler85]).

Definition 3.2.1. A *GH-poset* is a structure $\mathcal{P} = \langle P, \leqslant, 0, \mathrm{GL}_1, \mathrm{GL}_2, ..., \mathrm{GI}, ..., \mathrm{GH}_1\rangle$ where $\langle P, \leqslant\rangle$ is a partial ordering, $0 \in P$ and GL_1, GL_2,...,GI,...,GH_1 are unary relations such that

- for all n, GL_n, *GI* and GH_n are mutually disjoint,
- for all n, $\mathrm{GL}_n \subseteq \mathrm{GL}_{n+1}$ and $\mathrm{GH}_n \subseteq \mathrm{GH}_{n+1}$,
- 0 is the least element of $\mathcal{P}$, and
- $\mathrm{GL}_1(0)$ holds.

For $C \in \mathcal{G}^*$ and $x \in P$ we define $C(x)$ in the obvious way. A GH-poset $\mathcal{P}$ is *standard* if for all $x \in P$, there is a $C \in \mathcal{G}^*$ such that $C(x)$. A standard GH-poset can be represented as a quadruple $\langle P, \leqslant, 0, f\rangle$ where $f\colon P \to \mathcal{G}^*$ takes $x \in P$ to the unique $C \in \mathcal{G}^*$ such that $C(x)$. Note that every GH-poset can be extended to a standard GH-poset on the same universe.

Of course, the main example that we are interested in is the GH-poset of the Turing Degrees.

Lerman's standard H-posets can be represented by tuples $\langle P, \leqslant, 0, 1, f\rangle$ where $f\colon P \to \mathcal{C}^*$. (Actually Lerman also considers the symbols L_0^* and H_0^*, that we did not include in $\mathcal{C}^*$.) In contrast with GH-posets, H-posets have to satisfy that if $x \leqslant y$ then $f(x) \preceq f(y)$.

Theorem 3.2.2. *The existential theory of*

$$\mathcal{D} = \langle \mathbf{D}, \leqslant_T, \mathbf{0}, \mathrm{GL}_1, \mathrm{GL}_2, ..., \mathrm{GI}, ..., \mathrm{GH}_2, \mathrm{GH}_1\rangle$$

is decidable.

Proof: From the following proposition we get that an existential formula about $\mathcal{D}$ is true if and only if it does not contradict the definition of GH-poset. Because, if an existential formula φ holds in some finite GH-poset, then it hold in some finite standard GH-poset, and then, by the proposition, it holds in $\mathcal{D}$ too. It is not hard to show that one can check effectively whether φ contradicts the definition of GH-poset or not. $\square$

Proposition 3.2.3. *Every finite standard GH-poset can be embedded into $\mathcal{D}$. (Of course via a GH-poset embedding.)*

Proof: Let $\mathcal{P} = \langle P, \leqslant, ...\rangle$ be a finite GH-poset. The following lemma will allow us to consider only standard GH-posets where all the elements are either GL_1 or GH_1.

Bounding Lemma: *(Lerman, [Ler85, 2.8]) Let $\mathbf{a} \notin \mathrm{GL}_2$ be given, and fix $X \in \mathcal{G}^*$ such that $\mathbf{a} \in X$. Let $Y \prec X$ be given. Then there is a degree $\mathbf{b} \leqslant_T \mathbf{a}$ such that $\mathbf{b} \in Y$.*

Here $\prec$ refers to the following ordering on $\mathcal{G}^*$:

$$\mathrm{GL}_1^* \prec \mathrm{GL}_2^* \prec \mathrm{GL}_3^* \prec \cdots \prec GI^* \prec \cdots \prec \mathrm{GH}_3^* \prec \mathrm{GH}_2^* \prec \mathrm{GH}_1^*.$$

Corollary 3.2.4. *If $\mathbf{x} \leqslant_T \mathbf{y} \in \mathbf{D}$, $\mathbf{x} \in \mathrm{GL}_1$, $\mathbf{y} \in \mathrm{GH}_1$, and $X \in \mathcal{G}^*$, then there exists a degree $\mathbf{z}$ such that $\mathbf{x} \leqslant_T \mathbf{z} \leqslant_T \mathbf{y}$ and $\mathbf{z} \in X$.*

Proof: First, we observe that for $Y \in \mathcal{G}^*$, and $\mathbf{a} \geqslant_T \mathbf{x}$, since $\mathbf{x} \in \mathrm{GL}_1$, we have that $\mathbf{a} \in Y \Leftrightarrow \mathbf{a} \in Y(\mathbf{x})$. This is because $\mathbf{a} \vee \mathbf{x}' = \mathbf{a} \vee (\mathbf{x} \vee \mathbf{0}') = \mathbf{a} \vee \mathbf{0}'$. Then just apply the previous lemma relativized to $\mathbf{x}$. $\square$

Let $\mathcal{Q} = \langle Q, \leqslant\rangle$, where $Q = (P \smallsetminus \{0\}) \times \{0, 1\}$, and

$$\langle x, i\rangle \leqslant \langle y, j\rangle \quad \Leftrightarrow \quad x \leqslant y \vee (x = y \;\&\; i \leqslant j)$$

From the corollary above, we get that if we had an embedding $\psi\colon \mathcal{Q} \to \langle \mathbf{D}, \leqslant\rangle$, such that for all $x \in P$, $\psi(\langle x, 0\rangle) >_T 0$, $\psi(\langle x, 0\rangle)$ is GL_1 and $\psi(\langle x, 1\rangle)$ is GH_1, we could get an embedding $\varphi\colon \mathcal{P} \to \mathcal{D}$. Just let $\varphi(x)$ be some degree in between $\psi(\langle x, 0\rangle)$ and $\psi(\langle x, 1\rangle)$ which is in the class $f(x)$, and let $\varphi(0) = \mathbf{0} \in \mathbf{D}$. Now we have to show how to construct such a ψ.

Let $\{E_i : i \in P\}$ be a uniformly low, independent set of r.e. sets. For $F \subseteq P$, let $E_F = \bigoplus_{i\in F} E_i$ We will construct a sequence of sets $\{X_i\}_{i\in\omega}$ such that

X.1. For all i, X_{i+1} is r.e. in and above X_i.

X.2. For all i and $F \subseteq P$, $X_{2i} \oplus E_F$ is GL_1 and $X_{2i+1} \oplus E_F$ is GH_1.

X.3. For $i, j \in \omega$ and $F_1, F_2 \subseteq P$ we have that

$$X_i \oplus E_{F_1} \leqslant_T X_j \oplus E_{F_2} \Leftrightarrow i \leqslant j \;\&\; F_1 \subseteq F_2.$$

Then, we define $\psi\colon Q \to \mathbf{D}$ by

$$\psi(\langle x, i\rangle) = X_{2\,\mathrm{rk}(x)+i} \oplus E_{\{y\in P: y\leqslant x\}},$$

where rk is some increasing function from $\mathcal{P}$ to ω. It is not hard to check, using (X.2), and (X.3), that ψ is an embedding $\mathcal{Q} \to \langle \mathbf{D} \leqslant\rangle$ and that for all $x \in P \setminus \{0\}$, $\psi(\langle x, 0\rangle)$ is GL_1 and $\psi(\langle x, 1\rangle)$ is GH_1.

To construct the sequence $\{X_i\}_{i\in\omega}$, the main tool is the following proposition that we will prove in the next section.

Proposition 3.3.1: *Let $\{D_i : i \in G\}$ be a finite, uniformly low, independent set of r.e. sets. For $F \subseteq G$, let $D_F = \bigoplus_{j\in F} D_j$. Then, there exist an r.e. set A and an A-r.e. set B such that*

$$A' \equiv_T 0'' \equiv_T B \oplus 0' \equiv_T B'_G \text{ and}$$
$$\forall F \subset G\; \forall i \in G \setminus F\; \big(D_i \nleqslant_T B_F\big),$$

where $B_F = A \oplus B \oplus D_F$.

Let $P = G$, $\{D_i : i \in G\} = \{E_i : i \in P\}$ and A and B be as above. We let $X_0 = \emptyset$, $X_1 = A$ and $X_2 = A \oplus B$. Observe that for all $F \subseteq P$, $X_0 \oplus E_F$ is GL_1 (actually, it is low). We have that $X_1 \oplus E_F$ is GH_1 because it is r.e. and $(A \oplus D_F)' \geqslant_T 0''$. We have that $X_2 \oplus E_F$ is GL_1 because

$$X_2 \oplus E_F \oplus 0' \geqslant_T B \oplus 0' \equiv_T B'_G \geqslant_T (A \oplus B \oplus D_F)' = (X_2 \oplus E_F)'.$$

We construct the rest of the sequence by induction. Suppose we have defined the sequence up to X_{2n} satisfying the conditions (X.1)-(X.3). For each $i \in P$, let $D_i = E_i \oplus X_{2n}$. Since X_{2n} satisfies (X.2), we have that $\{D_i : i \in P\}$ is a finite, uniformly low, independent set of r.e. sets relative to X_{2n}. By the relativized version of the Proposition 3.3.1 we have sets A and B, both $\geqslant X_{2n}$, A r.e. in X_{2n} and B r.e. in A, such that

$$A' \equiv_T X''_{2n} \equiv_T B \oplus X'_{2n} \equiv_T (A \oplus B \oplus D_P)' \text{ and} \tag{3.2.1}$$

$$\forall F \subset G\ \forall i \in G \smallsetminus F\ \big(D_i \not\leqslant_T A \oplus B \oplus D_F\big). \tag{3.2.2}$$

Let $X_{2n+1} = A$ and $X_{2n+2} = A \oplus B$. As above, we get that $X_{2n+1} \oplus E_F \in \mathrm{GH}_1(X_{2n})$ and $X_{2n+2} \oplus E_F \in \mathrm{GL}_1(X_{2n})$. Since X_{2n} is GL_1 (by (X.2) with $F = \emptyset$), we have that $X_{2n+1} \oplus E_F \in \mathrm{GH}_1$ and $X_{2n+2} \oplus E_F \in \mathrm{GL}_1$.

Now, let us prove that (X.3) holds. It is clear that if $k \leqslant j$ and $F_1 \subseteq F_2$ then $X_k \oplus D_{F_1} \leqslant_T X_j \oplus D_{F_2}$. Now suppose that either $k \not\leqslant j$ or $F_1 \not\subseteq F_2$. In the latter case, from (3.2.2) we get that $X_k \oplus D_{F_1} \not\leqslant_T X_j \oplus D_{F_2}$. In the former case we divide into two possible cases. First assume that $j = 2n$. We cannot have that $X_{2n} \oplus D_{F_2} \geqslant_T X_{2n+1}$ because

$$(X_{2n} \oplus D_{F_2})'' \equiv_T X''_{2n} \equiv_T X'_{2n+1}.$$

Hence $X_k \oplus D_{F_1} \not\leqslant_T X_j \oplus D_{F_2}$. Second, assume that $j = 2n+1$. It cannot happen that $X_{2n+1} \oplus D_{F_2} \geqslant X_{2n+2}$ because $X_{2n+1} \oplus D_{F_2}$ is r.e. in X_{2n} but, since $X_{2n+2} \oplus X'_{2n} \equiv_T X''_{2n}$, $X_{2n+2} \not\leqslant_T X'_{2n}$.

We have proved that every finite GH-poset $\mathcal{P}$ can be embedded into $\mathcal{D}$. $\square$

3.3 The main Lemma.

In this section we prove the extension of Harrington's ZBC Lemma that we need to prove Proposition 3.2.3.

Harrington ZBC Lemma: *Given a set W, r.e. in and above Z', there exist sets B and C, such that, B is r.e. in Z, C is r.e. in B, and*

$$(Z \oplus B)' \equiv_T (Z \oplus B \oplus C)' \equiv_T Z' \oplus B \oplus C \equiv_T Z' \oplus W.$$

Proofs of Harrington's ZBC Lemma can be found in [Sim85, Lemma 2.1] and in [HS91, Theorem 2.5]. It consists of a finite injury construction on top of an infinite injury construction. Instead, to prove our extension, we needed two infinite injury tree constructions, one in top of the other.

Proposition 3.3.1. *Let $\{D_i : i \in G\}$ be a finite, uniformly low, independent set of r.e. sets. For $F \subseteq G$, let $D_F = \bigoplus_{j \in F} D_j$. Then, there exist an r.e. set A and an A-r.e. set B such that*

$$A' \equiv_T 0'' \equiv_T B \oplus 0', \tag{3.3.1}$$

$$B'_G \equiv_T 0'' \text{ and} \tag{3.3.2}$$

$$\forall F \subset G\ \forall i \in G \smallsetminus F\ \big(D_i \not\leqslant_T B_F\big), \tag{3.3.3}$$

where $B_F = A \oplus B \oplus D_F$.

We will do two constructions. First we show how to construct an r.e. operator, that, when applied to A, will give us B. Then, we show how to construct A. During the construction of A we use the r.e. operator constructed to guess how B is going to look at the end. Both constructions are going to be $0''$-priority arguments over a tree of strategies.

Although the proof we give does not formally assume knowledge of $0''$-priority arguments over a tree of strategies, familiarity with this kind of argument would be extremely useful in understanding the proof. The reader might look at [Soa87, Chapter XIV] for an introduction to tree constructions.

We have to satisfy various requirements. To get $A' \equiv_T 0''$, we will construct A such that $\forall n \in \omega\big(0''(n) = 1 - \lim_s A(\langle n, s\rangle)\big)$. Let E be an r.e. set such that if $n \in 0''$ then $E^{[n]} = m$ for some $m \in \omega$ and if $n \notin 0''$ then $E^{[n]} = \omega$. (We write $E^{[n]}$ for $\{x : \langle n, x\rangle \in E\}$ and by $E^{[n]} = m$ we mean $E^{[n]} = \{0, ..., m-1\}$.) Let $\{E_s\}_s$ be a recursive enumeration of E such that for all s and n, $E_s^{[n]}$ is an initial segment of ω. We will have that $A' \equiv_T 0''$ if, for every $n \in \omega$, the following requirement is satisfied:

$$P_n^A : A^{[n]} =^* E^{[n]}.$$

To get $0'' \equiv_T B \oplus 0'$, we will try to code the modulus of convergence of $A(\langle n, s\rangle)$ into B. We let $\tilde{A}$ be the A-r.e. set such that for all n, $\tilde{A}^{[n]} = k$ where k is the least such that $\forall x \geqslant k\big(A(\langle n, x\rangle) = A(\langle n, k\rangle)\big)$. The requirement P_n^B will try to enumerate the elements of $\tilde{A}^{[n]}$ into B, as long as it is permitted by higher priority negative requirements. We will prove later that, with the help of $0'$, we will be able to decode $\tilde{A}$ from B, and hence we will get that $0'' \leqslant_T 0' \oplus B$. To get $D_i \nleqslant_T B_F$, for $i \in G \smallsetminus F$, we have the negative requirements:

$$N_{\langle F,i,e\rangle} : \{e\}^{B_F} \neq D_i.$$

To satisfy these requirements we will use the Sacks preservation method (see [Soa87, VII.3]). Each requirement N_n is going to be split in two requirements N_n^A and N_n^B, the former working in the construction of A, and the latter in the construction of B. As in the Sacks jump theorem (see [Soa87, Remark VII.3.3]), these requirements help us keep the jump of B_F down, because they preserve computations of the form $\{e\}^{B_F}(0)\downarrow$. We will prove later that, because of this,

$$\forall F \subsetneq G\big(B_F' \equiv_T 0''\big). \tag{3.3.4}$$

Of course, we actually wanted $B_G' \equiv_T 0''$. There are two possible approaches to obtain this. The first one is to add requirements which preserve computations of the form $\{e\}^{B_G}(0)\downarrow$. The second one, is just to prove that $B_F' = 0''$ for all $F \subset G$. In the latter, we would be proving a weaker result, but it implies the statement of the theorem as follows: Let D_{-1} be an r.e. set such that $\{D_i : i \in G_1\}$ is an independent, uniformly low set, where $G_1 = G \cup \{-1\}$. To get D_{-1} construct a low r.e. set $D >_T D_G$ using the Sacks jump theorem (as in [Soa87, Remark VII.3.2]), and then construct $D_{-1} \leqslant_T D$ so that $\{D_i : i \in G_1\}$ is independent (using [Rob71, Corollary 6]). The weaker result we would be proving will give us an r.e. set A and an A-r.e. set B such that (3.3.1),(3.3.4) and (3.3.3) hold for G_1 instead of G. Since $G \subset G_1$, we have that $B'_G \equiv_T 0''$. We will take this second approach.

3.3.1 True Stages

Suppose that we are doing a construction using a tree of strategies and that γ is a node in the tree. For the strategy at γ, only the stages at which γ is accessible are relevant. Here we define the notion of being a true stage with respect to a given set of stages.

Given a recursive set S of stages and a recursive enumeration $\{D_s\}_s$ of an r.e. set D, we say that $s \in S$ is an *S-D-true stage* if $\exists x\big(s = \mu s' \in S(D_{s'} \restriction x = D \restriction x)\big)$. We are interested in true stages because of the following property. If $\sigma \in 2^{<\omega}$ is an initial segment of both D_s and $D_{p(s)}$, where $p(s) = \max t < s(t \in S)$, and s is S-D-true, then σ is an initial segment of D. Hence, if we have a computation with oracle $D_{p(s)}$ which remains unaltered if we change the oracle to D_s, it will remain unaltered if we change the oracle to D.

Note that the set of S-D-true stages is recursive in D. However, at a given stage $t \geqslant s$ we can guess recursively whether s is S-D-true as follows. We say that $s \in S$ *looks S-D-true at t*, and we write $s \preceq_S t$, if $\exists x\big(s = \mu s' \in S(s \leqslant t \ \&\ D_{s'} \restriction x = D_t \restriction x)\big)$. Note that $\langle S, \preceq_S\rangle$ is a partial order. Moreover, it is a tree in the sense that for all $s \in S$, $\langle\{s' : s' \preceq_S s\}, \preceq_S\rangle$ is a linear order. Also note that if s is S-D-true, then for all $t \geqslant s$, $s \preceq_S t$ and for all $t \leqslant s$ we have that $t \preceq_S s$ iff t is S-D-true.

3.3.2 Tree Constructions

Now, we show how to construct an r.e. set C using a tree of strategies the way we are going to construct A and B later. When we construct A and B, all we are going to do is to specify certain parameters of the construction of C.

Assume that we want to construct C satisfying certain positive and negative requirements. Suppose that there is a positive requirement P^C which wants to enumerate the elements of an r.e. set Y into C, for which we have a recursive enumeration $\{Y_s\}_s$. P^C is divided into infinitely many sub-requirements P^C_n, $n \in \omega$. Each P^C_n is in charge of enumerating the elements of $Y^{[n]}$ into $C^{[n]}$. We assume that the enumeration of Y satisfies that for all s and n, $Y^{[n]}_s$ is a finite initial segment of ω. Hence $Y^{[n]}$ is either ω or a finite initial segment of it.

We also have negative requirements N^C_n which want to preserve certain computations by imposing a restraint on the enumeration of C. At each stage s, N^C_n computes $l^C(n,s) \in \omega$. (In the constructions of A and B, $l^C(n,s)$ is an approximation to the length of agreement between $\{e\}^{B_F}$ and D_i.) When computing $l^C(n,s)$, N^C_n wants to approximate a computation which uses a certain r.e. set D_{F_n} as an oracle. So, N^C_n will be interested in D_{F_n}-true stages.

We arrange the strategies in a tree: $\mathbb{T} = (\{\mathtt{i}\} \cup \omega)^{<\omega}$. The nodes at level $2n$ work for N_n^C and the ones at level $2n+1$ work for P_n^C. The outcome of N_n^C is the restraint it imposes, and the outcome of P_n^C is $\mathtt{i}$ if $Y^{[n]}$ is infinite, and the first number not in $Y^{[n]}$ otherwise. We order each level as follows: $\mathtt{i} <_L 0 <_L 1 <_L \dots$. This induces a lexicographic order $<_L$ on $\mathbb{T}$ as in [Soa87, Definition XIV.1.1].

At each stage s we define $\gamma_s \in \mathbb{T}$, and we say that γ is *accessible* at s if $\gamma \subseteq \gamma_s$. We define:

- $S_\gamma^C = \{s : \gamma \subseteq \gamma_s\} \cup \{0\}$; we call the stages in S_γ, *γ-stages^C*.
- $T_\gamma^C = \{t : t$ is an S_γ^C-D_{F_n}-true stage$\}$ where $2n = |\gamma|$; we call the stages in T_γ^C, *γ-true stages^C*.
- We say that $t \prec_\gamma^C s$ if s looks S_γ^C-D_{F_n}-true at t, where $2n = |\gamma|$.
- $p_\gamma^C(t) = \max \bar{t} < t(\bar{t} \in S_\gamma^C)$, the last γ-stage^C before t.
- Let $\mathtt{TP}^C$ be maximal in $\mathbb{T} \cup [\mathbb{T}]$ such that for all $k < |\mathtt{TP}^C|$,

  $$\mathtt{TP}^C(k) = \liminf_{t \in S^C_{TP^C \restriction k}} \gamma_t(k);$$

  we call $\mathtt{TP}^C$, the *true path* of the construction of C.
- s is a *γ-expansionary stage^C* iff $s \in S_\gamma^C$ and $l^C(n,s) > l^C(n,t)$ for all γ-stages^C $t < s$.

The superscript C in S_γ^C, T_γ^C, stage^C, etc. denotes that these objects correspond to the C-construction. We include the superscript in the notation because later on we will be considering more than one construction at the same time. We might drop it if it is clear from the context which construction we are referring to.

Construction of C: Stage 0. Let $C_0 = \emptyset$ and $\gamma_0 = \emptyset$.

Stage $s+1$. Define $C_{s,0} = C_s$. For $k = 1, \dots, s$, run sub-stage k.

Substage k. Suppose we have already defined $\gamma_s \restriction k = \gamma$ and $C_{s,k-1}$.

$k = 2n+1$:

- Let $R(n,s) = \max\{\gamma_{s'}(2i) : s' \leqslant s \ \& \ (\gamma_{s'} \restriction 2i <_L \gamma \vee \gamma_{s'} \restriction 2i \subseteq \gamma)\}$, the maximum of all the restraints imposed by higher priority negative requirements.
- If, since the last γ-stage, something has been enumerated into $Y^{[n]}$, set $\gamma_s(k) = \mathtt{i}$ and enumerate all the elements of $\{n\} \times Y_s^{[n]}$ not less than $R(n,s)$ into $C_{s,k}$.
- Otherwise set $\gamma_s(k)$ to be the smallest number not in $Y_s^{[n]}$.

$k = 2n$:

- First, N_n^C computes $l^C(n,s)$, and hence it determines whether or not s is γ-expansionary.
- Let $\gamma_s(k)$ be the last γ-expansionary stage which is $\preceq_\gamma s$.

At the end of stage s define $C_{s+1} = C_{s,s}$.

Let $C = \bigcup_s C_s$. $\lozenge$

Note that, to construct an r.e. set C this way, all we have to do is specify $\{Y_s\}_s$, $l^C(n,s)$ and $\{D_{F_n,s}\}_s$ for each $n \in \omega$.

We define $\hat{C}_{s,k}$ as the best approximation to C that we have at sub-stage k of stage s:

$$\hat{C}_{s,k} = C_{s,k} \cup \bigcup \{(\{j\} \times \omega \smallsetminus R(j,s)) : j < \frac{k}{2} \ \&\ \gamma_s(2j+1)\downarrow = \mathtt{i}\}.$$

(Here we use $R(j,s)$ as $\{x : x < R(j,s)\}$.)

In the following lemma we state and prove some basic properties of this construction.

Lemma 3.3.2. *Suppose that $\gamma = \gamma_{s_0} \upharpoonright k \subseteq \mathit{TP}^C$.*

1. *For all $s \geqslant s_0$, $\gamma_s \not<_L \gamma$.*
2. *If $k = 2n+1$ then*
   (a) *for all $s \geqslant s_0$, $R(n,s) \geqslant R(n,s_0)$, and if $s \in S_\gamma$ then $R(n,s) = R(n,s_0)$;*
   (b) $\mathit{TP}^C(k) = \begin{cases} \mathtt{i} & \text{if } Y^{[n]} \text{ is infinite} \\ l & \text{if } Y^{[n]} = l < \omega; \end{cases}$
   (c) *if $\gamma_{s_0} \upharpoonright k+1 \subseteq \mathit{TP}^C$, then $\hat{C}^{[\leqslant n]}_{s_0,k} = C^{[\leqslant n]}$;*
   (d) *$C^{[\leqslant n]} =^* Y^{[\leqslant n]}$.*
3. *If $k = 2n$ then*
   (a) *If $\gamma_{s_0}(k) = s_1 \in T_\gamma$, then for all γ-stages $s \geqslant s_0$, $\gamma_s(k) \geqslant s_1$.*
   (b) *$\mathit{TP}^C(k) = \lim_{s \in T_\gamma} \gamma_s(k) =$ last γ-expansionary true stage, if such a stage exists.*
   (c) *$R(n,s_0) = \gamma_{s_0}(k)$.*
   (d) *If $\gamma_{s_0}(k) = s_1 \in T_\gamma$, then for all $s \geqslant s_0$, $s \in S_\gamma$, we have that $\hat{C}_{s,k} \upharpoonright s_1 = C \upharpoonright s_1$.*

Proof: The proof is by simultaneous induction on k. Suppose the lemma is true for all γ with $|\gamma| < k$. First suppose that $k = 2n + 1$ for some n. Let $\gamma' = \gamma \restriction k-1$. By part (1) of the induction hypothesis we have that, for all $s \geqslant s_0$, $\gamma_s \not<_L \gamma'$. By (3b), $\gamma(k-1)$ is a γ'-true stage, and then by (3a), for all $s \geqslant s_0$, if $s \in S_{\gamma'}$, $\gamma_s \not<_L \gamma$. This proves (1). Part (2a) follows from the previous one and the definition of R. Part (2b) is immediate from the construction. For (2c), we know, from the induction hypothesis, that $\hat{C}^{[<n]}_{s_0,k} = C^{[<n]}$. If $\gamma_{s_0}(k) = \mathtt{i}$, since $\liminf_s R(n,s) = R(n,s_0)$, $C^{[n]} = C^n_{s_0,k} \cup (\omega \setminus R(n,s_0)^{[n]}) = \hat{C}^{[n]}_{s_0,k}$, where $R(n,s_0)^{[n]} = \{x : \langle n,x\rangle < R(n,s_0)\}$. If $\gamma_{s_0}(k) = y_n = \mathtt{TP}^C(k) < \omega$, then nothing else is enumerated into $Y^{[n]}$ after s_0, and hence nothing is enumerated into $C^{[n]}$ after s_0. So $C^{[n]} = C^{[n]}_{s_0} = \hat{C}^{[n]}_{s,k}$. Part (2d) follows from the fact that for some s, $\gamma_s \restriction k+1 \subset \mathtt{TP}^C$, and that $\hat{C}^{[\leqslant n]}_{s,k} =^* Y^{[\leqslant n]}$.

Now suppose that $k = 2n$ for some n. To prove (1), assume that $k > 0$, (it is trivial otherwise) and let $\gamma' = \gamma \restriction k-1$. By induction hypothesis, we have that for all $s \geqslant s_0$, $\gamma_s \not<_L \gamma'$. If $\gamma_s(k-1) = \mathtt{i}$ we clearly never go left again. Otherwise, we do not enumerate anything in $Y^{[n]}$ any more, and hence, we never move left again either. For part (3a) observe that s_1 is γ-expansionary and that for all $s \geqslant s_1$, $s_1 \preceq_\gamma s$. Part (3b) follows from (3a). Now, let us prove (3c). Let $s_1 = \gamma_{s_0}(k)$. By (2a), $R(n,s_0) = R(n,s_1)$, and $R(n,s_1)$ can not be $> s_1$, but $R(n,s_1) \geqslant \gamma_{s_1}(k) = s_1$. So $R(n,s_0) = R(n,s_1) = s_1 = \gamma_{s_0}(k)$. For the last part we have that $\hat{C}^{[<n]}_{\gamma,s} = C^{[<n]}$ by (2c), and that $\hat{C}^{[\geqslant n]}_{\gamma,s} \restriction s_1 = C^{[\geqslant n]} \restriction s_1$ because for all $s \geqslant s_0$, $R(n,s) \geqslant s_1$. □

3.3.3 Construction of B

Now we construct an r.e. operator that, when applied to a set Z, returns a set $B[Z]$ r.e. in Z. Later, when we define A, we will let $B = B[A]$. We use the framework defined in Section 3.3.2.

Construction of $B[Z]$: All we need to do is to specify the parameters needed in the tree construction of 3.3.2. We let $\tilde{Z}$ be the set that P^B wants to enumerate. $\tilde{Z}$ has the following recursive enumeration:

$$\tilde{Z}_t = \{\langle e,x\rangle < t : \exists y > x\big(\langle e,y\rangle < t \;\&\; Z(\langle e,x\rangle) \neq Z(\langle e,y\rangle)\big)\}.$$

For each negative requirement N^B_n we have to define D_{F_n} and $l^{B[Z]}(n,t)$. For $n = \langle F,i,e\rangle$ let $D_{F_n} = D_F$. We will use the letter t for the stages in the B-construction and $\beta_t \in \mathbb{T}^B$ for the approximation to $\mathtt{TP}^B$ at t. Now suppose that we are at stage t, sub-stage k of the construction, where $k = 2n$ and $n = \langle F,i,e\rangle$. Assume we have already defined $B_{t,k-1}$ and $\beta = \beta_t \restriction k$. Define

$$l^{B[Z]}(n,t) = \begin{cases} x + \frac{1}{2} & \text{if } \{e\}^{B_{F,t,k}[Z]}_t(x)\downarrow = \{e\}^{B_{F,p_\beta(t)}[Z]}_{p_\beta(t)}(x) \text{ with} \\ & \text{the same computation} \\ x & \text{otherwise,} \end{cases}$$

where x is maximal such that $D_{i,t} \restriction x = \{e\}_t^{B_{F,t,k}[Z]} \restriction x = \{e\}_{p_\beta(t)}^{B_{F,p_\beta(t)}[Z]} \restriction x$ with the same computation. Recall that $p_\beta(t)$ is the last β-stage before t and that $B_{F,t,k}[Z] = Z \oplus B_{t,k}[Z] \oplus D_{F,t}$. $\Diamond$

First, observe that the construction is recursive in Z, and hence $B[Z]$ is Z-r.e. Second, observe that $\tilde{Z}_t$ depends only on $Z \restriction t$, and hence so do the first t stages of the construction of $B[Z]$.

Note that $l^{B[Z]}(n,t)$ is defined so that the initial segments of size $\lfloor l^{B[Z]}(n,t) \rfloor$ of $D_{i,t}$, $\{e\}_t^{B_{F,t,k}[Z]}$ and $\{e\}_{p_\beta(t)}^{B_{F,p_\beta(t)}[Z]}$ coincide, and the initial segments of size $\lceil l^{B[Z]}(n,t) \rceil$ of $\{e\}_t^{B_{F,t,k}[Z]}$ and $\{e\}_{p_\beta(t)}^{B_{F,p_\beta(t)}[Z]}$ coincide. (For $p \in \mathbb{Q}$, $\lfloor p \rfloor = \max q \in \mathbb{Z}(p \geqslant q)$ and $\lceil p \rceil = \mu q \in \mathbb{Z}(p \leqslant q)$.) The computation $\{e\}_t^{B_{F,t,k}[Z]} \restriction \lceil l^{B[Z]}(n,t) \rceil$ is the one we will want to preserve.

For the following definition and lemma fix Z and drop the suffix $[Z]$ from the notation.

Definition 3.3.3. For $n = \langle F, i, e \rangle$, let

$$l_n^B = \begin{cases} x + \frac{1}{2} & \text{if } \{e\}^{B_F}(x)\downarrow \\ x & \text{otherwise,} \end{cases}$$

where x is maximal such that $D_i \restriction x = \{e\}^{B_F} \restriction x$ (x might be ω).

Lemma 3.3.4. *Let $\beta = TP^B \restriction k$, where $k = 2n$ and $n = \langle F, i, e \rangle$, and assume that $\tilde{Z}^{[<n]}$ is finite.*

1. *If $\beta \subseteq \beta_{t_0}$, $\beta_{t_0}(k) = t_1 \in T_\beta$ and $l = \lceil l^B(\beta, t_0) \rceil$, then*

$$\{e\}_{t_0}^{B_{F,t_0,k}} \restriction l = \{e\}^{B_F} \restriction l.$$

2. *If $l \in \frac{1}{2} \cdot \omega = \{\frac{n}{2} : n \in \omega\}$ and $l \leqslant l_n^B$, then there exists a β-expansionary true stage t_0 such that $l^B(\beta, t_0) \geqslant l$. (Recall that l_n^B might equal ω.)*

3. *If $\{e\}^{B_F} \neq D_i$ then*

$$TP^B(k) = \lim_{t \in T_\beta} \beta_t(k) < \omega.$$

Proof: For part (1) we have to show that $B_{F,t_0,k}$ is preserved up to the use, u, of the computations $\{e\}_{t_0}^{B_{F,t_0,k}} \restriction \lceil l^B(\beta, t_0) \rceil$. By Lemma 3.3.2.3d and the assumption on $\tilde{Z}^{[<n]}$, we have that $B_{t_0,k} \restriction t_1 = \hat{B}_{t,k} \restriction t_1 = B \restriction t_1$. Note that $t_1 > u$. Since the computations $\{e\}_{t_0}^{B_{F,t_0,k}} \restriction \lceil l^B(\beta, t_0) \rceil$ were there at stage $p_\beta(t_1)$ too, we have that $D_{F,p_\beta(t_1)} \restriction u = D_{F,t_1} \restriction u$. Since $t_1 \in T_\beta$, nothing below u is enumerated into D_F after t_1. So we have that $B_{F,t_0,k} \restriction u = B_F \restriction u$. This proves (1). Part (2) is clear.

For part (3), suppose toward a contradiction, that there are infinitely many β-expansionary true stages. This implies that there is a stage $t_0 \in T_\beta$ with $l^B(\beta, t_0) >$

l_n^B. Let $l = \lfloor l_n^B \rfloor$. So we have that $\{e\}_{t_0}^{B_{F,t_0,k}}(l)\downarrow$, and that this computation is going to be preserved for ever by part (1). Since $D_i(l) \neq \{e\}^{B_F}(l)$, l had to be enumerated into D_i after stage t_0, and this disagreement is preserved for ever. So, for no t after that stage, we have that $l^B(\beta, t) > l^B(\beta, t_0)$. Hence, there are no more β-expansionary stages. □

3.3.4 Construction of A

We construct an r.e. set A satisfying the following requirements:

$$\begin{aligned} P_n^A : &\quad A^{[n]} =^* E^{[n]}, \\ N_{\langle F,i,e\rangle}^A : &\quad \{e\}^{B_F} \neq D_i. \end{aligned}$$

We will use a tree construction like the one in 3.3.2. At each stage s, N_n^A computes an approximation to

$$l_n = \begin{cases} x + \frac{1}{2} & \text{if } \{e\}^{B_F[A]}(x)\downarrow \\ x & \text{otherwise,} \end{cases}$$

where $x = \max x' \leqslant \omega\big(D_i \restriction x' = \{e\}^{B_F[A]} \restriction x'\big)$, and imposes a restraint on A to preserve these computations. To approximate $B_F[A]$, we run the construction of B for a few stages using $\hat{A}_{s,2n}$ as an oracle. (Recall that $\hat{A}_{s,2n}$ is the best approximation to A that N_n^A has at stage s.) So we have to decide for how many stages to run $B_t[\hat{A}_{s,2n}]$. For this purpose, along with the construction we define $\delta_s \in \mathbb{T}^B$ as an approximation to $\mathtt{TP}^B$.

Construction of A: All we need to do is to specify the parameters needed in the tree construction of 3.3.2. The set that P^A wants to enumerate is E, for which we have a recursive enumeration $\{E_s\}_s$. For each negative requirement N_n^A we have to define D_{F_n} and $l^A(n,s)$. For $n = \langle F, i, e\rangle$ let $D_{F_n} = D_F$. We will use the letter s for the stages of the A-construction and $\alpha_s \in \mathbb{T}^A$ for the approximation to $\mathtt{TP}^A$ at s. Now, for each s and $n < \frac{s}{2}$, we define $\delta_s \in \mathbb{T}^B$ and $l^A(n,s)$. Suppose that we are in stage s, sub-stage $k = 2n$ of the construction, and we have already defined $A_{s,k-1}$, $\alpha = \alpha_s \restriction k$ and $\delta = \delta_s \restriction k$.

- Let $t_{n,s} < p_\alpha^A(s)$ be maximal such that $t_{n,s} \prec_\delta^{B[\hat{A}_{s,k}]} s$. If there is no such $t_{n,s}$, let it be 0. (This is for how many stages we are going to run the computation of $B[\hat{A}_{s,k}]$.)

- $\delta_s(k) = \beta_{t_{n,s}}^{B[\hat{A}_{s,k}]}(k)$ (the last δ-expansionary stage$^{B[\hat{A}_{s,k}]}$ that is $< p_\alpha^A(s)$ and $\prec_\delta^{B[\hat{A}_{s,k}]} s$).

- $l^A(n,s) = l^{B[\hat{A}_{s,k}]}(n, t_{n,s})$ $(= l^{B[\hat{A}_{s,k}]}(n, \delta_s(k)))$.

- $\delta_s(k+1) = \mu x(x \notin \tilde{\hat{A}}_{s,k}^{[n]})$ (the place after which the nth column of $\hat{A}_{s,k}$ stabilizes).

$\Diamond$

Let $B = B[A]$. From now on, when we write the superscript B, we are referring to the construction of $B[A]$.

In the following lemma we show that δ_s is a good approximation to $\mathtt{TP}^B$.

Lemma 3.3.5. *Let $\alpha = \mathit{TP}^A \upharpoonright k$ and $\beta = \mathit{TP}^B \upharpoonright k$, where $k = 2n$ and $n = \langle F, i, e\rangle$.*

1. *If $\alpha \subseteq \alpha_s$, then $\beta \subseteq \delta_s$.*
2. *If s_0 is an α-expansionary true stage^A, then $t_0 = \delta_{s_0}(k)$ is a β-expansionary true stage^B. Moreover, for every α-stage^A $s \geqslant s_0$, $t_0 \leqslant \delta_s(k)$.*
3. *If s_0 is an α-expansionary true stage^A, and s_1 is any α-stage^A, $s_0 < s_1$, then $\delta_{s_0}(k) < \delta_{s_1}(k) \Leftrightarrow \alpha_{s_0}(k) < \alpha_{s_1}(k)$.*
4. *If s_0 is a true stage^A, then $t_0 = \delta_{s_0}(k)$ is a β-expansionary true stage^B.*
5. *If $\mathit{TP}^A \upharpoonright k+1 \subseteq \alpha_s$, then $\mathit{TP}^B \upharpoonright k+1 \subseteq \delta_s$.*

Proof: We prove the lemma by simultaneous induction on n. Let $\alpha' = \alpha \upharpoonright k-1$ and $\beta' = \beta \upharpoonright k-1$. From part (5) of the induction hypothesis, we have that if $\alpha' \subseteq \alpha_s$, then $\beta' \subseteq \delta_s$ (because $\alpha' = \mathtt{TP}^A \upharpoonright 2(n-1)+1$). If also $\alpha \subseteq \alpha_s$, then, by Lemma 3.3.2.2c, $\hat{A}^{[<n]}_{s,k} = A^{[<n]}$, and hence $\tilde{\hat{A}}^{[n-1]}_{s,k} = \tilde{A}^{[n-1]}$. Therefore, the computation of $\delta_s(k-1)$ is correct. This proves part (1).

To prove (2), we start by showing that t_0 is a β-expansionary true stage$^{B[\hat{A}_{s_0,k}]}$. Since $t_0 = \beta^{B[\hat{A}_{s_0,k}]}_{t_{n,s_0}}(k)$, it is clear that it is β-expansionary$^{B[\hat{A}_{s_0,k}]}$. It is a β-true stage$^{B[\hat{A}_{s_0,k}]}$ because $t_0 < p^A_\alpha(s_0)$, $t_0 \prec^{B[\hat{A}_{s_0,k}]}_\beta s_0$ and $s_0 \in T^A_\alpha$. The second observation is that since $\hat{A}_{s_0,k} \upharpoonright s_0 = A \upharpoonright s_0$ (this is by Lemma 3.3.2.3d), the first s_0 ($\geqslant t_0$) stages of the computations of $B[\hat{A}_{s_0,k}]$, and of $B[A]$ are the same. This implies that t_0 is a β-expansionary true stage^B. Moreover, if $s \in S^A_\alpha$ and $s > s_0$, then $\hat{A}_{s_0,k} \upharpoonright s_0 = \hat{A}_{s,k} \upharpoonright s_0 = A \upharpoonright s_0$. As above, this implies that t_0 is a β-expansionary true stage$^{B[\hat{A}_{s,k}]}$. Then, since $t_0 \prec^{B[\hat{A}_{s,k}]}_\beta s$ and $t_0 < p^A_\alpha(s)$, $t_0 \leqslant \delta_s(k)$.

Let us prove part (3). Let $t_0 = \delta_{s_0}(k)$ and $t_1 = \delta_{s_1}(k)$. From the proof of part (2) we get that t_0 is a β-expansionary true stage$^{B[\hat{A}_{\alpha,s_1}]}$ and $t_0 \leqslant t_1$. So, $t_1 > t_0$ iff $l^{B[\hat{A}_{\alpha,s_1}]}(n, t_1) > l^{B[\hat{A}_{\alpha,s_1}]}(n, t_0)$. Note that $l^A(n, s_1) = l^{B[\hat{A}_{\alpha,s_1}]}(n, t_1)$ and that, since the first s_0 stages in the computations of $B[\hat{A}_{s_0,k}]$, and of $B[\hat{A}_{s_1,k}]$ are the same, $l^{B[\hat{A}_{\alpha,s_1}]}(n, t_0) = l^{B[\hat{A}_{\alpha,s_0}]}(n, t_0) = l^A(n, s_0)$. So, $t_1 > t_0$ iff $l^A(n, s_1) > l^A(n, s_0)$. We have that $l^A(n, s_1) > l^A(n, s_0)$ iff there is an α-expansionary stage^A $\bar{\sigma}$, $s_0 \prec^A_\alpha \bar{\sigma} \preceq^A_\alpha s_1$, which happens iff $\alpha_{s_1}(k) > \alpha_{s_0}(k)$.

For part (4), let $s_1 = \alpha_{s_0}(k)$. s_1 is an α-expansionary stage^A, and it is $\preceq^A_\alpha s_0$. Since s_0 is α-true, so is s_1. Then, by part (2), $\delta_{s_1}(k)$ is a β-expansionary true stage^B. Since $\alpha_{s_0}(k) = s_1 = \alpha_{s_1}(k)$, by part (3), $t_0 = \delta_{s_0}(k) = \delta_{s_1}(k)$. So t_0 is a β-expansionary true stage^B.

For part (5), let $s_0 = \mathtt{TP}^A(k)$ and $t_0 = \delta_{s_0}(k)$. We claim that $t_0 = \mathtt{TP}^B(k)$. Since s_0 is an α-expansionary true stage^A, from (2), we have that t_0 is a β-expansionary true stage^B. We have to show that it is the last one. Suppose, toward a contradiction, that $t_1 > t_0$ is a β-expansionary true stage^B. Let $s_1 \in T^A_\alpha$ be such that $t_1 < p^A_\alpha(s_1)$ and $A_{s_1} \restriction t_1 = A \restriction t_1$. Then we have that t_1 is a β-expansionary true stage$^{B[\hat{A}_{s_1,k}]}$ and $t_1 \leqslant t_{n,s_1}$. So $\delta_{s_1}(k) \geqslant t_1 > t_0 = \delta_{s_0}(k)$, and hence $\alpha_{s_1}(k) > \alpha_{s_0}(k)$, which contradicts the fact that s_0 is the last α-expansionary true stage. Now, we have to show that for any $(\alpha^\wedge s_0)$-stage^A s, $\delta_s(k) = t_0$. Since $\alpha_s(k) = s_0 = \alpha_{s_0}(k)$, $\delta_s(k) = \delta_{s_0}(k) = t_0$. $\square$

Lemma 3.3.6. *Let $\alpha = \mathit{TP}^A \restriction k$ and $\beta = \mathit{TP}^B \restriction k$, where $k = 2n$ and $n = \langle F, i, e\rangle$, and assume that $\tilde{A}^{[<n]}$ is finite.*

1. *If $s_0 \in T_\alpha$, $t_0 = \delta_{s_0}(k)$ and $l = \lceil l^A(n, s_0)\rceil$, then*

$$\{e\}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]}_{t_0} \restriction l = \{e\}^{B_F} \restriction l.$$

2. *If $l \in \frac{1}{2} \cdot \omega$ and $l \leqslant l_n$, then there exists an α-expansionary true stage^A s_0 such that $l^A(n, s_0) \geqslant l$. (Recall that l_n could be ω.)*

Proof: Let $s_1 = \alpha_{s_0}(k)$. By Lemma 3.3.2.3d, $\hat{A}_{s_0,k} \restriction s_1 = A \restriction s_1$. Then, since $t_0 \leqslant s_1$ (this is because $\alpha_{s_0}(k) = \alpha_{s_1}(k)$, and hence, by Lemma 3.3.5.3, $t_0 = \delta_{s_0}(k) = \delta_{s_1}(k) \leqslant s_1$),

$$\{e\}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]}_{t_0} \restriction \lceil l^A(n, s_0)\rceil = \{e\}^{B_{F,t_0,k}}_{t_0} \restriction \lceil l^A(n, s_0)\rceil.$$

By Lemma 3.3.4.1, and since $t_0 \in T_\beta$ (this is by 3.3.5.4),

$$\{e\}^{B_{F,t_0,k}}_{t_0} \restriction \lceil l^B(n, t_0)\rceil = \{e\}^{B_F} \restriction \lceil l^B(n, t_0)\rceil.$$

Part (1) now follows, since $l^A(n, s_0) = l^B(n, t_0)$.

For part (2) we use Lemma 3.3.4.2. So we get a β-expansionary true stage^B t_0 such that $l^B(n, t_0) \geqslant l$. As in the proof of 3.3.5.5, there is a stage $s_0 \in T^A_\alpha$ such that the first t_0 stages of the computations of $B[\hat{A}_{s_0,k}]$, and of $B[A]$ are the same and $\delta_{s_0}(k) \geqslant t_0$. Therefore $l^A(n, s_0) = l^B(n, \delta_{s_0}(k)) \geqslant l^B(n, t_0) \geqslant l$. $\square$

3.3.5 Verifications

Now we show that A and $B = B[A]$ are as we wanted.

Lemma 3.3.7. *If $|\mathit{TP}^A| \geqslant 2n$, where $n = \langle F, i, e\rangle$, and $\tilde{A}^{[<n]}$ is finite, then $\{e\}^{B_F} \neq D_i$.*

PROOF: Suppose, toward a contradiction, that $\{e\}^{B_F} = D_i$. We will show that then $D_i \leqslant_T D_F$ contradicting the hypothesis. Let $\alpha = \mathtt{TP}^A \restriction k$, where $k = 2n$. Given $p \in \omega$, we want to find $D_i(p)$ recursively in D_F. Find $s_0 \in T^A_\alpha$, such that $l^A(\alpha, s_0) > p$. Such an s_0 exists because of Lemma 3.3.6.2, and we can find it recursively in $T^A_\alpha \leqslant D_F$. Then, by Lemma 3.3.6.1,

$$D_i(p) = \{e\}^{B_F}(p) = \{e\}_{t_0}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]},$$

where $t_0 = \delta_{s_0}(k)$. □

Lemma 3.3.8. *For all n, if $n = \langle F, i, e\rangle$, then*

1. $\{e\}^{B_F} \neq D_i$;
2. $|\mathit{TP}^B| \geqslant 2n + 1$.
3. $|\mathit{TP}^A| \geqslant 2n + 1$;
4. $A^{[n]} =^* E^{[n]}$ *and* $|\mathit{TP}^A| \geqslant 2n + 2$
5. $B^{[n]}$ *is finite,* $|\mathit{TP}^B| \geqslant 2n + 2$.

PROOF: We prove the lemma by induction on n. Suppose the lemma is true for all $m < n$. By part (4) of the inductive hypothesis we have that $|\mathtt{TP}^A| \geqslant 2n$ and $\tilde{A}^{[<n]}$ is finite. So, Lemma 3.3.7 implies (1). Then, Lemma 3.3.4.3 implies (2). To prove (3) we observe that if $s_0 < s_1$ are α-expansionary true stages^A, then $\delta_{s_0}(2n) < \delta_{s_1}(2n)$ (this is by 3.3.5.3). Then, by Lemma 3.3.5.2 both $\delta_{s_0}(2n)$ and $\delta_{s_1}(2n)$ are β-expansionary true stages^B. But then, since there are only finitely many β-expansionary true stages^B, there are only finitely many α-expansionary true stages^A. Hence $\mathtt{TP}^A(2n)$ exists. Part (4) is now implied by Lemma 3.3.2, parts 2d and 2b. This also implies that $\tilde{A}^{[n]}$ is finite and the last part follows, again by Lemma 3.3.2, parts 2d and 2b. □

From the previous lemma we get that $|\mathtt{TP}^A| = |\mathtt{TP}^B| = \omega$, and that all the requirements N_n and P^A_n are satisfied. What is left to show is that for all $F \subset G$, $B'_F \equiv_T 0''$ and that $0'' \equiv_T B \oplus 0'$.

Lemma 3.3.9. *For all $F \subset G$, $\mathit{TP}^A \equiv_T 0'' \equiv_T (B_F)'$.*

PROOF: Clearly $\mathtt{TP}^A \leqslant_T 0''$. We can compute $0''$, from $(B_F)'$, because $A \leqslant_T B_F$ and $0'' \leqslant_T A'$. Now we show how to compute $(B_F)'$ from $\mathtt{TP}^A$. Fix F, $i \in G \setminus F$ and $e \in \omega$. Let $n = \langle F, i, e\rangle$. From the definition of l_n, is follows that

$$\{e\}^{B_F}(0)\downarrow \Leftrightarrow l_n \geqslant \frac{1}{2}.$$

We can easily compute l_n from TP^A because $l_n = l_{n,s_0}$ where $s_0 = TP^A(2n)$. This is because $TP^A(2n)$ is the last $TP^A \restriction 2n$-expansionary true stage (see Lemma 3.3.2.3b). □

Lemma 3.3.10. *$TP^A \equiv_T 0'' \equiv_T B \oplus 0'$.*

PROOF: We know that $\mathtt{TP}^A \equiv_T 0''$ and that $B \oplus 0' \leqslant_T 0''$ because B is r.e. in an r.e. set. Now we prove that $\mathtt{TP}^A \leqslant_T B \oplus 0'$. By induction on n we compute

- $\mathtt{TP}^A(2n)$,
- $\mathtt{TP}^B(2n)$,
- $\mathtt{TP}^B(2n+1)$, and
- $\mathtt{TP}^A(2n+1)$,

recursively in $B \oplus 0'$. By Lemma 3.3.2.3b, we have that

$$\mathtt{TP}^A(2n) = \lim_{s \in T^A_\alpha} \alpha_s(2n),$$

where $\alpha = \mathtt{TP}^A \restriction 2n$. Since $T^A_\alpha \leqslant D_F$ (where F is such that $n = \langle F, i, e\rangle$), we can compute this recursively in $D'_F \equiv_T 0'$. Then we compute $\mathtt{TP}^B(2n) = \delta_s(2n)$, where $s = \mathtt{TP}^A(2n)$ (this is because of Lemma 3.3.5.5). Now, let

$$z_n = \mu z\big(\langle n, z\rangle \geqslant \mathtt{TP}^B(2n) \ \&\ z \notin B^{[n]}\big).$$

(The existence of z_n follows from Lemma 3.3.8.5.) From the construction, since $\mathtt{TP}^B(2n) = \liminf_t R^B(n,t)$, it has to be the case that $z_n \notin \tilde{A}^{[n]}$ (otherwise it would be eventually enumerated into $B^{[n]}$). Having this information, we can compute $\mathtt{TP}^B(2n+1) = \mu z(z \notin \tilde{A}^{[n]})$ recursively in $A \leqslant_T 0'$. Then we can compute $\lim_x E(\langle n, x\rangle) = \lim_x A(\langle n, x\rangle) = A(\langle n, z_n\rangle)$. If $\lim_x E(\langle n, x\rangle)$ is 1, then $\mathtt{TP}^A(2n+1) = \mathtt{i}$, and if it is 0, then $\mathtt{TP}^A(2n+1) = \mu x(x \notin E^{[n]})$ which can be computed from $E \leqslant_T 0'$. $\square$

This finishes the proof of Proposition 3.3.1.

Chapter 4
Generalized High degrees have the complementation property (*with Noam Greenberg and Richard A. Shore*).

Published in the Journal of Symbolic Logic, volume 69(4), year 2004, pages 1200-1220.

4.1 Introduction

A major theme in the investigation of the structure of the Turing degrees, $(\mathcal{D}, \leqslant_T)$, has been the relationship between the order theoretic properties of a degree and its complexity of definition in arithmetic as expressed by the Turing jump operator which embodies a single step in the hierarchy of quantification. For example, there is a long history of results showing that $\mathbf{0}'$ has many special order theoretic properties. To cite just a few: every countable partial order can be embedded below $\mathbf{0}'$ (Kleene and Post [KP54]); there are minimal degrees below $\mathbf{0}'$ (Sacks [Sac61]); $\mathbf{0}'$ cups to every degree above it (and so has the *cupping property*)(Friedberg [Fri57]); every degree below $\mathbf{0}'$ joins up to $\mathbf{0}'$ (and so has the *join property*)(Robinson [Rob72], Posner and Robinson [PR81]).

It was often hoped that some such property would distinguish either $\mathbf{0}'$ or some class of degrees closely related to it. For degrees below $\mathbf{0}'$, the notion of being close to $\mathbf{0}'$ (or $\mathbf{0}$ at the other end) was also measured by the jump operator via the high/low hierarchy: for $\mathbf{d} \leqslant_{\mathbf{T}} \mathbf{0}'$, $\mathbf{d} \in \mathbf{H}_n \Leftrightarrow \mathbf{d}^{(n)} = \mathbf{0}^{(n+1)}$ and $\mathbf{d} \in \mathbf{L}_n \Leftrightarrow \mathbf{d}^{(n)} = \mathbf{0}^{(n)}$. The questions then became at which levels of this hierarchy do the various properties of $\mathbf{0}'$ appear. The corresponding results for the above properties for $\mathbf{d} \leqslant_{\mathbf{T}} \mathbf{0}'$ are as follows: every countable partial order can be embedded below every $\mathbf{d} \notin \mathbf{L}_2$; there are minimal degrees below every $\mathbf{d} \in \mathbf{H}_1$; every $\mathbf{d} \notin \mathbf{L}_2$ cups to every degree above it; every degree below $\mathbf{d}$ joins up to $\mathbf{d}$ for $\mathbf{d} \in \mathbf{H}_1$. (Except for the last, these results are all known to be sharp in terms of the high/low hierarchy. The sharpness of the first follows from the existence of a minimal degree $\mathbf{a} \in \mathbf{L}_2 - \mathbf{L}_1$: There is a minimal $\mathbf{a} < \mathbf{0}'$ not in $\mathbf{L}_1$ by Sasso [Sas74] and it is in $\mathbf{L}_2$ by Jockusch and Posner [JP78]. This minimal degree is easily seen to be weakly recursive in the sense of Ishmukhametov [Ish99] and then, by that work, has a strong minimal cover $\mathbf{b}$, i.e. $\{\mathbf{0}, \mathbf{a}, \mathbf{b}\}$ is an initial segment of the degrees. This proves the sharpness of the third fact. The existence of such an initial segment with $\mathbf{b}$ also below $\mathbf{0}'$ follows by Lerman's methods as outlined in [Ler83, XII.5.8]. The sharpness of the second is by Lerman [Ler86].) The techniques used to prove all these positive results are tied up with approximation methods, rates of growth conditions and domination properties. Thus they are of independent interest for relating degree theoretic properties of functions with such conditions.

In the setting of the degrees as a whole the analogous measure of the strength of a degree in terms of its jumps is the generalized high/low hierarchy: $\mathbf{d} \in \mathrm{GH}_n \Leftrightarrow \mathbf{d}^{(n)} = (\mathbf{d} \vee \mathbf{0}')^{(n)}$ and $\mathbf{d} \in \mathrm{GL}_n \Leftrightarrow \mathbf{d}^{(n)} = (\mathbf{d} \vee \mathbf{0}')^{(n-1)}$. (We take $\mathrm{GL}_0 = \{\mathbf{0}\}$.) All of the results mentioned above for $\mathbf{d} \leqslant_{\mathbf{T}} \mathbf{0}'$, for example, are true for all degrees as long as we use the generalized hierarchy. (Jockusch and Posner [JP78]; (Cooper [Coo73]) for $\mathbf{H}_1$ and Jockusch [Joc77] for GH_1; Jockusch and Posner [JP78]; Posner [Pos77].) Once again approximations, rates of growth and domination properties play prominent roles in the constructions.

It was often hoped that these investigations would lead to a definition of $\mathbf{0}'$ in $\mathcal{D}$ or of some of the classes in the appropriate hierarchy in $\mathcal{D}$ or $\mathcal{D}(\leqslant \mathbf{0}')$. In fact, the jump operator has been defined in $\mathcal{D}$ by entirely different methods involving coding models of arithmetic and other arguments by Shore and Slaman [SS99] as have all of the high/low classes in $\mathcal{D}(\leqslant \mathbf{0}')$ with the exception of $\mathbf{L}_1$ by Nies, Shore and Slaman [NSS98]. The dream of a natural definition for any of these classes based on such order theoretic properties, however, still persists and can only be realized by investigations such as these. Moreover, the analysis of the relations between rates of growth and domination principles and the ordering of degrees remains intriguing on its own.

In this paper we consider the *complementation property* for degrees $\mathbf{d}$: for every $\mathbf{a} < \mathbf{d}$ there is a $\mathbf{b} < \mathbf{d}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{d}$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. The primary result here is that $\mathbf{0}'$ has the complementation property. This result has a long history. R. W. Robinson [Rob72, cf. [PR81]] showed that every $\mathbf{a} \in \mathbf{L}_2$ has a complement in $\mathcal{D}(\leqslant \mathbf{0}')$; Posner [Pos77], that every $\mathbf{a} \in \mathbf{H}_1$ has one and Epstein [Eps75], that every r.e. $\mathbf{a}$ has one. Finally, Posner [Pos81] showed that every $\mathbf{a} \notin \mathbf{L}_2$ has a complement in $\mathcal{D}(\leqslant \mathbf{0}')$ and so $\mathbf{0}'$ has the complementation property. By a further argument using relativization and an additional division into cases along with some known results, Posner also showed that every $\mathbf{d} \in \mathrm{GH}_0$, i.e. $\mathbf{d} \geqslant \mathbf{0}'$, also has the complementation property. At the end of that paper, he posed four questions:

1. Is there a uniform proof of the complementation property for $\mathbf{0}'$ (i.e. eliminate the split into cases and provide a single procedure that generates a complement for any nonrecursive $\mathbf{a} < \mathbf{0}'$)?

2. Does every nonrecursive $\mathbf{a} < \mathbf{0}'$ have a 1-generic complement?

3. Does every nonrecursive $\mathbf{a} < \mathbf{0}'$ have a complement of minimal degree?

4. Does every $\mathbf{d} \in \mathrm{GH}_1$ have the complementation property?

Slaman and Steel [SS89] answered questions 1 and 2 simultaneously by providing a uniform proof that every nonrecursive $\mathbf{a} < \mathbf{0}'$ has a 1-generic complement. Seetapun and Slaman circulated a sketch of a proposed affirmative answer to the third question around 1992 and that sketch has recently been extended and made into a proof by Lewis ([Lew03]). There have been several partial and related results about the fourth question. Epstein [Eps81], for example, showed that if $\mathbf{a} < \mathbf{h}$ are r.e. and $\mathbf{h} \in \mathbf{H}_1$ then $\mathbf{a}$ has a complement in $\mathcal{D}(\leqslant \mathbf{h})$. In this paper we supply a full affirmative answer:

Theorem 4.1.1. *Every degree* $\mathbf{d} \in \mathrm{GH}_1$ *has the complementation property, i.e. for every* $\mathbf{a} < \mathbf{d}$ *there is a* $\mathbf{b} < \mathbf{d}$ *such that* $\mathbf{a} \vee \mathbf{b} = \mathbf{d}$ *and* $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Like the original proof that $\mathbf{0}'$ has the complementation property, our proof of this theorem is nonuniform. Rather than two case ($\mathbf{a} \in \mathbf{L}_2$ and $\mathbf{a} \notin \mathbf{L}_2$) we here have three separate cases, depending on the "distance" between $\mathbf{a}$ and $\mathbf{d}$. The three cases are:

1. $\mathbf{a} \in \mathrm{GL}_2$.
2. $\mathbf{a} \notin \mathrm{GL}_2$, but there is some function recursive in $\mathbf{d}$ which is not dominated by any function recursive in $\mathbf{a}$.
3. Every function recursive in $\mathbf{d}$ is dominated by some function recursive in $\mathbf{a}$.

Note that as defined these three cases are disjoint. The point is that if $\mathbf{a} \in \mathrm{GL}_2$ then $\mathbf{d}' = (\mathbf{d} \vee \mathbf{0}')' \geqslant (\mathbf{a} \vee \mathbf{0}')' = \mathbf{a}''$ and so there is a function recursive in $\mathbf{d}$ which dominates every function recursive in $\mathbf{a}$. They are also each nonempty. For (2) just take $\mathbf{d}' \geqslant \mathbf{a}''$. For (3) choose $\mathbf{a} \geqslant \mathbf{0}'$ and $\mathbf{d}$ any degree above $\mathbf{a}$ which is hyperimmune free with respect to $\mathbf{a}$ such as is given by relativizing either Miller and Martin [MM68] or the standard construction of a Spector minimal degree using total recursive binary trees to $\mathbf{a}$. (The point here is that the usual forcing argument, even if only attempting to force minimality, decides totality of every ϕ_e^A. Thus, if ϕ_e^A is total, A lies on a recursive binary tree T every path through which makes ϕ_e total. So, for every n there is a level s of the tree such that $\phi_e^{T(\sigma)} \downarrow$ for every σ at level s. This guarantees that ϕ_e^A is dominated by the recursive function which, given n, finds such an s and then outputs the maximum value of $\phi_e^{T(\sigma)}$ for σ on level s.) We should also note, however, that the proof we give for case (1) only needs $\mathbf{d}' \geqslant \mathbf{a}''$ and so would cover some pairs in cases (2) as well. The constructions of a complement for $\mathbf{a}$ in $\mathcal{D}(\leqslant \mathbf{d})$ in each of the three cases are quite distinct. We describe the intuitions and the motivations for each now and provide the precise constructions and verifications in the following sections.

Case 1: The existence of a complement for $\mathbf{a}$ below $\mathbf{d}$ in this case was already proved in his thesis by Posner [Pos77]. Like the proof that $\mathbf{L}_2$ degrees have complements in $\mathcal{D}(\leqslant \mathbf{0}')$ in Posner and Robinson [PR81] that proof was indirect (see [Ler83, IV.15]) and based on a nonuniform proof of the join theorem for GH_1 (see [Ler83, IV.9]). It was also explicitly left open even in [Ler83] if the degree providing the join could be made GL_1. This was answered by Lerman [Ler85] but again nonuniformly. We supply a uniform proof for this sharper version of the join theorem for GH_1 which we then modify to get this case of the complementation theorem.

The basic idea of our proof of the join theorem for $\mathbf{d} \in \mathrm{GH}_1$ is, given $A <_T D \in \mathbf{d}$, to build a 1-generic set B in a construction recursive in D (so that $B <_T D$) that codes D and to have the choices we make at each stage depend on A, so that the construction itself (and hence D) can be recovered from $A \oplus B$. It combines the key idea of Jockusch and Shore's [JS84a] proof of the Posner-Robinson join theorem for $\mathbf{0}'$ with Jockusch's [Joc77] use of the recursion theorem with degrees in GH_1. This approach is maintained for the proof of the first case of the complementation theorem, except that to ensure that $A \wedge B \equiv_T 0$ we look for splits instead of strings forcing the jump. Since we are now looking for two strings rather than one, the coding is a little more complicated. Also, once we have found a split, we need the convergence of a functional with oracle A to let us know which string of the split we should take to diagonalize. This presents the problem that D cannot necessarily know if such a convergence will ever appear. However, by our case assumption, D can approximate whether a given functional with oracle A produces a total function (which is the only case we worry about). Thus, if we keep attacking some requirement, we will eventually believe the correct outcome and act appropriately.

Case 2: In this case, $\mathbf{a}$ is no longer in GL_2, hence $\mathbf{d}$ cannot approximate questions regarding totality. However, as was noted by Jockusch and Posner [JP78] (using Martin's [Mar66] characterization of the domination properties of $\mathbf{H}_1$), this loss is offset by $\mathbf{a}$'s ability to approximate bounds for searches conducted by $\mathbf{0}'$: for every function $f \leqslant_T \mathbf{0}'$ (even $\mathbf{a} \vee \mathbf{0}'$) there is a function $g \leqslant_T \mathbf{a}$, not dominated by f.

We would like to imitate a construction of complements below $\mathbf{0}'$, and as in the previous case, we use the fact that $\mathbf{d}$ being generalized high enables us to approximate constructions which are naturally recursive in $\mathbf{0}'$. However, another special aspect of $\mathbf{0}'$ is that it is *recursively enumerable* . Dominating the modulus (computation) function for a recursively enumerable (even Δ_2) set enables us to compute the set itself. In other words, there is a function $d \leqslant_T \mathbf{0}'$ which isn't dominated by any function of lesser degree. In the standard constructions, this property is used to bound searches for facts which are r.e. in $\mathbf{a}$. As this seems to be an essential element of all such constructions, we simply assume (this is the second case) that we are given a function $d \leqslant_T \mathbf{d}$ which isn't dominated by any function recursive in $\mathbf{a}$, and carry out the construction (with the necessary modifications, of course). Given such a function we can adapt the uniform construction of complements below $\mathbf{0}'$ of [SS89] to our second case. If such a function does not exist, we find ourselves in the third case for which a completely different approach seems to be required.

Case 3: The key observation which enables us to construct complements in the third case is the following. Suppose that an A falling under the third case is given, and we manage to construct some $B \leqslant_T D$ such that $A \wedge B \equiv_T 0$ and such that $D \leqslant_T (A \oplus B)'$. In this case we must have $D \equiv_T A \oplus B$; if not, then we have

$$A \oplus B \ <_T \ D \leqslant_T (A \oplus B)',$$

in other words, $D \in \Delta_2(A \oplus B)$ but is not $\Delta_1(A \oplus B)$, which implies (see [Ler83, III.5.9]) that the computation function for D (as a set which is $\Delta_2(A \oplus B)$) is not dominated by any function recursive in $A \oplus B$, so also not by any function recursive in A. This contradicts the assumption that we are in case (3).

To construct B as required, we simply construct a set B of *minimal* degree, satisfying the requirement $D \leqslant_T (A \oplus B)'$. Since we can assume that $D \nleqslant_T A'$ (by the same argument as above), we cannot have $B \leqslant_T A$. Since B is minimal, we get $A \wedge B \equiv_T 0$.

Now there is a mysterious aspect of our construction. We construct a set B *without reference* to the set A, and then use the assumed domination property of A to get our result. This implies that any B so constructed is automatically a complement for *every* A falling under the third case.

To accomplish this, we need to construct a B which will have properties which are fairly independent of the construction. We begin with a more controlled version of the construction of a minimal degree below a $\mathbf{d} \in \mathrm{GH}_1$ and then modify it to code in D in a way that can be recovered from B and any function g dominating a search function used in the construction. The case assumption for A will guarantee that there is such a function recursive in A.

We begin with the standard method of constructing (a Spector) minimal degree by forcing with perfect trees. One constructs a sequence of recursive perfect trees, each tree succeeded by a splitting subtree or an extension (full) subtree, forcing either that $B \leqslant_T \Phi(B)$ or that $\Phi(B)$ is recursive, respectively (of course, if total). To prove Sacks' [Sac61] result that there is a minimal degree below $\mathbf{0}'$, one introduces a priority argument version of this basic construction and uses partial recursive trees to get a minimal degree below $\mathbf{0}'$. The idea is to define the successor subtree as the splitting subtree, until we find an initial segment of the set we are building which does not split on our current tree, and then switch (permanently modulo action by higher priority requirements) to an extension (full) subtree.

The generalization of this technique to working below an arbitrary GH_1 degree was introduced by Jockusch in [Joc77] where he proved that below every GH_1 degree there is a minimal degree. Since, in this case, we do not have the power of $\mathbf{0}'$ to find whether a certain string splits on a tree or not, we keep looking for splits in hope of eventually finding one. The power of GH_1, together with the recursion theorem, allows us, as in the previous cases, to guess, eventually correctly, whether such a search will be successful. We use this guess to determine what kind of tree the successor will be. Since we may guess incorrectly for some time, we may switch back and forth between splitting and extension (full) subtrees, but only finitely many times, and so we maintain the finite injury character of the construction.

In such a construction, the limit tree in a given place in the sequence of final trees is determined not only by the set B we construct and the question whether it splits or not on the previous tree, but also on the time during the construction at which we finally made the correct guess - this is because that stage determines what the *stem* of the tree will be. The primary difference between our construction and Jockusch's is that we eliminate this dependency by making the stem of the tree independent of the stage and dependent only on B and on the previous tree. As we shall see, this change allows us to recover the final list of trees from $(B \oplus g)'$, where g is any function dominating the search in the construction for splits when needed or a notification from our D recursive approximation to the $(D \vee 0')'$ question as to whether there is a initial segment of B above which there are no splits. We will finally use the fact that A falls under the third case to see that such a g can be obtained from A.

Our solution to the complementation problem for GH_1 obviously leaves several natural open questions. Can the complements be taken to be either always 1-generic or always minimal? Is there a uniform procedure for computing a complement from D and $A = \{e\}^D$ for $0 <_T A <_T D \in \mathrm{GH}_1$?

Finally, before beginning the specific cases, we provide some general definitions. We fix a set $D \in \mathbf{d}$ and a set $A \in \mathbf{a}$ (later we may specify sets with particular properties). We construct a set B whose degree will be $\mathbf{a}$'s complement. The requirements we need to satisfy are:

- **C**: D is coded in B so that $D \leqslant_T A \oplus B$.
- $\mathbf{Z}_\Phi$: If $\Phi(A) = \Phi(B)$ is total, then it is recursive.

(Here Φ varies over Turing functionals; we identify a functional, and the requirement associated with it, with its index.) We think of a Turing functional as a recursive function $\Phi : 2^{<\omega} \to 2^{<\omega}$ which is extendable continuously to a function $\Phi : 2^\omega \to 2^{\leqslant\omega}$. If $q \in 2^{<\omega}$ and $n \in \mathrm{dom}\ \Phi(q)$ we sometimes write $\Phi(q;n)\downarrow$ and write $\Phi(q;n) = \Phi(q)(n)$.

We fix an implicit recursive bijection $\omega \leftrightarrow \omega^{<\omega}$. We let, for $n < \omega$,

$$\langle (n)_0, (n)_1, \ldots, (n)_{\ell(n)-1} \rangle$$

be the sequence associated with n.

In this paper, *tree* means a function tree, that is a partial function $T : 2^{<\omega} \to 2^{<\omega}$ which preserves order and nonorder and whose domain is closed under initial segments.

Suppose that Φ is a functional. A Φ-*split* is a pair of strings q_0, q_1 such that $\Phi(q_0) \perp \Phi(q_1)$, i.e. for some n, $\Phi(q_0;n)\downarrow \neq \Phi(q_1;n)\downarrow$. If (q_0, q_1) is a Φ-split, then we let $n_\Phi(q_0, q_1)$ be the least n witnessing this fact.

A split (q_0, q_1) *extends* p if both q_0 and q_1 extend p. If there is some Φ-split extending p, we say that p Φ-*splits* . If p Φ-splits, then we let $(q^0_\Phi(p), q^1_\Phi(p))$ be the least split extending p, and let $n_\Phi(p) = n_\Phi(q^0_\Phi(p), q^1_\Phi(p))$.

We say that a function $f : \omega \to \omega$ *dominates* the function $g : \omega \to \omega$ (and write $g <_* f$) if for all but finitely many n, $g(n) < f(n)$.

We use $\subset$ to denote containment whether proper or not.

4.2 Case One

We begin with a uniform proof of the join property (via a 1-generic) for $\mathbf{d} \in \mathrm{GH}_1$ that we then modify to produce a complement when A falls under Case 1.

Theorem 4.2.1. *If* $0 <_T A <_T D \in \mathrm{GH}_1$ *then there is a uniform procedure producing a 1-generic* B *such that* $A \vee B \equiv_T D$.

Proof: In addition to the coding the requirement $\mathbf{C}$, we here have just those for 1-genericity:

- $\mathbf{N}_\Phi$: $\Phi(B)\downarrow$ or $(\exists n)(\forall q \supseteq B \upharpoonright n)(\Phi(q)\uparrow)$.

(Here we diverge from the usage described above and assume that the Turing functionals Φ take no input).

Without loss of generality (or uniformity), we may assume that A is the set of (natural number) codes for the initial segments of a set $\tilde{A}$ of the same degree. The set B will be constructed as an increasing union of strings $\langle \beta_s \rangle$. However, for the sake of the applicability of the recursion theorem, we also present B as a finer union of initial segments $\langle p_t \rangle_{t<\omega}$. The construction is recursive in D, hence the complexity of the question

$$\exists t\, (p_t \text{ has no extension } q \text{ such that } \Phi(q)\downarrow)\ ?$$

is $(D \vee 0')' \equiv_T D'$. Thus its answer can be approximated recursively in D. By the recursion theorem, we assume that we have a function $G \leqslant_T D$ such that the limit $\lim_s G(\Phi, s)$ is the answer to the question. At stage s of the construction we proceed as follows:

Let $p_1 = \beta_1 = \langle\rangle$. We are given $\beta_s = p_t$ for some t and we work for requirement $\mathbf{N}_\Phi$ for $\Phi = (s)_0$.

At substage k of stage s, we let $p_{t+k} = p_t^\frown 0^k$. We find the least $r_0 > t+k$ such that either $G(\Phi, r_0) = \mathtt{yes}$ or a q extending p_{t+k} with $\Phi(q)\downarrow$ is found after r_0 many steps (this search halts, for if there is no such q, then p_{t+k} is a witness to $\lim G(\Phi, r) = \mathtt{yes}$).

Now if we didn't find a q as requested, we let l be the least element of A greater than k (of course A is infinite), and let $\beta_{s+1} = p_{t+k+1} = p_t^\frown 0^{l\frown} 1^\frown D(s)$.

Otherwise, we take the least $q \supset p_{t+k}$ such that $\Phi(q)\downarrow$ and such that for some l, $p_t{}^\frown 0^{l\frown} 1 \subset q$ (the string we find can always be extended to such a string). If $l \in A$ then we do nothing and move to the next substage. Otherwise, we let $\beta_{s+1} = p_{t+k+1} = q^\frown D(s)$.

To verify that this construction produces the desired B, we first need to show that there is no final stage s, in other words, that at every stage we eventually define β_{s+1} and move to the next stage. Assume the opposite holds and that s is the last stage. Thus at every substage k of s, we find the least q extending $\beta_s^\frown 0^k$ containing at least one additional 1 such that $\Phi(q)\downarrow$ and this q extends $\beta_s^\frown 0^{l\frown} 1$

for some $l \in A$. Thus we can recursively enumerate infinitely such l and so an infinite subset of A. As this would compute A recursively we have the desired contradiction.

To see that B is 1-generic consider any Φ. Find a stage s late enough such that the guess for $\mathbf{N}_\Phi$ is correct at stage s and $(s)_0 = \Phi$. If $G(\Phi) = \texttt{yes}$ then there is an initial segment of B with no extension q such that $\Phi(q)\downarrow$. If not, then we defined β_{s+1} such that $\Phi(\beta_{s+1})\downarrow$.

Finally to see that $D \leqslant_T A \oplus B$ it is clearly enough to show that $\langle \beta_s \rangle \leqslant_T A \oplus B$: Given β_s, we find the k such that $\beta_s{}^\frown 0^k{}^\frown 1 \subset B$. If $k \in A$ then $|\beta_{s+1}| = |\beta_s| + k + 2$. Otherwise, we can recursively find the least q extending $\beta_s{}^\frown 0^k{}^\frown 1$ such that $\Phi(q)\downarrow$. In this case $|\beta_{s+1}| = |q| + 1$. $\square$

We now turn to case 1 of the complementation theorem.

4.2.1 Construction

The situation is as in the proof of the join theorem above except that we have requirements $\mathbf{Z}_\Phi$ in place of $\mathbf{N}_\Phi$ and so our function G approximates the answer to the question

$$\exists t \left(p_t \text{ does not } \Phi\text{-split} \right) \ ?$$

Also, $A'' \leqslant_T D'$ so we can approximate $\mathrm{Tot}(A)$ recursively in D: we have a function $T \leqslant_T D$ such that for all functionals Φ, $\lim_s T(\Phi, s) = \texttt{total}$ if $\Phi(A)$ is total, and $\lim_s T(\Phi, s) = \texttt{nottotal}$ otherwise.

Construction of B: Let $p_1 = \beta_1 = \langle\rangle$. At stage s, we are given $\beta_s = p_t$ for some t and we work for requirement $\mathbf{Z}_\Phi$ for $\Phi = (s)_0$.

At substage k of stage s, we let $p_{t+k} = p_t{}^\frown 0^k$. We find the least $r_0 > s + k$ such that either $G(\Phi, r_0) = \texttt{yes}$ or a Φ-split extending p_{t+k} is found (the least one) after r_0 many steps (this search halts, for if there is no such split, then p_{t+k} is a witness to $\lim G(\Phi, r) = \texttt{yes}$). In the latter case, we find the least $r_1 > r_0$ such that either $T(\Phi, r_1) = \texttt{not total}$ or $\Phi(A; n_\Phi(p_{t+k}))\downarrow [r_1]$ (again, the search halts because no convergence witnesses $\lim T(\Phi, r) = \texttt{not total}$).

Now if we didn't find a split, or didn't find convergence, we let l be the least element of A greater than k (of course A is infinite), and let $\beta_{s+1} = p_{t+k+1} = p_t{}^\frown 0^l{}^\frown 1^\frown D(s)$.

Otherwise, we know which element of the split q is the one which gives the wrong answer about $\Phi(A; n_\Phi(p_{t+k}))$; we may assume it extends some $p_t{}^\frown 0^l{}^\frown 1$ for some $l \geqslant k$. If $l \in A$ then we do nothing and move to the next substage. Otherwise, we let $\beta_{s+1} = p_{t+k+1} = q'{}^\frown D(s)$, where q' is the least string $\subseteq q$ and extending $p_t{}^\frown 0^l{}^\frown 1$ such that $\Phi(q') \perp \Phi(A)$. $\Diamond$

4.2.2 Verifications

We first need to show that there is no final stage s, in other words, that at every stage we eventually define β_{s+1} and move to the next stage. Assume the opposite holds and that s is the last stage. Thus at every substage k of s, the least Φ-split (q_k^0, q_k^1) extending $\beta_s{}^\frown 0^k$ is found, and we know that at least one of q_k^0, q_k^1 extends $\beta_s{}^\frown 0^l 1$ for some $l \in A$.

Thus, we can recursively enumerate infinitely many pairs (i_k, j_k) (where both coordinates get larger and larger; $i_k, j_k \geqslant k$) such that at least one of i_k, j_k is in A.

Now we can compute A and get a contradiction. Recall that the elements of A are codes for initial segments of $\tilde{A}$. Let σ_k be the string coded by i_k and τ_k the string coded by j_k.

Now there are two cases. Suppose that for all $\sigma \subset \tilde{A}$ there is some k such that $\sigma \subset \sigma_k, \tau_k$. Then we can compute $\tilde{A}$ by enumerating

$$\{\sigma_k \cap \tau_k \,:\, k < \omega\}.$$

Otherwise, there is some $\sigma^* \subset \tilde{A}$ and some k^* such that for all $k > k^*$, σ^* is extended by exactly one of σ_k, τ_k, and this one has to be an initial segment of $\tilde{A}$. In this case we can compute $\tilde{A}$ by enumerating

$$\{\sigma_k \,:\, k > k^*, \sigma^* \subset \sigma_k\} \cup \{\tau_k \,:\, k > k^*, \sigma^* \subset \tau_k\}.$$

Lemma 4.2.2. *$A \wedge B \equiv_T 0$.*

Proof: Fix Φ such that $\Phi(A)$ is total. Find a stage s late enough such that the guesses for $T(\Phi)$ and $G(\Phi)$ are correct at stage s and $(s)_0 = \Phi$. If $G(\Phi) = \texttt{yes}$ then $\Phi(B)$, if total, is recursive. If not, then we must have diagonalized at stage s, so that $\Phi(A) \neq \Phi(B)$. $\square$

Lemma 4.2.3. *$\langle \beta_s \rangle \leqslant_T A \oplus B$.*

Proof: Given β_s, we find the k such that $\beta_s{}^\frown 0^k{}^\frown 1 \subset B$. If $k \in A$ then $|\beta_{s+1}| = |\beta_s| + k + 2$. Otherwise, from A we can find the q such that $\beta_{s+1} = q^\frown D(s)$. $\square$

Corollary 4.2.4. *$D \leqslant_T A \oplus B$.*

4.3 Case Two

The construction takes place on a tree T; B will be a branch on this tree. In Slaman and Steel's construction, a requirement $\mathbf{Z}_\Phi$ strives to find a Φ-split on T, and tries to let B extend that part of the split which gives the wrong answer about $\Phi(A)$. Since about $\mathrm{Tot}(A)$ we know naught, the strategy is to find such a split and then prevent weaker requirements from extending B beyond this split, until

$\mathbf{Z}_\Phi$ makes up its mind (due to a convergence of $\Phi(A)$) or decides to give up. In the original construction, the requirement gives up at the next stage it receives attention, at which it is either guaranteed a string which doesn't split on T (a *negative* win), or it is presented with a new split unto which its fortunes are now entrusted.

It is the domination property of d which is used to show that each requirement cannot be foiled for ever.

The first difficulty we come across is that in the construction of the tree, we wish to find out whether a certain node can be extended by some split. Instead of asking $\mathbf{0}'$, we construct the tree (ignoring the coding requirement for a while) recursively in A by looking, at level s, for splits, for $g(s)$ many steps, where $g \leqslant_T A$ is a function not dominated by some fixed $\tilde{f} \leqslant \mathbf{0}'$ which bounds the search needed to find all splits which exist. Thus in many levels devoted to some $\mathbf{Z}_\Phi$, the requirement falsely believes that no splits exist (and thus that it needs no action to succeed); but the domination properties of g ensure that, if there are infinitely many splits, splits will be found infinitely often, giving the requirement ample breathing space to act.

These changes create two new difficulties. As described earlier, a requirement $\mathbf{Z}_\Phi$ imposes its will on weaker requirements (which threaten to extend B beyond a split, while $\mathbf{Z}_\Phi$ is still waiting for convergence of $\Phi(A)$). This restraint is built into the construction of the tree, and is only imposed (for each split) for a single stage only, since at the next stage $\mathbf{Z}_\Phi$ looks ahead to find either a new split or no splits, in which case it is satisfied anyway. In our case, $\mathbf{Z}_\Phi$ does not know how long it needs to wait, thus restraints need to be made explicit. Fairness is maintained by observing that at the "true stages" (those stages at which $g > f$, meaning that the answers for the question "is there a split?" are correct) all unnecessary restraints are dropped.

However, each requirement needs to determine for *itself* whether to pursue positive satisfaction (by waiting for the next split) or to give up and let B extend a node beyond which no splits are found yet, hoping that indeed no splits will appear later. It turns out that D is sufficiently strong to enable $\mathbf{Z}_\Phi$ to guess which course to take (using the recursion theorem, of course). This solves this second problem.

We now give the details of the construction. For simplicity, we adopt a modular approach and ignore the coding requirement $\mathbf{C}$ for now, and later describe how to modify the construction to satisfy this requirement.

4.3.1 Construction of the Tree

We first define the function f which bounds the search for splits. Let

$$f_0(s) = \max\{\langle q^0_\Phi(p), q^1_\Phi(p)\rangle : p \in 2^{\leqslant s},\ \Phi \leqslant s \ \&\ p\ \Phi\text{-splits}\}.$$

Next, define f by recursion: $f(0) = f_0(0)$; $f(s+1) = f_0(f(s)+1)$. Finally, let $\tilde{f}(s) = f(\langle s, s\rangle)$. Observe that $\tilde{f}$ is recursive in $\mathbf{0}'$.

To level k of the tree we attach requirement $\mathbf{Z}_\Phi = (k)_0$. k is called a Φ-*level*.

We find some $g \leqslant_T A$, not dominated by $\tilde{f}$. We use g to define our tree T. $T(\langle\rangle) = \langle\rangle$. If $T(\sigma)$ is defined and equals p, and $|\sigma|$ is a Φ-level, then we look for a Φ-split extending p which appears before stage $g(s)$. If one is found, then we define $T(\sigma\frown i) = q^i_\Phi(p)$. Otherwise, we let $T(\sigma\frown 0) = p\frown 0$ and leave $T(\sigma\frown 1)$ undefined.

We say that a string p is a *node* if it is in range T. We let

$$L(k) = \{T(\sigma) : |\sigma| = k\},$$

and also write $lev(p) = k$ for all $p \in L(k)$.

It is easily verified that T is $\Delta_1(A)$ (so not only is T partial recursive in A but also dom T is recursive in A). We are also interested in verifying how closely this tree reflects the truth (about existence of splits).

Lemma 4.3.1. *For all k and $p \in L(k)$, $|p| < f(k)$.*

PROOF: This is true by induction; the inductive step holds because the elements of the $k+1^{st}$ level are least splits (or immediate successors) of strings of level k, and f bounds all such splits. □

Definition 4.3.2. A level k is called Φ-*true* if it is a Φ-level, and $g(k) > f(k)$.

Thus if k is a Φ-true level then for all $p \in L(k)$, if there is no Φ-split on T extending p then indeed there is no such split at all.

Lemma 4.3.3. *For every Φ there are infinitely many Φ-true levels.*

PROOF: We know that there are infinitely many $n \geqslant \Phi$ such that $g(n) \geqslant \tilde{f}(n)$. For each such n, if we let $k = \langle \Phi, n\rangle$, then

$$g(k) = g(\langle \Phi, n\rangle) \geqslant g(n) \geqslant f(\langle n, n\rangle) \geqslant f(\langle \Phi, n\rangle) = f(k).$$

□

4.3.2 The Construction

We now describe the construction of a set $B <_T D$ such that $A \wedge B \equiv_T 0$. We construct B as the union of an increasing sequence of initial segments $\langle \beta_s\rangle$ which are on T.

At stage s, each requirement $\mathbf{Z}_\Phi$ may impose a restraint $r(\Phi)[s]$. This is a level k of the tree (beyond the level of β_s), and the aim of the restraint is to prevent weaker requirements from extending β_{s+1} beyond the k^{th} level of T. If a stronger requirement $\mathbf{Z}_\Psi$ extends β_{s+1} beyond the k^{th} level of T then we say that it *injures* $\mathbf{Z}_\Phi$. We let

$$R(\Phi)[s] = \min\left(\{r(\Psi)[s] \,:\, \Psi < \Phi \;\&\; r(\Psi)[s]\downarrow\} \cup \{s\}\right).$$

This is the restraint imposed on requirement $\mathbf{Z}_\Phi$ at stage s. Thus at s, $\mathbf{Z}_\Phi$ is prohibited from defining β_{s+1} beyond the $R(\Phi)^{th}$ level of T.

There are various states in which a requirement $\mathbf{Z}_\Phi$ may find itself at a Φ-stage s. It may be *positively satisfied* at s if we have already forced that $\Phi(B) \perp \Phi(A)$, i.e. for some n we have

$$\Phi(A;n)\downarrow \neq \Phi(B;n)\downarrow [s].$$

If $\mathbf{Z}_\Phi$ is positively satisfied at some stage then it is satisfied until the end of time and doesn't need to act ever again.

We wish to describe when $\mathbf{Z}_\Phi$ is satisfied *negatively*. As explained above, this requirement will feel satisfied if it can make B extend some node p which doesn't Φ-split. Now without the power of $\mathbf{0}'$, this kind of satisfaction may be temporary, for the requirement may place its bets on some such p only to discover later some Φ-split extending p.

To make its decision whether to believe some negative satisfaction (or keep searching for splits), at various stages $\mathbf{Z}_\Phi$ will put some strings on a "testing list" ℓ_Φ (which can be thought of as a partial function defined by $\mathbf{Z}_\Phi$ during the construction). This will be the list of strings that $\mathbf{Z}_\Phi$ will test, to see if they may provide negative satisfaction.

We thus say that $\mathbf{Z}_\Phi$ is *negatively satisfied* at s if no Φ-splits extending the last element put on ℓ_Φ are found in $g(s)$ many steps.

[We note, that if $\mathbf{Z}_\Phi$ is negatively satisfied at a true stage s, then this satisfaction is correct and permanent; at true stages we find all possible splits above nodes on levels $\leqslant s$ on the tree.]

Now to decide whether it should even attempt to believe that negative satisfaction could come from some node p, $\mathbf{Z}_\Phi$ will try to find an answer to the question

Is there some node p which is put on ℓ_Φ and doesn't split?

This question can be answered by $(D \oplus 0')' \equiv_T D'$. So, by the recursion theorem and the limit lemma, there is a function $G \leqslant_T D$, that we can use during the construction, which approximates the answer (i.e. $\lim_s G(\Phi, s) = \texttt{yes}$ if the answer to the question is yes, and $\lim_s G(\Phi, s) = \texttt{no}$ otherwise.)

Let

$$L(k,p) = \{T(\sigma) : |\sigma| = k \text{ and } \mathrm{T}(\sigma) \supseteq \mathrm{p}\},$$

this is the collection of nodes of the k^{th} level of T which extend p. At stage s, let $L(k,s) = L(k,\beta_s)$.

We say that $p = T(\sigma)$ *splits on* T if both $T(\sigma^\frown 0)$ and $T(\sigma^\frown 1)$ are defined, i.e. if a relevant split exists and was discovered by step $g(|\sigma|)$ and put on T.

CONSTRUCTION OF B, WITH ORACLE D:

At stage 0, we let $\beta_1 = \langle\rangle$, we set the lists ℓ_Φ to be empty, and let $r(\Phi)$ be undefined for all Φ.

If Φ is initialized (at any stage), then $r(\Phi)$ is removed.

Stage s: Suppose that s is a Φ-stage.

If $\mathbf{Z}_\Phi$ believes it is satisfied (positively or negatively), then we skip this stage; $r(\Phi)$ is not defined.

Otherwise, $\mathbf{Z}_\Phi$ may attempt to act, starting ambitiously.

Positive action

If $\mathbf{Z}_\Phi$ can legally extend β_s to be permanently satisfied, it does so. Namely, if there is some p on T, extending β_s, on a level below $R(\Phi)[s]$, such that for some n we have $\Phi(p; n) \neq \Phi(A; n)[d(s)]$ then $\mathbf{Z}_\Phi$ lets $\beta_{s+1} = p$. $r(\Phi)$ is removed (as the requirement is satisfied). [d is the function recursive in D, not dominated by any function recursive in A.]

Negative attempts

Otherwise, we look for some p of level beneath $R(\Phi)[s]$ (but which extends β_s) which is not yet on ℓ_Φ and which is not yet seen to Φ-split. If such a p exists, we add the first one to ℓ_Φ. We now perform a *test* to see if we believe p can be a witness for negative success: we find the least $t > s$ such that either $G(\Phi, t) = \texttt{yes}$, or some Φ-split extending p is found by stage t. This search has to halt since if there is no split to be found, then p is a witness to the limit of $G(\Phi, t)$ being $\texttt{yes}$. If a Φ-split was found then we may try with a new node p (until we run out). Otherwise, $\mathbf{Z}_\Phi$ sets $\beta_{s+1} = p$, and removes $r(\Phi)$.

If we run out of p's without acting, the requirement does nothing at this stage, but we calculate

Restraint

If $r(\Phi)$ is not defined, then it is set to be $lev(\beta_s)$. Otherwise, if there is some Φ-level k such that $r(\Phi) < k \leqslant R(\Phi)[s]$ and such that *every* $p \in L(k, s)$ splits on T, then we update $r(\Phi)$ to be the maximal such k. If there is no such k, then $r(\Phi)$ is left unaltered.

Whenever $\mathbf{Z}_\Phi$ acts and extends β_s, it initializes all weaker requirements. $\Diamond$

4.3.3 Verifications

Since this is an oracle construction, we have $B \leqslant_T D$. We first check that the construction is fair.

Lemma 4.3.4. *Every requirement is initialized only finitely many times, and eventually stops acting itself.*

Proof: Since initialization occurs only by action of a stronger requirement, we assume by induction that after $\hat{s}$, no requirement stronger than $\mathbf{Z}_\Phi$ acts again and so $\mathbf{Z}_\Phi$ is never initialized again.

If at any time $\mathbf{Z}_\Phi$ acts positively, then its satisfaction is permanent and it will never act again. We now consider what happens to $\mathbf{Z}_\Phi$ after stage $\hat{s}$.

Infinite negative action is also impossible. For either $\lim G(\Phi, s) = \texttt{no}$, in which case after some stage we will never believe some string is a likely candidate for negative satisfaction; or $\lim G(\Phi, s) = \texttt{yes}$, by virtue of a witness p which was put on ℓ_Φ; when we put p on the list, we performed the test and acted for p (because it really doesn't split); and negative satisfaction is permanent from then on (no splits are ever discovered, and no new strings are put on the list). □

We digress to prove that the requirements succeed if they act.

Lemma 4.3.5. *If $\mathbf{Z}_\Phi$ ever acts positively, then it is met.*

Proof: Because then $\Phi(A) \neq \Phi(B)$. □

Lemma 4.3.6. *If $\mathbf{Z}_\Phi$ is eventually permanently satisfied negatively by some p then it is met.*

Proof: We have some $p \subset B$ such that no Φ-splits extend p. Then $\Phi(B)$, if total, is recursive. □

Now if a requirement is not eventually satisfied as in the last two lemmas, then there is a last stage at which the requirement acts, necessarily a negative action. Since this does not lead to permanent satisfaction, at some later stage the last action taken is discovered to be insufficient (a split is discovered, extending the last element of ℓ_Φ); after that stage, since the requirement never acts again, it also never believes it is satisfied. In this case we say that $\mathbf{Z}$ is *eventually unsatisfied.*

We let $s^*(\Phi)$ be the least Φ-stage after which neither $\mathbf{Z}_\Phi$ nor any stronger requirement ever acts again, and after which $\mathbf{Z}_\Phi$ is either permanently satisfied or eventually unsatisfied.

A requirement may also disturb weaker requirements by imposing restraint, so to check fairness, we need to verify that this restraint is not too prohibitive. If a requirement is permanently satisfied (positively or negatively) then it stops imposing any restraint.

Otherwise we have

Lemma 4.3.7. *Suppose that $\mathbf{Z}_\Phi$ is not eventually satisfied. Then at $s^*(\Phi)$, $r(\Phi)$ is defined; thereafter it is never removed and cannot decrease. For all $s \geqslant s^*(\Phi)$, $lev(\beta_s) \leqslant r(\Phi)[s]$.*

Proof: The first part is because $\mathbf{Z}_\Phi$ is never initialized again, or acts again. The second follows because weaker requirements respect $r(\Phi)$. □

We need to examine what happens to the restraint if a certain requirement is eventually unsatisfied. For this we use the true levels.

Lemma 4.3.8. *Suppose that $\mathbf{Z}_\Phi$ is eventually unsatisfied, $k > s^*(\Phi)$ is a Φ-true level and that $s > k$ is a Φ-stage such that $lev(\beta_s) \leqslant k \leqslant R(\Phi)[s]$. Then at s, $r(\Phi)$ is increased to at least k.*

PROOF: If not, then we must have some $p \in L(k, s)$ which does not split on T. But k is a true stage; thus p doesn't split, and $\mathbf{Z}_\Phi$ would act negatively at stage s. $\square$

We get

Lemma 4.3.9. $\lim_s R(\Phi)[s] = \infty$.

PROOF: Assume (by induction) that $\lim R(\Phi)[s] = \infty$. Suppose, for contradiction, that $\lim r(\Phi, s) = r$ is finite. We can choose some Φ-true level $k > r$ and wait for a later Φ-stage $s > s^*(\Phi)$ such that $R(\Phi)[s] > k$. $lev(\beta_s) \leqslant r < k$; then we must have $r(\Phi)[s] \geqslant k$, contradiction. $\square$

Lemma 4.3.10. $\lim_s |\beta_s| = \infty$.

PROOF: There are infinitely many Φ such that, for all X, $\Phi(X) = \langle\rangle$, so there are no Φ-splits. For all these Φ, $\mathbf{Z}_\Phi$ will act negatively the first time it receives attention. Therefore, β_s is extended infinitely often. $\square$

All that is left is to verify that $\mathbf{Z}_\Phi$ is met, even if it is eventually unsatisfied.

Lemma 4.3.11. *If $\mathbf{Z}_\Phi$ is eventually unsatisfied, then $\Phi(A)$ is not total.*

PROOF: Suppose otherwise, and let $h(n)$ be the stage at which $\Phi(A; n)\downarrow$; $h \leqslant_T A$. We show that h dominates d for a contradiction.

Let $s_1, s_2, \ldots$ be the stages, after $s^*(\Phi)$, when $r(\Phi)[s]$ is increased. Let $r_i = r(\Phi)[s_i]$, and let p_i be the element of the r_i^{th} level of T which is an initial segment of B. Since $p_i \in L(r_i, s_i)$ ($\beta_{s_i} \subset B$), by the instructions of the construction, we must have that p_i splits on T. We let n_i be the splitting point, i.e. $n_i = n_\Phi(p_i)$.

We claim that for all $i < \omega$, $h(n_i) \geqslant d(n_{i+1})$. At stage s_{i+1}, $\mathbf{Z}_\Phi$ did not act positively. However, the Φ-split of p_i was still available for $\mathbf{Z}_\Phi$ to pick for positive satisfaction; the restraint until s_{i+1} was r_i and $r_i \leqslant R(\Phi, s_i) \leqslant R(\Phi, s_{i+1})$ (the latter inequality holds because after s_i no requirement stronger than $\mathbf{Z}_\Phi$ acts). This implies that $\Phi(A; n_i)\uparrow [d(s_{i+1})]$, in other words,

$$d(s_{i+1}) < h(n_i).$$

Of course, $n_i < s_i$ so $d(n_{i+1}) < d(s_{i+1}) < h(n_i)$.

Both h and d are increasing functions, so the last inequality implies that h dominates d: for every $n > n_1$, there exits i such that $n_i < n \leqslant n_{i+1}$. Then

$$d(n) \leqslant d(n_{i+1}) < h(n_i) \leqslant h(n).$$

$\square$

4.3.4 Coding D into B

In this section we show how to modify the construction to satisfy the coding requirement **C**; this will ensure that indeed B is a complement for A below D.

To do this, we specify an infinite, recursive set of levels on f (say all k such that $(k)_0 = 0$) which will be devoted to **C** (by the padding lemma this does not disturb the previous construction). Let $\langle n_i \rangle_{i<\omega}$ be an increasing enumeration of this set. We define a tree $T_D \subset T$ (as functions) by removing all nodes extending $T(\sigma\frown(1 - D(i)))$ for $\sigma \in 2^{n_i}$. Namely,

$$T_D = T \restriction \{\sigma : \forall n_i < \operatorname{dom} \sigma : \sigma(n_i) = D(i)\}.$$

Lemma 4.3.12. *$T_D \leqslant_T D$ and for every $X \in [T_D]$, $D \leqslant_T X \oplus A$.*

Proof: To find whether $i \in D$, A can find (on T) the unique σ on level $n_i + 1$ such that $T(\sigma) \subset X$. Since $X \in [T_D]$ we have $\sigma(n_i) = D(i)$. □

Thus all we need to do is repeat the previous construction, using T_D instead of T. This can be done because the construction is recursive in D; the construction follows through verbatim.

4.4 Case Three

4.4.1 A minimal degree below D

We first give our version of the construction of a minimal degree below D. As in Sacks's and Jockusch's constructions, we will construct a sequence of partial recursive trees $\langle T_k \rangle$ and an increasing sequence of strings β_s whose union B is a path on all of the trees. Later we will show how to modify this construction so that D can be recovered from $\langle T_k \rangle$. Now we just build B such that $\langle T_k \rangle \leqslant_T (A \oplus B)'$.

We follow Lerman's [Ler83] definitions of splitting subtrees. If T is a tree and Φ is a functional, then a Φ-splitting subtree of T is a subtree S such that for all $p \in \operatorname{dom} T$, $T(p\frown 0)$ and $T(p\frown 1)$, if defined, are a Φ-split. The canonical splitting subtree above a node $T(p)$ (denoted $S = Sp(T, \Phi, p)$) is defined by induction, letting the stem be $T(p)$; given $S(q)$ on T, $S(q\frown 0), S(q\frown 1)$ is defined to be the least Φ-split on T extending $S(q)$, and undefined if none is found.

We say that $X \in [T]$ *Φ-splits* on T if for all $\sigma \subset X$, there is a Φ-split on T extending σ.

The requirement to be satisfied is

$\mathbf{M}_\Phi$ If B Φ-splits on every tree T_k then it lies on a Φ-splitting tree.

Associating requirements with numbers as usual, the strategy to satisfy $k = \mathbf{M}_\Phi$ is to let $T_{k+1} = T_k$ if B does not Φ-split on T_k, and $Sp(T_k, \Phi, p)$ for some p otherwise.

We make p depend on B directly, by letting p be the *shortest string* such that $B \in Sp(T_k, \Phi, p)$.

As usual, the construction will have approximations $T_k[s]$ to T_k and an increasing sequence $\langle \beta_s \rangle$ of strings whose union is B. During each stage of the construction the trees $T_k[s]$ are enumerated a certain number of steps to find splits above β_s. We will make sure that once T_k has stabilized, if B splits on T_k then at every s there is a split found on T_k extending β_s. Thus from the construction we can calculate a function f which bounds the search needed to find splits on the trees. Namely, we have a function $f \leqslant_T D$ such that for all $k = \mathbf{M}_\Phi$, if B Φ-splits on T_k then for almost all n, a Φ-split extending $B \upharpoonright n$ will be found on T_k after enumerating the tree for $f(n)$ many steps.

[We note that it is not necessary to assume that $\operatorname{dom} \beta_s \geqslant s$ to get such a function; to calculate $f(n)$ we can simply wait for a stage at which $\operatorname{dom} \beta_s > n$. This will be helpful in the full construction we give at the end].

First, we would like to ensure that the above conditions on the construction indeed produce the desired result. Suppose, then, that B, $\langle T_k \rangle$ and f are given as described.

Lemma 4.4.1. *B has minimal Turing degree.*

Proof: See [Ler83, V.2.6]. □

Suppose that g is a function which dominates f.

Lemma 4.4.2. $\langle T_k \rangle \leqslant_T (B \oplus g)'$.

Proof: Take $k = \mathbf{M}_\Phi$. Suppose we have already determined what T_k is. First, we wish to find out what kind of tree T_{k+1} is, which is the same as finding whether B Φ-splits on T_k or not.

We first ask $(B \oplus g)'$ if there is some n such that after running T_k for $g(n)$ many steps, no Φ-splits above $B \upharpoonright n$ on T_k are found. If not, B splits on T_k. If there is such an n, then a search (recursive in $B \oplus g$) can find the least one n_0. We now ask $0'$ (which is below $(B \oplus g)'$) if there are any Φ-splits extending $B \upharpoonright n_0$ on T_k. If not, then B does not split on T_k. If such a split exists, we repeat the process, asking if there is some $n_1 > n_0$ such that after $g(n_1)$ many steps no Φ-split extending $B \upharpoonright n_1$ is found on T, and if one exists, we find it and ask $0'$, etc. This process must stop. For if B doesn't split on T_k then eventually we find some n such that there are no Φ-splits on T_k extending $B \upharpoonright n$ (and of course none before $g(n)$.) If B does split on T_K, then the properties of f ensure that eventually, a Φ split extending $B \upharpoonright n$ is always found before step $f(n)$, hence $g(n)$.

Now if B doesn't Φ-split on T_k then $T_{k+1} = T_k$. Otherwise, we need to find the least q such that $B \in [\mathrm{Sp}(T_k, \Phi, q)]$; We then know that $T_{k+1} = \mathrm{Sp}(T_k, \Phi, q)$.

Suppose that $T_k(q) \subset B$. Consider the following process: Let $T = \mathrm{Sp}(T_k, \Phi, q)$. $T(\langle\rangle) = T_k(q) \subset B$. Since every initial segment of B splits on T_k, both $T(0)$ and

$T(1)$ are defined. We find them. If neither are initial segments of B, then the process halts. Otherwise, pick the one (say $T(0)$) such that $T(0) \subset B$. By the same reasoning, both $T(01)$ and $T(11)$ are defined, etc.

We see that this process is recursive in B. Thus B' can tell us whether it eventually terminates or not. However, a quick inspection shows that the process terminates iff $B \notin [T]$. Thus

$$\{r : B \in [\mathrm{Sp}(T_k, \Phi, q)]\}$$

is recursive in $B' \leqslant (B \oplus g)'$. We can thus inductively (on q such that $T_k(q) \subset B$) ask whether q is an element of the above set (this search is recursive in $B \oplus B' \equiv_T B'$). Eventually we will find the least one which is the desired q. $\square$

Now recall that we had a set A, falling under the third case, which excludes the second case. This means that for every function f recursive in D, there is some g recursive in A which dominates f. Thus we can go back and write A in each place we wrote g to get $\langle T_k \rangle \leqslant_T (A \oplus B)'$.

4.4.2 Construction I

The construction which yields the B, $\langle T_k \rangle$ and f with the desired properties is not difficult, and in fact differs only slightly from Jockusch's construction. We describe it now.

We construct a sequence β_s.

By the recursion theorem, we have a function $G(T, \Phi, s) \leqslant_T D$ such that for all (partial recursive function) trees T and Turing functionals Φ, $\lim_s G(T, \Phi, s)$ is the answer to the question

> Is there some t such that β_t is on T, and above β_t there are no Φ-splits on T?

At stage 0, we let $k(0) = 0$, $T_0[0] = \mathrm{id}$, $\beta_0 = \langle\rangle$ and $f(0) = 0$.

At stage $s+1$, we are given a sequence of trees $T_0[s], \ldots, T_{k(s)}[s]$ (each $T_{k+1}[s]$ is either equal to $T_k[s]$ or is a splitting subtree of $T_k[s]$), and β_s which is on every $T_k[s]$.

Using G we can define a function H, which tells us how far a requirement has to search until its guess (about whether B splits or not) has evidence in reality. Namely, if $k = \mathbf{M}_\Phi$, then $H(k, s)$ is defined to be the least stage $t > s$ such that either $G(T_k[s], \Phi, t) = \texttt{yes}$, or a Φ-split extending β_s is found on $T_k[s]$. By the properties of G, we know that such a t must exist. We also let $F(k, s) = \texttt{split}$ if a Φ-split above β_s is found on $T_k[s]$ at stage $H(k, s)$; otherwise we let $F(k, s) = \texttt{no split}$. Thus F gives us k's guess about whether B Φ-splits on $T_k[s]$. There is a *discrepancy* between reality and k's belief if $F(k, s) = \texttt{split}$ and $T_{k+1}[s] = T_k[s]$, or if $F(k, s) = \texttt{no split}$ but $T_{k+1}[s]$ is a splitting subtree of $T_k[s]$. If there is such a discrepancy, then it must be fixed by k by changing the tree T_{k+1} according to belief.

We thus say that k *requires attention* at s if there is a discrepancy between k's beliefs and reality. If there is some k requiring attention, we act as follows:

- If $F(k,s) = \texttt{no split}$, we let $T_{k+1}[s+1] = T_k[s]$.
- If $F(k,s) = \texttt{split}$, we want to define $T_{k+1}[s+1]$ to be $\mathrm{Sp}(T_k[s], \Phi, q)$ for some $q \subset p = T_k[s]^{-1}(\beta_s)$; as discussed above, we pick q to be the least such that β_s is *extended by some string* on $\mathrm{Sp}(T_k[s], \Phi, q)$.

  How do we effectively find this q? This is similar to the process in the proof of lemma 4.4.2: we inductively check $q = \langle\rangle, q = p \restriction 1, q = p \restriction 2, \ldots$. We know there is a split extending β_s; thus we can search for the least split on the tree $T_k[s]$ extending $T_k[s](q)$. If neither of the two strings of the split are compatible with β_s, this was the wrong q, and we try the next one. If one is, we repeat the process for that part of the split which is an initial segment of β_s. We halt with a "yes" answer (for the q checked) if we get to a string extending β_s.

In either case we let $k(s+1) = k+1$ and $T_l[s+1] = T_l[s]$ for $l \leqslant k$.

If no requirement asks for attention, we let $k(s+1) = k(s)+1$, $T_l[s+1] = T_l[s]$ for $l \leqslant k(s)$ and $T_{k(s)+1}[s+1] = T_{k(s)}[s]$.

Now either β_s is not on $T_{k(s+1)}[s+1]$, but properly extended by some string which is on this tree; or β_s is on $T_{k(s+1)}[s+1]$ and not terminal on it. Thus we can properly extend β_s to some β_{s+1} on that tree. That's the end of stage s.

We verify that the objects constructed indeed have the desired properties. We first show that the sequence of trees stabilizes.

Lemma 4.4.3. *For all k there is a stage $s^*(k)$ such that for all $t \geqslant s^*(k)$, $k(t) > k$.*

By the construction, we have that for all $t > s^*(k)$, $T_k[t] = T_k[s^*(k)]$; we let T_k be this final tree.

PROOF: $s^*(0) = 1$. Assume $s^*(k)$ exists, we will show that $s^*(k+1)$ exists too. Suppose that $k = M_\Phi$. Suppose that after $t > s^*(k)$, $G(T_k, \Phi, s)$ has stabilized on the correct answer. Then we can let $s^*(k+1) = t$, because neither k, nor any stronger requirement, will seek attention after t. □

Corollary 4.4.4. *Each M_Φ succeeds.*

PROOF: The guesses are eventually correct. Thus if B splits on T_k, then T_{k+1} is some splitting subtree of B; if B doesn't split on T_k then we're done anyway. □

Also as a corollary, we have that if B splits on T_k then eventually, at every s, a split above β_s on T_k is found at stage s of the construction; this allows us to calculate the function f as required.

Finally,

Lemma 4.4.5. *Assume that $k = M_\Phi$ and that B splits on T_k. Then $T_{k+1} = \mathrm{Sp}(T_k, \Phi, q)$ where q is the least string such that $B \in [\mathrm{Sp}(T_k, \Phi, q)]$.*

PROOF: Say $T_{k+1} = \mathrm{Sp}(T_k, \Phi, q)$, and was finally defined at stage s. Of course $B \in [T_{k+1}]$. If $r \subsetneq q$ then at stage s we verified that β_s went off $\mathrm{Sp}(T_k, \Phi, r)$ (no string on that tree extends β_s), thus $B \notin [\mathrm{Sp}(T_k, \Phi, r)]$. $\square$

4.4.3 Coding D

We now add more subtrees to our sequence T_k so that we will be able to recover D from $\langle T_k \rangle$.

We have new requirements $\mathbf{C}_n$, whose aim is to code the answer to $n \in D$? into the sequence of trees.

We organize all $\mathbf{C}_n$ and $\mathbf{M}_\Phi$ requirements effectively and identify each requirement with its place on the list.

For $\mathbf{C}_n$, we use two different kinds of narrow subtrees.

Definition 4.4.6. For $p \in 2^{<\omega}$, let $\mathrm{nar}_p, \mathrm{row}_p \colon 2^{<\omega} \to 2^{<\omega}$ be defined as follows

$$\begin{aligned} \mathrm{nar}_p(\sigma) &= p^\frown 00\sigma(0)00\sigma(1)00\sigma(2)\ldots 00\sigma(|\sigma|-1) \\ \mathrm{row}_p(\sigma) &= p^\frown 11\sigma(0)11\sigma(1)11\sigma(2)\ldots 11\sigma(|\sigma|-1). \end{aligned}$$

Now, suppose that T is a tree and that $p \in \operatorname{dom} T$. We let $\mathrm{Nar}(T, p) = T \circ \mathrm{nar}_p$ and $\mathrm{Row}(T, p) = T \circ \mathrm{row}_p$.

It is easy to see that if T is partial recursive, then so are $\mathrm{Nar}(T, p)$ and $\mathrm{Row}(T, p)$, and an index for each can be obtained uniformly from p and an index for T.

Lemma 4.4.7. *Suppose that T is a tree, and that $p, q \in \operatorname{dom} T$. Then*

$$[\mathrm{Nar}(T, p)] \cap [\mathrm{Row}(T, q)] = 0.$$

This property motivates us to satisfy $k = \mathbf{C}_n$ by letting $T_{k+1} = \mathrm{Nar}(T_k, p)$ for some p if $n \in D$ and $T_{k+1} = \mathrm{Row}(T_k, p)$ for some p otherwise. As we did for the splitting subtrees, to be able to recover T_{k+1} from T_k and $(A \oplus B)'$ we eliminate the element of chance in the game by ensuring that $T_{k+1} = \mathrm{Nar}(T_k, p)$ for the least p such that $B \in [\mathrm{Nar}(T_k, p)]$ (or Row, as D decides).

Lemma 4.4.8. *Suppose that T is partial recursive and $X \in [T]$. Then*

$$\{p \in \operatorname{dom} T \,:\, X \in [\operatorname{Nar}(T,p)]\}$$

is recursive in X' (similarly for Row*).*

Proof: From X we can determine the $h \in 2^\omega$ such that $X \supset T[p^\frown h]$ (if $T(p) \not\subset X$ then X knows that p is not in the set). Then we simply ask if for such h and all $n \neq 0 \mod 3$, $h(n) = 0$ (1 for Row). □

So indeed if we code as promised, then B' can find T_{k+1} from T_k by inductively asking, for longer and longer p, such that $T_k(p) \subset B$, whether $B \in [\operatorname{Nar}(T_k, p)]$ or $B \in \operatorname{Row}(T_k, p)$. The first "yes" answer we get is the correct one (and also tells us whether $n \in D$).

It would seem that all we need to do now is add this last element to the construction, namely if at stage s, $k = k(s) = \mathbf{C}_n$ and no stronger requirement asks for attention, then we should define $T_{k+1}[s]$ to be $\operatorname{Nar}(T_k[s], p)$ (or $\operatorname{Row}(T_k[s], p)$ if $n \notin D$) for the shortest p such that β_s is extended by some string on $\operatorname{Nar}(T_k[s], p)$. However, this is not so easy, for the reason that β_s may be nonterminal on $T_k[s]$ for some $k < k(s)$, but terminal on the narrow subtree $T_{k+1}[s]$; for $p = T_k^{-1}(\beta_s)$, $T_k(p^\frown 0)$ may be defined but $T_k[s](p^\frown 00)$ or $T_k[s](p^\frown 000)$ may be undefined. In this case we would not be able to extend β_s and would be stuck.

We note, however, that the reason that $T_k[s](p^\frown 00)$ or $T_k[s](p^\frown 000)$ are undefined, is that some split was not found on an earlier tree $T_l[s]$. The strategy to extricate ourselves from this situation is to try and extend β_s on $T_l[s]$ until we find the necessary split or guess that such a split does not exist. In the former case we can go on with the construction; in the latter we change our guess about T_{l+1} and remove later trees.

4.4.4 Construction II

We first describe the idea of the construction. The new construction attempts to follow in the footsteps of the previous construction; requirements $k = \mathbf{M}_\Phi$ define subtrees T_{k+1} which are splitting subtrees or not, and amend the trees according to their guesses. New requirements $k = \mathbf{C}_n$ will define T_{k+1} to be some narrow subtree of T_k, the type of which will be determined by whether $n \in D$ or not. All goes well as long as our initial segment $\beta = \beta_s$ is nonterminal on the trees. Suppose, however, that β is terminal on $T_{k[s]}[s]$, even after the action of $\mathbf{M}$ requirements. We then look at the greatest $l \leqslant k(s)$ such that $T_l[s]$ is a splitting subtree of $T_{l-1}[s]$. We note that β must be nonterminal on $T_l[s]$; otherwise, $l-1$ would have redefined T_l to be equal to T_{l-1}.

Let $p = T_l^{-1}\beta$. The reason that β is terminal on $T_{k(s)}$ is that for some q, $T_l(p^\frown q)$ is undefined; the latter is undefined because sufficiently many splits above β have not been discovered on T_{l-1}. If some extension γ of β on T_{l-1} really doesn't split on

T_{l-1}, then we would like to extend β to γ and redefine $T_l = T_{l-1}$; this would resolve the issue. Otherwise, we will find more and more splits until β will be discovered to not be terminal on $T_{k(s)}$ after all. The question is how to tell whether there is such γ. After all, our guessing function G only answers questions about initial segments of B, not arbitrary nodes on the tree.

The solution is to gradually extend β, at substages of the stage s. As β is not terminal on T_l, we can extend it for one step on T_l. Now we search for splits on T_{l-1} as usual; if we believe there are none T_{l-1} can act and set $T_l = T_{l-1}$. Otherwise, a split is found which means we can extend β one step further on T_l. The process continues until β is nonterminal on $T_{k(s)}$, or a node γ as described is found and set as an initial segment of B.

We note a notational decision. It could be the case that at the beginning of the stage, some $k \in [l, k(s))$ redefines $T_{k+1} = T_k$ because β is terminal on T_k. Also, it could be the case that at some some substage we believe that β can no longer be extended on T_{l-1} (and redefine $T_l = T_{l-1}$); no previous tree is acting, which means that the problem which was just described has just reappeared, for a smaller pair $(l, k(s))$. The above actions will have to be repeated, but we do this at the next stage $s+1$ (we can move on because β is on all trees, even if terminal on some; between substages we will not change the trees, as the indexing would become cumbersome.)

We now give the full construction. At stage s we are given a sequence of trees $\langle T_k[s] \rangle_{k \leqslant k(s)}$ and a string β_t on all of those trees.

The guessing function $G(T, \Phi, s)$ is defined as before; it approximates the answer to the same question:

> Is there some t such that β_t is on T and there is no Φ-split extending β_t on T?

The elements of the construction are as follows:

Guessing Splitting

We define guessing functions H and F. At stage s, at a substage at which we have β_r as an initial segment of B, and for $k = \mathbf{M}_\Phi$, $H(k, s, r)$ is defined to be the least stage $u > r + s$ such that either $G(T_k[s], \Phi, u) = \texttt{yes}$, or a Φ-split extending β_r is found on $T_k[s]$ after running the tree for u many steps. $F(k, s, r)$ is defined analogously; it is `split` in the former outcome and `no split` in the latter.

Discrepancy between belief and reality is defined as in the previous construction; at the particular stage and substage, k feels the discrepancy if $F(k, s, r) = \texttt{split}$ but $T_{k+1}[s] = T_k[s]$, or if $F(k, s, r) = \texttt{no split}$ but $T_{k+1}[s+1]$ is a splitting subtree of $T_k[s]$.

At any stage and substage, if there is some requirement $k = \mathbf{M}_\Phi < k(s)$ asking for attention, then the strongest such requirement receives it and redefines T_{k+1}

according to its belief F. As before, it lets $T_{k+1}[s+1] = T_k[s]$ if $F(k,s,t) =$ `no split`, and $T_{k+1}[s+1] = \mathrm{Sp}(T_k[s], \Phi, p)$ for the shortest p such that β_r is extended by some string on the latter tree. It lets $k(s+1) = k+1$ and leaves the trees T_l for $l \leqslant k$ unchanged.

If β_r is not on $T_{k+1}[s+1]$ then it is extended by some string on that tree; we let β_{r+1} be such a string. Otherwise, we hand β_r to the next stage.

Action of the last tree

At some substage of stage s, we may instruct the last requirement $k(s)$ to act (this may happen if no other requirement asks for attention and if β_r is not terminal on $T_{k(s)}[s]$). $k(s)$ first extends β_r to some string β_{r+1} on $T_{k(s)}[s]$. Then,

- If $k(s) = \mathbf{M}_\Phi$, it lets $T_{k+1}[s+1] = T_k[s]$.
- If $k(s) = \mathbf{C}_n$ and $n \in D$, then $k(s)$ lets $T_{k+1}[s+1] = \mathrm{Nar}(T_k[s], p)$ for p the shortest substring of $T_k[s]^{-1}(\beta_{r+1})$ such that β_{r+1} is extended by some node on $\mathrm{Nar}(T_k[s], p)$. If necessary, β_{r+1} is extended again to β_{r+2} so that it actually lies on the new tree.
- If $k(s) = \mathbf{C}_n$ and $n \notin D$, we act as in the previous case, but with Row replacing Nar.

In both cases, we let $k(s+1) = k(s)+1$, and leave the trees T_l for $l \leqslant k(s)$ unchanged.

Construction

The instructions at stage s are as follows. If some $\mathbf{M}$ requirement asks for attention, we let the strongest one act and end the stage. If no $\mathbf{M}$ requirement asks for attention at the beginning of the stage, and if β_t is not terminal on $T_{k(s)}[s]$, then $k(s)$ is asked to act and to end the stage.

Otherwise, we let $T_l[s]$ be the last tree in our sequence which is a splitting subtree of its predecessor $T_{l-1}[s]$. There is a function h which is either the identity or a composition of functions of the form nar_q and row_q such that $T_{k(s)}[s] = T_l[s] \circ h$. Let $p = T_l^{-1}(\beta_t)$, $q = T_{k(s)}[s]^{-1}(\beta_t) = h^{-1}(p)$ and $r = h(q^\frown 0)$. Our aim is to extend β_t to be $T_l[s](r) = T_{k(s)}[s](q^\frown 0)$. This is done in substages.

As noted above, we must have β_t not terminal on $T_{l-1}[s]$, for we found a split on T_{l-1} extending β_t (or $l-1$ would have acted). Therefore also β_t is not terminal on $T_l[s]$. Thus we can extend it and let $\beta_{t+1} = T_l[s](r \upharpoonright |p|+1)$. We move to the first substage.

In general, at the beginning of substage i we have $\beta_{t+i} = T_l[s](r \upharpoonright |p|+i)$, already discovered to be on $T_l[s]$. If no $\mathbf{M}$ requirement wishes to act at this substage, then it must be that a split extending β_{t+i} was found on T_{l-1}; we can thus further extend this string on T_l and move to the next substage.

If at some substage an **M** requirement wishes to act, then that action ends the stage. Otherwise, at the end of the $i = |r| - |p|^{th}$ substage we have $\beta_{t+i} = T_l[s](r) = T_{k(s)}[s](p^\frown 0)$. (Note that during substages, β_{t+j} may be not a string on $T_{k(s)}[s]$, but at this last substage it is.) If β_{t+i} is terminal on $T_{k(s)}[s]$ we do not change any of the trees and move on to stage $s+1$ (we let $k(s+1) = k(s)$). Otherwise, we can let $k(s)$ act and end the stage.

4.4.5 Verifications II

The proof of lemma 4.4.3 goes through unaltered, noticing that if $k = \mathbf{C}_n$ then once T_k has stabilized, so will T_{k+1}, since k never acts to change the next tree.

As a corollary, we get that β_t gets extended infinitely often (so we do get a set B at the end). This is because if $k(s+1) > k(s)$ (i.e. if $k(s)$ gets to act), β_t has been extended at stage s (in fact, β_t has been extended at every stage except perhaps at stages at which some **M** requirement acts at the beginning of the stage, making $k(s+1) \leqslant k(s)$.)

We now may conclude that the construction succeeds; the rest of the verifications are as above: we showed that $\langle T_k \rangle \leqslant (A \oplus B)'$ and that D can be obtained from $\langle T_k \rangle$.

Chapter 5
Embedding and coding below a 1-generic degree (*with Noam Greenberg*).

Published in the Notre Dame Journal of Formal Logic, volume 44 (4), year 2003, pages 200-216.

5.1 Introduction

The complexity of the theory of degree structures (as partial orderings) has been for a long time a focus of attention of researchers. Among the noted results we can mention are that the theory of all Turing degrees Th($\mathcal{D}$) is undecidable (Lachlan [Lac68]); the theory of $\mathcal{D}(\leqslant \mathbf{0}')$ is undecidable (Lerman [Ler78]); the theory of the recursively enumerable degrees Th($\mathcal{R}$) is undecidable (Harrington and Shelah [HS82]).

A particular method for proving undecidability is embedding models of arithmetic in the degree structure with parameters. If one finds a first order condition on the parameters which ensures that the coded model is the standard one, then the theory of the structure interprets first-order true arithmetic. For structures which are interpretable in arithmetic this shows that the theory is as complicated as possible. Such results were obtained for $\mathcal{D}(\leqslant \mathbf{0}')$ (Shore [Sho81], where the result is extended to $\mathcal{D}(\leqslant \mathbf{a})$ for many other arithmetic degrees $\mathbf{a}$); and for $\mathcal{R}$ (Harrington and Slaman, and also Slaman and Woodin; see [NSS98]). Another important similar result is that Th($\mathcal{D}$) is recursively isomorphic to true second-order arithmetic (Simpson [Sim77]). We show in this paper that if $\mathbf{g}$ is 2-generic, or if it is a 1-generic degree below $\mathbf{0}'$, then this method can be employed in $\mathcal{D}(\leqslant \mathbf{g})$ and so we get the same result.

We code models of arithmetic below a 1-generic degree in a direct way, using coding schemes defined in [NSS98]. Further, this coding, together with the technique of comparison maps (again from [NSS98]), shows that if the 1-generic degrees are downward dense in the structure $\mathcal{D}(\leqslant \mathbf{g})$ then the standard models can be isolated. We then quote results of Chong and Jockusch ([CJ84]) and Jockusch ([Joc80]) which show that this condition holds if $\mathbf{g}$ is a 1-generic degree below $\mathbf{0}'$ or if $\mathbf{g}$ is 2-generic. (In fact, Haught, in [Hau86], showed that every nonzero degree below a 1-generic degree below $\mathbf{0}'$ is 1-generic.) We note that this technique cannot be extended to all 1-generic degrees; Both Kumabe ([Kum90]) and Chong and Downey ([CD90]) show that there is a 1-generic degree which bounds a minimal degree.

The coding tool we use is the coding introduced by Slaman and Woodin ([SW86]). One of the questions connected with this coding is where can one find the parameters needed for the coding, relative to the structure coded. It follows from the proof of [SW86, Prop. 2.5], Slaman and Woodin show that a 2-generic suffices. Their claim that parameters can be found below the jump of the coded structure was covered in detail in Odifreddi and Shore [OS91]. In order to code models below 1-generic degrees, we show here that a 1-generic filter suffices.

The requirement that standard models can be identified in a first order way is quite stringent. If we drop this requirement we get structures in which a class of models satisfying some finite part of arithmetic T is interpreted; this class contains the standard model. Then the theory of the structure can effectively separate the theorems of T and their negations; for sufficiently complicated T this shows that the structure is undecidable. Using our results concerning the coding parameters, we show that if $\mathbf{a}$ bounds a 1-generic degree then $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{a}))$ is undecidable.

This result, though, can be deduced from earlier work. Jockusch ([Joc80]) showed that every 1-generic degree is recursively enumerable in a strictly lower degree. Relativizing, one can apply the undecidability results of Shore ([Sho81]) which use techniques related to r.e. degrees, to get the aforementioned result. We mention our proof because it is straightforward in its use of genericity and does not appeal to recursive enumerability.

Embeddings of algebraic objects into degree structures have a close connection with undecidability results; indeed all the early undecidability results are established by coding some class of algebraic objects (such as linear orderings, partial orderings and graphs) into the degree structure. A striking example is Lerman's work, [Ler78, Ler83], which showed that every countable upper semi-lattice can be embedded in $\mathcal{D}$ as an initial segment; so the question about the theories of initial segments involves the theories of such semi-lattices. Further, before Slaman and Woodin introduced their coding, Lerman's results were used in undecidability proofs by using lattices to code models of arithmetic (Nerode and Shore [NS80a, NS80b, Sho81]). Later, Shore ([Sho82]) found a simpler method of embedding lattices below any r.e. degree (not as initial segments though). He applied Jockusch's result mentioned earlier to embedding techniques below r.e. degrees and showed that every recursive lattice can be embedded below any 1-generic degree (the power of the technique lies in embedding non-distributive lattices; the result for distributive lattices follows from the fact that the countable atomless Boolean algebra is embeddable below a 1-generic degree, even preserving $\mathbf{0}$ and $\mathbf{1}$). As we did before, we give a direct proof; we are, however, able to improve it to show that embeddings can be found which preserve $\mathbf{0}$ and $\mathbf{1}$.

We remark that Downey, Jockusch and Stob ([DJS96]) showed that every recursive lattice with least and greatest element can be embedded into $\mathcal{D}(\leqslant \mathbf{a})$ preserving $\mathbf{0}$ and $\mathbf{1}$, where $\mathbf{a}$ is any array nonrecursive degree. These are the degrees which bound pb-generic degrees; this is a notion of genericity which is intermediate between 1 and 2-genericity. Unlike the 1-generics, the array nonrecursive degrees are upward closed. Our theorem cannot be improved in this direction; there is a degree $\mathbf{a}$ which is a strong minimal cover of a 1-generic degree (Kumabe, [Kum00]). Hence, for example, the diamond lattice cannot be embedded in $\mathcal{D}(\leqslant \mathbf{a})$ preserving $\mathbf{1}$.

5.1.1 Notation

Given $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \frown \tau$ for the string π of length $\max\{|\sigma|, |\tau|\}$ such that for all $i < |\pi|$

$$\pi(i) = \begin{cases} \sigma(i) & \text{if } i < |\sigma| \\ \tau(i) & \text{if } |\sigma| \leqslant i < |\tau|. \end{cases}$$

If $\sigma, \tau \in 2^{<\omega}$ and $E \subseteq \omega$, we say that σ and τ are *E-equivalent*, and we write $\sigma \equiv_E \tau$, if for all $x \in E \cap \operatorname{dom} \sigma \cap \operatorname{dom} \tau$, $(\sigma(x) = \tau(x))$.

We assume we have a fixed recursive bijection between ω and V_ω. In particular we identify finite sequences of natural numbers with the number coding the sequence. For $A \subseteq \omega$ and $n < \omega$ we let the n^{th} *column* of A be

$$A^{[n]} = \{x \in \omega : \langle n, x\rangle \in A\}.$$

If $F \subseteq \omega$ and for every $i \in F$ we have a set $A_i \subseteq \omega$ then we let

$$\bigoplus_{i \in F} A_i = \bigcup_{i \in F} \{i\} \times A_i.$$

Thus if $i \in F$ then the i^{th} column of $\bigoplus_{i \in F} A_i$ is again A_i.

If $A \subseteq \omega$ then we denote its Turing degree $\deg_T A$ by $\mathbf{a}$. $\mathcal{D}$ is the collection of all Turing degrees. A nonempty set of degrees $\mathcal{J}$ is an *ideal* if it is closed downwards and with respect to the join operation. For example, if $\mathbf{a}$ is a Turing degree then

$$\mathcal{D}(\leqslant \mathbf{a}) = (\mathbf{a}) = \{\mathbf{b} \in \mathcal{D} : \mathbf{b} \leqslant \mathbf{a}\}$$

is an ideal.

If $\phi(\bar{x})$ is a formula in the language of upper semi-lattices, then we say that ϕ is *absolute for ideals* if for every ideal $\mathcal{J}$ and every tuple $\bar{\mathbf{a}} \in \mathcal{J}$,

$$(\mathcal{J}, \leqslant_T) \models \phi(\bar{\mathbf{a}}) \quad \Leftrightarrow \quad (\mathcal{D}, \leqslant_T) \models \phi(\bar{\mathbf{a}}).$$

A formula ϕ in the language of upper semi-lattices is *bounded* if all quantifiers appearing in ϕ are bounded, i.e. of the form $\exists x \leqslant t$, $\forall x \leqslant t$, where t is a term not containing x. Every bounded formula is absolute for ideals.

5.1.2 1-Genericity

We consider the notion of 1-generic filters with regards to various forcing notions.

Definition 5.1.1. Let $\mathbb{P}$ be a partial ordering on ω (we regard $\mathbb{P}$ as a *forcing notion*). Let $C \subset \omega$. A filter $G \subset \mathbb{P}$ is *C-1-generic* if for every $W \subset \mathbb{P}$ which is recursively enumerable in C, either $G \cap W \neq 0$ or there is some $p \in G$ such that for all $q \leqslant_{\mathbb{P}} p$, $q \notin W$.

A *1-generic degree* is a Turing degree which contains a filter which is 1-generic for *set Cohen forcing* ($2^{<\omega}$, ordered by reverse inclusion).

Let $\mathbb{P}, \mathbb{Q}$ be partial orderings on ω. We say that an injection $i : \mathbb{Q} \to \mathbb{P}$ is a *dense embedding* if i preserves $\leqslant, \perp$ and for every $p \in \mathbb{P}$ there is a $q \in \mathbb{Q}$ such that $i(q) \leqslant_{\mathbb{P}} p$ (see [Kun80, VII.7]).

The following is a recursive analogue of a familiar theorem of set theory.

Proposition 5.1.2. *Let $i : \mathbb{Q} \to \mathbb{P}$ be a dense embedding. Let $C \geqslant_T i \oplus \mathbb{P} \oplus \mathbb{Q}$.*

1. *Suppose that $G \subset \mathbb{Q}$ is 1-generic over C. Let H be the upward closure of i“G in $\mathbb{P}$. Then $H \subset \mathbb{P}$ is a C-1-generic filter, and $H \leqslant_T G \oplus C$.*

2. *Suppose that $H \subset \mathbb{P}$ is 1-generic over C. Let $G = i^{-1}H$. Then $G \subset \mathbb{Q}$ is C-1-generic filter, and $G \leqslant_T H \oplus C$.*

Proof: (1) Let $W \subset \mathbb{P}$ be recursively enumerable in C. Without loss of generality, assume that W is closed downwards (i.e. open). $i^{-1}W$ is also recursively enumerable in C and is open in $\mathbb{Q}$. The fact that i is dense implies that the upward closure of i“$i^{-1}W$ in $\mathbb{P}$ is W.

If $G \cap i^{-1}W \neq 0$ then $H \cap W \neq 0$. Otherwise, there is some $p \in G$ such that no extension of p in $\mathbb{Q}$ is in $i^{-1}W$; so $p \perp_{\mathbb{Q}} r$ for all $r \in i^{-1}W$. This implies that $i(p) \perp_{\mathbb{P}} s$ for all $s \in i$“$i^{-1}W$, so $i(p)$ has no extension in W. It is immediate to check that H is a filter, so H is indeed C-1-generic.

Given any $p \in \mathbb{P}$, by genericity we can find some $q \in G$ such that either $i(q) \leqslant_{\mathbb{P}} p$ or $i(q) \perp_{\mathbb{P}} p$, and this decides whether $p \in H$.

(2) is easier.

$\square$

It is well-known that Cohen forcing is universal for all countable forcings: every (nontrivial) countable notion of forcing embeds densely into Cohen forcing (see [Kan94, Prop. 10.20]). Further, for each forcing $\mathbb{P}$ there is a dense embedding $i : \mathbb{P} \to 2^{<\omega}$ which is recursive in $\mathbb{P}$; this is shown, for example, in [SWb]. For completeness, we show that *function Cohen forcing* ($\omega^{<\omega}$, ordered by reverse inclusion) embeds into set Cohen forcing. We will later (5.2.9) see another example of this universality.

Proposition 5.1.3. *There is a recursive, dense embedding of function Cohen forcing into set Cohen forcing.*

Proof: For $\sigma \in \omega^n$, let $i(\sigma) = 0^{\sigma(0)}10^{\sigma(1)}1 \cdots 10^{\sigma(n-1)}1$. i is dense because for every $\tau \in 2^{<\omega}$, $\tau^\frown 1 \in \mathrm{range}(i)$. It is clear that i preserves $\subset$ and $\perp$. $\square$

Thus a degree is 1-generic iff it contains some $G \subset \omega^{<\omega}$ which is 1-generic.

5.2 Slaman and Woodin Coding

Let $\mathcal{J} = \{\mathbf{c}_i : i \in I\}$ be an antichain of Turing degrees. (I could be either ω or some finite set.) We want to find degrees $\mathbf{c}$, $\mathbf{g}_0$ and $\mathbf{g}_1$ such that the elements of $\mathcal{J}$ are the minimal solutions below $\mathbf{c}$ of the following inequality in $\mathbf{x}$

$$(\mathbf{g}_0 \cup \mathbf{x}) \cap (\mathbf{g}_1 \cup \mathbf{x}) \neq (\mathbf{x}). \tag{5.2.1}$$

For each $i \in I$, let $\hat{C}_i$ be an element of $\mathbf{c}_i$ and let

$$C_i = \{\alpha \in 2^{<\omega} : \alpha \subset \hat{C}_i\}.$$

Let $C = \bigoplus_{i \in I} C_i$ and let $\mathbf{c} = \deg_T C$. Given $F \subseteq I$, we let $C_F = \bigoplus_{i \in F} C_i$.

Let $\mathbb{P}$ be Slaman and Woodin's notion of forcing for their coding. The elements of $\mathbb{P}$ are triples $p = \langle p_0, p_1, F_p \rangle$ where $p_0, p_1 \in 2^{<\omega}$, F_p is a finite subset of I, and $|p_0| = |p_1|$. We call $|p_0|$ the *length* of p and write $|p|$ instead of $|p_0|$. The partial ordering of $\mathbb{P}$ is defined as follows: $q \leqslant_{\mathbb{P}} p$ if:

- $p_0 \subseteq q_0$ and $p_1 \subseteq q_1$;
- $F_p \subseteq F_q$; and
- for all $y = \langle i, x \rangle$ such that $i \in F_p$, $x \in C_i$ and $|p| \leqslant y < |q|$, we have that $q_0(y) = q_1(y)$. In other words, it is required that q_0 and q_1 are $C_{F_p} \smallsetminus |p|$-equivalent.

Note that $\langle \mathbb{P}, \leqslant_{\mathbb{P}}, \perp_{\mathbb{P}} \rangle \leqslant_T C$.

Given a filter $G \subseteq \mathbb{P}$, let $G_0 = \bigcup\{p_0 : p \in G\}$ and $G_1 = \bigcup\{p_1 : p \in G\}$; let $\mathbf{g}_0 = \deg_T G_0$ and $\mathbf{g}_1 = \deg_T G_1$ be their degrees. Recall that a filter $G \subseteq \mathbb{P}$ is C-1-generic if for every set $W \subseteq \mathbb{P}$ which is recursively enumerable in C, either $G \cap W \neq \emptyset$, or there is a $p \in G$ such that $\forall q \leqslant_{\mathbb{P}} p (q \notin W)$. Observe that for every C-1-generic G, $G_0, G_1 \in 2^\omega$ and $\forall k \exists p \in G (k \in F_p)$.

Theorem 5.2.1. *Let G be a C-1-generic filter on $\mathbb{P}$, and let $\mathbf{g}_0$ and $\mathbf{g}_1$ be defined from G as above. Then $\mathcal{J}$ is the collection of minimal solutions of equation (5.2.1) below* $\mathbf{c}$.

In order to prove Theorem 5.2.1, it is sufficient to show that the following requirements are satisfied. Here k varies over I, Φ varies over all Turing functionals, and X varies over all sets which are recursive in C.

- P_k: $C_k \not\equiv_T (G_0 \oplus C_k) \wedge (G_1 \oplus C_k)$ (if the latter exists).
- $M_{X,\Phi}$: If $\Phi^{G_0 \oplus X} = \Phi^{G_1 \oplus X} = D$ are total and equal, and if $D \nleq_T X$, then for some k, $C_k \leqslant_T X$.

The P_k requirements ensure that the C_ks are solutions to (5.2.1) and the $M_{X,\Phi}$ requirements ensure that the C_ks are minimal solutions, and that no other minimal solutions exist below C.

Lemma 5.2.2. *For every k, P_k is met. Therefore all the sets C_k satisfy equation (5.2.1).*

This is exactly as in the proof of [SW86, Prop. 2.5], but for completeness, we present the proof.

Proof: Let $E_k = C_k \cap G_0^{[k]}$. It is immediate that $E_k \leqslant_T G_0 \oplus C_k$. However, we also have $E_k \leqslant_T G_1 \oplus C_k$. In fact

$$G_0^{[k]} \cap C_k \ =^* \ G_1^{[k]} \cap C_k,$$

because there is some $p \in G$ such that $k \in F_p$; for all $\langle k, x\rangle > |p|$ with $x \in C_k$, we have $G_0^{[k]}(x) = G_1^{[k]}(x)$.

It remains to show that $E_k \nleqslant_T C_k$. Consider a Turing functional Φ and let

$$S_{k,\Phi} = \{q \in \mathbb{P} : \exists x \in C_k(q_0(k, x)\downarrow \neq \Phi^{C_k}(x)\downarrow)\}.$$

Since $S_{k,\Phi}$ is C-r.e., there has to be some $p \in G$ such that either $p \in S_{k,\Phi}$, or $\forall q \leqslant_{\mathbb{P}} p(q \notin S_{k,\Phi})$. In the former case we have $\Phi^{C_k} \neq E_k$. In the latter case, we claim that $\Phi^{C_k}(x)\uparrow$ for all $x \in C_k$ such that $\langle k, x\rangle \geqslant |p|$; for if $\Phi^{C_k}(x)\downarrow$ for some such x then one can easily extend p to a condition in $S_{k,\Phi}$.

Therefore, we have that for all Φ, $\Phi^{C_k} \neq E_k$, and hence $E_k \nleqslant_T C_k$. □

Minimality Requirements

Now fix $X \leqslant_T C$ and Φ such that $D = \Phi^{G_0 \oplus X} = \Phi^{G_1 \oplus X}$ and such that $D \nleqslant_T X$. We want to show that for some k, $C_k \leqslant_T X$. The general idea of the proof (as done by Slaman and Woodin) is as follows. A *split* of a condition $p \in \mathbb{P}$ is a pair of strings $\sigma, \tau \supseteq p_0$ such that $\Phi^{\sigma \oplus X}$ and $\Phi^{\tau \oplus X}$ are contradictory. Clearly no such split can be a condition in the generic, so by genericity there is some condition $\bar{p}$ which is not extended by splits. Now, every condition has some split, as D is not recursive. So the reason that such a split is not an extension of $\bar{p}$ is that σ and τ contain some contradictory information about $x \in C_k$ for some x and k such that $k \in \bar{F} = F_{\bar{p}}$. The idea is to read off information about C_k by searching for such splits.

Now the way we go about fulfilling this strategy is the new part of the proof so we describe it more closely. As discussed, we will find (in Lemma 5.2.5) $\bar{F}$ and $\bar{p}$ as above such that for every split (σ, τ) of $\bar{p}$ there is some $k \in \bar{F}$ and some $\gamma \in C_k$ such that $\sigma(\langle k, \gamma\rangle) \neq \tau(\langle k, \gamma\rangle)$. Further, we will look for "special" splits (σ, τ) of $\bar{p}$, which means that for some $k \in \bar{F}$ and $\alpha \in 2^{<\omega}$, if σ and τ differ on some $\langle i, \gamma\rangle$ with $i \in \bar{F}$, then necessarily $i = k$ and $\gamma \supseteq \alpha$. As we are guaranteed such a difference for *some* i and γ, we have $\gamma \in C_k$; as C_k is the set of initial segments of the set $\hat{C}_k$, we must have $\alpha \in C_k$. We will show that recursively in X, for some k, one can enumerate infinitely many such special splits with α arbitrarily long, and thus is able to enumerate infinitely many elements of C_k. As C_k is recursive in any of its infinite subsets, this gives us a method of calculating C_k from X.

Definition 5.2.3. We call a condition $q \in \mathbb{P}$ *contradictory* if for some x,

$$\Phi^{q_0 \oplus X}(x)\downarrow \neq \Phi^{q_1 \oplus X}(x)\downarrow.$$

Being contradictory is a C-r.e. condition, so, by C-1-genericity, there is some $p \in G$ such that either p is contradictory or no extension of it is contradictory. The former case cannot hold because $\Phi^{G_0 \oplus X} = \Phi^{G_1 \oplus X}$, so the latter is the case.

Definition 5.2.4. Given $p \in \mathbb{P}$ and a set E, an E-split of p is a pair $\langle \sigma, \tau \rangle$ such that

- $\sigma \supseteq p_0$ and $\tau \supseteq p_0$;
- $\Phi^{\sigma \oplus X}(m)\downarrow \neq \Phi^{\tau \oplus X}(m)\downarrow$ for some m.
- $|\sigma| = |\tau|$.
- $\sigma \equiv_E \tau$.

If $\langle \sigma, \tau \rangle$ is a split, we let $m(\sigma, \tau)$ be the least m such that

$$\Phi^{\sigma \oplus X}(m)\downarrow \neq \Phi^{\tau \oplus X}(m)\downarrow.$$

Lemma 5.2.5. *There is a finite $F \subseteq I$ and a condition $p \in G$ which has no C_F-split.*

Proof: Let $\bar{p}$ be a condition in G which has no contradictory extensions and let $F = F_{\bar{p}}$. Consider the set

$$S = \{q \leqslant_{\mathbb{P}} \bar{p} : \exists \sigma \in 2^{|q|}(\langle \sigma, q_0 \rangle \text{ is a } C_F\text{-split of } \bar{p}\}$$

Since S is C-recursive, by C-1-genericity, there is some $p \in G$ such that either p is in S or no extension of p is in S. Observe that if $p \leqslant_{\mathbb{P}} \bar{p}$ has any C_F split, then we can easily construct some extension of p in S, so it suffices to show that $G \cap S = \emptyset$.

Suppose that $p \in S \cap G$ and let σ be a string such that $\langle \sigma, p_0 \rangle$ is a C_F-split of $\bar{p}$. Let $m = m(\sigma, p_0)$. By our assumptions on X and Φ, there is some extension q of p such that $\Phi^{q_1 \oplus X}(m)\downarrow$. $q_0 \supseteq p_0$ and so $\Phi^{q_0 \oplus X}(m)\downarrow = \Phi^{p_0 \oplus X}(m)\downarrow$. Also, q is not contradictory. To sum it up, we have

$$\Phi^{q_1 \oplus X}(m)\downarrow = \Phi^{q_0 \oplus X}(m) = \Phi^{p_0 \oplus X}(m) \neq \Phi^{\sigma \oplus X}(m)\downarrow.$$

Let $\bar{\sigma} = \sigma \frown q_0$. Then $\langle \bar{\sigma}, q_1, F \rangle$ is a contradictory extension of $\bar{p}$ contradicting our choice of $\bar{p}$. $\square$

Lemma 5.2.6. *Let E_0, E_1 be recursive sets. Suppose that every $p \in G$ has a $(E_0 \cap E_1)$-split. Then either every $p \in G$ has a E_0-split or every $p \in G$ has a E_1-split.*

Proof: Suppose, toward a contradiction, that there is some condition in G which has no E_0-split and some condition in G which has no E_1-split. Then, by taking a lower bound, we find some $\bar{p} \in G$ which has neither any E_0-split nor any E_1-split. We can also assume that $\bar{p}$ has no contradictory extensions. Consider

$$S = \{q \leqslant_{\mathbb{P}} \bar{p} : \exists \sigma, \tau \in 2^{|q|}(\sigma \equiv_{E_0} q_0 \equiv_{E_1} \tau \;\&\; \langle \sigma, \tau \rangle \text{ is a } E_0 \cap E_1\text{-split of } \bar{p}\}$$

Since S is C-recursive, there is some $p \in G$ such that either p is in S or no extension of p is in S. We note that every $p \in G$ has an extension in S: Take any $p \in G$; without loss of generality $p \leqslant \bar{p}$. Let $\langle \sigma, \tau \rangle$ be a B-split of p, and let q_0 be defined as follows:

$$q_0(x) = \begin{cases} \sigma(x) & \text{if } x \in E_0 \\ \tau(x) & \text{if } x \in E_1 \\ p_0(x) & \text{otherwise.} \end{cases}$$

This definition makes sense because $\sigma \equiv_{E_0 \cap E_1} \tau$. We have $\sigma \equiv_{E_0} q_0$ and $\tau \equiv_{E_1} q_0$. Then $\langle q_0, p_1 \curvearrowright q_0, F_p \rangle$ extends p and is in S.

Thus we have some $p \in S \cap G$. Let σ and τ witness that $p \in S$ and let $m = m(\sigma, \tau)$. There is some extension q of p such that $\Phi^{q_0 \oplus X}(m)\downarrow$. Let $\bar{\sigma} = \sigma \curvearrowright q_0$ and $\bar{\tau} = \tau \curvearrowright q_0$. Then, either $\langle \bar{\sigma}, q_0 \rangle$ is an E_0-split of $\bar{p}$, or $\langle \bar{\tau}, q_0 \rangle$ is an E_1-split of $\bar{p}$ (according to the value of $\Phi^{q_0 \oplus X}(m)$), contradicting the definition of $\bar{p}$. □

Lemma 5.2.7. *Let E_0,..., E_{n-1} be recursive sets. Suppose that every $p \in G$ has a $(E_0 \cap E_1 \cap \cdots \cap E_{n-1})$-split. Then, for some $i < n$, every $p \in G$ has a E_i-split.*

Proof: The magic word is 'induction'. □

Lemma 5.2.8. *For every finite set $S \subset \omega$, every $p \in G$ has an S-split.*

Proof: If $\max S < |p|$, then the notions of an S-split of p and of a $\emptyset$-split of p coincide. Since we can make $p \in G$ large, if the lemma fails then there is some $p \in G$ with no $\emptyset$-splits. We show that this assumption implies that $D \leqslant_T X$, which contradicts our previous assumptions.

Pick some $p \in G$ which has no $\emptyset$-splits. To compute $D(x)$ recursively in X, one looks for some $\sigma \supseteq p_0$ such that $\Phi^{\sigma \oplus X}(x)\downarrow$. Since p has no $\emptyset$-splits, necessarily $\Phi^{\sigma \oplus X}(x) = \Phi^{G_0 \oplus X}(x) = D(x)$. □

Now we show that for some k, $C_k \leqslant_T X$. By Lemma 5.2.5, fix a finite $\bar{F}$ and $\bar{p} \in G$ such that $\bar{p}$ has no $C_{\bar{F}}$-splits. Given $\alpha \in 2^{<\omega}$ and $k \in \bar{F}$ let

$$\begin{aligned} E_{k,\alpha} &= \{\langle i, \beta \rangle : i \in \bar{F} \;\&\; (i \neq k \vee \beta \not\supseteq \alpha)\} \\ &= (\bar{F} \times \omega) \setminus \{\langle k, \beta \rangle : \beta \supseteq \alpha\}. \end{aligned}$$

First observe that if there is an $E_{k,\alpha}$-split, $\langle \sigma, \tau \rangle$ of $\bar{p}$, then $\alpha \in C_k$. This is because, since $\bar{p}$ has no $C_{\bar{F}}$-split, σ and τ differ on some $\langle i, \gamma \rangle \in C_{\bar{F}} \setminus E_{k,\alpha}$, and hence $i = k$, $\gamma \supseteq \alpha$ and $\gamma \in C_k$. Therefore $\alpha \in C_k$. So

$$Y := \{\langle k, \alpha \rangle : \text{there is a } E_{k,\alpha}\text{-split of } \bar{p}\}$$

is subset of $C_{\bar{F}}$.

Now, fix $n \in \omega$, and observe that

$$E_n := \bigcap_{k \in \bar{F}, \alpha \in 2^n} E_{k,\alpha} = \{\langle i, \beta \rangle : i \in \bar{F} \ \& \ |\beta| < n\}$$

is finite. So, by Lemma 5.2.8 every $p \in G$ has a E_n-split. Then, by Lemma 5.2.7, for some $k \in \bar{F}$ and $\alpha \in 2^n$ there is a $E_{k,\alpha}$-split $\langle \sigma, \tau \rangle$ of $\bar{p}$. Hence, Y is infinite. Then, for some $k \in \bar{F}$,

$$Y_k := \{\alpha : \langle k, \alpha \rangle \in Y\}$$

is an infinite subset of C_k. Note that Y_k is r.e. in X, and therefore $C_k \leqslant_T X$.

5.2.1 Coding Countable Sets

To find the parameters for coding countable sets, we first need to relate genericity for $\mathbb{P}$ with genericity for Cohen forcing. Let $\mathbb{Q} = \omega^{<\omega}$ be function Cohen forcing.

Proposition 5.2.9. *There is a dense embedding $i : \mathbb{Q} \to \mathbb{P}$ which is recursive in C.*

Proof: Let $\{p_i\}$ be a recursive enumeration of the elements of $\mathbb{P}$. We say that a condition $p \in \mathbb{P}$ decides G up to p_n if for all $i \leqslant n$, $p \leqslant_{\mathbb{P}} p_i$ or $p \perp_{\mathbb{P}} p_i$. For every n, the collection of conditions which decide G up to p_n is dense in $\mathbb{P}$; we denote this collection by Ψ_n.

We claim that there is a process, uniformly recursive in C, which, given $p \in \mathbb{P}$ and $n < \omega$, enumerates an infinite maximal antichain below p, recursive in C, of conditions which decide G up to p_n. First, we find an infinite maximal antichain below p. For each $k < \omega$, let $p^k = (p_0 {}^\frown 0^k 1, p_1 {}^\frown 0^k 1, F_p)$; note that $p^k \leqslant_{\mathbb{P}} p$. Now define A_p by inductively deciding whether $p_i \in A_p$: $p_i \leqslant_{\mathbb{P}} p$ is added to A_p if it is one of the p^ks, or if it is incompatible with all of the p^ks and with all elements previously decided to be in A_p. $q \leqslant_{\mathbb{P}} p$ is incompatible with all p^ks iff $\min\{l \geqslant |p| : q_0(l) \neq 0\} \neq \min\{l \geqslant |p| : q_1(l) \neq 0\}$; this shows that A_p is recursive in C, and it is immediate that A_p is an infinite, maximal antichain below p.

For every $q \in A_p$ we find a maximal antichain B_q below q contained in Ψ_n in much the same manner; we don't mind if B_q is finite, so we simply apply the inductive process, restricted to elements of Ψ_n. Note that we indeed get a maximal antichain below q, because Ψ_n is dense open below q. Now

$$B_{p,n} := \bigcup_{q \in A_p} B_q$$

is an infinite, maximal antichain below p, is contained in Ψ_n, and can be enumerated recursively in C (uniformly in p and n), by enumerating A_p and B_q for q enumerated in A_p, dovetailing of course.

We can now easily define $i(\sigma)$ by induction on σ; $i(\langle\rangle)$ is the empty condition of $\mathbb{P}$. If $i(\sigma)$ is defined, then $i(\sigma^\frown\{n\})$ is the n^{th} element enumerated in $B_{i(\sigma),|\sigma|}$. Clearly i is recursive in C, and i is an embedding of $\mathbb{Q}$ into $\mathbb{P}$ preserving $\perp$. To see that i is dense, take any p_n. $i``\omega^n$ is a maximal antichain in $\mathbb{P}$, so for some $\sigma \in \omega^n$, $i(\sigma)$ is compatible with p_n. Since $i(\sigma)$ decides G up to p_n, we must have $i(\sigma) \leqslant_{\mathbb{P}} p_n$. □

The following is well known.

Proposition 5.2.10. *Suppose that G is 1-generic over B. Suppose that $A_0, A_1 \leqslant_T B$ and that $n, m < \omega$. Then*

$$A_0 \oplus G^{[n]} \leqslant_T A_1 \oplus G^{[m]}$$

iff $A_0 \leqslant_T A_1$ and $n = m$. □

We finally show how to code countable sets. This follows [SW86, Prop. 2.15].

Theorem 5.2.11. *There is a bounded formula $\psi(x, \bar{y})$ in the language of upper semi-lattices such that whenever we have a sequence of reals $\langle C_i \rangle$, a real $C \geqslant_T \bigoplus_i C_i$ and some G which is 1-generic over C, then there is a tuple $\bar{\mathbf{a}}$ of degrees below $\mathbf{g} \cup \mathbf{c}$ such that*

$$\mathbf{x} \in \{\mathbf{c}_i : i < \omega\} \Leftrightarrow \mathcal{D} \models \psi(\mathbf{x}, \bar{\mathbf{a}}).$$

Of course, $\mathbf{c}_i = \deg_T C_i$, $\mathbf{c} = \deg_T C$ and $\mathbf{g} = \deg_T G$.

Proof: By Theorem 5.2.1 and Proposition 5.1.2, there is a bounded formula $\phi(x, y, z_0, z_1)$ such that for every countable antichain of degrees $\mathcal{C} = \{\mathbf{c}_i\}$ and every G which is 1-generic over $C = \bigoplus C_i$, there are $G_0, G_1 \leqslant_T C \oplus G$ such that $\mathcal{C}$ is definable by the formula $\phi(x, \mathbf{c}, \mathbf{g}_0, \mathbf{g}_1)$.

Let $\mathcal{C} = \{\mathbf{c}_i : i < \omega\}$. Let $G_i = (G^{[0]})^{[i]}$, and let $\mathcal{G} = \{\mathbf{g}_i : i < \omega\}$; $\mathcal{G}$ is an antichain, as $G^{[0]}$ is 1-generic over C. Let $\mathcal{I} = \{\mathbf{c}_i \cup \mathbf{g}_i : i < \omega\}$; $\mathcal{I}$ is an antichain. Note that $\bigoplus G_i$ and $\bigoplus(C_i \oplus G_i)$ are both recursive in $C \oplus G^{[0]}$. As $G^{[1]}$ is 1-generic over $C \oplus G^{[0]}$, there are parameters below $G \oplus C$ coding $\mathcal{I}$ and $\mathcal{G}$ as above.

Now $\mathcal{C}$ is definable from the above parameters and $\mathbf{c}$ by the bounded formula

$$x < \mathbf{c} \,\&\, \exists(g \in \mathcal{G}, z \in \mathcal{I})(z = g \cup x)$$

□

Porism 5.2.12. *The function taking $\mathbf{c}_i$ to $\mathbf{g}_i$ is definable with the same parameters by the formula*

$$x \in \mathcal{C} \;\&\; y \in \mathcal{G} \;\&\; (x \cup y) \in \mathcal{I}.$$

We now code countable functions.

Theorem 5.2.13. *Suppose that* $\mathcal{B} = \{\mathbf{b}_i : i < \omega\}$ *and* $\mathcal{C} = \{\mathbf{c}_i : i < \omega\}$ *are sets of degrees. Let* $B_i \in \mathbf{b}_i$, $C_i \in \mathbf{c}_i$, $B = \bigoplus B_i$ *and* $C = \bigoplus C_i$. *Suppose that* G *is 1-generic over* $B \oplus C$. *Then the function taking* $\mathbf{b}_i$ *to* $\mathbf{c}_i$ *is definable with parameters found below* $B \oplus C \oplus G$.

As for sets, the coding is done uniformly by a bounded formula.

PROOF: Let $E = B \oplus C \oplus G$. We can find parameters below $\mathbf{e}$ coding the sets $\mathcal{B}$ and $\mathcal{C}$. Again split G: let $G_n = \left(G^{[0]}\right)^{[n]}$. As $G^{[1]}$ is 1-generic over $C \oplus D \oplus G^{[0]}$, we saw in purism 5.2.12 that the relations $\{(\mathbf{b}_i, \mathbf{g}_i) : i < \omega\}$ and $\{(\mathbf{g}_i, \mathbf{c}_i) : i < \omega\}$ are both definable with parameters below E. Now composition gives the desired function. $\square$

Remark 5.2.14. Both theorems 5.2.11 and 5.2.13 hold if the sets of degrees are finite.

5.3 Interpreting Arithmetic

To get undecidability results, we code models of arithmetic into $\mathcal{D}(\leqslant \mathbf{g})$. Let T be a finitely axiomatizable theory in the language of arithmetic which is hereditarily undecidable and ensures that every model of T has a standard part and of course, which holds in the standard model. We can pick T to be Robinson arithmetic, Shoenfield's theory N ([Sho67, Ch. 6]), or PA$^-$.

We use the terminology of [NSS98] concerning coding schemes. In particular, we use their scheme for coding models of arithmetic in partial orders. Rather than repeat the definitions, we review the needed properties. We have formulas $\phi_{\text{dom}}, \phi_0, \phi_S, \phi_+$ and $\phi_\times$ in the language of partial orderings. If $\mathcal{L} = (L; \leqslant_\mathcal{L})$ is a partial ordering, then the interpretation of arithmetic in $\mathcal{L}$ is the structure

$$N_\mathcal{L} = (\phi_{\text{dom}}(\mathcal{L}); \phi_0(\mathcal{L}), \phi_S(\mathcal{L}), \phi_+(\mathcal{L}), \phi_\times(\mathcal{L}))$$

for the language of arithmetic. Moreover, the scheme (the defining formulas) can be chosen such that there is a recursive partial ordering $\mathcal{L}^*$ such that $N_{\mathcal{L}^*}$ is isomorphic to the standard model of arithmetic.

This scheme can be transformed into a scheme of coding arithmetic in a degree structure such as $\mathcal{D}(\leqslant \mathbf{r})$ via the coding of countable sets; namely, given a tuple of parameters $\bar{\mathbf{a}}$ for ψ (of Theorem 5.2.11) we let $L_{\bar{\mathbf{a}}}$ be the set coded (defined) by $\psi(x, \bar{\mathbf{a}})$ (we saw that the parameters code the same set or relation in any local degree structure $\mathcal{D}(\leqslant \mathbf{r})$ which contains the parameters) and let $\mathcal{L}_{\bar{\mathbf{a}}}$ be the model $(L_{\bar{\mathbf{a}}}; \leqslant_T)$. Having found a partial ordering we can use the scheme above to interpret arithmetic: we let $M_{\bar{\mathbf{a}}} = N_{\mathcal{L}_{\bar{a}}}$. The correctness condition $\chi(\bar{\mathbf{a}})$ states that $M_{\bar{\mathbf{a}}} \models T$. All formulas involved are bounded, and so $M_{\bar{\mathbf{a}}}$ (and the correctness of $\bar{\mathbf{a}}$) is well-defined and doesn't depend on the ideal in which we're working.

Proposition 5.3.1. *Suppose that* $\mathbf{g}$ *is 1-generic. Then there are* $\bar{\mathbf{a}} \in D(\leqslant \mathbf{g})$ *such that* $M_{\bar{\mathbf{a}}}$ *is isomorphic to the standard model of arithmetic. (In particular,* $\bar{\mathbf{a}}$ *satisfy the correctness condition.)*

Proof: Let $G \in \mathbf{g}$ be a 1-generic set. We know if H is 1-generic and $\mathcal{L} = (\{p_i\}_{i<\omega}, <_{\mathcal{L}})$ is a recursive partial ordering, then there are sets $\{P_i\}_{i<\omega}$ such that $\bigoplus_n P_n \leqslant_T H$ and $p_i \to P_i$ is an embedding of $\mathcal{L}$ into the degrees. Thus the recursive ordering $\mathcal{L}^*$ which was discussed above can be embedded below $G^{[0]}$ in such a uniform way. Theorem 5.2.11 shows that there is some tuple $\bar{\mathbf{a}}$ below $G^{[0]} \oplus G^{[1]}$ which codes (via ψ) the copy of $\mathcal{L}^*$ embedded below $G^{[0]}$. Then $M_{\bar{\mathbf{a}}} \cong N_{\mathcal{L}^*}$ is isomorphic to the standard model. □

This gives a direct proof of the following corollary, which, as mentioned in the introduction, can be deduced from work of Shore ([Sho81]) and Jockusch ([Joc80]).

Corollary 5.3.2. *Suppose that* $\mathbf{c}$ *is a degree which bounds a 1-generic degree. Then* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{c}))$ *is undecidable.*

We now employ the technique of comparison maps from [NSS98]. Let φ be the formula coding binary relations. Let $\theta(\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1, \bar{\mathbf{c}})$ be a correctness condition stating that $\varphi(x, y; \bar{\mathbf{c}})$ codes an injective function $h_{\bar{\mathbf{c}}}$ from an initial segment of $M_0 = M_{\bar{\mathbf{a}}_0}$ to an initial segment of $M_1 = M_{\bar{\mathbf{a}}_1}$ which preserves the arithmetical structure. Let $\xi(x, y; \bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1)$ say that there is some $\bar{\mathbf{c}}$ such that $\theta(\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1, \bar{\mathbf{c}})$ holds and $h_{\bar{\mathbf{c}}}(x) = y$. If both tuples $\bar{\mathbf{a}}_i$ satisfy the correctness condition χ, then ξ defines a relation $R_{\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1}$ between M_0 and M_1, which restricted to the standard part of M_0 is a partial isomorphism, defined on a not necessarily proper initial segment of this standard part. Note that R depends heavily on the ideal $\mathcal{J}$ in which we are working, as the quantification of $\bar{\mathbf{c}}$ is unbounded. Given a large enough ideal, $R_{\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1}$ will be total on the standard part of M_0; what we need is that all finite partial isomorphisms of initial segments of M_0 to initial segments of M_1 can be coded by parameters $\bar{\mathbf{c}}$ in $\mathcal{J}$.

If $\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1 \leqslant \mathbf{b}$ and $\mathbf{g}$ is 1-generic over $\mathbf{b}$, then Theorem 5.2.13 shows that in fact $\bar{\mathbf{c}}$ can be found below $\mathbf{b} \cup \mathbf{g}$.

Proposition 5.3.3. *Suppose that* $\mathcal{J}$ *is an ideal and suppose that the 1-generic degrees are downward dense in* $\mathcal{J}$ *(that is, every nonzero* $\mathbf{a} \in \mathcal{J}$ *bounds a 1-generic degree). Then there is a correctness condition* χ^* *such that* $\chi^*(\mathcal{J})$ *is non-empty, and for all* $\bar{\mathbf{a}} \in \mathcal{J}$ *such that* $\mathcal{J} \models \chi^*(\bar{\mathbf{a}})$*,* $M_{\bar{\mathbf{a}}}$ *is isomorphic to the standard model of arithmetic.*

It follows that $\mathcal{J}$ interprets the standard model (without parameters) and so that first order true arithmetic is reducible to $\mathrm{Th}(\mathcal{J}, \leqslant_T)$.

Proof: Let $\chi^*(\bar{\mathbf{a}})$ say that the correctness condition $\chi(\bar{\mathbf{a}})$ holds, and that there is some nonzero $\mathbf{b}$ such that whenever $\bar{\mathbf{a}}' \leqslant \mathbf{b}$ is a tuple such that $\chi(\bar{\mathbf{a}}')$ holds, then $\operatorname{dom} R_{\bar{\mathbf{a}}, \bar{\mathbf{a}}'} = M_{\bar{\mathbf{a}}}$ (i.e. $R_{\bar{\mathbf{a}}, \bar{\mathbf{a}}'}$ is total).

If $\chi^*(\bar{\mathbf{a}})$ holds in $\mathcal{J}$, let $\mathbf{b}$ witness this fact. Since there is a 1-generic degree below $\mathbf{b}$, there is a standard model $M_{\bar{\mathbf{a}}'}$ with $\bar{\mathbf{a}}' \leqslant \mathbf{b}$. Totality of $R_{\bar{\mathbf{a}},\bar{\mathbf{a}}'}$ implies that $M_{\bar{\mathbf{a}}}$ must be standard.

Now we show existence. Let $\mathbf{g} \in \mathcal{J}$ be 1-generic. Let $G_i = G^{[i]}$. If $\bar{\mathbf{a}} \leqslant \mathbf{g}_0$ codes a standard model then $\chi^*(\bar{\mathbf{a}})$ holds, with witness $\mathbf{b} = \mathbf{g}_1$. This is because parameters $\bar{\mathbf{c}}$ coding the finite comparison maps from $M_{\bar{\mathbf{a}}}$ to any models coded below $\bar{\mathbf{g}}_1$ can be found in $\mathcal{J}$, as $\mathbf{g}_2$ is 1-generic over $\mathbf{g}_0 \cup \mathbf{g}_1$. □

This establishes our main theorems:

Theorem 5.3.4. *If* $\mathbf{g} < \mathbf{0}'$ *is 1-generic, then* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ *is recursively isomorphic to true arithmetic.*

Proof: Chong and Jockusch [CJ84] show that the 1-generic degrees are downward dense in $\mathcal{D}(\leqslant \mathbf{g})$ whenever $\mathbf{g}$ is 1-generic and below $\mathbf{0}'$. □

Theorem 5.3.5. *If* $\mathbf{g}$ *is 2-generic, then true arithmetic is 1-reducible to* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$.

Proof: Martin (see [Joc80, Thm. 4.1]) showed that the 2-generic degrees are downward dense in $\mathcal{D}(\leqslant \mathbf{g})$ whenever $\mathbf{g}$ is 2-generic. □

Remark 5.3.6. We remark that this shows that the set of reals A for which $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{a}))$ computes $\mathbf{0}^{(\omega)}$ is comeager. We also remark that if $\mathbf{g}$ is arithmetic, then $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ can be interpreted in first order true arithmetic; thus for every arithmetic 2-generic degree $\mathbf{g}$, $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ is recursively isomorphic to true arithmetic.

Remark 5.3.7. $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ is constant for arithmetically generic $\mathbf{g}$ (see [Ler83, Ex. IV 2.13]). In fact, the n-quantifier part of $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ can be uniformly decided by $\mathbf{0}^{(\omega)}$. It follows that the theory of $\mathcal{D}(\leqslant \mathbf{g})$ for arithmetically generic degrees $\mathbf{g}$ is recursively isomorphic to true arithmetic.

We get a little more. A degree $\mathbf{a}$ is 1-*REA* if it is recursively enumerable. A degree $\mathbf{a}$ is $n+1$-*REA* if it is r.e. in some $\mathbf{b} \leqslant \mathbf{a}$. We remark that every n-r.e. degree is n-REA ([JS84b]).

Lemma 5.3.8. *If* $\mathbf{a}$ *is* n*-REA for some* $n < \omega$*, then for all* $\mathbf{b} < \mathbf{a}$ *there is some* $\mathbf{g} \in (\mathbf{b}, \mathbf{a})$ *which is 1-generic over* $\mathbf{b}$.

Proof: We show that if $\mathbf{a}$ is n-REA then for all $\mathbf{b} < \mathbf{a}$ there is some $\mathbf{c} \in [\mathbf{b}, \mathbf{a})$ such that $\mathbf{a}$ is r.e. in $\mathbf{c}$. The lemma follows by relativizing to $\mathbf{b}$ the fact that every r.e. degree bounds a 1-generic degree (see [Soa87, Ex. VI 3.9]).

Let $\mathbf{a}$ be n-REA; let this be witnessed by $\mathbf{0} = \mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \cdots < \mathbf{a}_n = \mathbf{a}$ (i.e. $\mathbf{a}_{i+1}$ is r.e. in $\mathbf{a}_i$). Let $\mathbf{b} < \mathbf{a}$. Let $i < n$ be the least such that $\mathbf{b} \cup \mathbf{a}_{i+1} = \mathbf{a}$. Since $\mathbf{a}_0 = \mathbf{0} < \mathbf{a}$, we have $\mathbf{b} \cup \mathbf{a}_i < \mathbf{a}$. We claim that $\mathbf{a}$ is r.e. in $\mathbf{b} \cup \mathbf{a}_i$. Since $\mathbf{a} = \mathbf{b} \cup \mathbf{a}_{i+1}$, it is sufficient to show that $\mathbf{b} \cup \mathbf{a}_i$ can enumerate $\mathbf{a}_{i+1}$. But $\mathbf{a}_{i+1}$ is r.e. in $\mathbf{a}_i$. □

Theorem 5.3.9. *If* $\mathbf{c}$ *is n-REA then* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{c}))$ *is recursively isomorphic to true arithmetic.*

PROOF: The correctness condition $\chi^*(\bar{\mathbf{a}})$ will say that $\chi(\bar{\mathbf{a}})$ holds, that $\cup\bar{\mathbf{a}} < \mathbf{c}$ and that for every $\bar{\mathbf{a}}'$ such that $(\cup\bar{\mathbf{a}}') \cup (\cup\bar{\mathbf{a}}) < \mathbf{c}$ (and such that $\chi(\bar{\mathbf{a}}')$ holds), $R_{\bar{\mathbf{a}},\bar{\mathbf{a}}'}$ is total.

If $\chi^*(\bar{\mathbf{a}})$ holds then there is some 1-generic $\mathbf{g} \in (\cup\bar{\mathbf{a}}, \mathbf{a})$ and so some standard model $M_{\bar{\mathbf{a}}'}$ coded below $\mathbf{g}$; it follows that $M_{\bar{\mathbf{a}}}$ must be standard.

$\chi^*(\mathcal{D}(\leqslant \mathbf{c}))$ is not empty. Let $\mathbf{g}_0 < \mathbf{c}$ be some 1-generic degree, and let $\mathbf{a} < \mathbf{g}_0$ code a standard model. For every $\mathbf{b} \in (\mathbf{g}_0, \mathbf{c})$ which bounds some $\bar{\mathbf{a}}'$ which code a model $M_{\bar{\mathbf{a}}'}$, and for every final initial segment of $M_{\bar{\mathbf{a}}}$, there are $\bar{c}$ which code the isomorphism between this initial segment and its copy in $M_{\bar{\mathbf{a}}'}$; this is because there is some $\mathbf{g}_1 < \mathbf{c}$ which is 1-generic over $\mathbf{b}$. It follows that $\chi^*(\bar{\mathbf{a}})$ holds. □

We are left with a couple of questions for which we do not yet know an answer.

Question 5.3.10. Is there a 1-generic degree $\mathbf{g}$ such that $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ does not interpret true arithmetic? Is there one such that $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ is more complicated than true arithmetic?

Question 5.3.11. Suppose that $\mathbf{a}$ bounds a 1-generic degree. Does $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{a}))$ interpret true arithmetic?

5.4 Lattice Embeddings

In this section we show how to embed lattices into $\mathcal{D}(\leqslant \mathbf{g})$, where $\mathbf{g}$ is 1-generic, preserving $\mathbf{0}$ and $\mathbf{1}$ (we consider only lattices with $\mathbf{0}$ and $\mathbf{1}$, where $\mathbf{0} \neq \mathbf{1}$).

We start by defining lattice tables. Whitman [Whi46] observed that every lattice can be embedded in a lattice of equivalence relations. Then, Jónsson showed how to construct a lattice table that also satisfies (3) below in [Jón53] (he maintained the notation of equivalence relations). Shore, in [Sho82] pointed out that, using Jónsson's construction, for every recursive lattice we can get a uniformly recursive lattice table, also satisfying (3). We modify Shore's proof a bit to get a recursive lattice table that also satisfies (4).

Definition 5.4.1. Let L be a lattice and T be a set of functions from L to ω. Given $\alpha, \beta \in T$ and $p \in L$, we write $\alpha \sim_p \beta$ if $\alpha(p) = \beta(p)$ (observe that $\sim_p$ is an equivalence relation and that one can think of $\alpha(p)$ as a name for the equivalence class of α.) We say that T is a *lattice table* for L if for all $p, q, r \in L$

1. $p \leqslant q \Leftrightarrow \forall \alpha, \beta \in T(\alpha \sim_q \beta \Rightarrow \alpha \sim_p \beta)$

2. $p \cup q = r \Rightarrow \forall \alpha, \beta \in T(\alpha \sim_p \beta \;\&\; \alpha \sim_q \beta \Rightarrow \alpha \sim_r \beta)$

3. $p \wedge q = r \;\&\; \alpha \sim_r \beta \Rightarrow \exists \gamma_1, \gamma_2, \gamma_3 \in T \;\; (\alpha \sim_p \gamma_1 \sim_q \gamma_2 \sim_p \gamma_3 \sim_q \beta)$.

4. $\forall \alpha\, \beta \in T, (\alpha \neq \beta \Rightarrow \alpha \sim_{\mathbf{0}} \beta \;\&\; \alpha \not\sim_{\mathbf{1}} \beta)$.

The definition follows [Sho82, Thm. 7] but adds condition (4).

We say that a lattice table T is recursive if there is some numbering of the elements of T which makes them uniformly recursive.

Proposition 5.4.2. *Every recursive lattice L has a recursive lattice table.*

Shore [Sho82, Thm. 7] constructs a lattice table satisfying (1)-(3). We show how to modify Shore's construction to add condition (4).

SKETCH OF PROOF: One first defines a set of functions $T_0 = \{\beta_{p,i} : p \in L, i < 2\}$ by letting

$$\beta_{p,0}(q) = \begin{cases} \langle p, 0\rangle & \text{if } q \neq \mathbf{0} \\ 0 & \text{if } q = \mathbf{0} \end{cases} \qquad \beta_{p,1}(q) = \begin{cases} \beta_{p,0}(q) & \text{if } q \leqslant p \\ \langle p, 1\rangle. & \text{if } q \nleqslant p \end{cases}$$

Note that $\beta_{\mathbf{1},0} = \beta_{\mathbf{1},1}$. It is easy to check that (1), (2) and (4) are satisfied for T_0. Now suppose that a set of functions T_i which satisfies (1), (2) and (4), and such that $\bigcup_{\alpha \in T_i} \alpha\text{``}L$ is coinfinite, is given. Suppose that $p, q, r \in L$ and $\alpha, \beta \in T_i$ are such that $p \wedge q = r$ and $\alpha \sim_r \beta$, and such that (3) fails in this situation. Then we enlarge T_i to T_{i+1} by adding three functions $\gamma_0, \gamma_1, \gamma_2$ defined as follows. Let w, x, y and z be new numbers not in the range of any of the functions in T_i.

$$\begin{aligned}
\gamma_0(s) &= \begin{cases} \alpha(s) & \text{if } s \leqslant p \\ w & \text{if } s \nleqslant p \end{cases} \\
\gamma_1(s) &= \begin{cases} \gamma_0(s) & \text{if } s \leqslant q \\ x & \text{if } s \leqslant p \;\&\; s \nleqslant q \\ y & \text{otherwise} \end{cases} \\
\gamma_3(s) &= \begin{cases} \beta(s) & \text{if } s \leqslant q \\ x & \text{if } s \leqslant p \;\&\; s \nleqslant q \\ z & \text{otherwise.} \end{cases}
\end{aligned}$$

Then T_{i+1} satisfies the induction hypothesis, and also (3) for p, q, r, α, β. (To check (4) one has to note that neither $p = \mathbf{1}$ nor $q = \mathbf{1}$.) Also note that a recursive index for T_{i+1} can be uniformly obtained from a recursive index for T_i. By bookkeeping we get a uniformly recursive lattice table as desired. □

Suppose that T is a lattice table for a lattice L. For $p \in L$ and $\sigma \in T^{\leqslant\omega}$ we define $h_p^\sigma \in \omega^{|\sigma|}$ by letting $h_p^\sigma(i) = (\sigma(i))(p)$ (in the language of equivalence relations, h_p^σ gives the sequence of $\sim_p$-equivalence classes of the elements of σ). If $\sigma, \tau \in T^{\leqslant\omega}$, write $\sigma \sim_p \tau$ if $\forall i < \min\{|\sigma|, |\tau|\}\, (\sigma(i) \sim_p \tau(i))$. Observe that if $|\sigma| = |\tau|$, then $\sigma \sim_p \tau$ iff $h_p^\sigma = h_p^\tau$.

In [Sho82, Thm. 8], Shore constructs, given a recursive lattice table T for a lattice L, a function $g \leqslant 0'$ such that $p \to \deg_T(h_p^g)$ is an embedding of L into $\mathcal{D}(\leqslant 0')$. Moreover, given an r.e. degree $\mathbf{a}$, he constructs $g \leqslant \mathbf{a}$ such that $p \to \deg_T(h_p^g)$ is an embedding of L into $\mathcal{D}(\leqslant \mathbf{a})$. These embeddings preserve neither $\mathbf{0}$ nor $\mathbf{1}$.

Here we prove the following:

Proposition 5.4.3. *If T is a recursive lattice table for a lattice L, and if $g \in T^\omega$ is 1-generic, then the map $p \mapsto h_p^g$ is a lattice embedding of L into $\mathcal{D}(\leqslant_T \mathbf{g})$ preserving $\mathbf{0}$ and $\mathbf{1}$.*

This last proposition implies the following theorem.

Theorem 5.4.4. *If $\mathbf{g}$ is 1-generic, then every recursive lattice can be embedded in $\mathcal{D}(\leqslant \mathbf{g})$ preserving $\mathbf{0}$ and $\mathbf{1}$.*

By 1-generic we mean a 1-generic filter for the forcing $T^{<\omega}$ which can be identified with function Cohen forcing by the numbering of T which makes T recursive. Lemmas 5.4.5 and 5.4.6 below follow from Shore's proof. Lemma 5.4.7 is new.

Let $g \in T^\omega$ be 1-generic. We start by proving the facts about h^g which do not need genericity.

Lemma 5.4.5.

1. *$h_{\mathbf{1}}^g \equiv_T g$.*
2. *$h_{\mathbf{0}}^g$ is a constant function, and hence recursive.*
3. *if $p \leqslant q$ then $h_p^g \leqslant_T h_q^g$.*
4. *if $p \cup q = r$, then $h_p^g \oplus h_q^g \equiv_T h_r^g$*

PROOF: (1) and (2) follow from 5.4.1(4); $g(i)$ is the unique $\alpha \in T$ such that $\alpha(\mathbf{1}) = h_{\mathbf{1}}^g(i)$. For part (3) consider $p \leqslant q$. Take $i \in \omega$; we want to compute $h_p^g(i)$ using h_q^g. Find $\alpha \in T$ such that $\alpha(q) = h_q^g(i)$. Since $\alpha \sim_q g(i)$, $\alpha \sim_p g(a)$, so $h_p^g(i) = \alpha(p)$.

For part (4), we already have from (3) that $h_p^g \oplus h_q^g \leqslant_T h_r^g$. Take $i \in \omega$; we want to compute $h_r^g(i)$ using h_p^g and h_q^g. Find $\alpha \in T$ such that $\alpha(p) = h_p^g(i)$ and $\alpha(q) = h_q^g(i)$. Then, since $\alpha \sim_p g(i)$ and $\alpha \sim_q g(i)$, we have $\alpha \sim_r g(i)$, so $h_r^g(i) = \alpha(r)$. ☐

Now we show that h^g is a poset embedding.

Lemma 5.4.6. *If $p \not\leqslant q$, then $h_p^g \not\leqslant_T h_q^g$.*

Proof: Consider a Turing functional Φ and suppose that $\Phi^{h_q^g}$ is total. We want to show that $h_p^g \neq \Phi^{h_q^g}$. Let

$$S = \{\tau \in T^{<\omega} : \exists x \ \ (h_p^\tau(x) \neq \Phi^{h_q^\tau}(x)\downarrow)\}.$$

By 1-genericity, there is a $\tau_0 \subset g$ such that either $\tau_0 \in S$ or $\forall \sigma \supseteq \tau_0(\sigma \notin S)$. The former case clearly implies that $h_p^g \neq \Phi^{h_q^g}$. We show that the latter case is impossible. Assume, toward a contradiction, that $\tau_0 \subset g$ and $\forall \sigma \supseteq \tau_0(\sigma \notin S)$. Let α and β be such that $\alpha \sim_q \beta$ but $\alpha \not\sim_p \beta$. By 1-genericity there is some $\tau_1 \subset g$, such that for some $x \geqslant |\tau_0|$, $\tau_1(x) = \alpha$. Since $\Phi^{h_q^g}$ is total, there is a $\tau_2 \subset g$ extending τ_1 such that $\Phi^{h_q^{\tau_2}}(x)\downarrow$. Let σ be obtained from τ_2 just by changing the value at x to β. Then $\tau_2 \sim_q \sigma$, so $\Phi^{h_q^\sigma}(x)\downarrow = \Phi^{h_q^{\tau_2}}(x)$ but $h_p^\sigma(x) = \beta(p) \neq \alpha(p) = h_p^{\tau_2}(x)$. So either σ or τ_2 is in S and both extend τ, contradicting our assumption. $\square$

Finally we prove that h^g preserves meet.

Lemma 5.4.7. *If $p \wedge q = r$, then $h_p^g \wedge h_q^g \equiv_T h_r^g$.*

Proof: ¿From Lemma 5.4.5 we have that $h_p^g, h_q^g \geqslant_T h_r^g$. We may assume that $p \neq q$ and use Posner's trick. Suppose that $D = \Phi^{h_p^g} = \Phi^{h_q^g}$. We want to show that $D \leqslant_T h_r^g$. First consider

$$S_0 = \{\tau : \exists x(\Phi^{h_p^\tau}(x)\downarrow \neq \Phi^{h_q^\tau}(x)\downarrow)\}.$$

Clearly g does not meet S_0, so there is a $\tau_0 \subset g$ such that $\forall \sigma \supseteq \tau_0 \ (\sigma \notin S_0)$. Now consider

$$S_1 = \{\tau \supseteq \tau_0 : \exists \sigma_0, \sigma_1, \sigma_2, \sigma_3 \in O_{\tau_0, |\tau|}, \exists x \in \omega \\ (\Phi^{h_p^{\sigma_0}}(x)\downarrow \neq \Phi^{h_q^{\sigma_3}}(x) \ \& \ \sigma_0 \sim_p \sigma_1 \sim_q \tau \sim_p \sigma_2 \sim_q \sigma_3)\},$$

where $O_{\tau,n} = \{\sigma \in T^n : \sigma \supseteq \tau\}$. We claim that no $\tau \subset g$ is in S_1. Suppose $\tau \subset g$ is in S_1 and σ_0, σ_1, σ_2, σ_3 and x witness it. Extend τ to $\bar{\tau}$ such that $\Phi^{h_p^{\bar{\tau}}}(x)\downarrow = \Phi^{h_q^{\bar{\tau}}}(x)\downarrow$. For $i = 0, 1, 2, 3$, let $\bar{\sigma}_i = \sigma_i \curvearrowright \bar{\tau}$. Either $\Phi^{h_p^{\bar{\sigma}_0}}(x) \neq \Phi^{h_q^{\bar{\tau}}}(x)$ or $\Phi^{h_p^{\bar{\tau}}}(x) \neq \Phi^{h_q^{\bar{\sigma}_3}}(x)$. Suppose $\Phi^{h_p^{\bar{\sigma}_0}}(x) \neq \Phi^{h_q^{\bar{\tau}}}(x)$. Since $\bar{\sigma}_0 \sim_p \bar{\sigma}_1$, $\Phi^{h_p^{\bar{\sigma}_1}}(x)\downarrow = \Phi^{h_p^{\bar{\sigma}_0}}(x)$, and since $\bar{\tau} \sim_q \bar{\sigma}_1$, $\Phi^{h_q^{\bar{\sigma}_1}}(x)\downarrow = \Phi^{h_q^{\bar{\tau}}}(x)$. Therefore $\bar{\sigma}_1 \in S_0$ and extends τ_0. This contradicts the definition of τ_0 and proves our claim. So there is some $\tau_1 \subset g$ such that $\forall \sigma \supseteq \tau_1 \ (\sigma \notin S_1)$.

Now we claim that for all $\sigma \supseteq \tau_1$ such that $\sigma \sim_r g$ and for all x such that $\Phi^{h_p^\sigma}(x)\downarrow$, we have $\Phi^{h_p^\sigma}(x) = D(x)$. Otherwise, find some $\sigma \sim_r g$ which extends τ_1 and find an x such that $\Phi^{h_p^\sigma}(x)\downarrow \neq D(x) = \Phi^{h_q^g}(x)$. Let $\sigma_3 \subset g$ be such that $\Phi^{h_q^{\sigma_3}}(x)\downarrow$ and $|\sigma_3| \geqslant \sigma$. Let $\sigma_0 = \sigma \curvearrowright \sigma_3$. Since $\sigma_0 \sim_r \sigma_3$ and both extend τ_1, by definition 5.4.1.(3), there exist σ_1, σ_2 and τ, extending τ_1, such that

$$\sigma_0 \sim_p \sigma_1 \sim_q \tau \sim_p \sigma_2 \sim_q \sigma_3.$$

But then τ is an extension of τ_1 in S_1. This contradiction proves our second claim.

Finally we show that $D \leqslant_T h_r^g$. Take $x \in \omega$. To compute $D(x)$ recursively in h_r^g look for $\sigma \supseteq \tau_1$ such that $\Phi^{h_p^\sigma}(x)\downarrow$ and $\forall i < |\sigma|\ (\sigma(i)(r) = h_r^g(y))$. (Notice that $\forall i < |\sigma|(\sigma(i)(r) = h_r^g(y))$ is equivalent to $\sigma \sim_r g$.) Some initial segment of g serves as such a σ, so the search will end. Then $D(x) = \Phi^{h_p^\sigma}(x)$. $\square$

Part II

Reverse Mathematics

Chapter 6
Equivalence between Frass's conjecture and Jullien's theorem.

This chapter, except for the last section, will be published in the Annals of Pure and Applied Logic.

6.1 Introduction

We compare the strength of two known theorems about linear orderings. We will conclude that, in some sense that we specify below, these two theorems are equally hard to prove.

The two theorems

On the one hand, we have Fraïssé's conjecture. A binary relation $\leqslant_P$ on a set P is a *quasiordering* if it is reflexive and transitive. A quasiordering is a *well quasiordering* if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of P, there exists $i < j$ such that $x_i \leqslant_P x_j$. An equivalent definition of well quasiordering, that might be easier to visualize, is that $\leqslant_P$ contains no infinite descending chains and no infinite antichain. The proof of the equivalence follows from Ramsey's theorem. FRA is the statement that says that the countable linear orderings form a well quasiordering under the relation of embeddablity. Roland Fraïssé conjectured in [Fra48] that there are no sequences of countable linear orderings which are strictly descending under embeddablity. Although this statement is slightly different from FRA, FRA became known as Fraïssé's conjecture. Moreover, FRA is still known as Fraïssé's conjecture even though it is not a conjecture anymore. Richard Laver proved FRA in [Lav71] using Nash-Williams complicated notion of better quasiordering [NW68].

On the other hand, we have Jullien's Theorem and the study of the extendibility linear orderings. A *linearization* of a partial ordering $\mathcal{P} = \langle P, \leqslant_P\rangle$ is a linear ordering $\langle P, \leqslant_L\rangle$ such that $\forall x, y \in P(x \leqslant_P \Rightarrow x \leqslant_L y\rangle)$. A linear ordering $\mathcal{L}$ is *extendible*[1] if every countable partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a linearization which does not embed $\mathcal{L}$ either. For example, the extendibility of ω^* (the linear ordering of the negative integers) is a well known result and it can be translated as every well founded partial ordering has a well ordered

[1]This property is sometimes called *weakly extendability* and extendibility refers to the same property but considering all partial orderings $\mathcal{P}$, and not only the countable ones. A characterization of these linear orderings has been given by Bonnet [BP82]. Since we are only interested in countable objects, we omit the word "weakly". Other names given to this property in the literature are *enforceable* and *Szpilrajn*.

linearization. (We give a proof of this in Lemma 6.6.2.) But for instance, **2**, the linear ordering with two elements, is not extendible. Other linear orderings which are not extendible are the ones of the form $\langle\rightarrow,\leftarrow\rangle$. We say that $\mathcal{L}$ is of the form $\langle\rightarrow,\leftarrow\rangle$ if $\mathcal{L}$ can be written as a sum of two linear orderings, $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A}$ embeds in every final segment of itself and $\mathcal{B}$ embeds in every initial segment of itself; for example $\mathcal{L} = \omega + \omega^*$. The extendibility of η, the order type of the rational numbers, was proved by Bonnet and Pouzet in [BP69] (see also [BP82, p. 140]). Linear orderings which do not contain a copy of η are called *scattered*. A characterization of exactly which linear orderings are extendible has been given by Jullien in his Ph.D. thesis [Jul69]. There, he proved that every scattered linear ordering has a unique minimal decomposition, and then he gave a characterization of the extendible linear orderings which depends on the minimal decomposition of the linear ordering (see Definition 6.3.8 and Statement 6.5.8). Here, we will study Jullien's result and also an equivalent formulation that is simpler to state because it does not use minimal decompositions. This new equivalent formulation, that we call JUL, says that a linear ordering is not extendible if and only if contains a linear ordering of the form either **2** or $\langle\rightarrow,\leftarrow\rangle$ in an essential way (see Statement 6.5.2).

Reverse Mathematics

What we would like to know is exactly which set existence axioms are needed to prove these two theorems. The questions of what axioms are necessary to do mathematics is of great importance in Foundations of Mathematics and is the main question behind Friedman and Simpson's program of Reverse Mathematics. Old known examples along this line of investigations are Euclid's question of whether the fifth postulate was necessary to do geometry and the question of the necessity of the Axiom of Choice to do mathematics. To analyze this question formally it is necessary to fix a logic system. Reverse Mathematics deals with subsystems of Z_2, the system of second-order arithmetic. Second-order Arithmetic, even though it is a lot weaker than set theory, is rich enough to be able to express an important fragment of classical mathematics. This fragment includes number theory, calculus, countable algebra, real and complex analysis, differential equations and combinatorics among others. Almost all of mathematics that can be modeled with, or coded by, countable objects can be done in Z_2.

It happens often that the analysis of theorems from the viewpoint of reverse math gives a deeper understanding of the theorems and sometimes leads to new proofs. This is definitely the case in this paper.

The idea of Reverse Mathematics is as follows. We start by fixing a basic system of axioms. The most commonly used system is RCA_0 which closely related to Computable Mathematics. When this program started, RCA, which is slightly stronger than RCA_0, was often used as the basic system. In RCA, as in RCA_0, the only sets we can assume exist are the ones that we can describe via an effective algorithm. Now, given a theorem of "ordinary" mathematics, the question is what axioms do we need to add to the basic system to prove this theorem. Moreover, we want the least set of axioms needed. It is often the case in Reverse Mathematics that we can prove that a certain set of axioms is needed to prove a theorem by proving the axioms from the theorem using some basic system. Because of this idea this program is called Reverse Mathematics. When we have that a theorem can be proved from a certain system of axioms and that the axioms can be proved from the theorem using for example RCA, we say that the theorem and the system are *equivalent over* RCA. Many different system of axioms have been defined and studied. But a very interesting fact is that most of the theorems that have been analyzed, have been proved equivalent over RCA_0 to one of five systems. These five systems are RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0, listed in increasing order of strength. The basic reference for this subject is [Sim99].

The language of second order arithmetic is the usual language of first order arithmetic (which contains non-logical symbols 0, 1, $+$, $\times$ and $\leqslant$) augmented with set variables and a membership relation $\in$. (We use the letters $x, y, z, n, m, ...$ for number variables and capital letters $X, Y, Z, A, ...$ for set variables.) The axioms of Z_2, are divided in three groups. First we have the *Basic axioms* which say that the natural numbers form an ordered semiring. Then we have the *Induction axioms.* Given a formula $\varphi(x)$ of second-order arithmetic we have the axiom:

$$\varphi(0) \ \& \ \forall x(\varphi(x) \Rightarrow \varphi(x+1)) \Rightarrow \forall x \varphi(x). \tag{$\mathsf{IND}(\varphi)$}$$

Last, we have the *Comprehension axioms.* These axioms are *set existence axioms* in the sense that they say that sets with certain properties exist. Again, we have one for each formula $\varphi(x)$:

$$\exists X \forall x(x \in X \Leftrightarrow \varphi(x)). \tag{$\mathsf{CA}(\varphi)$}$$

The formula φ above may have first or second order, free variable other than x. In that case, (6.1.1) and (6.1.2) are the universal closure of the formulas shown above. Subsystems of Z_2 are obtained by restricting the induction and comprehension axioms to certain classes of formulas. The basic system RCA_0 consist of the basic axioms, and the schemes of Σ^0_1-induction and Δ^0_1-comprehension. Σ^0_1-induction is the scheme of axioms that contains a sentence (6.1.1) for each Σ^0_1 formula $\varphi(x)$. The *Recursive Comprehension Axiom scheme* or Δ^0_1-comprehension consist of the axioms of the form

$$\forall x(\varphi(x) \Leftrightarrow \neg\psi(x)) \Rightarrow \exists X \forall x(x \in X \Leftrightarrow \varphi(x)).$$

where φ and ψ are Σ^0_1 formulas. (A formula ψ is Σ^0_0 if it contains no set quantifiers and all the first order quantifiers are bounded, that is, of the form either $(\forall y < t)$ or $(\exists y < t)$. A formula φ is Σ^0_1 if it is of the form $\exists z \psi(z)$, where ψ is a Σ^0_0 formula.) Another important system is ACA_0. Its axioms are the ones of RCA_0 plus the *Arithmetic Comprehension Axiom scheme*, which consist of the sentences (6.1.2) for arithmetic formulas $\varphi(x)$. (A formula is *arithmetic* if it contains no second order quantifiers.) The scheme of arithmetic comprehension is equivalent to the sentence that says that for every set X, there exists a set X' which is the Turing jump of X. For other classes, Γ, of formulas, like Π^1_1 for example, the system Γ-CA_0 is defined analogously. A system that will be important in this paper is ATR_0. It consist of RCA_0 and the axiom scheme of *Arithmetic Transfinite Recursion*. The scheme of Arithmetic Transfinite Recursion is a little technical so we omit the details. What it says is that arithmetic comprehension can be iterated along any ordinal, which is equivalent to say that the Turing jump can be iterated along any ordinal. For example, ATR_0 is equivalent to the fact that any two ordinals are comparable.

All the systems we have described have restricted induction. The subindex 0 in the notation of a system means that the induction scheme it contains is Σ^0_1-induction. If we drop the subindex 0, and for example get RCA or ATR, is because we are adding the Full induction scheme to the system. The *Full induction scheme* consists of the sentences (6.1.1), for all formulas $\varphi(x)$. A subindex $*$, as in ATR_*, indicates that the system contains the scheme of Σ^1_1-induction. (Σ^1_1-induction, also called Σ^1_1-IND, is defined analogously to Σ^0_1-induction. A formula φ is Σ^1_1 if it is of the form $\exists X \psi(X)$, where ψ is an arithmetic formula.)

Fraïsé's conjecture

The theory of well quasiorderings has been of interest to people studying reverse math because it contains results that seem to be very difficult to prove in comparison with results from other areas of mathematics. Most of the proof seem to require Π^1_2-CA_0, which is more that what is usually needed. However, none of these theorems have been proved to be equivalent to Π^1_2-CA_0 and for most of them the exact proof theoretic strength is unknown. A very interesting example is Kruskal's theorem [Kru60] which says that the class of finite trees is well quasiordered under embeddablity (preserving greatest lower bounds). Harvey Friedman proved that Kruskal's theorem can not be proved in ATR_0. (See [Sim85] for a proof of Friedman's result and [RW93] for an analysis of the exact proof theoretic strength of Kruskal's theorem.) The reader can find a survey on the theory of well quasiorderings studied from the viewpoint of reverse mathematics in [Mar].

The exact proof theoretic strength of FRA is also unknown. It is known that Laver's proof of FRA can be carried out in Π^1_2-CA_0, and that since FRA is a true Π^1_2 statement, it cannot imply Π^1_1-CA_0. (Because every true Π^1_2 sentence holds in every β-model, but Π^1_1-CA_0 does not.) Shore [Sho93] proved that the fact that the class of well orderings is well quasiordered under embeddablity implies ATR_0, getting as a corollary that FRA implies ATR_0. But we still do not know whether FRA could be proved using just ATR_0 (not even Π^1_1-CA_0), as has been conjectured by Peter Clote [Clo90], Stephen Simpson [Sim99, Remark X.3.31] and Alberto Marcone [Mar].

Along with FRA, we study two other statements equivalent to it over RCA_0. One, that we call $\mathsf{WQO(ST)}$, says that the class of signed trees is well quasiordered. A signed tree is a well founded tree which has each node labeled with either a $+$ or a $-$. Given signed trees T and $\check{T}$, we say that $T \preccurlyeq \check{T}$ if there is a homomorphism from T to $\check{T}$ (see Definition 6.2.1). A useful property of signed trees is that if there exists a homomorphism between two recursive signed trees, then there is one that is hyperarithmetic (Lemma 6.2.5 says even more than this). This helps us reduce the quantifier complexity of certain formulas talking about them when working in ATR_0. It might also be useful when trying to prove FRA in ATR_0. We are interested in signed trees because they can be used to represent certain linear orderings that we will call *h-indecomposable*. We will show that, under certain assumptions, every indecomposable linear ordering is equimorphic to an h-indecomposable one. We are also interested in signed trees because they give us a better understanding of the embeddablity relation on linear orderings. For example, in Chapter 9 we use signed trees to prove that every hyperarithmetic linear ordering is equimorphic to a recursive one.

The other statement we prove equivalent to FRA is the *Finite decomposability of linear orderings*, that we call FINDEC. A version of FINDEC was proved by Laver in [Lav71]. It says that every scattered linear ordering can be decomposed, up to equimorphism, as a sum of h-indecomposable linear orderings. (A partial ordering is *scattered* if it does not contain a copy of the rational numbers. Two linear orderings are equimorphic if each one can be embedded into the other.) The representation of the scattered linear orderings that FINDEC gives us will allow us to prove properties about them as, for example, extendibility. We will also look at minimal decompositions of scattered linear orderings. A *minimal decomposition* is a finite decomposition of minimal length. The interesting feature of minimal decomposition is that they are unique up to equimorphism. We will prove that the existence of minimal decompositions for every scattered linear ordering is also equivalent to FRA.

Jullien's Theorem

In the case of extendibility of linear orderings, people have been interested not only in its reverse mathematical strength, but also in the effective content of certain theorems. For example, Szpilrajn proved in [Szp30] that every partial ordering has a linearization. This can be done in an effective way; that is, for every partial ordering we can effectively construct a linearization of it (see [Dow98, Observation 6.1]). The effectiveness of the extendibility of ω^* has also been studied: Rosenstein and Kierstead proved that every recursive well founded partial ordering has a recursive well founded linearization; and Rosenstein and Statman proved that there is a recursive partial ordering without recursive descending sequences which has no recursive linearization without recursive descending sequences. (For proofs of these results and other related ones see [Ros84] and see [Ros82] for more background.) The proof theoretic strength of the fact that ω^* is extendible was studied by Rod Downey, Denis Hirschfeldt, Steffen Lempp and Reed Solomon in [DHLS03]. They showed that the extendibility of ω^* can be proved in ACA_0, that it implies WKL_0, and that it is not implied by WKL_0. It is not known whether it is equivalent to ACA_0, or it is strictly in between WKL_0 and ACA_0. In that same paper they studied the extendibility of ζ, the order type of the integers, and of η, the order type of the rationals. They prove that the extendibility of ζ is equivalent to ATR_0 over RCA_0. For η, they adapted Bonnet and Pouzet's proof of its extendibility to work in Π^1_2-CA_0 and then they give a modification of their proof, due to Howard Becker, that uses only Π^1_1-CA_0. Joseph Miller [Mil] proved that the extendibility of η implies WKL_0 and that over Σ^1_1-AC_0, it implies ATR_0. We prove in this paper that the extendibility of η is provable in ATR_*, which is strictly weaker than Π^1_1-CA_0, using a completely different proof. Our proof is based on a general analysis of the extendibility of h-indecomposable linear orderings and on the fact that if a partial ordering does not emebed η, there is some h-indecomposable linear ordering that is does not embed either.

Rod Downey and R. B. Remmel asked about the effective content of Bonnet-Jullien result (that here we call Jullien's theorem) in [DR00, Question 4.1] and also in [Dow98, Question 6.1]. In [DR00] they observe that Jullien's proof requires Π^1_2-CA_0, and they mention that it would be remarkable if Jullien's theorem was equivalent to Π^1_2-CA_0. It will follow from our results that this is not the case. (Because it is implied by RCA_*+FRA which does not imply Π^1_2-CA_0.)

As we said above, Jullien's theorem, as stated in his thesis, says that a scattered linear ordering is extendible if and only if it has a minimal decomposition of a certain kind. The first problem that we have here is that the existence of minimal decompositions is proof theoretically too strong (it implies FINDEC, which implies FRA). Therefore, the statement that we call JUL(min-dec), and asserts that a linear ordering which has a minimal decomposition is extendible if and only if a certain property of the decomposition holds, does not completely characterize the extendible linear orderings. However, we do study the proof theoretic strength of JUL(min-dec), and we prove that it is equivalent of ATR_* over RCA_*. This proof is divided in to parts. In one we prove that every h-indecomposable linear ordering is extendible. Moreover, we prove that for all h-indecomposable linear orderings, $\mathcal{L}$, any partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a linearization, hyperarithmetic in $\mathcal{P} \oplus \mathcal{L}$, which does not embed $\mathcal{L}$. In the other part we use this result to prove that every linear ordering, $\mathcal{L}$, which is a finite sum of h-indecomposable ones satisfying a certain property is extendible. We also get that the linearizations can be taken to be hyperarithmetic in $\mathcal{L}$ and the partial ordering. The fact that we are getting hyperarithmetic linearizations not only is interesting in itself from the viewpoint of effective mathematics, but also it is useful to reduce the complexity of some formulas we need to prove by induction. We will use the fact that existential quantification over the hyperarithmetic sets is, in certain cases, equivalent to universal second order quantification. This will allow us to transform some complicated formulas into Π^1_1 equivalents and then prove them by Σ^1_1-IND. The extendibility of η will follow from the extendibility of h-indecomposable linear orderings and the fact that if a partial ordering does not embed η, there is some h-indecomposable linear ordering which it does not embed either.

Because of the problem we mentioned earlier is that we study the equivalent formulation, JUL, of Jullien's theorem. We will show that one of the directions of JUL can be proved in RCA_0; it is the other direction that is proof theoretically strong. It will also not be hard to show that JUL follows from JUL(min-dec) and the existence of minimal decompositions for every scattered linear ordering. Using this, we show that JUL follows from FRA and Σ^1_1-IND. We will also prove that JUL implies FRA over RCA_0, getting that JUL and FRA are equivalent over RCA_*.

We have to note that we are not proving the equivalence of FRA and JUL over RCA_0. Instead we prove it over RCA_*, which in addition to RCA_0 has Σ^1_1-IND. RCA_* is still a very weak system and, as RCA_0 and RCA, is closely related to to Computable mathematics. From our work, one can still get that the amount of set existence axioms needed to prove JUL and FRA is the same.

Laver's partition theorem

In [Lav73], Laver used his previous work on FRA to prove some partition results about scattered linear ordering. The theorem we are interested in, that we call LAV, when restricted to the class countable linear orderings, says the following. For every countable linear ordering $\mathcal{L}$ there exists a natural number $n_{\mathcal{L}}$ such that for every coloring of $\mathcal{L}$ with finitely many colors, there exists a subset of $\mathcal{L}$ which is equimorphic to $\mathcal{L}$ and is colored with at most $n_{\mathcal{L}}$ many colors. We show that RCA+LAV implies FRA but we do not know whether the other implication holds or not. (See the end of Section 6.7 for a short discussion about this reversal.)

Simpson claimed in [Sim99, pag. 176] that Friedman's system, ATR_0, is the weakest set of axioms which permits the development of a decent theory of countable ordinals. Then, we could conclude from our work that, over RCA_*, FRA (which could still be equivalent to ATR_*) is the weakest set of axioms which permits the development of a decent theory of countable linear orderings modulo equimorphisms.

6.1.1 Basic Definitions

We use $\mathbb{N}$ for the set of all the natural numbers and ω for the linear ordering $\omega = \langle \mathbb{N}, \leqslant_{\mathbb{N}} \rangle$. Some authors use ω for the standard first order model of the natural numbers. Since we are not dealing with models at all, this will not cause confusion.

Even though our language only let us talk about natural numbers, we can encode pairs and finite sequences of natural numbers as natural numbers. We have a recursive pairing function $\langle \cdot, \cdot \rangle$, and recursive projection functions $(\cdot)_0$ and $(\cdot)_1$ such that $(\langle x, y \rangle)_0 = x$ and $(\langle x, y \rangle)_1 = y$. The same for triplets of elements, $\langle x, y, z \rangle$, and strings $\langle x_0, ..., x_{n-1} \rangle$ of any finite length. Given a set X, we denote by Seq_X the set of strings of elements of X. We use Seq for $\mathrm{Seq}_{\mathbb{N}}$ and Seq_2 for $\mathrm{Seq}_{\{0,1\}}$, the set of binary stings. For a string $\sigma = \langle x_0, ..., x_{n-1} \rangle$ we define $|\sigma| = n$, $\sigma(i) = x_i$, $last(\sigma) = x_{n-1}$, $\sigma^- = \langle x_0, ..., x_{n-2} \rangle$, $\sigma^\frown x = \langle x_0, ..., x_{n-1}, x \rangle$, $\sigma^\frown \langle y_0, ..., y_{m-1} \rangle = \langle x_0, ..., x_{n-1}, y_0, ..., y_{m-1} \rangle$ and $\sigma \restriction m = \langle x_0, ..., x_{m-1} \rangle$.

Orderings

A binary relation $\leqslant_P$ on a set P is a *quasiordering* if it is reflexive and transitive. It is a *partial ordering* if it is also antisymmetric, and a *linear ordering* if it is also total (i.e: $\forall x, y \in P(y \leqslant_P x \vee x \leqslant_P y)$). If a partial ordering is called $\mathcal{P}$, we will usually use the letter P for its domain and $\leqslant_P$ for its relation. An *embedding* from a partial ordering $\mathcal{P} = \langle P, \leqslant_P\rangle$ to another partial ordering $\mathcal{Q} = \langle Q, \leqslant_Q\rangle$ is a one-to-one map $f\colon P \to Q$ such that $\forall x, y \in P(x <_P y \Leftrightarrow f(x) <_Q f(y))$. If this is the case, we write $f\colon \mathcal{P} \hookrightarrow \mathcal{Q}$. When such an f exists, we say that $\mathcal{P}$ *embeds* in $\mathcal{Q}$, and write $\mathcal{P} \preccurlyeq \mathcal{Q}$. Two linear orderings $\mathcal{L}_1$ and $\mathcal{L}_2$ are *equimorphic* if $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$. We write $\mathcal{L}_1 \sim \mathcal{L}_2$ when $\mathcal{L}_1$ and $\mathcal{L}_2$ are equimorphic. An *equimorphism* between $\mathcal{L}_1$ and $\mathcal{L}_2$ is a pair $\langle f_1, f_2\rangle$, where $f_1\colon \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ and $f_2\colon \mathcal{L}_2 \hookrightarrow \mathcal{L}_1$. If the embeddings f_1 and f_2 are inverses of each other, we have an *isomorphism*, we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *isomorphic* and we write $\mathcal{L}_1 \cong \mathcal{L}_2$. A linearization of a partial ordering $\langle P, \leqslant_P\rangle$ is a relation $\leqslant_Q$ on P such that $\langle P, \leqslant_Q\rangle$ is a linear ordering and $\leqslant_Q$ extends $\leqslant_P$ in the sense that $\forall x, y \in P(x \leqslant_P y \Rightarrow x \leqslant_Q y)$.

Some examples of linear orderings are: $\mathbf{1}$, the linear ordering with one element; $\mathbf{m}$, the linear ordering with m many elements; ω, the order type of the natural numbers; ζ, the order type of the integers; η, the order type of the rationals; and ω_1^{CK}, the first non-recursive ordinal. A partial ordering which does not embed η is said to be *scattered*.

We have some operations on the class of orderings. The *reverse* partial ordering of $\mathcal{P} = \langle P, \leqslant_P\rangle$ is $\mathcal{P}^* = \langle P, \geqslant_P\rangle$. The *product*, $\mathcal{P} \times \mathcal{Q}$, of two partial orderings $\mathcal{P}$ and $\mathcal{Q}$ is obtained by substituting a copy of $\mathcal{P}$ for each element of $\mathcal{Q}$. That is: $\mathcal{P} \times \mathcal{Q} = \langle P \times Q, \leqslant_{P\times Q}\rangle$ where $\langle x, y\rangle \leqslant_{P\times Q} \langle x', y'\rangle$ iff $y <_Q y'$ or $y = y'$ and $x \leqslant_P x'$. The *sum*, $\sum_{i\in\mathcal{P}} \mathcal{P}_i$, of a set of partial orderings $\{\mathcal{P}_i\}_{i\in P}$ indexed by another partial ordering $\mathcal{P}$, is constructed by substituting a copy of $\mathcal{P}_i$ for each element $i \in P$. So, for example, $\mathcal{P} \times \mathcal{Q} = \sum_{i\in\mathcal{Q}} \mathcal{P}$. When $\mathcal{P} = \mathbf{m}$, we sometimes write $\mathcal{P}_0 + \ldots + \mathcal{P}_{m-1}$, $\sum_{i<m} \mathcal{P}_i$ or $\sum_{i=0}^{m-1} \mathcal{P}_i$ instead of $\sum_{i\in\mathbf{m}} \mathcal{P}_i$. When $\mathcal{P} = \omega$, we sometimes write $\sum_{i=k}^{\infty} \mathcal{P}_i$ or $\sum_{i\in\omega, i\geqslant k} \mathcal{P}_i$ instead of $\sum_{i\in\omega} \mathcal{P}_{i+k}$. The *direct sum*, $\bigoplus_{i\in I} \mathcal{P}_i$, of a set of partial orderings $\{P_i\}_{i\in I}$ indexed by a set I, is constructed by taking the disjoint union of the $\mathcal{P}_i$ and letting elements from different P_i's be incomparable. So $\bigoplus_{i\in I} \mathcal{P}_i = \sum_{i\in\mathcal{I}} \mathcal{P}_i$, where $\mathcal{I}$ is the partial ordering with domain I where all the elements are incomparable.

Given a linear ordering $\mathcal{L} = \langle L, \leqslant_L\rangle$, we can order Seq_L in various ways. The first ordering we have is the one given by inclusion. For two strings, σ and τ, we use the word *incompatible* when they are incomparable under inclusion, and write $\sigma|\tau$. The most common linear ordering on Seq_L is the *lexicographic ordering*, $\leqslant_{\mathrm{Seq}_L}$: Given $\sigma_0, \sigma_1 \in \mathrm{Seq}_L$, we let $\sigma_0 \leqslant_{\mathrm{Seq}_L} \sigma_1$ iff either $\sigma_0 \subseteq \sigma_1$ or $\sigma_0|\sigma_1$ and $x_0 \leqslant_L x_1$, where x_0 and x_1 are such that for some τ, $\tau^\frown x_0 \subseteq \sigma_0$, $\tau^\frown x_1 \subseteq \sigma_1$, and $x_0 \neq x_1$. On Seq_2 we also have the *Left-to-right ordering*, $\leqslant_{LR}$. It coincides with the lexicographic ordering on incompatible stings. When $\sigma \subset \tau$ we let $\sigma \leqslant_{LR} \tau$ if $\tau(|\sigma|) = 1$ and $\sigma \geqslant_{LR} \tau$ if $\tau(|\sigma|) = 0$. Observe that $\langle \mathrm{Seq}_2, \leqslant_{LR}\rangle$ has order type η.

Given a partial ordering $\mathcal{P} = \langle P, \leqslant_P\rangle$, and $x \in P$, we let $P_{(<x)} = \{y \in P :$

$y <_P x\}$ and $\mathcal{P}_{(<x)} = \langle P_{(<x)}, \leqslant_P \rangle$. Analogously we define $\mathcal{P}_{(>x)}$, $\mathcal{P}_{(\leqslant x)}$, and $\mathcal{P}_{(\geqslant x)}$. We let $(x, y)_P$ be the interval $\{z : x <_P z <_P y\}$.

A linear ordering, $\mathcal{L}$, is *indecomposable* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{B}$ or $\mathcal{L} \preccurlyeq \mathcal{A}$. $\mathcal{L}$ is *indecomposable to the right (left)* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \neq \emptyset$ ($\mathcal{A} \neq \emptyset$), $\mathcal{L} \preccurlyeq \mathcal{B}$ ($\mathcal{L} \preccurlyeq \mathcal{A}$). Sometimes, instead of saying that $\mathcal{L}$ is indecomposable to the right (left), we say that $\mathcal{L}$ is $\rightarrow$ (is $\leftarrow$).

Lemma 6.1.1. *(RCA_0) If $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$, then $\eta \preccurlyeq \mathcal{A}$.*

PROOF: Assume $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. Observe that then $\mathcal{A} + \mathbf{1} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. So we have two embeddings $f_0, f_1 \colon A \hookrightarrow A$ and an $a \in \mathcal{A}$ such that $\forall x, y \in A(f_0(x) <_A a <_A f_1(y))$. Now, given $\sigma \in \mathrm{Seq}_2$ define

$$f(\sigma) = f_{\sigma(0)}(f_{\sigma(1)}(...(f_{\sigma(|\sigma|-1)}(a))...)).$$

f is an embedding of $\langle \mathrm{Seq}_2, \leqslant_{LR} \rangle \cong \eta$ into $\mathcal{A}$. □

Lemma 6.1.2. *(RCA_0) If $\mathcal{A}$ is scattered, indecomposable to the right, and different from $\mathbf{1}$, then $\mathbf{1} + \mathcal{A} \sim \mathcal{A}$ but $\mathcal{A} + \mathbf{1} \not\preccurlyeq \mathcal{A}$.*

PROOF: For the first part decompose $\mathcal{A}$ as $\mathcal{B} + \mathcal{C}$ with $\mathcal{B}$ and $\mathcal{C}$ non-empty. Then $\mathbf{1} \preccurlyeq \mathcal{B}$ and $\mathcal{A} \preccurlyeq \mathcal{C}$.

For the second part, if $\mathcal{A} + \mathbf{1} \preccurlyeq \mathcal{A}$, we have that for some $a \in \mathcal{A}$, $\mathcal{A} \preccurlyeq \mathcal{A}_{(<a)}$. Since $\mathcal{A} \preccurlyeq \mathcal{A}_{(\geqslant a)}$, we have that $\mathcal{A} + \mathcal{A} \preccurlyeq A$. By the previous lemma, this contradicts the assumption that $\mathcal{A}$ is scattered. □

6.2 Signed trees and h-indecomposable linear orderings

In this section we introduce signed trees and h-indecomposable linear orderings. An h-indecomposable (or *hereditarily indecomposable*) linear ordering is an indecomposable linear ordering that is built up recursively from simpler h-indecomposable linear orderings (Definition 6.2.6). We are interested in them because, since we have a nice way of representing them, it is easier to prove properties about them. It can be proved (in classical mathematics) that every indecomposable scattered linear ordering is equimorphic to an h-indecomposable one (Lemma 6.3.3). Therefore, since we are only interested in the class of linear orderings up to equimorphism, we are not loosing generality by only considering the h-indecomposable linear orderings. To represent h-indecomposable linear ordering we use signed trees. Signed trees are easy to deal with (see for example Lemma 6.2.5) and they encode the whole structure of the h-indecomposable linear orderings.

6.2.1 Signed trees

Definition 6.2.1. A *signed tree* is pair $\langle T, s_T \rangle$, where T is a well founded subtree of Seq and s_T is a map, called a *sign function*, from T to $\{+, -\}$. We will usually write T instead of $\langle T, s_T \rangle$. A *homomorphism* from a signed tree T to another signed tree $\check{T}$ is map $f \colon T \to \check{T}$ such that

- for all $\sigma \subset \tau \in T$ we have that $f(\sigma) \subset f(\tau)$ and
- for all $\sigma \in T$, $s_{\check{T}}(f(\sigma)) = s_T(\sigma)$.

In the class of signed trees, we define a binary relation $\preccurlyeq$. We let $T \preccurlyeq \check{T}$ if there exists a homomorphism $f\colon T \to \check{T}$.

We will also consider the empty tree with the empty sign function $\langle \emptyset, \emptyset \rangle$ as a signed tree. We will denote it by $\emptyset$.

Remark 6.2.2. For f to be a homomorphism, we do not require that $\sigma|\tau$ implies $f(\sigma)|f(\tau)$.

Notation 6.2.3. For $\sigma \in T$, we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $s_{T_\sigma}(\tau) = s_T(\sigma^\frown\tau)$.

Statement 6.2.4. Let $\mathsf{WQO(ST)}$ be the statement that says that the class of signed trees is well quasiordered under $\preccurlyeq$: For every sequence $\langle T_i \rangle_{i\in\mathbb{N}}$ of signed trees there are $i < j$ such that $T_i \preccurlyeq T_j$.

It will follow from Proposition 6.2.13 that $\mathsf{WQO(ST)}$ follows from Fraïsé's conjecture, and therefore it is provable in classical mathematics. We will prove in the next section that FRA and $\mathsf{WQO(ST)}$ are actually equivalent over RCA_0. $\mathsf{WQO(ST)}$ seems to be a statement that is easier to deal with than Fraïsé's conjecture, and it might be useful for the study of the latter one.

The following lemma is an important property about signed trees that we will use later.

Lemma 6.2.5. *(ATR_0) Given recursive signed trees T and $\check{T}$ we can decide whether $T \preccurlyeq \check{T}$ recursively in $0^{2\alpha+2}$ where α is the rank of T. Moreover, if $T \preccurlyeq \check{T}$, then we can find a homomorphism recursively in $0^{2\alpha+2}$.*

PROOF: $T \preccurlyeq \check{T}$ if and only if there is a $\sigma \in \check{T}$ such that $s_T(\emptyset) = s_{\check{T}}(\sigma)$ and for each n there is an m such that $T_{\langle n\rangle} \preccurlyeq \check{T}_{\sigma^\frown m}$. Then, by effective transfinite recursion we can construct a $\Sigma^0_{2\alpha+2}$-computable formula which says $T \preccurlyeq \check{T}$. (See [AK00, Chapter 7] for a definition of Σ^0_α-computable formulas.) More specifically, given $\tau \in T$, $\tau' \in \check{T}$, define a formula $\varphi_{\tau,\tau'}$ by effective transfinite recursion as follows:

$$\varphi_{\tau,\tau'} \equiv \exists \sigma \in \check{T}(\tau' \subseteq \sigma \;\&\; s_T(\tau) = s_{\check{T}}(\sigma) \;\&\; \forall n(\tau^\frown n \in T \Rightarrow \exists m(\varphi_{\tau^\frown n, \sigma^\frown m}))).$$

By transfinite induction we can prove that $\varphi_{\tau,\tau'}$ is a $\Sigma^0_{2\,\mathrm{rk}(T_\tau)+2}$-computable formula. Then, $0^{2\alpha+2}$ can compute the truth value of these formulas. We claim that $T \preccurlyeq \check{T}$ if and only if $\varphi_{\emptyset,\emptyset}$ holds. It $f\colon T \to \check{T}$ is a homomorphism, then we can prove by transfinite induction that for every $\tau \in T$, $\varphi_{\tau,f(\tau)}$ holds, and then that $\varphi_{\emptyset,\emptyset}$ holds too. On the other hand, we can prove, also by transfinite induction, that if $\varphi_{\tau,\tau'}$ holds, there is a homomorphism $f_\tau\colon T_\tau \to \check{T}_{\tau'}$ recursive in $0^{2\,\mathrm{rk}(T_\tau)+2}$. To define the homomorphism we have to search for a $\sigma \in \check{T}_{\tau'}$, and then for each n find an m_n and a homomorphism $f_n\colon T_{\tau^\frown n} \hookrightarrow \check{T}_{\sigma^\frown m_n}$; $0^{2\,\mathrm{rk}(T_\tau)+2}$ can do this uniformly. Then let $f_\tau(\emptyset) = \sigma$ and $f_\tau(n^\frown\pi) = \sigma^\frown m_n{}^\frown f_n(\pi)$. $\square$

6.2.2 H-indecomposable linear orderings

We associate to each signed tree T, a linear ordering $\mathrm{lin}(T)$. The idea is the following: If $T = \{\emptyset\}$, then we let $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$ depending on whether $s_T(\emptyset) = +$ or $s_T(\emptyset) = -$. Now suppose $T \supsetneq \{\emptyset\}$. For $i \in \mathbb{N}$, let T_i be the tree $\{\sigma : i^\frown\sigma \in T\}$, and consider the signed function over T_i defined by $s_{T_i}(\sigma) = s_T(i^\frown\sigma)$. If $s_T(\emptyset) = +$, we want $\mathrm{lin}(T)$ to be an ω sum of copies of T_0, T_1,..., where each T_i appears infinitely often in the sum. So, we let

$$\mathrm{lin}(T) = \sum_{n\in\omega} \mathrm{lin}(T_{(n)_0}).$$

If $s_T(\emptyset) = -$, we let

$$\mathrm{lin}(T) = \sum_{n\in\omega^*} \mathrm{lin}(T_{(n)_0}).$$

Now we give the formal definition of $\mathrm{lin}(T)$. It is not hard two see that the two definitions coincide.

Definition 6.2.6. To each signed tree T we assign a linear ordering $\mathrm{lin}(T) = \langle L, \leqslant_T\rangle$. Given $\sigma \in \mathrm{Seq}$, let $(\sigma)_0 = \langle(\sigma(0))_0, (\sigma(1))_0, ..., (\sigma(|\sigma|-1))_0\rangle$. Let $\widehat{T} = \{\sigma \in \mathrm{Seq} : (\sigma)_0 \in T\}$. Let L be the set of strings $\sigma^\frown m \in \mathrm{Seq}$ such that σ is an end node of $\widehat{T}$ and $m \in \mathbb{N}$. Let $\sigma_1{}^\frown m_1$ and $\sigma_2{}^\frown m_2$ be distinct elements in L, let $\tau \in \widehat{T}$ and $n_1 \neq n_2 \in \mathbb{N}$ be such that $\tau^\frown n_1 \subseteq \sigma_1{}^\frown m_1$ and $\tau^\frown n_2 \subseteq \sigma_2{}^\frown m_2$. We define

$$\sigma_1{}^\frown m_1 <_T \sigma_2{}^\frown m_2 \Leftrightarrow \begin{cases} n_1 < n_2 \ \&\ s_T((\tau)_0) = + \text{ or} \\ n_1 > n_2 \ \&\ s_T((\tau)_0) = -. \end{cases}$$

lin at the empty signed tree is defined to be $\mathbf{1}$.

We say that a linear ordering, $\mathcal{L}$, is *h-indecomposable* if it is of the form $\mathrm{lin}(T)$ for some signed tree T. $\mathcal{L}$ is *h-indecomposable to the right* if $s_T(\emptyset) = +$ and *h-indecomposable to the left* otherwise.

Remark 6.2.7. One should observe that the definition of $\mathrm{lin}(T)$ depends on the pairing function used, which is something that, usually, one would like to avoid. But, in this paper, we are only interested in linear orderings up to equimorphisms. It is not hard to see that if we use another paring function, as long as it satisfies that

$$\forall i \exists^\infty n (i = (n)_0),$$

we will get an equimorphic linear ordering.

Example 6.2.8. We show how the function lin behaves on small signed trees. We represent the signed trees with a picture, where the root is on top and on every node we put a $+$ or $-$ depending on the value of s_T on it.

$$\mathrm{lin}({\scriptstyle +}) = \omega; \qquad \mathrm{lin}\begin{pmatrix} - \\ | \\ - \\ | \\ - \end{pmatrix} = ... + (... + \omega^* + \omega^*) + (... + \omega^* + \omega^*);$$

$$\mathrm{lin}\begin{pmatrix} + \\ | \\ - \end{pmatrix} = \omega^* + \omega^* + \omega^* + ...; \qquad \mathrm{lin}\begin{pmatrix} & + & \\ \swarrow & & \searrow \\ - & & + \end{pmatrix} \sim \omega + \omega^* + \omega + \omega^* ...$$

In the rest of this section we will prove that h-indecomposable linear orderings are indecomposable and scattered, and that the quasi-ordering $\preccurlyeq$ on signed trees coincides with the quasi-ordering $\preccurlyeq$ on h-indecomposable linear orderings. In the last subsection of this section we will prove that $\mathsf{WQO(ST)}$ implies ATR_0. In a first reading of the paper, the reader could assume these results and move on to the next section.

Lemma 6.2.9. *(RCA_0) Every h-indecomposable linear ordering, $\mathcal{L}$, is indecomposable. Moreover, if $\mathcal{L}$ is h-indecomposable to the right (left), for every $x \in L$ we can find an embedding $f\colon \mathcal{L} \hookrightarrow \mathcal{L}_{(>x)}$, $(f\colon \mathcal{L} \hookrightarrow \mathcal{L}_{(<x)})$, uniformly recursively in x and $\mathcal{L}$.*

Proof: $\mathbf{1}$ is both, h-indecomposable and indecomposable. So suppose that $\mathcal{L}$ is h-indecomposable to the right. Think of the domain of $\mathcal{L}$ as $\{\langle m, y\rangle : y \in L_m\}$, where, if $\mathcal{L} = \text{lin}(T)$, then $\mathcal{L}_m = \text{lin}(T_{(m)_0})$. Say $x = \langle \bar{m}, y\rangle$, $y \in L_{\bar{m}}$. Consider an increasing function $h\colon \mathbb{N} \to \mathbb{N}$ such that $\forall n(h(n) > \bar{m}\ \&\ (h(n))_0 = (n)_0)$. (For example $h(0) = \langle 0, \bar{m}+1\rangle$ and $h(n) = \langle (n)_0, h(n-1)\rangle$.) Define $f(\langle m, y\rangle) = \langle h(m), y\rangle$. It is not hard to see that $f\colon \mathcal{L} \hookrightarrow \mathcal{L}_{(>x)}$. □

A version of the converse of this lemma will be proved in 6.3.3 using stronger assumptions.

Lemma 6.2.10. *(RCA_0) Every h-indecomposable linear ordering is scattered.*

Proof: Suppose that we have an embedding $f : \mathbb{Q} \hookrightarrow \mathcal{L}$, where $\mathcal{L} = \text{lin}(T)$ is h-indecomposable. Given $\sigma \in T$, let $\mathcal{L}_\sigma = \text{lin}(T_\sigma)$. By recursion on n, we define a_n and $b_n \in \mathbb{Q}$ and $\sigma_n \in T$, such that $a_n <_{\mathbb{Q}} b_n$, and $f(a_n)$ and $f(b_n)$ belong to the same copy of $\mathcal{L}_{\sigma_n}$ in $\mathcal{L}$. Let $\sigma_0 = \emptyset$ and a_0 and b_0 be any two different elements of $\mathbb{Q}$. Suppose we have already defined a_n and $b_n \in \mathbb{Q}$ and $\sigma_n \in T$. So, we have that

$$f((a_n, b_n)_{\mathbb{Q}}) \subseteq \mathcal{L}_{\sigma_n} = \sum_{m \in \omega (\text{or } \omega^*)} \mathcal{L}_{\sigma_n{}^\frown(m)_0}.$$

Since $(a_n, b_n)_{\mathbb{Q}}$ does not embed in either ω or ω^*, there have to be some $m \in \mathbb{N}$, and some a_{n+1} and $b_{n+1} \in \mathbb{Q}$, with $a_n \leqslant_{\mathbb{Q}} a_{n+1} <_{\mathbb{Q}} b_{n+1} \leqslant_{\mathbb{Q}} b_n$, such that $f(a_{n+1}), f(b_{n+1}) \in \mathcal{L}_{\sigma_n{}^\frown(m)_0}$. Note that we can find m, a_{n+1} and b_{n+1} recursively. Let $\sigma_{n+1} = \sigma_n{}^\frown(m)_0$. We have just defined partial recursively sequences $\langle \sigma_n\rangle_n$, $\langle a_n\rangle_n$ and $\langle b_n\rangle_n$, and proved by induction that $f(a_n)$ and $f(b_n)$ belong to the same copy of $\mathcal{L}_{\sigma_n}$ and that for every n, σ_n, a_n and b_n are defined. We can also show by induction that $\forall n < m(\sigma_n \subsetneq \sigma_m)$. Therefore, we have constructed a an infinite path in T, contradicting the fact that it is well founded. □

Before proving that the quasi-ordering $\preccurlyeq$ on signed trees coincides with the ordering $\preccurlyeq$ on h-indecomposable linear orderings we need to prove the following lemma.

Lemma 6.2.11. *(RCA_0) If $\mathcal{L}$ is h-indecomposable to the right and $\mathcal{L} \preccurlyeq \sum_{i\in\alpha^*} \mathcal{A}_i$, where α is well ordered, then for some $i \in \alpha$, $\mathcal{L} \preccurlyeq \mathcal{A}_i$.*

(ACA_0) Moreover, given recursive indices for $\mathcal{L}$, $\langle \mathcal{A}_i : i \in \alpha\rangle$, and the embedding $f\colon \mathcal{L} \hookrightarrow \sum_{i\in\alpha^*} \mathcal{A}_i$ we can find an i and a recursive index for an embedding $g\colon \mathcal{L} \hookrightarrow \mathcal{A}_i$, uniformly recursively in $0'$.

Proof: Consider $f\colon \mathcal{L} \hookrightarrow \sum_{i\in\alpha^*} \mathcal{A}_i$. Write $\mathcal{L}$ as $\sum_{m\in\omega} \mathcal{L}_m$, and for each m let x_m be a member of $\mathcal{L}_m$ (say the least one in the order of the natural numbers). Note that the sequence $\langle x_m\rangle_{m\in\mathbb{N}}$ is co-final in $\mathcal{L}$. For each m, let $a_m \in \alpha^*$ be such that $f(x_m) \in \mathcal{A}_{a_m}$. The sequence $\langle a_m\rangle_{m\in\mathbb{N}}$ is decreasing in α (increasing in α^*). Since α is well ordered, there is some m_0 such that $\forall m \geqslant m_0(f(x_m) \in \mathcal{A}_{a_{m_0}})$. Let $i = a_{m_0}$. (Observe that if $0'$ exists, it can find i.) Therefore f maps $\sum_{j=m_0+1}^{\infty} \mathcal{L}_j$ into $\mathcal{A}_i$. Then, we can construct g by composing f with an embedding of $\mathcal{L}$ into $\sum_{j=m+1}^{\infty} \mathcal{L}_j$, that we have by Lemma 6.2.9. □

Corollary 6.2.12. *(RCA_0) If $\mathcal{L}$ is h-indecomposable to the right and $\mathcal{L} + \mathbf{1} \preccurlyeq \sum_{i\in\omega} \mathcal{A}_i$, then for some $i \in \omega$, $\mathcal{L} \preccurlyeq \mathcal{A}_i$.*

(ACA_0) Moreover, given recursive indices for $\mathcal{L}$, $\langle \mathcal{A}_i : i \in \omega\rangle$, and the embedding $f\colon \mathcal{L} + \mathbf{1} \hookrightarrow \sum_{i\in\omega} \mathcal{A}_i$ we can find an i and a recursive index for an embedding $g\colon \mathcal{L} \hookrightarrow \mathcal{A}_i$, uniformly recursively in $0'$.

Proof: If we have an embedding of $\mathcal{L} + \mathbf{1}$ into $\sum_{i\in\omega} \mathcal{A}_i$, we have an embedding of $\mathcal{L}$ into $\sum_{i<n} \mathcal{A}_i$ for some n. Since the linear ordering $\mathbf{n} \cong \mathbf{n}^*$ is well ordered, the corollary follows from the previous lemma. □

Proposition 6.2.13. *(ACA_0) Let T and $\check{T}$ be signed trees. Then*

$$T \preccurlyeq \check{T} \Leftrightarrow \operatorname{lin}(T) \preccurlyeq \operatorname{lin}(\check{T}).$$

Proof: If either T or $\check{T}$ is empty, then the result is trivial. So suppose neither is empty. First assume that f is a homomorphism witnessing $T \preccurlyeq \check{T}$. Without lost of generality, we can assume that T, $\check{T}$ and f are recursive. Because if they are not, we can relativize the proof. We use effective transfinite recursion to construct an embedding $g\colon \operatorname{lin}(T) \to \operatorname{lin}(\check{T})$. Since for each n, $T_{\langle n\rangle}$ has rank less than T, we can assume that for each n, we have uniformly defined an embedding $g_n\colon \operatorname{lin}(T_{\langle n\rangle}) \to \operatorname{lin}(\check{T}_{f(\langle n\rangle)})$. For each n, let $a_n \in \mathbb{N}$ be such that $f(\emptyset)^\frown a_n \subseteq f(\langle n\rangle)$. We can easily modify each g_n and assume that $g_n\colon \operatorname{lin}(T_{\langle n\rangle}) \to \operatorname{lin}(\check{T}_{f(\emptyset)^\frown a_n})$. Let $h\colon \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\forall n((h(n))_0 = a_{(n)_0}$. (For example, let $h(n+1) = \langle a_{(n)_0}, h(n)\rangle$.) We know that $s_T(\emptyset) = s_{\check{T}}(f(\emptyset))$. Assume, without lost of generality, that $s_T(\emptyset) = +$. Now, use the embeddings g_n to construct an embedding

$$\operatorname{lin}(T) = \sum_{n\in\omega} \operatorname{lin}(T_{\langle (n)_0\rangle}) \preccurlyeq \sum_{n\in\omega} \operatorname{lin}(\check{T}_{f(\emptyset)^\frown (h(n))_0}),$$

and then use the obvious embeddings

$$\sum_{n\in\omega} \operatorname{lin}(\check{T}_{f(\emptyset)^\frown (h(n))_0}) \preccurlyeq \sum_{n\in\omega} \operatorname{lin}(\check{T}_{f(\emptyset)^\frown (n)_0}) = \operatorname{lin}(\check{T}_{f(\emptyset)}) \preccurlyeq \operatorname{lin}(\check{T}).$$

For the other direction, consider $g\colon \operatorname{lin}(T) \hookrightarrow \operatorname{lin}(\check{T})$. Again, we can assume that T, $\check{T}$ and g are recursive. We will define $\sigma \in \check{T}$ such that $s_{\check{T}}(\sigma) = s_T(\emptyset)$ and assign to each $n \in \mathbb{N}$ an $m_n \in \mathbb{N}$ and a recursive index for an embedding $g_n\colon \operatorname{lin}(T_{\langle n\rangle}) \hookrightarrow \operatorname{lin}(\check{T}_{\sigma^\frown m_n})$. We do it uniformly recursively in $0'$ so that we can use $0'$-effective transfinite recursion to define f as follows: From the embeddings g_n, we can get homomorphisms $f_n\colon T_{\langle n\rangle} \to \check{T}_{\sigma^\frown m_n}$. Then, define $f(\emptyset) = \sigma$ and $f(\langle n\rangle^\frown \tau) = \sigma^\frown m_n{}^\frown f_n(\tau)$.

We start by defining σ and $\bar{g}\colon \operatorname{lin}(T) \hookrightarrow \operatorname{lin}(\check{T}_\sigma)$. For this purpose, we define a sequence $\bar{\sigma}_0$, $\bar{g}_0$, $\bar{\sigma}_1$, $\bar{g}_1$,..., $\bar{\sigma}_n$, $\bar{g}_n$ by recursion. Let $\bar{\sigma}_0 = \emptyset$, and $\bar{g}_0 = g$. Suppose now, we have already defined $\bar{\sigma}_j$ and $\bar{g}_j$. If $s_T(\emptyset) = s_{\check{T}}(\bar{\sigma}_j)$, let $n = j$, $\sigma = \bar{\sigma}_j$ and $\bar{g} = \bar{g}_j$. Otherwise, suppose that $s_T(\emptyset) = +$ and $s_{\check{T}}(\bar{\sigma}_j) = -$. (The other case is analogous.) Then, by Lemma 6.2.11, we can find $i \in \mathbb{N}$ and $\bar{g}_{j+1}\colon \mathcal{L} \hookrightarrow \mathcal{L}_{\bar{\sigma}_j{}^\frown (i)_0}$. Let $\bar{\sigma}_{j+1} = \bar{\sigma}_j{}^\frown (i)_0 \in T$. Since T is well founded, this process cannot go for ever. So, at some point we have to find a j with $s_T(\emptyset) = s_{\check{T}}(\sigma_j)$ and define σ and $\bar{g}$.

Suppose that $s_T(\emptyset) = s_{\check{T}}(\sigma) = +$. (The other case is analogous.) For every $n \in \mathbb{N}$ we have

$$\operatorname{lin}(T_{\langle n\rangle}) + \mathbf{1} \preccurlyeq \operatorname{lin}(T) \preccurlyeq \operatorname{lin}(\check{T}_\sigma) = \sum_{m\in\omega} \operatorname{lin}(\check{T}_{f(\sigma)^\frown (m)_0}).$$

So, by Corollary 6.2.12, for some m_n, we have a recursive index for an embedding, g_n, of $\operatorname{lin}(T_{\langle n\rangle})$ into $\operatorname{lin}(\check{T}_{f(\sigma)^\frown (m_n)_0})$. $0'$ can find these uniformly. □

6.2.3 $\mathsf{WQO(ST)}$ implies ATR_0

Shore proved in [Sho93] that the fact that the class of well orderings is well quasiordered under embeddablity implies ATR_0. We will use Shore's result to prove the following proposition.

Proposition 6.2.14. *(RCA_0) $\mathsf{WQO(ST)}$ implies ATR_0.*

The Proposition will follow from the following three lemmas.

Lemma 6.2.15. *(ACA_0) $\mathsf{WQO(ST)}$ implies ATR_0.*

Proof: We work in ACA_0 and assume $\mathsf{WQO(ST)}$. We will prove that for every sequence $\langle\alpha_i\rangle_{i\in\mathbb{N}}$ of ordinals, there are $i < j$ such that α_i embeds in α_j. By Shore's result, this implies ATR_0. For each i we construct a tree T_i as follows: Let T_i be the tree of descending sequences $\langle a_0, ..., a_n\rangle$ with entries in α_i. Consider T_i as a signed tree using the constant function equal to $+$ as the sign function s_{T_i}. By $\mathsf{WQO(ST)}$, there are $i < j$ such that $T_i \preccurlyeq T_j$. We claim that this implies that α_i embeds in α_j. Let f be a homomorphism $T_i \to T_j$. We define $g : \alpha_i \to \alpha_j$ as follows: Given $a \in \alpha_i$, let

$$g(a) = \min\{b \in \alpha_j : \exists\sigma \in T_i(a = last(\sigma) \;\&\; b = last(f(\sigma)))\}$$

where $last(\tau)$ is the last entry of τ. Note that ACA_0 can prove the existence of g. We have to show that $a_0 < a_1 \in \alpha_i$ implies $g(a_0) < g(a_1)$. Let $\sigma \in T_i$ be such that $last(\sigma) = a_1$ and $last(f(\sigma)) = g(a_1)$. Consider $\tau = \sigma^\frown a_0 \in T_i$ and let $b_0 = last(f(\tau))$. Necessarily $f(\tau) \supset f(\sigma)$, and hence, b_0 is smaller than $last(f(\sigma)) = g(a_1)$. So $g(a_0) \leqslant b_0 < g(a_1)$. □

Now we have to prove that $\mathsf{WQO(ST)}$ implies ACA_0 over RCA_0. We first prove that $\mathsf{WQO(ST)}$ implies ACA_0 over RCA_2, and then prove that $\mathsf{WQO(ST)}$ implies RCA_2. RCA_2 is the system that consist of RCA_0 together with the axiom scheme of Σ^0_2-induction.

Lemma 6.2.16. *(RCA_2) $\mathsf{WQO(ST)}$ implies ACA_0.*

Proof: We will prove that $\mathsf{WQO(ST)}$ implies that $K = 0'$ exists. Then, by relativizing the proof, as usual, we can get that for all set X, X' exists, and hence ACA_0.

Let T be the tree of sequences $\langle s_0, ..., s_n\rangle \in \mathrm{Seq}$ such that $\{s_0 < ... < s_n\}$ is the set of stages that look true at stage s_n for the enumeration of K. We say that a stage t *looks true at stage* s *for the enumeration of* K if for all u between t and s, $k_u \geqslant k_t$, where $\{k_u\}_{u\in\mathbb{N}}$ is an enumeration of K. t is a *true* stage if it looks true at every $s \geqslant t$. Note that if T has a path, it is unique and is the set of the true stages of the enumeration of K. So, from that path we would be able to compute $0'$. Also note that using Σ^0_2-induction we can prove that for every m there is an s_m which is a true stage for the enumeration of K and for which there are m many true stages before s_m. (Σ^0_2-induction is needed because a statement that says that there exits a stage that is a true stage which satisfies some recursive predicate is Σ^0_2.) Assume, toward a contradiction, that $0'$ does not exists as a set. Then we would have that T is well founded. For each $n \in \mathbb{N}$, let T_n be the signed tree $\langle T, s_{T_n}\rangle$ where

$$s_{T_n}(\sigma) = \begin{cases} + & \text{if } \sigma \in T \ \&\ |\sigma| \neq n \\ - & \text{if } \sigma \in T \ \&\ |\sigma| = n. \end{cases}$$

Now use $\mathsf{WQO(ST)}$ to get $n < m$ such that $T_n \preccurlyeq T_m$. Let f be an homomorphism from T_n into T_m. Let s be a true stage such that there are $n-1$ many true stages before s. Let σ be the corresponding tuple $\in T$. (i.e.: $\sigma = \langle s_0, ..., s_{n-1}\rangle$, where $\{s_0 < ... < s_{n-1} = s\}$ is the set of stages that look true at s.) Since $s_{T_n}(\sigma) = -$, we have to have that $s_{T_m}(f(\sigma)) = -$, and hence $|f(\sigma)| = m > n$.

We claim that $f(\sigma) \supset \sigma$. Let t be the last element of $f(\sigma)$. If $t > s$, then, since s is a true stage, we would have that $\sigma \subseteq f(\sigma)$. Then $\sigma \subset f(\sigma)$ because $|\sigma| < |f(\sigma)|$. Suppose then, that $t < s$, and σ is incomparable with $f(\sigma)$. There are at most $s-t-1$ many $\tau \in T$ extending $f(\sigma)$. Consider the $s+(s-t)$th true stage and the corresponding sequence in T. We can construct a sequence $\{\sigma_i\}_{i<s-t}$ of nodes of T, such that

$$\sigma \subset \sigma_1 \subset \sigma_2 \subset ... \subset \sigma_{s-t-1}.$$

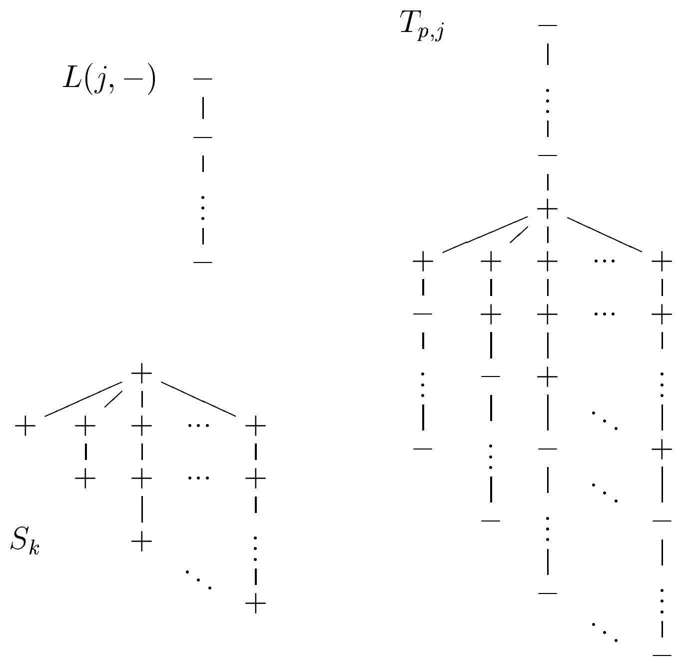
Then, for every $i < j < s - t$ we have to have that $f(\sigma) \subset f(\sigma_i) \subset f(\sigma_j)$. But there are not $s - t$ different nodes on T above $f(\sigma)$. This contradiction to the Pigeon-Hole Principle proves our claim.

Now we can prove by induction that for every n, $f^n(\sigma) \subset f^{n+1}(\sigma)$. Therefore, using f, we can compute the infinite path of T and hence $0'$ too. $\square$

Lemma 6.2.17. *(RCA_0) $\mathsf{WQO(ST)}$ implies RCA_2.*

Proof: Let $\psi(x) = \exists u \forall v \phi(x, u, v)$ be a Σ^0_2 formula. To verify the instance of the induction scheme for ψ, it suffices to prove that, for each $n \in \mathbb{N}$, there exists a set $Z = \{x < n : \psi(x)\}$. Because we can then employ the induction axiom with Z as a parameter, and get induction for ψ up to any n. For each $j < n$ there is a $u_j \leqslant \omega$ such that, if $\psi(j)$, u_j is the first witness for $\exists u \forall v \phi(j, u, v)$, and if $\neg\psi(j)$, $u_j = \omega$. (Each u_j exists by bounded Σ^0_1-comprehension. Note that we are not claiming the existence of the tuple $\langle u_j : j < n \rangle$.)

We will construct a sequence $\langle T_i \rangle_{i \in \mathbb{N}}$ of signed trees and then apply $\mathsf{WQO(ST)}$ to it. Each T_i will have n branches $T_{i,j}$, $j = 0, \dots, n-1$. Given $k \in \mathbb{N}$ and $* \in \{+, -\}$, let $L(k, *)$ bet the signed tree which is linearly ordered, has size k and all its nodes have sign $*$. Given $l \in \omega + 1$, let S_l be the sign tree which has a root signed $+$ and for each $i < l$ there is a branch of the form $L(i, +)$. To construct $T_{p,j}$ attach a copy of $S_{u_j - p}$ after the end node of $L(j, -)$ and then attach a copy of $L(n - j, -)$ after each end node of $S_{u_j - p}$. (If $p > u_j$ let $u_j - p = 0$ and if $u_j = \omega$, let $u_j - p = \omega$.) See pictures of $L(k, *)$, $S_{u_j - p}$ and $T_{p,j}$ below. It is not hard to see how to construct $T_{p,j}$ recursively.



By $\mathsf{WQO(ST)}$, there exists $p < q$ such that $T_p \preccurlyeq T_q$. Then, for every $j_0 < n$ there is a $j_1 < n$ such that $T_{p,j_0} \preccurlyeq T_{q,j_1}$. We claim that necessarily $j_0 = j_1$. Every path though T_{p,j_0} consists of $n-j_0$ nodes signed $-$, then some nodes signed $+$ and then j_0 nodes signed $-$. Every path though T_{p,j_1} consists of $n-j_1$ nodes signed $-$, then some nodes signed $+$ and then j_1 nodes signed $-$. The n nodes signed $-$ in a path though T_{p,j_0} have to be mapped into the n nodes signed $-$ in a path though T_{p,j_1}, and the nodes signed $+$ have to be mapped to nodes signed $+$. Therefore, it has to be the case that $j_0 = j_1$. We have also proved that necessarily $S_{u_{j_0}-p} \preccurlyeq S_{u_{j_1}-q}$

The second observation is that if $\psi(j)$ and $T_{p,j} \preccurlyeq T_{q,j}$, then $u_j \leqslant p$. This is because, to have that $S_{u_j-p} \preccurlyeq S_{u_j-q}$, we need to have that $u_j - p \leqslant u_j - q = 0$.

So we have that $\psi(j) \Leftrightarrow (\exists u \leqslant p)\forall v \psi(j,u,v)$. Therefore, Z can be proved to exists in RCA_0 by bounded Σ^0_1 comprehension. (See [Sim99, Definition II.3.8 and Theorem II.3.9] for the technical definition and proof of this principle in RCA_0.)

□

6.3 Finite decomposability

Definition 6.3.1. A *finite decomposition* of a linear ordering, $\mathcal{L}$, is a finite tuple signed trees $\langle T_0, ..., T_n\rangle$, such that

$$\mathcal{L} \sim \sum_{i=0}^{n} \mathrm{lin}(T_i).$$

If $\mathcal{F}_i = \mathrm{lin}(T_i)$, we may abuse notation and say that the tuple of h-indecomposable linear orderings $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ is a finite decomposition of $\mathcal{L}$. In this case we will implicitly assume that the sequence $\langle T_0, ..., T_n\rangle$ is also given.

Statement 6.3.2. Let FINDEC be the statement that says that every scattered linear ordering has a finite decomposition.

FINDEC gives us a nice representation of scattered linear orderings up to equimorphism. This representation will be very useful in the proof of Jullien's Theorem. A proof of FINDEC can be extracted from [Jul69] using Fraïssé's conjecture and Π^1_1-DC_0 (which is equivalent to Σ^1_2-DC_0 and to Δ^1_2-CA_0 plus Σ^1_2-induction [Sim99, Theorem VII.6.9.2], and strictly stronger than, for example, Π^1_1-CA plus Σ^1_2-induction). In this section we prove that FINDEC is equivalent to $\mathsf{WQO(ST)}$and in the next section that it is equivalent to Fraïssé's conjecture. We also analyze finite decompositions of minimal length. We show in ATR_0that, if finite decompositions exists, then minimal decompositions also exists and are unique modulo equimorphisms.

The next lemma uses FINDEC to show that h-indecomposability is the same as indecomposability, modulo equimorphism.

Lemma 6.3.3. *FINDEC implies that every scattered indecomposable linear ordering is equimorphic to an h-indecomposable linear ordering.*

Proof: Let $\mathcal{L}$ be scattered and indecomposable, say to the right. By FINDEC, $\mathcal{L} \sim \sum_{i=0}^{n} \mathcal{F}_i$, where each $\mathcal{F}_i$ is h-indecomposable. Since $\mathcal{L}$ is indecomposable to the right, $\mathcal{L} \preccurlyeq \mathcal{F}_n$. Obviously $\mathcal{F}_n \preccurlyeq \mathcal{L}$, therefore $\mathcal{L} \sim \mathcal{F}_n$ which is h-indecomposable. □

6.3.1 FINDEC and WQO(ST)

We prove that FINDEC is equivalent to WQO(ST) over RCA_0.

Lemma 6.3.4. *(RCA_0) WQO(ST) implies FINDEC.*

Proof: Clote proved that ATR_0 implies that every scattered linear ordering, $\mathcal{L}$, can be embedded in $\mathbb{Z}^\alpha$ for some ordinal α [Clo89, Theorem 16]. ($\mathbb{Z}^\alpha$ is defined by effective transfinite induction as follows: $\mathbb{Z}^0 = \mathbf{1}$, $\mathbb{Z}^{\alpha+1} = \mathbb{Z}^\alpha \times \mathbb{Z}$ and $\mathbb{Z}^{\lim_m \alpha_m} = \sum_{m\in\omega} \mathbb{Z}^{\alpha_m} \times \omega^* + \sum_{m\in\omega^*} \mathbb{Z}^{\alpha_m} \times \omega$.) The least α such that $\mathcal{L}$ embeds in a finite sum of $\mathbb{Z}^\alpha$s is the *rank* of $\mathcal{L}$. The rank can be defined in ATR_0 as in [Clo89]. Recall that WQO(ST) implies ATR_0, so we can use ATR_0 here. By arithmetic transfinite induction we prove that for every ordinal α the following holds: Every recursive linear ordering $\mathcal{L}$ of rank α, for which there is a recursive embedding $\mathcal{L} \preccurlyeq \mathbb{Z}^\alpha \times \mathbf{m}$ for some $m \in \mathbb{N}$, is equimorphic to a finite sum, $\sum_{i=0}^{n} \mathcal{F}_i$, of h-indecomposable linear orderings such that

- each $\mathcal{F}_i$ has of rank $\leqslant \alpha$,
- each $\mathcal{F}_i$ is recursive in $0^{2(\alpha+1)^2}$, and
- the equimorphism is recursive in $0^{2(\alpha+1)^2}$.

To do this using only arithmetic transfinite induction (which we have in ATR_0, even in ACA_0; see [Sim99, Lemma V.2.1]) we need to fix a big ordinal α_0 and prove that the statement above holds for every $\alpha < \alpha_0$ by induction on α. Note that ATR_0 implies that $0^{2\alpha_0^2}$ exists as a set. This is why the sentence that we are proving by transfinite induction is just arithmetic. To get finite decomposability for every scattered linear ordering $\mathcal{L}$, the proof has to work for every ordinal α_0 and relative to every set X.

We can write $\mathcal{L}$ as a finite sum of ω or ω^* sums of linear orderings of rank less than α. Clearly, it suffices to consider the case that $\mathcal{L}$ is equal to one of these sums, $\sum_{i\in\omega} \mathcal{L}_i$. By inductive hypothesis, for each i there is a equimorphism recursive in $0^{2\alpha^2}$ between $\mathcal{L}_i$ and a finite sum of h-indecomposable linear orderings of rank $< \alpha$ recursive in $0^{2\alpha^2}$. So now, we have that $\mathcal{L} \sim \sum_{i\in\omega} G_i$, where each $G_i = \mathrm{lin}(T_i)$ is h-indecomposable. Recursively in $0^{2\alpha^2+2}$ we can find these equimorphisms uniformly, and hence the equimorphism $\mathcal{L} \sim \sum_{i\in\omega} G_i$. By Lemma 6.2.5, recursively uniformly in $0^{2\alpha^2+2\alpha}$, we can tell, for each i and j, whether $G_i \preccurlyeq G_j$ or not. Moreover, if $G_i \preccurlyeq G_j$, we can find the embedding. By WQO(ST) and Proposition 6.2.13, it cannot happen that

$$\forall k \exists i, j \geqslant k \forall l > j (G_i \not\preccurlyeq G_l).$$

Otherwise we could define a subsequence $\langle G_{k_i}\rangle_{i\in\mathbb{N}}$ such that $\forall i<j(G_i \not\preccurlyeq G_j)$. Let k_0 be such that $\forall i,j \geqslant k_0 \exists l > j(G_i \preccurlyeq G_l)$. Let $T=\{i^\frown\sigma : \sigma \in T_{i+k_0}\}$, $s_T(\emptyset)=+$ and $s_T(i^\frown\sigma)=s_{T_{i+k_0}}(\sigma)$. We claim that

$$L \sim \sum_{i=0}^{k_0-1} G_i + \mathrm{lin}(T).$$

We have to construct an equimorphism between

$$\sum_{i=k_0}^{\infty} G_i \quad \text{and} \quad \mathrm{lin}(T)=\sum_{m\in\omega} G_{(m)_0+k_0}.$$

The equimorphism can be easily constructed given the pairs $\langle i,j\rangle$ such that $G_i \preccurlyeq G_j$ and the embeddings $f_{ij}\colon G_i \hookrightarrow G_j$, which we have recursively in $0^{2\alpha^2+2\alpha}$. Note that $2(\alpha+1)^2 \geqslant 2\alpha(\alpha+1)=2\alpha^2+2\alpha$. Then $\langle T_0,\dots,T_{k_0-1},T\rangle$ is a finite decomposition of $\mathcal{L}$. □

The proof of the other direction is divided in two steps.

Lemma 6.3.5. *(ACA_0) FINDEC implies $\mathsf{WQO(ST)}$.*

Proof: Suppose $\mathsf{WQO(ST)}$ is false. Then, using Proposition 6.2.13, there is a sequence $\langle\mathcal{L}_i\rangle_{i\in\mathbb{N}}$ of h-indecomposable linear orderings such that for all $i<j$, $\mathcal{L}_i \not\preccurlyeq \mathcal{L}_j$. By taking an infinite subsequence, we can assume that all the $\mathcal{L}_i$ are h-indecomposable in the same direction. Let us assume they are all h-indecomposable to the right. Let $\mathcal{L}=\sum_{i\in\omega}\mathcal{L}_i$. We claim that $\mathcal{L}$ is scattered but it can not be decomposed as a finite sum of h-indecomposables and therefore that FINDEC does not hold. By Lemma 6.2.10, each $\mathcal{L}_i$ is scattered, so $\mathcal{L}$ is scattered too. Suppose, toward a contradiction, that $\mathcal{L}\sim\sum_{j=0}^{n}\mathcal{F}_j$, where each $\mathcal{F}_j$ is h-indecomposable.

First we show that for some $k\in\mathbb{N}$, $\mathcal{F}_n \sim \sum_{i=k}^{\infty}\mathcal{L}_i$. Let f and g be an embeddings, $f\colon \mathcal{L}\hookrightarrow \sum_{j=0}^{n}\mathcal{F}_j$ and $g\colon \sum_{j=0}^{n}\mathcal{F}_j \hookrightarrow \mathcal{L}$. Let $h\colon\mathcal{L}\hookrightarrow\mathcal{L}$ be the composition of g and f. We claim that for every $k\in\mathbb{N}$, the image of $\sum_{i=k}^{\infty}\mathcal{L}_i$ under h is included in $\sum_{i=k}^{\infty}\mathcal{L}_i$. The proof of the claim is a straightforward induction using that each $\mathcal{L}_i$ is indecomposable to the right and hence cannot be embedded into a proper initial segment of it. Now, let k_0 be such that $f^{-1}(F_n)=\mathcal{G}_{k_0}+\sum_{i=k_0+1}^{\infty}\mathcal{L}_i$, where $\mathcal{G}_{k_0}$ is a non-empty final segment of $\mathcal{L}_{k_0}$, and let k_1 be the greatest k such that $g(F_n)\subseteq\sum_{i=k}^{\infty}\mathcal{L}_i$. Since then $h(G_{k_0}+\sum_{i=k_0+1}^{\infty}\mathcal{L}_i)\cap\mathcal{L}_{k_1}\neq\emptyset$, by the claim above, $k_0\leqslant k_1$. Therefore

$$\sum_{i=k_0}^{\infty}\mathcal{L}_i \preccurlyeq \mathcal{G}_{k_0}+\sum_{i=k_0+1}^{\infty}\mathcal{L}_i \preccurlyeq \mathcal{F}_n \preccurlyeq \sum_{i=k_1}^{\infty}\mathcal{L}_i \preccurlyeq \sum_{i=k_0}^{\infty}\mathcal{L}_i.$$

So, $\mathcal{F}_n\sim\sum_{i=k_0}^{\infty}\mathcal{L}_i$. Let $k=k_0$.

Since $\mathcal{F}_n$ is indecomposable, either $\mathcal{F}_n\preccurlyeq\mathcal{L}_k$ or $\mathcal{F}_n\preccurlyeq\sum_{i=k+1}^{\infty}\mathcal{L}_i$. The former case is not possible because we would have that $\mathcal{L}_k+1\preccurlyeq\mathcal{L}_k$, which contradicts

Lemma 6.1.2. In the latter case we would have that $\mathcal{L}_k + 1 \preccurlyeq \sum_{i=k+1}^{\infty} \mathcal{L}_i$. Then, by Corollary 6.2.12, $\mathcal{L}_k \preccurlyeq \mathcal{L}_m$ for some $m \geqslant k+1$, contradicting our initial assumption. □

Lemma 6.3.6. *(RCA_0) FINDEC implies ACA_0.*

Proof: We will prove that FINDEC implies that $K = 0'$ exists. Then, by relativizing the proof, as usual, we can get that for all set X, X' exists, and hence ACA_0.

Let $\{k_0, k_1, ...\}$ be a recursive enumeration of K. For each $s \in \mathbb{N}$ let $K_s = \{k_0, ..., k_s\}$ and $\sigma_s = K_s \restriction k_s + 1$. Consider the following ordering of $\mathbb{N}$.

$$s <_B t \Leftrightarrow \sigma_s <_{KB} \sigma_t,$$

where $<_{KB}$ is the Kleene-Brouwer ordering of Seq_2. ($\sigma <_{KB} \tau$ iff $\sigma \supseteq \tau$ or $\sigma | \tau$ and $\sigma \leqslant_{\mathsf{Seq}_\omega} \tau$.) Let $\mathcal{B} = \langle \mathbb{N}, \leqslant_B \rangle$. For each s we have that either for some $t > s$, $k_t < k_s$, in which case we have that $\forall t' \geqslant t (s <_B t')$, or that for every $t > s$, $k_t > k_s$ (in other words, s is a *true stage*), in which case we have that $\forall t' > s (t' <_B s)$. In the former case we say that s is in the *left side* of $\mathcal{B}$, and in the latter case that s is in the *right side.* Just for the sake of giving some intuition about the shape of $\mathcal{B}$, we observe that ACA_0 proves that $\mathcal{B}$ has order type $\omega + \omega^*$. RCA_0 cannot prove this fact. Furthermore, if we had an order preserving map from ω^* into $\mathcal{B}$, then we could compute infinitely many true stages and hence K.

FINDEC implies that $\mathcal{B}$ is equimorphic to a finite sum of h-indecomposable linear orderings. Since $\mathcal{B}$ is infinite, at least one of the summands has to be infinite. Because of the fact that every element has finitely many elements either to the right or to the left, we are left with three possible decomposition of $\mathcal{B}$:

$$\mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega + \mathbf{1} + ... + \mathbf{1};$$
$$\mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega * + \mathbf{1} + ... + \mathbf{1};$$
$$\mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega + \omega * + \mathbf{1} + ... + \mathbf{1}.$$

We can eliminate the first possibility by proving that there is no embedding $\mathcal{B} \preccurlyeq \mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega + \mathbf{1} + ... + \mathbf{1}$. To do this all we have to show is that every element in the right side has to be mapped to one of the $\mathbf{1}$s at the left of the copy of ω, and then that there are infinitely many elements in the right side of $\mathcal{B}$, or in other words, infinitely many true stages. (The second possibility can be eliminated too. But we do not need to do it.) Therefore, we have a map from ω^* to $\mathcal{B}$ as we needed to compute K. □

Corollary 6.3.7. *$\mathsf{WQO(ST)}$ and FINDEC are equivalent over RCA_0.*

Proof: Use the previous three lemmas. □

6.3.2 Minimal decomposition

Finite decompositions of a linear ordering are not unique. For example, $\langle\omega^2\rangle$ and $\langle\omega, \mathbf{1}, \omega^2\rangle$ are two finite decompositions of ω^2. This is why we are interested in considering minimal finite decompositions of linear orderings.

Jullien proved that every scattered linear ordering has a minimal decomposition, and, in a certain sense, a unique one [Jul69]. His definitions of finite and minimal decompositions were, although essentially the same, a bit different from ours. Because of this, our proof of uniqueness is simpler than his. The existence of minimal decompositions follows easily from the existence of finite decompositions and Σ^1_2-induction. To prove it using just ATR_0, a little work is required.

Definition 6.3.8. A *minimal decomposition* of a linear ordering is a finite decomposition of minimal length.

Lemma 6.3.9. *(RCA_2) If $\langle\mathcal{F}_0, ..., \mathcal{F}_n\rangle$ and $\langle\check{\mathcal{F}}_0, ..., \check{\mathcal{F}}_m\rangle$ are finite decomposition of $\mathcal{L}$, then there exists a set $X \subseteq \{0, ..., m\}$ of size at most $n+1$ such that $\sum_{i\in X}\check{\mathcal{F}}_i \sim \mathcal{L}$. Moreover, there exists an embedding*

$$g\colon \sum_{i=0}^{n}\mathcal{F}_i \hookrightarrow \sum_{i\in X}\check{\mathcal{F}}_i.$$

such that for each $i \leqslant n$, there is a $j \in X$, such that the image of $\mathcal{F}_i$ under g is contained in $\check{\mathcal{F}}_j$.

Proof: Let f be an embedding $f\colon \sum_{i=0}^{n}\mathcal{F}_i \hookrightarrow \sum_{i=0}^{m}\check{\mathcal{F}}_i$. As in the proof of Lemma 6.2.11, for each $i \leqslant n$, there is an $x_i \in F_i$ and a $j_i \leqslant m$ such that $\check{\mathcal{F}}_{j_i}$ contains the the image under f of $\mathcal{F}_{i(\geqslant x_i)}$ if $\mathcal{F}_i$ is $\rightarrow$, and of $\mathcal{F}_{i(\leqslant x_i)}$ if $\mathcal{F}_i$ is $\leftarrow$. (If $\mathcal{F}_i$ is $\mathbf{1}$, let x_i be the the only element of F_i.) The sequence $\langle x_0, ..., x_n\rangle$ exists by Σ^0_2-induction. Let $X = \{j_i : i \leqslant n\}$. Now, using Lemma 6.2.9 we can construct embeddings $g_i\colon \mathcal{F}_i \hookrightarrow \check{\mathcal{F}}_{j_i}$, uniformly in i, such that the image of g_i is contained in the image of $\mathcal{F}_i$ under f. Then, putting all the g_is together, we can construct $g\colon \sum_{i=0}^{n}\mathcal{F}_i \hookrightarrow \sum_{i\in X}\check{\mathcal{F}}_i$. So, we have that

$$\mathcal{L} \preccurlyeq \sum_{i=0}^{n}\mathcal{F}_i \preccurlyeq \sum_{i\in X}\check{\mathcal{F}}_i \preccurlyeq \mathcal{L}.$$

□

Proposition 6.3.10. *(ATR_0) If a linear ordering $\mathcal{L}$ has a finite decomposition, then it has a minimal decomposition. Moreover, this minimal decomposition is unique up to equimorphism.*

Proof: The uniqueness of the minimal decomposition follows from the previous lemma: If $\langle\mathcal{F}_0, ..., \mathcal{F}_n\rangle$ and $\langle\check{\mathcal{F}}_0, ..., \check{\mathcal{F}}_n\rangle$ are minimal decomposition of $\mathcal{L}$, then the X given by the previous lemma has to be the whole set $\{0, ..., n\}$. Then, necessarily

$j_i = i$ for all $i \leqslant n$, and hence $\mathcal{F}_i \preccurlyeq \check{\mathcal{F}}_i$. Analogously we get $\check{\mathcal{F}}_i \preccurlyeq \mathcal{F}_i$ for each i, and therefore $\mathcal{F}_i \sim \check{\mathcal{F}}_i$.

Now, let $\langle \mathcal{F}_0, ..., \mathcal{F}_n \rangle$ be a finite decomposition of $\mathcal{L}$. We will prove that $\mathcal{L}$ has a minimal decomposition. We consider the least m such that there is a subset X of $\{0, ..., n\}$ of size $m+1$ such that

$$\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i \in X} \mathcal{F}_i.$$

The existence of such an m requires induction. We will prove that the formula $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i \in X} \mathcal{F}_i$ is Σ^0_1 over some parameters that, using ATR_0, we can prove exist. Let $\langle T_0, ..., T_n \rangle$ be a sequence of signed trees such that $\mathrm{lin}(T_i) = \mathcal{F}_i$. Let α be the maximum of the ranks of the T_is plus 1. We claim that we can decide whether $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i \in X} \mathcal{F}_i$ recursively in $Z^{(2\alpha+2)}$, where Z is some set that computes $\langle T_0, ..., T_n \rangle$. Let $\{j_0 < ... < j_m\} = X$. If $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i \in X} \mathcal{F}_i$, then, by the previous lemma, there exists an embedding g such that for each $i \leqslant n$, there is a $j \in X$, such that the image under g of each $\mathcal{F}_i$ is contained in $\mathcal{F}_j$. So $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i \in X} \mathcal{F}_i$ is equivalent to

$$\bigvee_{0=i_0 \leqslant ... \leqslant i_m \leqslant n} \left(\bigwedge_{k \leqslant m} \mathcal{F}_{i_k} + \mathcal{F}_{i_k+1} + ... + \mathcal{F}_{i_{k+1}-1} \preccurlyeq \mathcal{F}_{j_k} \right).$$

Observe that in general, if $\mathcal{C}$ is $\rightarrow$, then $\mathcal{A} + \mathcal{B} \preccurlyeq \mathcal{C}$ if and only if $\mathcal{A} + \mathbf{1} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$. Also observe that $\mathcal{A} + \mathbf{1} \preccurlyeq \mathcal{C}$ if and only if $\mathcal{A} \times \omega \preccurlyeq \mathcal{C}$. So, now, the question "$\mathcal{F}_{i_k} + \mathcal{F}_{i_k+1} + ... + \mathcal{F}_{i_{k+1}-1} \preccurlyeq \mathcal{F}_{j_k}$?", supposing $\mathcal{F}_{j_k}$ is $\rightarrow$, becomes a conjunction of formulas of the forms $\mathcal{F}_i \preccurlyeq F_j$ and $\mathcal{F}_i \times \omega \preccurlyeq F_j$. Since, by Lemma 6.2.5, $Z^{(2\alpha+2)}$ can answer all these questions, it can tell whether $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i \in X} \mathcal{F}_i$. This proves our claim.

Now, by Σ^0_1-induction, there is an m and an X as required above. We claim that $\langle \mathcal{F}_i : i \in X \rangle$ is a minimal decomposition of $\mathcal{L}$. Suppose, toward a contradiction, that $\langle \check{\mathcal{F}}_0, ..., \check{\mathcal{F}}_l \rangle$ is a finite decomposition of $\mathcal{L}$ of length $l+1 < m+1$. But then, by the Lemma above, there is some $Y \subset X$ such that $\sum_{i \in Y} \mathcal{F}_i$ is equimorphic to $\mathcal{L}$, contradicting the minimality of m. $\square$

Since FINDEC implies ATR_0, we obtain the following equivalence.

Corollary 6.3.11. *The following are equivalent over RCA_0:*

1. *FINDEC.*

2. *Every scattered linear ordering has a minimal decomposition.*

6.4 Fraïssé's conjecture

Statement 6.4.1. Fraïssé's conjecture, FRA, is the statement that says that the class of linear orderings is well quasiordered under embeddablity.

As we said in the introduction, the exact proof theoretic strength of FRA is unknown. All we know it that it is provable in Π^1_2-CA_0, that it implies ATR_0 (Shore [Sho93]) but that it does not imply Π^1_1-CA_0. We prove in this section that it is equivalent to the two statements studied above.

Theorem 6.4.2. *The following are equivalent over RCA_0:*

1. *WQO(ST)*
2. *FINDEC*
3. *FRA*

Proof: We have already proved that WQO(ST) and FINDEC are equivalent. Obviously FRA implies that the class of h-indecomposable linear orderings is well quasiordered. It follow from Proposition 6.2.13, and the fact that FRA implies ACA_0, that FRA implies WQO(ST).

Now we show that WQO(ST) implies FRA. Recall that WQO(ST) implies ATR_0, so we can use ATR_0 here. Consider a sequence $\langle \mathcal{L}_i : i \in \mathbb{N}\rangle$ of linear orderings. For some set X and ordinal α, we have that these linear orderings are all recursive in X and have rank less than α. (The rank of a scattered linear ordering is defined at the beginning of the proof of Lemma 6.3.4.) By relativization, assume X is recursive.

The idea is like the one in the proof of [Fra00, 7.5.4], but we have to be a little bit more careful. We prove that, for every ordinal α, the set of recursive linear orderings of rank less that α is well quasiordered. We use Higman's theorem which is provable in ACA_0; see, for example, [Mar]. Higman's theorem says that if $\mathcal{P}$ is well quasiordered, then $\langle \mathrm{Seq}_P, \preccurlyeq_P\rangle$ is well quasiordered too, where $\sigma \preccurlyeq_P \tau$ if there is a strictly increasing $f : \{0, ..., |\sigma| - 1\} \to \{0, ..., |\tau| - 1\}$ such that $\forall i < |\sigma| (\sigma(i) \leqslant_P \tau(f(i)))$.

Let $\mathcal{H}_\alpha$ be the set of h-indecomposable linear orderings of rank less than α, which are recursive in $0^{2\alpha^2}$. It follows from WQO(ST) that $\mathcal{H}_\alpha$ is well quasiordered, and then, by Higman's theorem, that $\langle \mathrm{Seq}_{\mathcal{H}_\alpha}, \preccurlyeq_{\mathcal{H}_\alpha}\rangle$ is well quasiordered too. For each i, let $S_i = \langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle \in \mathrm{Seq}_{\mathcal{H}_\alpha}$ be such that $\mathcal{L}_i = \sum_{j=0}^n \mathcal{F}_j$. ($S_i$ exists by Lemma 6.3.4 and its proof.) Then, for some $i < j$, $S_i \preccurlyeq_{\mathcal{H}_\alpha} S_j$. Hence $\mathcal{L}_i \preccurlyeq \mathcal{L}_j$. □

The following lemma gives us another statement equivalent to FRA. We will use it later.

Lemma 6.4.3. *The following are equivalent over ACA_0:*

1. *FRA.*
2. *There is no infinite strictly descending sequence of linear orderings which are h-indecomposable to the right.*

Proof: Clearly FRA implies (2). Let us prove that (2) implies $\mathsf{WQO(ST)}$, and hence FRA. Suppose, toward a contradiction, that $\langle T_i \rangle_{i\in\mathbb{N}}$ is a sequence of signed trees such that for all $i < j$, $T_i \not\preccurlyeq T_j$. For each n, define a signed tree $S_n = \{i^\frown\sigma : \sigma \in T_i, i \geqslant n\}$ and $s_{S_n}(\emptyset) = +$ and $s_{S_n}(i^\frown\sigma) = s_{T_i}(\sigma)$. We claim that for all $n < m$, $S_n \succ S_m$. Take $n < m$. Clearly $S_n \succcurlyeq S_m$. If $S_n \preccurlyeq S_m$, then for some $j \geqslant m$, $T_n \preccurlyeq T_j$, contradicting our assumption on $\langle T_i \rangle_{i\in\mathbb{N}}$. Therefore $\langle \mathrm{lin}(S_n)\rangle_{n\in\mathbb{N}}$ is a strictly descending sequence of linear orderings h-indecomposable to the right. □

6.5 Jullien's theorem

In his doctoral dissertation [Jul69] Jullien characterized all the extendible linear orderings. We want to analyze the proof theoretic strength of Jullien's theorem. The first problem we have is that, as formulated in [Jul69], Jullien's theorem does not make sense if FINDEC does not hold. We formulate Jullien's theorem in two different ways which do not need FINDEC to make sense.

Definition 6.5.1. A segment $\mathcal{B}$ of a linear ordering $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$ is *essential* if whenever we have $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}' + \mathcal{C}$ for some linear ordering $\mathcal{B}'$, it has to be the case that $\mathcal{B} \preccurlyeq \mathcal{B}'$.

Statement 6.5.2. JUL is the statement: A scattered linear ordering $\mathcal{L}$ is extendible if and only if it does not have an essential segment $\mathcal{B}$ of either of the following forms:

- $\mathcal{B} = \mathcal{R} + \mathcal{Q}$ where R is indecomposable to the right and $\mathcal{Q}$ is indecomposable to the left, or
- $\mathcal{B} = \mathbf{2}$.

This version of Jullien's theorem is different from the ones that appear in the literature. We find it more natural than the usual formulations and it does not require the notion of minimal decompositions. We will describe Jullien's formulation of his theorem in subsection 6.5.3. The fact that the two formulations are equivalent follows from an analysis of the essential segments of a linear ordering given a minimal decomposition of it. See, for example, Lemma 6.5.9 below.

Notation 6.5.3. We say that a linear ordering $\mathcal{B}$ has the form $\langle\rightarrow,\leftarrow\rangle$ if $\mathcal{B} = \mathcal{R} + \mathcal{Q}$ where $\mathcal{R}$ is indecomposable to the right and $\mathcal{Q}$ is indecomposable to the left.

6.5.1 Proof of the easy direction

We start by proving, using just RCA_0, that if $\mathcal{L}$ has an essential segment of the form either $\mathbf{2}$ or $\langle\rightarrow,\leftarrow\rangle$, then it is not extendible.

Lemma 6.5.4. *(RCA_0) If $\mathcal{L}$ has an essential segment which is not extendible, then $\mathcal{L}$ is not extendible.*

Proof: Write $\mathcal{L}$ as $\mathcal{A}+\mathcal{B}+\mathcal{C}$ where $\mathcal{B}$ is an essential, not extendible segment of $\mathcal{L}$. There is some partial ordering $\mathcal{P}$ such that $\mathcal{B} \not\preccurlyeq \mathcal{P}$, but $\mathcal{B}$ embeds in any linearization of $\mathcal{P}$. Let $\mathcal{Q}=\mathcal{A}+\mathcal{P}+\mathcal{C}$. First note that $\mathcal{L} \not\preccurlyeq \mathcal{Q}$: This is because any embedding $\mathcal{L} \preccurlyeq \mathcal{Q}$, induces an embedding of $\mathcal{L}$ into $\mathcal{A}+\mathcal{B}'+\mathcal{C}$, where $\mathcal{B}'$ is a chain in $\mathcal{P}$, and hence $\mathcal{B} \not\preccurlyeq \mathcal{B}'$, contradicting the essentially of $\mathcal{B}$. On the other hand, $\mathcal{L}$ embeds in any linearization of $\mathcal{Q}$, because a linearization of $\mathcal{Q}$ is of the form $\mathcal{A}+\mathcal{D}+\mathcal{C}$, where $\mathcal{D}$ is a linearization of $\mathcal{P}$, and $\mathcal{B}$ embeds in any linearization of $\mathcal{P}$. □

The proof of the following lemma is exactly the one in [Jul69, Lemma V.2.2]. Since Jullien's thesis [Jul69] was never published, we include the proof here.

Lemma 6.5.5. *(RCA_0) The following linear orderings are not extendible.*

- $\mathbf{2}$,
- *any linear ordering of the form $\langle \rightarrow, \leftarrow \rangle$.*

Proof: To see that $\mathbf{2}$ is not extendible consider the poset which consist of two incomparable elements.

For the other case, let $\mathcal{A}=\mathcal{B}+\mathcal{C}$ be such that $\mathcal{B}$ is $\rightarrow$ and $\mathcal{C}$ is $\leftarrow$. We will define a partial ordering $\mathcal{P}$ such that $\mathcal{A} \not\preccurlyeq \mathcal{P}$, but $\mathcal{A}$ embeds in every linearization of $\mathcal{P}$.

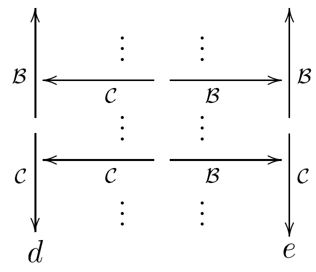
First, suppose that $\mathcal{B} \not\preccurlyeq \mathcal{C}$ and $\mathcal{C} \not\preccurlyeq \mathcal{B}$. Let $\mathcal{D}=\mathcal{C}+\mathcal{B}$ and $\{d,e\}$ be two elements not in D. We first define a set P:

$$P=(\{d\}\cup D\cup\{e\})\times D.$$

Now we define and ordering $\leqslant_P$ on P.

$$\langle w,x\rangle \leqslant_P \langle y,z\rangle \Leftrightarrow \begin{cases} w=d \;\&\; x\leqslant_D z, \text{ or}\\ y=e \;\&\; x\leqslant_D z, \text{ or}\\ w\leqslant_D y \;\&\; x=z.\end{cases}$$

See picture of $\mathcal{P}$ below. (In the picture, an element of $\mathcal{P}$ is grater than another if it is above of to the right of it.)



We claim that $\mathcal{A} \not\preccurlyeq \mathcal{P}$, but $\mathcal{A}$ embeds in every linearization of $\mathcal{P}$. Every maximal chain in $\mathcal{P}$ is either of the form $\mathcal{C} + \mathcal{B}$, of the form $\mathcal{C}_1 + \mathcal{C} + \mathcal{B} + \mathcal{C}_0 + \mathcal{B}$ where $\mathcal{C}_1 + \mathcal{C}_0 = \mathcal{C}$, or of the form $\mathcal{C} + \mathcal{B}_0 + \mathcal{C} + \mathcal{B} + \mathcal{B}_1$ where $\mathcal{B}_0 + \mathcal{B}_1 = \mathcal{B}$. In any case, any chain of $\mathcal{P}$ can be embedded into a linear ordering of the form $\mathcal{C}+\mathcal{B}_0+\mathcal{C}+\mathcal{B}+\mathcal{C}_0+\mathcal{B}$, where $\mathcal{B}_0$ is a proper initial segment of $\mathcal{B}$ and $\mathcal{C}_0$ a proper final segment of $\mathcal{C}$. From these six summands, $\mathcal{B}$ only embeds in the ones isomorphic to $\mathcal{B}$ and $\mathcal{C}$ in the ones isomorphic to $\mathcal{C}$. Therefore, if we had an embedding of $\mathcal{A}$ into $\mathcal{C} + \mathcal{B}_0 + \mathcal{C} + \mathcal{B} + \mathcal{C}_0 + \mathcal{B}$, we should have that a final segment of $\mathcal{B}$ is mapped into one of the copies of $\mathcal{B}$ and that an initial segment of $\mathcal{C}$ into one of the copies of $\mathcal{C}$, which is impossible. Now let $\mathcal{Q} = \langle P, \leqslant_Q \rangle$ be a linearization of $\mathcal{P}$. If for every $x, y \in D$, $\langle d, x \rangle \leqslant_Q \langle e, y \rangle$, then $\{d\} \times B \cup \{e\} \times C$ is a subset of $\mathcal{Q}$ of type $\mathcal{A}$. Otherwise, there exists $x, y \in D$ such that $\langle d, x \rangle \geqslant_Q \langle e, y \rangle$, then $B \times \{y\} \cup C \times \{x\}$ is a subset of $\mathcal{Q}$ of type $\mathcal{A}$. In any case, $\mathcal{A} \preccurlyeq \mathcal{Q}$.

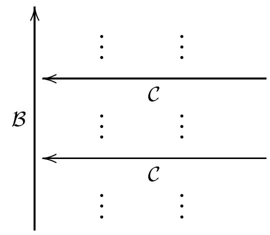
The second case is that $\mathcal{B} \preccurlyeq \mathcal{C}$ but $\mathcal{C} \not\preccurlyeq \mathcal{B}$.

$$P = (\{d\} \cup C) \times B,$$

where d is a new element. Now we define and ordering $\leqslant_P$ on P.

$$\langle w, x \rangle \leqslant_P \langle y, z \rangle \Leftrightarrow \begin{cases} w = d \;\&\; x \leqslant_D z, \text{ or} \\ w \leqslant_C y \;\&\; x = z. \end{cases}$$

See picture of $\mathcal{P}$ below. (In the picture, again, an element of $\mathcal{P}$ is grater than another if it is above of to the right of it.)



We claim that $\mathcal{A} \not\preccurlyeq \mathcal{P}$ but $\mathcal{A}$ embeds in every linearization of $\mathcal{P}$. Every chain in $\mathcal{P}$ is can be embedded in $\mathcal{B}_0 + \mathcal{C}$, where $\mathcal{B}_0$ is an initial segment of $\mathcal{B}$. If f is an embedding $\mathcal{A} \hookrightarrow \mathcal{B}_0 + \mathcal{C}$, then, since $\mathcal{B} \not\preccurlyeq \mathcal{B}_0$, there is some $x \in B$ such that $f(x) \in C$. But then, we have an embedding of $\mathbf{1} + \mathcal{C}$ into $\mathcal{C}$ contradicting Lemma 6.1.2. So $\mathcal{A} = \mathcal{B} + \mathcal{C} \not\preccurlyeq \mathcal{B}_0 + \mathcal{C}$. Now let $\mathcal{Q} = \langle P, \leqslant_Q \rangle$ be a linearization of $\mathcal{P}$. If for every $x, z \in B$ and $y \in C$, $\langle d, x \rangle \leqslant_Q \langle y, z \rangle$, then $\{d\} \times B \cup C \times \{x\}$ for some $x \in B$ is a subset of $\mathcal{Q}$ of type $\mathcal{A}$. Otherwise, there exists $x, z \in B$ and $y \in C$ such that $\langle d, x \rangle \geqslant_Q \langle y, z \rangle$. Then $\mathcal{B}+\mathcal{C}$ embeds into $\langle C_{(<y)} \times \{z\} \cup C \times \{x\}, \leqslant_Q \rangle$. In any case, $\mathcal{A} \preccurlyeq \mathcal{Q}$.

The case where $\mathcal{B} \not\preccurlyeq \mathcal{C}$ and $\mathcal{C} \preccurlyeq \mathcal{B}$ is analogous. It cannot be the case that $\mathcal{B} \preccurlyeq \mathcal{C}$ and $\mathcal{C} \preccurlyeq \mathcal{B}$, because we would have $\mathcal{B} + \mathbf{1} \preccurlyeq \mathcal{C} + \mathbf{1} \preccurlyeq \mathcal{C} \preccurlyeq \mathcal{B}$, contradicting Lemma 6.1.2. $\square$

Corollary 6.5.6. *The implication from left to right in* JUL *is provable in* RCA_0.

Proof: Immediate from the previous two lemmas. $\square$

6.5.2 Implications of JUL

Now we show that FRA is necessary to prove the right to left direction of JUL.

Lemma 6.5.7. *(RCA_0) JUL implies FRA.*

Proof: First we prove that JUL implies ATR_0. For this observe that ζ, the linear ordering of the integers does not have essential intervals of the form $\mathbf{2}$, or $\langle\rightarrow,\leftarrow\rangle$. Then, by JUL, ζ has to be extendible. Downey, Hirschfeldt, Lempp and Solomon proved in [DHLS03] that the extendibility of ζ implies ATR_0.

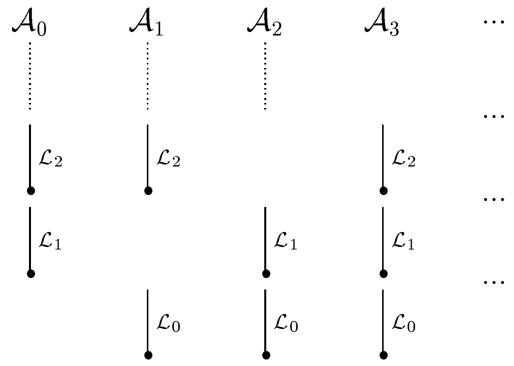
Suppose that FRA does not hold. Then, by Lemma 6.4.3, there is a sequence $\langle\mathcal{L}_i\rangle_{i\in\mathbb{N}}$ of linear orderings which are h-indecomposable to the right such that for all $i<j$, $\mathcal{L}_i \succ \mathcal{L}_j$. Assume that each $\mathcal{L}_i$ has a first element 0_{L_i}; otherwise add a first element to $\mathcal{L}_i$. Let

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \cdots + \mathcal{L}_n + \cdots,$$

and, for each $n\in\mathbb{N}$, define

$$\mathcal{A}_n = \mathcal{L}_0 + \cdots + \mathcal{L}_{n-1} + \mathcal{L}_{n+1} + \cdots.$$

Let $\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$. (See the diagram of $\mathcal{P}$ below. In the picture, an element of $\mathcal{P}$ is grater than another if it is above it.)



We think of the domain of $\mathcal{L}$ as $\{\langle i,x\rangle : x\in L_i, i\in\mathbb{N}\}$, the domain of $\mathcal{A}_n$ as $\{\langle i,x\rangle : x\in L_i, i\in\mathbb{N}, i\neq n\}$ and the domain of $\mathcal{P}$ as $\{\langle n,i,x\rangle : x\in L_i, i,n\in\mathbb{N}, i\neq n\}$.

The first claim is that $\mathcal{L}\not\preccurlyeq\mathcal{P}$. Suppose that there is an embedding $f\colon\mathcal{L}\hookrightarrow\mathcal{P}$. Then, for some n, f is an embedding $\mathcal{L}\preccurlyeq\{n\}\times\mathcal{A}_n$. Think of f as en embedding into $\mathcal{A}_n$. We prove, by induction on $i<n$, that for every $x\in L_i$, $\langle i,x\rangle <_{\mathcal{A}_n}$

$f(\langle i+1, 0_{L_{i+1}}\rangle)$. Suppose it is true for $i-1$, but that $f(\langle i+1, 0_{L_{i+1}}\rangle) \leqslant_{\mathcal{A}_n} \langle i, x\rangle$ for some $x \in \mathcal{L}_i$. So,

$$f(\{i\} \times L_i \cup \{\langle i+1, 0_{L_{i+1}}\rangle\}) \subseteq \{i\} \times L_i.$$

But, since $\mathcal{L}_i$ is h-indecomposable to the right, $\mathcal{L}_i + \mathbf{1} \not\preccurlyeq \mathcal{L}_i$. Contradiction. This implies that

$$f(\{n\} \times L_n \cup \{\langle n+1, 0_{L_{n+1}}\rangle\}) \subseteq \sum_{j \geqslant n+1} L_j,$$

which, by Corollary 6.2.12, implies that for some $j > n$, $\mathcal{L}_n \preccurlyeq \mathcal{L}_j$, contradicting our assumptions.

The second claim is that $\mathcal{L}$ embeds in every linearization of $\mathcal{P}$. Let $\leqslant_Q$ be a linearization of $\leqslant_P$ and $\mathcal{Q} = \langle P, \leqslant_Q\rangle$. We consider three possible cases. First, suppose that for every $n > 0$ and every $x \in L_{n-1}$, $\langle n, n-1, x\rangle \leqslant_Q \langle n+1, n, 0_{L_n}\rangle$. Then, $f(\langle i, x\rangle) = \langle i+1, i, x\rangle$ is an embedding of $\mathcal{L}$ into $\mathcal{Q}$. Second, if for some n, for every $y \in L_n$, $\langle n+1, n, y\rangle \leqslant_Q \langle n, n+1, 0_{L_{n+1}}\rangle$, then

$$f(\langle i, y\rangle) = \begin{cases} \langle n+1, i, y\rangle & \text{if } i \leqslant n \\ \langle n, i, y\rangle & \text{if } i > n \end{cases}$$

is an embedding of $\mathcal{L}$ into $\mathcal{Q}$. Last, suppose that neither of the above is the case. Then, for some $n > 0$ and $x \in L_{n-1}$, $\langle n, n-1, x\rangle \geqslant_Q \langle n+1, n, 0_{L_n}\rangle$, and for some $y \in L_n$ $\langle n+1, n, y\rangle \geqslant_Q \langle n, n+1, 0_{L_{n+1}}\rangle$. Therefore, for all $z \in L_{n-1}$, $z \geqslant_{L_{n-1}} x$,

$$\langle n+1, n, 0_{L_n}\rangle \leqslant_Q \langle n, n-1, z\rangle \leqslant_Q \langle n+1, n, y\rangle$$

Let h_n be an embedding of $\mathcal{L}_n$ into $\mathcal{L}_{n-1(>x)}$ and h_{n+1} be an embedding of $\mathcal{L}_{n+1}$ into $\mathcal{L}_{n(>y)}$. Now, define $f \colon \mathcal{L} \to \mathcal{Q}$ as follows

$$f(\langle i, z\rangle) = \begin{cases} \langle n+1, i, z\rangle & \text{if } i < n \\ \langle n, n-1, h_n(z)\rangle & \text{if } i = n \\ \langle n+1, n, h_{n+1}(z)\rangle & \text{if } i = n+1 \\ \langle n+1, i, z\rangle & \text{if } i > n+1. \end{cases}$$

The reader can check that f is an embedding of $\mathcal{L}$ into $\mathcal{Q}$.

The third claim, needed to get a contradiction to JUL, is that $\mathcal{L}$ does not have an essential segment which is either $\mathbf{2}$, or of the form $\langle \rightarrow, \leftarrow\rangle$. If $\mathcal{A}$ is segment of $\mathcal{L}$ of order type $\mathbf{2}$, then $\mathcal{A} \subset \{i\} \times L_i$ for some i. But, since for all $x \in L_i$,

$$\mathcal{L} \sim \mathcal{L}_0 + \cdots + \mathcal{L}_{i-1} + \mathcal{L}_{i(>x)} + \mathcal{L}_{i+1} + \cdots,$$

$\mathcal{A}$ cannot be essential. Now suppose that $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}$, where $\mathcal{B}$ is indecomposable to the right and $\mathcal{C}$ is indecomposable to the left and $\mathcal{B} + \mathcal{C}$ is an essential segment. Let i be least such that $C \cap L_i \neq \emptyset$. $\mathcal{C}$ cannot contain a final segment of

$\mathcal{L}_i$, because otherwise $\mathcal{L}_i+\mathbf{1} \preccurlyeq \mathcal{C}+\mathbf{1} \preccurlyeq \mathcal{C} \preccurlyeq \mathcal{L}_i$. So $\mathcal{C}$ is contained in a proper initial segment of $\mathcal{L}_i$. Let j be maximal such that $B \cap L_j \neq \emptyset$. j could be either i or $i-1$. $\mathcal{B}$ cannot contain a final segment of $\mathcal{L}_{j-1}$, because otherwise $\mathcal{L}_{j-1}+\mathbf{1} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{L}_j$. So $\mathcal{B} \subseteq \mathcal{L}_j$. If $j=i$, then $\mathcal{B}+\mathcal{C}$ is contained in a proper initial segment of $\mathcal{L}_i$, and therefore $\mathcal{L} \preccurlyeq \mathcal{A}+\mathcal{D}$. If $j=i-1$, $\mathcal{B}$ is a final segment of $\mathcal{L}_j$, and hence $\mathcal{B} \sim \mathcal{L}_j$ and $\mathcal{C}$ is a proper initial segment of $\mathcal{L}_i$. So, we have that $\mathcal{L} \preccurlyeq \mathcal{A}+\mathcal{B}+\mathcal{D}$ Then, since $\mathcal{B}+\mathcal{C}$ is essential, $\mathcal{B}+\mathcal{C} \preccurlyeq \mathcal{B}$, and therefore $\mathcal{L}_j+\mathbf{1} \preccurlyeq \mathcal{B}+\mathcal{C} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{L}_j$. This contradicts Lemma 6.1.2. □

6.5.3 Minimal decomposition and the proof of Jullien's theorem

Our next goal is to prove JUL in the system RCA_*+FRA.

What Jullien did in [Jul69] is to prove that every scattered linear ordering has a unique minimal decomposition, and then characterize the extendible linear orderings by putting conditions on their minimal decompositions:

Statement 6.5.8. JUL(min-dec) is the statement that says that if $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ is a minimal decomposition of $\mathcal{L}$, then $\mathcal{L}$ is extendible if and only if there is no i such that either $\mathcal{F}_i = \mathcal{F}_{i+1} = \mathbf{1}$ or $\mathcal{F}_i$ is indecomposable to the right and $\mathcal{F}_{i+1}$ is indecomposable to the left.

The problem with this statement is that, without knowing that minimal decompositions always exists, JUL(min-dec) is not enough to classify all the extendible linear orderings, as Jullien did. So, from the viewpoint of reverse math, this is not a satisfactory formulation of Jullien's classification of the extendible linear orderings. We could say that Jullien's theorem, as stated in [Jul69], is the conjunction of JUL(min-dec) and the sentence that says that every scattered linear ordering has a minimal decomposition (which is equivalent to FRA; see Corollary 6.3.11 and Theorem 6.4.2).

We will prove that JUL(min-dec) is equivalent to ATR_* over RCA_*. Then, use this result to prove that that FRA implies JUL.

Lemma 6.5.9. *(RCA_0) If $\langle \mathcal{F}_i : i \leqslant n\rangle$ is a minimal decomposition of $\mathcal{L}$, and either $\mathcal{F}_i = \mathcal{F}_{i+1} = \mathbf{1}$ or $\mathcal{F}_i$ is h-indecomposable to the right and $\mathcal{F}_{i+1}$ is h-indecomposable to the left, then $\mathcal{F}_i + \mathcal{F}_{i+1}$ is an essential segment of $\mathcal{L}$.*

Proof: Suppose, toward a contradiction, that f is an embedding,

$$f\colon \mathcal{L} \hookrightarrow (\mathcal{F}_0+\cdots+\mathcal{F}_{i-1})+\mathcal{A}+(\mathcal{F}_{i+2}+\cdots+\mathcal{F}_n),$$

and $\mathcal{F}_i+\mathcal{F}_{i+1} \not\preccurlyeq \mathcal{A}$. If $\mathcal{F}_i = \mathcal{F}_{i+1} = 1$, then $\mathcal{A}$ has to be either $\emptyset$ or $\mathbf{1}$, so $\mathcal{F}_0+\cdots+\mathcal{F}_{i-1}+\mathcal{A}+\mathcal{F}_{i+2}+\cdots+\mathcal{F}_n$ is a decomposition of $\mathcal{L}$ with less less than $n+1$ terms. This contradicts the minimality of the decomposition of $\mathcal{L}$. Now suppose that $\mathcal{F}_i$ is h-indecomposable to the right and $\mathcal{F}_{i+1}$ is h-indecomposable to the left. If there exist $x \in F_i$ and $y \in F_{i+1}$ such that both $f(x)$ and $f(y)$ belong to A, then

$$\mathcal{F}_i + \mathcal{F}_{i+1} \preccurlyeq \mathcal{F}_{i(>x)} + \mathcal{F}_{i+1(<y)} \preccurlyeq \mathcal{A}.$$

So, either $\forall x \in \mathcal{F}_i(f(x) \notin \mathcal{A})$ or $\forall x \in \mathcal{F}_{i+1}(f(x) \notin \mathcal{A})$. Suppose the former is the case. The other case is analogous. If, there is some $x \in \mathcal{F}_i$ such that $f(x) \in \mathcal{F}_{i+2}+\cdots+\mathcal{F}_n$, then, since $\mathcal{F}_i \sim \mathcal{F}_{i(>x)}$, we have that $\mathcal{F}_i+\cdots+\mathcal{F}_n \preccurlyeq \mathcal{F}_{i+2}+\cdots+\mathcal{F}_n$. Hence

$$\mathcal{L} \sim \mathcal{F}_1 + \cdots + \mathcal{F}_{i-1} + \mathcal{F}_{i+2} + \cdots \mathcal{F}_n,$$

contradicting the minimality of $\langle \mathcal{F}_i : i \leqslant n\rangle$. So, for every $x \in F_i$, $f(x) \in F_0 + \cdots + F_{i-1}$. Then

$$\mathcal{L} \sim \mathcal{F}_1 + \cdots + \mathcal{F}_{i-1} + \mathcal{F}_{i+1} + \cdots \mathcal{F}_n,$$

contradicting, again, the minimality of $\langle \mathcal{F}_i : i \leqslant n\rangle$. □

Corollary 6.5.10. *The direction from left to right of* JUL*(min-dec) is provable in* RCA_0.

PROOF: Use the previous lemma and Corollary 6.5.6. □

Now we want to prove the other direction of JUL(min-dec) using ATR_*. We will use that ATR_* proves that $\mathcal{L}$ and $\mathbf{1}+\mathcal{L}+\mathbf{1}$ are extendible when $\mathcal{L}$ is h-indecomposable, and not $\mathbf{1}$, which we will prove in the next section. Moreover, in the next section, in Proposition 6.6.18, we will prove that every partial ordering, $\mathcal{P}$, which does not embed $\mathbf{1}+\mathcal{L}+\mathbf{1}$, has a linearization which is hyperarithmetic in $\mathcal{L}$ and $\mathcal{P}$, and does not embed $\mathbf{1}+\mathcal{L}+\mathbf{1}$. We could get JUL(min-dec), using ATR and the results of the next section, using a proof similar to Jullien's. But, since we want to use ATR_*, we have to make some modifications.

One important fact that we use to lower the complexity of certain formulas is the following.

Lemma 6.5.11. *[Sim99, Theorem VIII.3.20] For any Σ^1_1 formula $\varphi(X,Y)$, we can find a Σ^1_1 formula $\varphi'(X)$ such that* ATR_0 *proves*

$$\varphi'(X) \Leftrightarrow \forall Y(Y \text{ hyperarithmetic in } X \Rightarrow \varphi(X,Y)).$$

The plan of the proof is as follows. First, we prove that every scattered linear ordering of the right form has a finite decomposition of a certain kind:

Lemma 6.5.12. *(ATR_0) Let $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ be a minimal decomposition of a linear ordering $\mathcal{L}$, such that $\mathcal{F}_i + \mathcal{F}_{i+1}$ is neither $\mathbf{2}$ nor $\langle \rightarrow, \leftarrow\rangle$ for any $i < n$. Then, $\mathcal{L}$ has a finite decomposition of one of the following forms:*

$$\langle \mathbf{1}, \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m, \mathbf{1}\rangle,$$
$$\langle \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m, \mathbf{1}\rangle,$$
$$\langle \mathbf{1}, \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m\rangle, \qquad \text{or}$$
$$\langle \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m\rangle,$$

where each $\check{\mathcal{F}}_i$ is h-indecomposable, either to the left or to the right, but not $\mathbf{1}$.

Next, we use this decomposition of $\mathcal{L}$ to reduce the problem of the extendibility of $\mathcal{L}$ to the extendibility of $\mathbf{1}+\check{\mathcal{F}}_i+\mathbf{1}$ for each i:

Lemma 6.5.13. *(ATR_*) Suppose that $\mathcal{L}$ has a finite decomposition of the form $\langle \mathbf{1}, \mathcal{F}_0, \mathbf{1}, \mathcal{F}_1, \mathbf{1}, ..., \mathcal{F}_m, \mathbf{1}\rangle$, where each $\mathcal{F}_i$ is h-indecomposable but not $\mathbf{1}$. Consider a partial ordering $\mathcal{P} = \langle P, \leqslant_P\rangle$ such that $\mathcal{L} \not\preccurlyeq \mathcal{P}$. Then there exists a partition $\langle P_i : i \leqslant m\rangle$ of P such that*

- *if $x \in P_i$, $y \in P_j$ and $x \leqslant_P y$, then $i \leqslant j$, and*
- *for all $i \leqslant m$, $\mathbf{1}+\mathcal{F}_i+\mathbf{1} \not\preccurlyeq \mathcal{P}_i$, where $\mathcal{P}_i = \langle P_i, \leqslant_P\rangle$.*

Then, we will use the results in the next section to linearize each $\mathcal{P}_i$ and get a linear ordering which does not embed $\mathbf{1}+\mathcal{F}_i+\mathbf{1}$. We will show that Σ^1_1-IND is enough to get all these linearization simultaneously and construct a linearization of $\mathcal{P}$ which does not embed $\mathcal{L}$.

Proof of Lemma 6.5.12: All we need to observe is that if $\mathcal{F}_i$ is $\leftarrow$, then $\mathcal{F}_i \sim \mathcal{F}_i + \mathbf{1}$, and if $\mathcal{F}_i$ is $\rightarrow$, then $\mathcal{F}_i \sim \mathbf{1}+\mathcal{F}_i$. Therefore, if none of the $\mathcal{F}_i+\mathcal{F}_{i+1}$ is of the form $\langle\rightarrow,\leftarrow\rangle$ or $\mathbf{2}$, then $\mathcal{F}_i+\mathcal{F}_{i+1} \sim \mathcal{F}_i+\mathbf{1}+\mathcal{F}_{i+1}$. Apply this to insert $\mathbf{1}$s in the finite decomposition $\langle\mathcal{F}_0, ..., \mathcal{F}_n\rangle$ to get the desired decomposition. □

Proof of Lemma 6.5.13: We prove by induction on $i \leqslant m$ that there exists a partition $\langle P_0, ..., P_{i-1}, \bar{P}_i\rangle$ of P, hyperarithmetic in $\mathcal{P}$, such that

- for $j, k < i$, if $x \in P_j$, $y \in P_k$ and $x \leqslant_P y$, then $j \leqslant k$,
- for $j < i$, if $x \in P_j$, $y \in \bar{P}_i$ then $y \not\leqslant_P x$,
- for all $j < i$, $\mathbf{1}+\mathcal{F}_j+\mathbf{1} \not\preccurlyeq \mathcal{P}_j = \langle P_j, \leqslant_P\rangle$, and
- $\mathbf{1}+\mathcal{F}_i+\mathbf{1}+...+\mathbf{1}+\mathcal{F}_m+\mathbf{1} \not\preccurlyeq \langle\bar{P}_i, \leqslant_P\rangle$.

The case $i = m$ will give us the Lemma. By Lemma 6.5.11, the formula we are proving by induction is equivalent to a Π^1_1 one. (Π^1_1-induction is equivalent to Σ^1_1-IND [Sim99, Lemma VIII.4.9].) The base case $i = 0$ is trivial; just take the trivial partition $\langle P\rangle$. Now suppose we have $\langle P_0, ..., P_{i-1}, \bar{P}_i\rangle$ satisfying the conditions above. Let $\phi_+(x)$ be the Σ^1_1-formula that says that

$$x \in \bar{P}_i \text{ and } \mathbf{1}+\mathcal{F}_i+\mathbf{1} \preccurlyeq \bar{P}_{i(\leqslant x)}$$

and $\phi_-(x)$ be the Σ^1_1-formula that says that

$$x \in \bar{P}_i \text{ and } \mathbf{1}+\mathcal{F}_{i+1}+...+\mathcal{F}_m+\mathbf{1} \preccurlyeq \mathcal{P}_{(\geqslant x)}.$$

Since $\mathbf{1}+\mathcal{F}_i+\mathbf{1}+...+\mathbf{1}+\mathcal{F}_m+\mathbf{1} \not\preccurlyeq \bar{P}_i$, there is no x such that $\phi_+(x)$ & $\phi_-(x)$. Then, by Σ^1_1-separation (which is equivalent to ATR_0; see [Sim99, Theorem V.5.1]), there is a set $Q \subseteq \bar{P}_i$ such that

$$\forall x(\phi_-(x) \Rightarrow x \in Q \;\&\; \phi_+(x) \Rightarrow x \in \bar{P}_i \setminus Q).$$

Moreover, Q can be taken hyperarithmetic in $\mathcal{P}$. (Let f be a recursive map that assigns to each x a recursive linear ordering such that $\neg\phi_+(x)$ iff $f(x)$ is a well ordering [Sim99, Proof of Lemma VII.3.4]. By the Σ^1_1 bounding principle [Sim99, Lemma V.6.2], there is an ordinal α such that for all x with $\phi_-(x)$, $f(x) \leqslant \alpha$. Now, let Q be the set of x's such that α has an initial segment isomorphic to $f(x)$. Q is hyperarithmetic (see the proof of [Sim99, Lemma VII.3.19]).) Let P_i be the downward closure of Q in $\bar{P}_i$. (i.e.: $P_i = \{x \in \bar{P}_i : \exists y \in Q(x \leqslant_P y)\}$.) Since for no $x \in Q$, $\mathbf{1} + \mathcal{F}_i + \mathbf{1} \preccurlyeq \mathcal{P}_{(\leqslant x)}$, we have that that $\mathbf{1} + \mathcal{F}_i + \mathbf{1} \not\preccurlyeq \mathcal{P}_i$. Analogously $\mathbf{1} + \mathcal{F}_{i+1} + ... + \mathcal{F}_m + \mathbf{1} \not\preccurlyeq \bar{P}_i \setminus P_i$. Let $\bar{P}_{i+1} = \bar{P}_i \setminus P_i$. It is not hard to see that $\langle P_0, ..., P_i, \bar{P}_{i+1}\rangle$ satisfies the conditions above. □

Theorem 6.5.14. *JUL(min-dec) is equivalent to ATR$_*$ over RCA$_*$.*

Proof: Assume JUL(min-dec). Since $\omega^* + \omega$ is a minimal decomposition of ζ, we have that ζ is extendible. It is proved in [DHLS03, Theorem 3] that the extendibility of ζ implies ATR_0 over RCA_0.

Let us prove JUL(min-dec) from ATR_*. The direction from left to right was proved in Corollary 6.5.10. We now prove the other direction. Let $\langle \check{\mathcal{F}}_i : i \leqslant n\rangle$ be a minimal decomposition of $\mathcal{L}$ such that for no i, $\check{\mathcal{F}}_i = \check{\mathcal{F}}_{i+1} = \mathbf{1}$ or $\check{\mathcal{F}}_i + \check{\mathcal{F}}_{i+1}$ is $\langle \rightarrow, \leftarrow\rangle$. Let $\mathcal{P}$ be a partial ordering which does not embed $\mathcal{L}$. By Lemma 6.5.12, $\mathcal{L}$ has a finite decomposition of one of four possible forms. Suppose $\mathcal{L}$ has a decomposition of the form $\langle \mathbf{1}, \mathcal{F}_0, \mathbf{1}, \mathcal{F}_1, \mathbf{1}, ..., \mathcal{F}_m, \mathbf{1}\rangle$. The other cases are similar to this one. Then, consider a partition, $\{P_i : i \leqslant m\}$, of P as in Lemma 6.5.13.

Now, by induction on $i \leqslant m$, we prove that there exists a sequence $\langle \mathcal{Q}_0, ..., \mathcal{Q}_i\rangle$, hyperarithmetic in $\mathcal{P}$ and $\mathcal{L}$, such that for each $j \leqslant i$, $\mathcal{Q}_i = \langle P_i, \leqslant_Q\rangle$ is a linearization of $\mathcal{P}_i$ which does not embed $\mathbf{1} + \mathcal{F}_i + \mathbf{1}$. The formula we are proving by induction is equivalent to a Π^1_1 one by Lemma 6.5.11, so we can do this using Σ^1_1-IND. The base case and induction step follow immediately from Proposition 6.6.18.

Define $\mathcal{Q} = \sum_{i \leqslant m} Q_i$, and think of the domain of $\mathcal{Q}$ as P. So, $\mathcal{Q}$ is a linearization of $\mathcal{P}$. We claim that $\mathcal{Q}$ does not embed $\mathcal{L}$. Suppose, toward a contradiction, that we have an embedding $f\colon \mathbf{1} + \mathcal{F}_0 + ... + \mathcal{F}_m + \mathbf{1} \hookrightarrow \mathcal{Q}$. Let $x_0 \leqslant_Q x_1 \leqslant_Q ... \leqslant_Q x_{m+1}$ be the image under f of the $\mathbf{1}$s in $\mathbf{1} + \mathcal{F}_0 + ... + \mathcal{F}_m + \mathbf{1}$. For each $i \leqslant m+1$ let $j_i \leqslant m$ be such that $x_i \in P_{j_i}$. Note that for every $i \leqslant m$, $j_i \leqslant j_{i+1}$. We claim that for some i, $j_i = j_{i+1} = i$. It can be easily proved by induction on i that, if the claim is not true, then for every i, $j_i \geqslant i$. We then get a contradiction when we let $i = m+1$. So, there exists some i such that $x_i, x_{i+1} \in P_i$. But then, f maps $\mathbf{1} + \mathcal{F}_i + \mathbf{1}$ into $\mathcal{Q}_i$, contradicting the definition of the Q_is. □

Corollary 6.5.15. *JUL is equivalent to FRA over RCA$_*$.*

Proof: We have proved, in Lemma 6.5.7, that JUL implies FRA. Now assume FRA holds, and hence FINDEC too. Recall that FRA implies ATR_0, so from the theorem above, we have JUL(min-dec). Let $\mathcal{L}$ a scattered linear ordering which does not have any essential segment of the form $\mathbf{2}$ or $\langle\rightarrow,\leftarrow\rangle$. Using FINDEC and Proposition 6.3.10, we get that $\mathcal{L}$, has a minimal decomposition $\langle \mathcal{F}_i : i \leqslant n\rangle$. From Lemma 6.5.9, we get that for no i, $\mathcal{F}_i + \mathcal{F}_{i+1}$ is of the form $\mathbf{2}$ or $\langle\rightarrow,\leftarrow\rangle$. Using JUL(min-dec) we get that $\mathcal{L}$ is extendible. $\square$

6.6 Extendibility of h-indecomposable linear orderings

This section is devoted to proving the following theorem in ATR_*.

Theorem 6.6.1. *(ATR_*) Every h-indecomposable linear ordering is extendible.*

Every result in this section is going to be proved in ATR_*. So, unless otherwise stated, we will be working in ATR_*.

ATR_* it is not strong enough to prove the existence of ω_1^{CK}. But it can prove the existence of a linear ordering which contains ω_1^{CK}. Let ξ be a recursive linear ordering such that every hyperarithmetic well ordering embeds into ξ as an initial segment. We write $x \in \omega_1^{CK}$ as an abbreviation for $x \in \xi$ and $\xi_{(<x)}$ is well ordered. The existence of such a ξ in ATR_0 follows from [Sim99, Lemma VIII.3.14 and Theorem VIII.3.15].

6.6.1 Extendibility of ω^* and $(\omega^2)^*$

Before we prove the extendibility of an arbitrary h-indecomposable linear ordering, we provide two examples. These examples will illustrate some key ideas used in the general case.

Theorem 6.6.2. *ω^* is extendible.*

A stronger version of this theorem is proved in [DHLS03]. They prove that ω^* is extendible in ACA_0. Our proof, even though it uses ATR_0, is easier to understand and incorporates an idea that we will generalize later.

Proof: Consider a recursive partial ordering $\mathcal{P}$ which does not embed ω^*, or equivalently, which is well founded. If $\mathcal{P}$ is not recursive, relativize. Consider the rank function, $\mathrm{rk}_{\mathcal{P}}$, on $\mathcal{P}$. Let $\alpha \in \omega_1^{CK}$ be the rank of $\mathcal{P}$. Define a linearization, $\leqslant_Q$, of $\mathcal{P}$ as follows: let $x \leqslant_Q y$ iff $\mathrm{rk}_{\mathcal{P}}(x) < \mathrm{rk}_{\mathcal{P}}(y)$ or $\mathrm{rk}_{\mathcal{P}}(x) = \mathrm{rk}_{\mathcal{P}}(y)$ and $x \leqslant_{\mathbb{N}} y$ (where $\leqslant_{\mathbb{N}}$ is the ordering of the natural numbers; recall that the domain of $\mathcal{P}$ is a subset of $\mathbb{N}$). Observe now that $\omega^* \not\preccurlyeq \langle P, \leqslant_Q\rangle$. $\square$

Using Proposition 6.6.7, we get as a corollary of the previous theorem that $\mathbf{1} + \omega^*$ is extendible too. We will use this in the next theorem.

Theorem 6.6.3. *(ATR_0) $(\omega^2)^*$ is extendible.*

In ATR_0, this is a new result. The key idea is the use of the trees $T_{x,\omega^{2*}}$ defined below. It allows us to prove this theorem in ATR_0, and is going to be very useful in the more general case.

We write ω^{2*} for $(\omega^2)^*$.

Proof: Consider a partial ordering $\mathcal{P}$ which does not embed ω^{2*}. Assume $\mathcal{P}$ is recursive; otherwise relativize. Let $T_{\mathcal{P},\omega^{2*}}$ be the set of all $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle$ such that:

- for every $i < n$, π_i is a $(n-i)$-tuple from $\mathcal{P}$;
- for every $i < n$ and $j < k < |\pi_i|$, $\pi_i(j) >_P \pi_i(k)$;
- for every $i, i' < n$, $j < |\pi_i|$ and $j' < |\pi_{i'}|$, if $i < i'$ then $\pi_i(j) >_P \pi_{i'}(j')$.

We claim that $T_{\mathcal{P},\omega^{2*}}$ is well founded. Indeed, a path f though $T_{\mathcal{P},\omega^{2*}}$ codes a sequence $\langle f_0, f_1, ...\rangle$ such that each f_i is a descending sequence in $\mathcal{P}$ and for all x, y and $i < j$, $f_i(x) >_P f_j(y)$. Therefore, f codes a embedding $\omega^{2*} \hookrightarrow \mathcal{P}$. Let $\alpha \in \omega_1^{CK}$ be the rank of $T_{\mathcal{P},\omega^{2*}}$. Now, for each $x \in P$ let $T_{x,\omega^{2*}}$ be the subtree of $T_{\mathcal{P},\omega^{2*}}$ which consist of the $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle$ such that $\forall i < n \forall j < n - i(\pi_i(j) \leqslant_P x)$. So $T_{x,\omega^{2*}}$ is $T_{\mathcal{P}(\leqslant x),\omega^{2*}}$ Let r_x be the rank of $T_{x,\omega^{2*}}$, and for each $\gamma < \alpha$, let

$$Q_\gamma = \{x \in P : r_x = \gamma\}.$$

We claim that for each γ, $\mathbf{1} + \omega^* \not\preccurlyeq Q_\gamma$. Suppose, toward a contradiction, that there exists an $f\colon \omega^* \hookrightarrow Q_\gamma$ and an $x \in Q_\gamma$ such that for all $n \in \omega^*$, $x \leqslant_P f(n)$. Let $y = f(0) \in P$. We will prove that $r_x < r_y$ contradicting the fact that both x and y are in Q_γ. In order to prove this, we use f to construct an embedding, g, of $T_{x,\omega^{2*}}$ into $T_{y,\omega^{2*}}$ such that $g(\emptyset) \supsetneq \emptyset$. Given $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle \in T_{x,\omega^{2*}}$, let $g(\sigma) = \langle f \restriction n+1, \pi_0, ..., \pi_{n-1}\rangle \in T_{y,\omega^{2*}}$. Clearly g is an embedding and $g(\emptyset) = \langle\langle f(0)\rangle\rangle$. Therefore, the rank of $\langle\langle f(0)\rangle\rangle$ in $T_{y,\omega^{2*}}$ is grater than or equal the rank of $T_{x,\omega^{2*}}$, and hence $r_y > r_x$. This proves our claim.

Now we want to linearize each Q_γ so that $\mathbf{1} + \omega^*$ does not embed in the linearization. To do this consider $\bigoplus_{\gamma<\alpha} Q_\gamma$ and observe that $\mathbf{1} + \omega^*$ does not embed in it. Therefore, it has a linearization that does not embed $\mathbf{1} + \omega^*$. For each γ, let $\leqslant_{Q_\gamma}$ be the restriction of this linearization to Q_γ. Now define a linearization $\leqslant_Q$ of $\mathcal{P}$ as follows: let $x \leqslant_Q y$ iff $r_x < r_y$ or $r_x = r_y$ and $x \leqslant_{Q_{r_x}} y$. Note that this is the same as defining

$$\langle P, \leqslant_Q\rangle = \sum_{\gamma<\alpha} \langle Q_\gamma, \leqslant_{Q_\gamma}\rangle.$$

Observe that there cannot be an embedding of ω^{2*} into $\langle P, \leqslant_Q\rangle$ because we would have an embedding of ω^{2*} into some $\langle Q_\gamma, \leqslant_{Q_\gamma}\rangle$ when not even $\mathbf{1} + \omega^*$ embeds in $\langle Q_\gamma, \leqslant_{Q_\gamma}\rangle$. □

6.6.2 Extendibility of $\mathbf{1} + \mathcal{L} + \mathbf{1}$

Let $\mathcal{L}$ be an h-indecomposable linear ordering. We study here the relation between the extendibility of $\mathcal{L}$ and the extendibility of $\mathbf{1} + \mathcal{L} + \mathbf{1}$. Assume that $\mathcal{L}$ is h-indecomposable to the left. Note that then, $\mathbf{1} + \mathcal{L} + \mathbf{1} \sim \mathbf{1} + \mathcal{L}$.

The general ideas in this subsection come from [Jul69, Lemma V.2.4].

Lemma 6.6.4. *If $\mathbf{1} + \mathcal{L} + \mathbf{1}$ is extendible, then so is $\mathcal{L}$.*

Proof: Let $\mathcal{P}$ be a partial ordering such that $\mathcal{L} \not\preccurlyeq \mathcal{P}$. Let $\mathcal{Q} = \mathbf{1} + \mathcal{P} + \mathbf{1}$. Then $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{Q}$, hence $\mathcal{Q}$ has a linearization $\mathcal{R} = \langle Q, \leqslant_R \rangle$, such that $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{R}$. The restriction of $\leqslant_R$ to P is a linearization of $\mathcal{P}$ which does not embed $\mathcal{L}$. □

Now consider a poset $\mathcal{P}$ such that $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$ and assume that $\mathcal{L}$ is extendible. We will show how to linearize $\mathcal{P}$, so that $\mathbf{1} + \mathcal{L} + \mathbf{1}$ does not embed in the linearization. We will partition $\mathcal{P}$ into infinitely many pieces $\{\mathcal{P}_m : m \in \omega\}$ such that for each m, $\mathcal{L} \not\preccurlyeq \mathcal{P}_m$. The idea is that then we can use the extendibility of $\mathcal{L}$ to linearize each $\mathcal{P}_m$ and get a linearization of $\mathcal{P}$ as the one required.

Definition 6.6.5. If $\mathcal{P}$ has a least element a, let $P_0 = P \smallsetminus \{a\}$, $P_1 = \{a\}$ and $P_n = \emptyset$ for $n > 1$. Suppose now that $\mathcal{P}$ has no least element and that we have already defined P_i for $i < n$. Let a_n be the least, in the order of the natural numbers, element of $P \smallsetminus (\bigcup_{i<n} P_i)$. (We are assuming that the domain of P is a subset of the natural numbers.) Now, let $P_n = \{x \in P \smallsetminus (\bigcup_{i<n} P_i) : x >_P a_n\}$.

Lemma 6.6.6. *If $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$, then*

1. *$\{\mathcal{P}_m\}_{m\in\omega}$ is a partition of $\mathcal{P}$.*
2. *if $x \leqslant_P x'$, $x \in P_m$ and $x' \in P_{m'}$, then $m \geqslant m'$.*
3. *For each m, $\mathcal{L} \not\preccurlyeq \mathcal{P}_m$.*

Proof: The first two parts follow easily from the definitions. For the last part, note that if $\mathcal{L} \preccurlyeq \mathcal{P}_m$, then $\mathbf{1} + \mathcal{L} + \mathbf{1} \preccurlyeq \mathbf{1} + \mathcal{L} \preccurlyeq \mathcal{P}$. □

Proposition 6.6.7. *Given an h-indecomposable linear ordering $\mathcal{L}$, $\mathcal{L}$ is extendible if and only if $\mathbf{1} + \mathcal{L} + \mathbf{1}$ is.*

Proof: We have already shown the implication from right to left. Now assume $\mathcal{L}$ is extendible and consider $\mathcal{P}$ such that $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$. Let $\{\mathcal{P}_m\}_{m\in\mathbb{N}}$ be as defined above. For each m let $\mathcal{Q}_m = \langle P_m, \leqslant_{Q_m} \rangle$ be a linearization of $\mathcal{P}_m$ which does not embed $\mathcal{L}$. To get all the linearizations $\{\mathcal{Q}_m\}_{m\in\mathbb{N}}$ uniformly, consider $\mathcal{Q} = \bigoplus_{m\in\mathbb{N}} \mathcal{P}_m$. Observe that $\mathcal{L} \not\preccurlyeq \mathcal{Q}$, and linearize $\mathcal{Q}$ so that $\mathcal{L}$ does not embed in the linearization. Observe now that $\sum_{m\in\omega^*} \mathcal{Q}_m$ is a linearization of $\mathcal{P}$ which does not embed $\mathbf{1} + \mathcal{L}$ (by Corollary 6.2.12, substituting left for right and ω^* for ω). □

Remark 6.6.8. Note that the results we have proved so far in this subsection could have been proved using only RCA_0. But ATR_* is enough for our purposes.

6.6.3 Extendibility of $\sum_{m\in\omega^*}\mathcal{L}_m$

Now suppose we are given a partial ordering $\mathcal{P}$ such that $\mathcal{L}\not\preccurlyeq\mathcal{P}$. Again assume that $\mathcal{L}=\operatorname{lin}(T)$ is h-indecomposable to the left, and also assume that $\mathcal{L}\neq\omega^*$. Let $\mathcal{L}_k=\operatorname{lin}(T_{\langle(k)_0\rangle})$. (Since $\mathcal{L}\neq\omega^*$, $T_{\langle m\rangle}$ exists.) So $\mathcal{L}=\sum_{k\in\omega^*}\mathcal{L}_k$. We will partition $\mathcal{P}$ into $\{\mathcal{P}_{m,\gamma}\}_{m\in\omega,\gamma\in\omega_1^{CK}}$ such that for each m and γ, $\mathbf{1}+\mathcal{L}_m+\mathbf{1}\not\preccurlyeq\mathcal{P}_{m,\gamma}$. Note that if we could uniformly linearize each $\mathcal{P}_{m,\gamma}$ into a linear ordering $\mathcal{Q}_{m,\gamma}$ such that $\mathbf{1}+\mathcal{L}_m+\mathbf{1}\not\preccurlyeq\mathcal{Q}_{m,\gamma}$, then $\sum_{\langle m,\gamma\rangle\in\omega\times\omega_1^{CK}}\mathcal{Q}_{m,\gamma}$ would be a linearization of $\mathcal{P}$ which does not embed $\mathcal{L}$.

We will construct the partition in a similar way as in the proof that ω^{2*} is extendible. But the fact that ω^{2*} is an ω^*-sum of terms which are all equal (all terms are ω^*) made that proof easier. In the general case, instead of considering one tree $T_{\mathcal{P},\mathcal{L}}$, we have to consider a tree $T^m_{\mathcal{P},\mathcal{L}}$ for each $m\in\mathbb{N}$. This modification is needed for the proof of Lemma 6.6.11(4) below.

Definition 6.6.9. Given a poset $\mathcal{P}$ and $m\in\mathbb{N}$, define $T^m_{\mathcal{P},\mathcal{L}}$ to be the set of all $\sigma=\langle\pi_0,\dots,\pi_{n-1}\rangle$ such that:

- for every $i<n$, π_i is a $(n-i)$-tuple from $\mathcal{P}$;
- for every $i<n$ and $j,k<|\pi_i|$, if $j<_{L_{m+i}}k$, then $\pi_i(j)<_P\pi_i(k)$;
- for every $i,i'<n$, $j<|\pi_i|$ and $j'<|\pi_{i'}|$, if $i<i'$ then $\pi_i(j)>_P\pi_{i'}(j')$.

Lemma 6.6.10. *If $\mathcal{L}\not\preccurlyeq\mathcal{P}$, then for all m, $T^m_{\mathcal{P},\mathcal{L}}$ is well founded.*

Proof: Suppose that $T^m_{\mathcal{P},\mathcal{L}}$ is not well founded. A path f though $T^m_{\mathcal{P},\mathcal{L}}$ codes a sequence $\langle f_0,f_1,f_2,\dots\rangle$ such that each f_i is an embedding of $\mathcal{L}_{m+i}$ into $\mathcal{P}$ and for all x,y, if $i<j$, then $f_i(x)>_P f_j(y)$. So, we have an embedding

$$\sum_{i\in\omega^*,i\geqslant m}\mathcal{L}_i\preccurlyeq\mathcal{P}.$$

Since $\mathcal{L}$ is h-indecomposable to the left, we have an embedding $\mathcal{L}\preccurlyeq\mathcal{P}$, contradicting the hypothesis. $\square$

When $\mathcal{L}\not\preccurlyeq\mathcal{P}$, we have that for each $x\in P$ and $m\in\omega$, $T^m_{\mathcal{P}_{(\leqslant x)},\mathcal{L}}$ is well founded, and uniformly recursive in x and m (and $\mathcal{P}$ and $\mathcal{L}$). Let $T^m_{x,\mathcal{L}}=T^m_{\mathcal{P}_{(\leqslant x)},\mathcal{L}}$ So, each tree $T^m_{x,\mathcal{L}}$ has a rank $r_{x,m}\in\omega_1^{CK}$. For each x, let r_x be the least of $\{r_{x,m}:m\in\mathbb{N}\}$ and m_x be the least m such that $r_{x,m}=r_x$. Define $\operatorname{rk}_{\mathcal{P},\mathcal{L}}(x)=\langle m_x,r_x\rangle$. Given $\gamma\in\xi$, and $m\in\mathbb{N}$, let $P_{m,\gamma}=\{x\in P:\operatorname{rk}_{\mathcal{P},\mathcal{L}}(x)=\langle m,\gamma\rangle\}$.

Lemma 6.6.11. *Assume that $\mathcal{L}\not\preccurlyeq\mathcal{P}$ and that $\mathcal{P}$ is hyperarithmetic, then*

1. *For $\gamma\in\xi\smallsetminus\omega_1^{CK}$, $P_{m,\gamma}=\emptyset$.*
2. *$\{P_{m,\gamma}\}_{\gamma\in\xi,m\in\mathbb{N}}$ is a partition of P.*

3. *If $x \leqslant_P x'$, $x \in P_{m,\gamma}$ and $x' \in P_{m'\gamma'}$, then $\langle m, \gamma\rangle \leqslant_{\omega\times\xi} \langle m', \gamma'\rangle$. i.e. $\gamma <_\xi \gamma'$ or $\gamma = \gamma'$ and $m \leqslant m'$.*

4. $\mathbf{1} + \mathcal{L}_m + \mathbf{1} \not\preccurlyeq \mathcal{P}_{m,\gamma}$.

Proof: Part (1) is because the trees $T^m_{\mathcal{P},\mathcal{L}}$ are well founded and hyperarithmetic. Part (2) is clear because for all $x \in P$, $\mathrm{rk}_{\mathcal{P},\mathcal{L}}(x) \in \mathbb{N}\times\xi$. To prove part (3) we show that if $x \leqslant_P y$, then $\mathrm{rk}_{\mathcal{P},\mathcal{L}}(x) \leqslant_{\omega\times\xi} \mathrm{rk}_{\mathcal{P},\mathcal{L}}(y)$: Since for each m, $T^m_{x,\mathcal{L}} \subseteq T^m_{y,\mathcal{L}}$, we have that $r_{x,m} \leqslant_\xi r_{y,m}$. Therefore, $r_x \leqslant_\xi r_y$, and if $r_x = r_y$, then $m_x \leqslant m_y$. For the last part consider $x, y \in P_{m,\gamma}$, and suppose, toward a contradiction, that there is an embedding $f\colon \mathcal{L}_m \hookrightarrow (x, y)_P$. We shall define an embedding, g, of $T^{m+1}_{x,\mathcal{L}}$ into $T^m_{y,\mathcal{L}}$ such that $g(\emptyset) \supsetneq \emptyset$. This will imply that the rank of $T^{m+1}_{x,\mathcal{L}}$ is strictly smaller than the rank of $T^m_{y,\mathcal{L}}$, and therefore $r_x \leqslant_\xi r_{x,m+1} <_\xi r_{y,m} = r_y$. This would contradict the assumption that $r_x = r_y = \gamma$. Given $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle \in T^{m+1}_{x,\mathcal{L}}$, let

$$g(\sigma) = \langle f \restriction n + 1, \pi_0, ..., \pi_{n-1}\rangle \in T^m_{y,\mathcal{L}}.$$

It is not hard to check that g is as wanted. □

6.6.4 One step iteration

Now we join the previous two constructions into one. The partition we define in this subsection is the one that we will iterate later to construct a linearization of $\mathcal{P}$.

Let $\mathcal{L} = \mathrm{lin}(T)$ be h-indecomposable to the left. The case when $\mathcal{L}$ is $\rightarrow$ is analogous. First suppose that $\mathcal{L} \neq \omega^*$ and that $\mathcal{L} = \sum_{m\in\omega^*} \mathcal{L}_m$, where $\mathcal{L}_m = \mathrm{lin}(T_{(m)_0})$.

Definition 6.6.12. For $m, n \in \mathbb{N}$ and $\gamma \in \xi$, let

$$P_{m,\gamma,n} = \{x \in P_n : \mathrm{rk}_{\mathcal{P}_n,\mathcal{L}}(x) = \langle m, \gamma\rangle\},$$

where $\mathcal{P}_n$ is as defined in 6.6.5. Note that the definition of $P_{m,\gamma,n}$ depends also on $\mathcal{L}$.

Lemma 6.6.13. *If $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$ and $\mathcal{P}$ is hyperarithmetic, then*

1. *For $\gamma \in \xi \setminus \omega_1^{CK}$, $P_{m,\gamma,n} = \emptyset$.*

2. *$\{P_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$ is a partition of P.*

3. *if $x \leqslant_P x'$, $x \in P_{m,\gamma,n}$ and $x' \in P_{m'\gamma'n'}$ then $\langle m, \gamma, n\rangle \leqslant_{\omega\times\xi\times\omega^*} \langle m', \gamma', n'\rangle$. i.e. $n \geqslant n'$ or $n = n'$ and either $\gamma <_\xi \gamma'$ or $\gamma = \gamma'$ and $m \leqslant m'$.*

4. $\mathbf{1} + \mathcal{L}_m + \mathbf{1} \not\preccurlyeq \mathcal{P}_{m,\gamma,n}$.

Proof: For each part, first apply Lemma 6.6.6 and then Lemma 6.6.11. □

The case $\mathcal{L} = \omega$ or $= \omega^*$ is a little different. Suppose $\mathcal{L} = \omega^*$. First define $\langle P_n \rangle_{n\in\mathbb{N}}$ exactly as in Definition 6.6.5. So, we have that if $\mathbf{1} + \omega^* \not\preccurlyeq \mathcal{P}$, then $\omega^* \not\preccurlyeq \mathcal{P}_n$ for any n. Let $\mathrm{rk}_{\mathcal{P}_n,\omega^*}(x) = \langle x, \mathrm{rk}(\mathcal{P}_{n(\leqslant x)})\rangle \in \omega \times \xi$. (We are using here that $P \subseteq \mathbb{N}$.) Here, $\mathrm{rk}(\mathcal{P}_{n(<x)})$ is the usual rank of the well founded partial ordering $\mathcal{P}_{n(<x)}$. Since $\mathcal{P}_n$ is hyperarithmetic, $\mathrm{rk}(\mathcal{P}_{n(<x)}) \in \omega_1^{CK}$. Let $P_{m,\gamma,n} = \{x \in P_n : \mathrm{rk}_{\mathcal{P}_n,\omega^*}(x) = \langle m, \gamma\rangle\}$. In other words, $P_{m,\gamma,n} = \{m\}$ if $m \in \mathcal{P}_n$ and $\mathrm{rk}(\mathcal{P}_{n(\leqslant x)}) = \gamma$, and $P_{m,\gamma,n} = \emptyset$ otherwise.

Lemma 6.6.14. *If $\mathbf{1} + \omega^* + \mathbf{1} \sim \mathbf{1} + \omega^* \not\preccurlyeq \mathcal{P}$ and $\mathcal{P}$ is hyperarithmetic, then*

1. *For $\gamma \in \xi \smallsetminus \omega_1^{CK}$, $P_{m,\gamma,n} = \emptyset$.*
2. *$\{P_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$ is a partition of P.*
3. *if $x \leqslant_P x'$, $x \in P_{m,\gamma,n}$ and $x' \in P_{m'\gamma'n'}$ then $\langle m,\gamma,n\rangle \leqslant_{\omega\times\xi\times\omega^*} \langle m',\gamma',n'\rangle$.*
4. *Each $P_{m,\gamma,n}$ has at most one element.*

Proof: Parts (1), (2) and (4) follow from the fact that for all x and n, $\mathrm{rk}_{\mathcal{P}_n,\omega^*}(x) \in \omega_1^{CK}$ and that $P_{m,\gamma,n} \subseteq \{m\}$. Part (3) it is also immediate from the definition of the sets $P_{m,\gamma,n}$. □

The idea now, to linearize $\mathcal{P}$, is to keep on partitioning each piece we get in this fashion. First we partition $\mathcal{P}$ into $\{\mathcal{P}_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$. Then, we partition each $\mathcal{P}_{m,\gamma,n}$, which is not a singleton, into $\{\mathcal{P}_{\langle\langle m,\gamma,n\rangle,\langle m',\gamma',n'\rangle\rangle}\}_{\gamma'\in\xi,m',n'\in\mathbb{N}}$ so that $\mathbf{1}+\mathcal{L}_{\langle(m)_0,(m')_0\rangle}+\mathbf{1} \not\preccurlyeq \mathcal{P}_{\langle\langle m,\gamma,n\rangle,\langle m',\gamma',n'\rangle\rangle}$, where $\mathcal{L}_\sigma = \mathrm{lin}(T_\sigma)$. We keep on doing this until we get a partition of $\mathcal{P}$ into singletons. The problem is that, to iterate this process, we need a uniform way of getting $\{\mathcal{P}_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$ from $\mathcal{P}$. Note that the definition we gave of $\mathcal{P}_{m,\gamma,n}$ only makes sense when $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\hookrightarrow \mathcal{P}$. Using the fact that the rank function is Δ^1_1 we get that $\{\mathcal{P}_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$ is Δ^1_1 in $\mathcal{P}$. So, there is a Σ^1_1 formula $\varphi^\Sigma(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x)$ and a Π^1_1 formula $\varphi^\Pi(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x)$ such that if $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{P}$, then for all m,γ, n, and x,

$$\varphi^\Sigma(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x) \Leftrightarrow \varphi^\Pi(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x) \Leftrightarrow x \in P_{m,\gamma,n}.$$

We will use these formulas later to define the iteration process.

6.6.5 The complement of a linear ordering

Now we construct the structure over which we are going to iterate the process of partitioning $\mathcal{P}$.

For each h-indecomposable linear ordering $\mathcal{L} = \mathrm{lin}(T)$ we will define another linear ordering $\mathrm{com}(T)$, and a Π^1_1 subclass of it, $\mathrm{com}^{CK}(T)$, that we call *the complement of* $\mathbf{1}+\mathcal{L}+\mathbf{1}$. The name of "complement" is inspired by the following property. Suppose that T is recursive, then for every recursive linear ordering $\mathcal{A}$ we have that

$$\mathcal{A} \preccurlyeq \overset{CK}{\mathrm{com}}(T) \Leftrightarrow \mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{A}.$$

The implication from left to right will follow from Lemma 6.6.17, and the other direction from the main result of this section, Proposition 6.6.18.

The idea of the definition of com(T) is like the one of the definition of lin(T) (Definition 6.2.6), but instead of taking ω (or ω^*) sums we take $\omega^* \times \xi^* \times \omega$ (or $\omega \times \xi \times \omega^*$) sums. Thus, for example, if $s_T(\emptyset) = +$, then

$$\mathrm{com}(T) = \sum_{\langle m,\gamma,n\rangle \in \omega^* \times \xi^* \times \omega} \mathrm{com}(T_{(m)_0}),$$

and if $s_T(\emptyset) = -$, then

$$\mathrm{com}(T) = \sum_{\langle m,\gamma,n\rangle \in \omega \times \xi \times \omega^*} \mathrm{com}(T_{(m)_0}).$$

Definition 6.6.15. Given a recursive signed tree T, let

$$\mathrm{com}(T) = \{\sigma \in \mathrm{Seq}_{\omega^* \times \xi^* \times \omega} : \sigma \neq \emptyset \;\&\; l(\sigma^-) \text{ is an end node of } T\},$$

where $l(\langle\langle m_0, a_0, n_1\rangle, ..., \langle m_k, a_k, n_k\rangle\rangle) = \langle (m_0)_0, ..., (m_k)_0\rangle$ and $\sigma^- = \sigma \restriction |\sigma| - 1$. Now we define an ordering on com(T). Consider $\sigma_1 \neq \sigma_2 \in \mathrm{com}(T)$. Let $\tau \in \mathrm{Seq}_{\omega^* \times \xi^* \times \omega}$ and $x_1 \neq x_2 \in \omega^* \times \xi^* \times \omega$ be such that $\tau^\frown x_1 \subseteq \sigma_1$ and $\tau^\frown x_2 \subseteq \sigma_2$. We define

$$\sigma_1 \leqslant_{\mathrm{com}(T)} \sigma_2 \Leftrightarrow \begin{cases} x_1 \leqslant_{\omega^* \times \xi^* \times \omega} x_2 & \& \; s_T(l(\tau)) = + \quad \text{or} \\ x_1 \geqslant_{\omega^* \times \xi^* \times \omega} x_2 & \& \; s_T(l(\tau)) = -. \end{cases}$$

Let com$^{CK}(T)$ be the class of all $\sigma \in \mathrm{com}(T)$ such that for all $i < |\sigma|$, $(\sigma(i))_1 \in \omega_1^{CK}$. Let $\widetilde{\mathrm{com}}(T)$ be the downward closure of com(T), i.e.

$$\widetilde{\mathrm{com}}(T) = \{\sigma \in \mathrm{Seq}_{\omega^* \times \xi^* \times \omega} : \exists \tau \supseteq \sigma (\tau \in \mathrm{com}(T))\}.$$

Observe that $\widetilde{\mathrm{com}}(T)$ is a tree and com(T) is the set of end nodes of $\widetilde{\mathrm{com}}(T)$.

Example 6.6.16. Let us look at one of the simplest cases. $T = \{\emptyset\}$ and $s_T(\emptyset) = -$. So lin$(T) = \omega^*$, com$^{CK}(T)$, the complement of $\mathbf{1}+\omega^*+\mathbf{1} \sim \mathbf{1}+\omega^*$, is $\omega \times \omega_1^{CK} \times \omega^* \sim \omega_1^{CK} \times \omega^*$. On the one hand, observe that $\mathbf{1}+\omega^*$ does not embed in $\omega \times \omega_1^{CK} \times \omega^*$. Because otherwise we would have an embedding of ω^* into a popper final segment of $\omega \times \omega_1^{CK} \times \omega^*$, but every proper final segment of it is well ordered, since it is included in a segment of the form $\omega \times \omega_1^{CK} \times \mathbf{n}$. Therefore, if $\mathbf{1}+\omega^* \preccurlyeq \mathcal{A}$, then $\mathcal{A} \not\preccurlyeq \mathrm{com}^{CK}(T)$. On the other hand, consider a recursive linear ordering $\mathcal{A}$ such that $\mathbf{1}+\omega^* \not\preccurlyeq \mathcal{A}$. We can decompose $\mathcal{A}$ into a sum $\sum_{i\in\omega^*} \mathcal{A}_i$ such that each $\mathcal{A}_i$ is recursive and well ordered. Decompose $\mathcal{A}$ in the same way we partitioned $\mathcal{P}$ in Definition 6.6.5, but now we get $\mathcal{A} = \sum_{i\in\omega^*} \mathcal{A}_i$ because $\mathcal{A}$ is linearly ordered.) Then, each $\mathcal{A}_i$ embeds in ω_1^{CK}, so we have an embedding of $\mathcal{A}$ into $\omega \times \omega_1^{CK} \times \omega^*$.

Lemma 6.6.17. *Let T be a recursive signed tree and $\mathcal{L} = \mathrm{lin}(T)$. Then $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathrm{com}^{CK}(T)$.*

PROOF: Suppose g is an embedding of $\mathbf{1}+\mathcal{L}+\mathbf{1}$ into $\mathrm{com}(T)$ such that for all $x \in \mathbf{1}+\mathcal{L}+\mathbf{1}$, $g(x) \in \mathrm{com}^{CK}(T)$. For each n, we define $\sigma_n \in T$ and a recursive embedding $g_n\colon \mathbf{1}+\mathcal{L}_{\sigma_n}+\mathbf{1} \hookrightarrow \mathrm{com}(T_{\sigma_n})$, uniformly in $0''$ (where $\mathcal{L}_\sigma = \mathrm{lin}(T_\sigma)$). We will define the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that for all n, $\sigma_n \subsetneq \sigma_{n+1}$, contradicting the well-foundedness of T. Let $\sigma_0 = \emptyset$ and $g_0 = g$. Suppose we have defined σ_n and $g_n\colon \mathbf{1}+\mathcal{L}_{\sigma_n}+\mathbf{1} \hookrightarrow \mathrm{com}(T_{\sigma_n})$. If for some n we have that $T_{\sigma_n} = \{\emptyset\}$ and $\mathcal{L}_{\sigma_n}$ is ω^*, then we get a contradiction because we have an embedding of $\mathbf{1}+\omega^*$ into $\omega \times \omega_1^{CK} \times \omega^*$. Analogously if for some n we have that $\mathcal{L}_{\sigma_n}$ is ω. To fix ideas assume that $s_T(\sigma_n) = -$. So

$$L_{\sigma_n} = \sum_{m\in\omega^*} \mathcal{L}_{\sigma_n{}^\frown(m)_0} \quad \text{and} \quad \mathrm{com}(T_{\sigma_n}) = \sum_{\langle m,\gamma,n\rangle\in\omega\times\xi\times\omega^*} \mathrm{com}(T_{\sigma_n{}^\frown(m)_0}).$$

Think of $\omega \times \xi \times \omega^*$ and $\omega^* \times \xi^* \times \omega$ as having the same domain but opposite orderings. For each m, let x_m be a member of $\mathcal{L}_{\sigma_n{}^\frown(m)_0}$, the mth term in the first sum above. So $\langle x_m\rangle_{m\in\mathbb{N}}$ is co-initial in $\mathcal{L}$. Let $a_m \in \omega^* \times \xi^* \times \omega$ be the first entry of the sequence $g_n(x_m) \in \mathrm{Seq}_{\omega^*\times\xi^*\times\omega}$. So $g_n(x_m)$ belongs to the a_mth term in the second sum above. Let $b \in \omega^* \times \xi^* \times \omega$ be the first entry of $g_n(x) \in \mathrm{Seq}_{\omega^*\times\xi^*\times\omega}$, where x is the first element of $\mathbf{1}+\mathcal{L}_{\sigma_n}+\mathbf{1}$. Note that

$$a_0 \leqslant_{\omega^*\times\xi^*\times\omega} a_1 \leqslant_{\omega^*\times\xi^*\times\omega} a_2 \leqslant_{\omega^*\times\xi^*\times\omega} \cdots \leqslant_{\omega^*\times\xi^*\times\omega} b.$$

Let $a = \lim_m(a_m)$ (with the discrete topology). The limit has to exist, because otherwise we would have an embedding of $\omega+\mathbf{1}$ into $\omega^* \times \xi^* \times \omega$, or equivalently of $\mathbf{1}+\omega^*$ into $\omega \times \xi \times \omega^*$, contradicting what is said in the example above. Let $\sigma_{n+1} = \sigma_n{}^\frown((a)_0)_0$. Find $\bar{m}$ such that $\forall m \geqslant \bar{m}(a_m = a)$. Then, we have that

$$\sum_{m\in\omega^*, m>\bar{m}} \mathcal{L}_{\sigma_n{}^\frown(m)_0} \preccurlyeq \mathrm{com}(T_{\sigma_{n+1}})$$

Now, pick a copy of $\mathbf{1}+\mathcal{L}_{\sigma_{n+1}}+\mathbf{1}$ inside $\sum_{m\in\omega^*,m>\bar{m}} \mathcal{L}_{\sigma_n{}^\frown(m)_0}$ and construct g_{n-1} as the restriction of g_n to it.

□

6.6.6 The linearization

Now we describe the partition process that we mentioned earlier. Let $\mathcal{L} = \mathrm{lin}(T)$ be a recursive h-indecomposable linear ordering. Consider $\mathcal{P}$, a recursive partial ordering such that $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{P}$. We will define a hyperarithmetic family $\{P_\sigma\}_{\sigma\in\widetilde{\mathrm{com}}(T)}$ of subsets of P indexed by $\widetilde{\mathrm{com}}(T)$, such that

C1. If $\sigma \in \mathrm{com}(T) \smallsetminus \mathrm{com}^{CK}(T)$, then $P_\sigma = \emptyset$.

C2. $\{P_\sigma\}_{\sigma\in\mathrm{com}^{CK}(T)}$ is a partition of P.

C3. If $\sigma, \tau \in \mathrm{com}(T)$, $x \in P_\sigma$, $y \in P_\tau$ and $x \leqslant_P y$, then $\sigma \leqslant_{\mathrm{com}(T)} \tau$.

C4. For $\sigma \in \widetilde{\text{com}}(T) \setminus \text{com}(T)$, $\mathbf{1} + \mathcal{L}_{l(\sigma)} + \mathbf{1} \not\preccurlyeq \mathcal{P}_\sigma$ and $\{P_{\sigma^\frown x}\}_{x \in \omega^* \times \xi^* \times \omega}$ is the partition of P_σ given by Definition 6.6.12 with respect to $L_{l(\sigma)}$.

C5. For $\sigma \in \text{com}(T)$, P_σ is either empty or a singleton.

Then we can construct a map from $\mathcal{P}$ to $\text{com}^{CK}(T)$ which preserves order. Just map $x \in P$ to the $\sigma \in \text{com}(T)$ such that $P_\sigma = \{x\}$. Therefore we have a linearization of $\mathcal{P}$ which, by Lemma 6.6.17, does not embed $\mathbf{1} + \mathcal{L} + \mathbf{1}$. This will prove the following proposition.

Proposition 6.6.18. *Given a recursive h-indecomposable linear ordering $\mathcal{L} = \text{lin}(T)$ and a recursive partial ordering $\mathcal{P}$ such that $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$, there is a hyperarithmetic linearization $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathcal{Q} \preccurlyeq \text{com}^{CK}(T)$, and therefore $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{Q}$.*

Theorem 6.6.1 now follows from the relativized version of the previous proposition and Proposition 6.6.7.

The obvious definition of $\{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T)}$ by recursion using the construction of 6.6.4, would use a too complicated recursion that is not available in ATR_*. The problem is that the definition of the partition of each P_σ only makes sense when we know that $\mathbf{1} + \mathcal{L}_{l(\sigma)} + \mathbf{1} \not\preccurlyeq \mathcal{P}_\sigma$. But to prove that, we have to have already defined P_{σ^-} and proved that $\mathbf{1} + \mathcal{L}_{l(\sigma^-)} + \mathbf{1} \not\preccurlyeq \mathcal{P}_\sigma$.

The main tool to construct this partition of $\mathcal{P}$ is the following lemma.

Lemma 6.6.19. *(ATR_*) Let $\psi^\Sigma(X, x)$ be a Σ^1_1 formula and $\psi^\Pi(X, x)$ and $\chi(X)$ be Π^1_1 formulas. Suppose that we know that for every set X,*

$$\begin{aligned} \chi(X) \Rightarrow \quad & \forall y(\psi^\Sigma(X, y) \Leftrightarrow \psi^\Pi(X, y)) \ \& \\ & \forall Y(Y = \{y : \psi^\Sigma(X, y)\} \Rightarrow \chi(Y)). \end{aligned} \tag{X}$$

Let X_0 be a given set such that $\chi(X)$. Then, there exists a sequence $\langle R_n : n \in \mathbb{N}\rangle$ such that

1. *$R_0 = X_0$,*
2. *for every n, $R_{n+1} = \{y : \psi^\Sigma(R_n, y)\} = \{y : \psi^\Pi(R_n, y)\}$, and*
3. *for every n, $\chi(R_n)$.*

First we show how this implies Proposition 6.6.18.

Proof of Proposition 6.6.18: We have to construct a family $\{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T)}$ of subsets of P indexed by $\widetilde{\text{com}}(T)$ satisfying conditions (C1)-(C5). We have already seen how this implies the proposition. We apply the Lemma above to construct a sequence $\langle R_n : n \in \mathbb{N}\rangle$ such that $R_n = \{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T), |\sigma| = n}$. All we need to do is to define ψ^Σ, ψ^Π, χ and X_0. Let $X_0 = \{P\}$. Let Γ be either Σ or Π. We let $\psi^\Gamma(X, x)$ be the formula that says the following: X is of the form $\{Q_\sigma : \sigma \in \widetilde{\text{com}}(T), |\sigma| = n\}$

for some n, x is of the form $\langle \tau, y\rangle$ for some $\tau \in \widetilde{\mathrm{com}}(T)$ with $|\tau| = n+1$ and $y \in P$, and

$$\varphi^{\Gamma}(Q_{\tau^-}, \mathcal{L}_{l(\tau^-)}, \tau(n), y).$$

(The formulas $\varphi^{\Gamma}(Q, \mathcal{L}, x, y)$ were defined at the end of Subsection 6.6.4.) Let $\chi(X)$ be the formula that says that X is hyperarithmetic, it is of the form $\{Q_\sigma : \sigma \in \widetilde{\mathrm{com}}(T), |\sigma| = n\}$ for some n, and for each $\tau \in \widetilde{\mathrm{com}}(T)$ with $|\tau| = n$, $\mathbf{1}+\mathcal{L}_{l(\tau)}+\mathbf{1} \not\preccurlyeq Q_\tau$.

Note that ψ^{Σ} is Σ^1_1 and ψ^{Π} and χ are Π^1_1. Condition (X) follows from Lemmas 6.6.13 and 6.6.14 and the comments on φ^{Σ} and φ^{Π} at the end of subsection 6.6.4. $\square$

Proof of Lemma 6.6.19: Let Γ be either Σ or Π and $\bar{\Gamma}$ be the other one. We say that a sequence $\bar{R} = \langle R_i : i < n\rangle$, with $n \leqslant \omega + 1$, is *acceptable*$^{\Gamma}$ if $R_0 = X_0$, and for all $i < n - 1$

$$\forall y(y \in R_{i+1} \Rightarrow \psi^{\Gamma}(R_i, y)) \text{ and } \forall y(\psi^{\bar{\Gamma}}(R_i, y) \Rightarrow y \in R_{i+1}).$$

We say that $\bar{R}$ *satisfies* χ if $\forall i < n(\chi(R_i))$. We make three observations.

The first observation is that if $\bar{R}$ is acceptable$^{\Pi}$ and satisfies χ, then it is also acceptable$^{\Sigma}$: For each i, since $\chi(R_i)$, $\forall y(\psi^{\Sigma}(R_i, y) \Leftrightarrow \psi^{\Pi}(R_i, y))$, and therefore $R_{i+1} = \{y : \psi^{\Sigma}(R_i, y)\} = \{y : \psi^{\Pi}(R_i, y)\}$.

The second observation is that if $\bar{R}$ is acceptable$^{\Pi}$ and satisfies χ, $\bar{Q}$ is either acceptable$^{\Sigma}$ or acceptable$^{\Pi}$ and $|\bar{R}| = |\bar{Q}|$, then $\bar{R} = \bar{Q}$: Use arithmetic induction. If $R_i = Q_i$, since $\chi(R_i)$ we have that

$$Q_{i+1} = \{y : \psi^{\Pi}(Q_i, y)\} = \{y : \psi^{\Pi}(R_i, y)\} = R_{i+1}.$$

These two observations imply that if there is an $\bar{R}$ which is acceptable$^{\Pi}$ and satisfies χ, then it is the unique acceptable$^{\Pi}$ sequence and also the unique acceptable$^{\Sigma}$ sequence.

The last observation is that for every n there exists a $\bar{R}$ of length n which is hyperarithmetic in R_0, acceptable$^{\Pi}$ and satisfies χ. We prove this using Σ^1_1-IND. By Lemma 6.5.11, the formula we are proving by induction is equivalent to a Π^1_1 one. For the induction basis consider $\langle R_0\rangle$. For the induction step assume we have $\bar{R}$ of length $n \geqslant 1$ which is hyperarithmetic in R_0, acceptable$^{\Pi}$ and satisfies χ. Since $\chi(R_{n-1})$, because of condition (X) we can define

$$R_n = \{y : \psi^{\Sigma}(R_{n-1}, y)\} = \{y : \psi^{\Pi}(R_{n-1}, y)\},$$

by Δ^1_1-CA (which holds in ATR_*; [Sim99, Lemma VII.4.1]). Since R_{n-1} is hyperarithmetic, R_n is too. Now, $\bar{R}^\frown R_n$ has length $n + 1$, is hyperarithmetic in R_0, is acceptable$^{\Pi}$ and satisfies χ.

Now we want to define $\bar{R}$ of length ω, acceptable$^{\Pi}$, acceptable$^{\Sigma}$ and satisfying χ. We define it by Δ^1_1-CA as follows: We let $\langle n, x\rangle \in \bar{R}$ if and only if there exists a sequence $\langle Q_0, ..., Q_n\rangle$, hyperarithmetic in R_0 and acceptable$^{\Pi}$, such that $x \in Q_n$, which is equivalent to a Π^1_1 formula by Lemma 6.5.11. Equivalently, $\langle n, x\rangle \in \bar{R}$ if and only if there exists a sequence $\langle Q_0, ..., Q_n\rangle$, acceptable$^\Sigma$ such that $x \in Q_n$, which is a Σ^1_1 formula. It follows from the observations above that these two definitions are equivalent and that $\bar{R}$ is as required. $\square$

6.6.7 Extendibility of η

The proof theoretic strength of the fact that η^* is extendible was studied by Downey, Hirschfeldt, Lempp and Solomon in [DHLS03]. They showed the extendibility of η in Π^1_2-CA_0 and give a modification of their proof, due to Howard Becker, that uses only Π^1_1-CA_0. Becker's modification is based in the observation that if $\eta \not\preccurlyeq \mathcal{P}$ and $\mathcal{P}$ is recursive, then $\mathcal{P}$ has a hyperarithmetic linearization which does not embed η. This observations allowed him to use Lemma 6.5.11 to reduce the complexity of certain formulas used in the proof. We prove now that the extendibility of η is provable in ATR_*. Notice that ATR_* is strictly weaker than Π^1_1-CA_0. (It is weaker because Π^1_1-CA_0 implies ATR_0 and Σ^1_1-IND. It is strictly weaker because every β-model is a model of ATR_* but there is a β-model which is not a model of Π^1_1-CA_0. See [Sim99, Chapters VI and VII].) Joseph Miller [Mil] proved that the extendibility of η implies WKL_0 and that over Σ^1_1-AC_0, it implies ATR_0. Whether the extendibility of η is equivalent to ATR_0 over RCA_0 is still an open question.

Theorem 6.6.20. *(ATR_*) η is extendible.*

PROOF: Take a partial ordering $\mathcal{P}$ such that $\eta \not\preccurlyeq \mathcal{P}$. Consider the class of all the recursive trees T such that, if $s_T\colon T \to \{+,-\}$ is the constant function equal to $+$, then $\mathrm{lin}(T) = \mathrm{lin}(\langle T, s_T\rangle) \preccurlyeq \mathcal{P}$. (Note that the definition of $\mathrm{lin}(T)$ did not require T to be well founded.) Only consider the trees T that also satisfy that for every $\sigma \in T$, σ has an extension which is an end node of T. This is a Σ^1_1 class of trees, and therefore different from the class of well founded recursive trees (see [Sim99, Theorem V.1.9]). We claim that there is no tree T in this class with $\mathrm{lin}(T) \preccurlyeq \mathcal{P}$ which is not well founded. Suppose, toward a contradiction that $\mathrm{lin}(T) \preccurlyeq \mathcal{P}$ and $\langle a_i\rangle_{i\in\mathbb{N}}$ is a path through T. We will show that then, there is an embedding of η into $\mathcal{P}$. Consider the left-to-right ordering, $\leqslant_{LR}$, on Seq_2 which has order type η. Given $\sigma \in \mathrm{Seq}_2$, define $\bar{\sigma} \in \mathrm{Seq}_3$ of length $|\sigma|+1$ by letting, for $i < |\sigma|$, $\bar{\sigma}(i) = 0$ if $\sigma(i) = 0$ and $\bar{\sigma}(i) = 2$ if $\sigma(i) = 1$ and let $\bar{\sigma}(|\sigma|) = 1$. Now define $f(\sigma)$ to be a string in $\mathrm{lin}(T) \subseteq \mathrm{Seq}$ extending

$$\langle\langle a_0, \bar{\sigma}(0)\rangle, \langle a_1, \bar{\sigma}(1)\rangle, ..., \langle a_{|\sigma|}, \bar{\sigma}(|\sigma|)\rangle \in \widehat{T},$$

which exist by our assumption on T. Note that if $\sigma <_{LR} \tau$, then $f(\sigma) <_{\mathrm{lin}(T)} f(\tau)$. So, we have that $\eta \preccurlyeq \mathrm{lin}(T) \preccurlyeq \mathcal{P}$, contradicting our assumptions.

Hence, there has to be some well founded T such that $\mathrm{lin}(T) \not\preccurlyeq \mathcal{P}$. By Theorem 6.6.1, $\mathrm{lin}(T)$ is extendible, and therefore, there is a linearization of $\mathcal{P}$ which does not embed $\mathrm{lin}(T)$. But then, this linearization cannot embed η either. $\square$

6.7 Another equivalent statement

After Laver proved Fraïssé's conjecture, he used his results to prove some partition results about linear orderings. The most interesting partition result he proved, that we call LAV, is the following.

Theorem 6.7.1. *[Lav73] For every countable linear ordering $\mathcal{L}$ there exists a natural number $n_{\mathcal{L}}$ such that, if C is a finite set (of colors) and $f\colon \mathcal{L} \to C$ is any function (i.e., a coloring of $\mathcal{L}$), then there exists a set $F \subseteq C$ of size at most $n_{\mathcal{L}}$ such that $f^{-1}[F]$ is equimorphic to $\mathcal{L}$.*

We think that it is very likely that LAV is equivalent to FRA over RCA_0. But we have only proved one of the implications.

Lemma 6.7.2. *LAV implies FRA over RCA_0.*

Proof: Suppose, toward a contradiction, that FRA does not hold. Then, by Lemma 6.4.3, we get that there exists an inifinite sequence of linear orderings $\mathcal{L}_0 \succ \mathcal{L}_1 \succ \mathcal{L}_2 \succ ...$, all of them indecomposable to the right. Let

$$\mathcal{L} = \sum_{k\in\omega} \mathcal{L}_k.$$

Let $n = n_{\mathcal{L}}$ be as in the statement of LAV. Let $C = \{0, ..., n\}$. Consider the following coloring $f\colon \mathcal{L} \to \mathcal{C}$. Given a color $i \in C$ and $x \in \mathcal{L}_k \subset \mathcal{L}$, let $f(x) = i$ if and only if k can be written as a multiple of $n+1$ plus i, or, in other words, if k is congruent to i modulo $n+1$. Now, let $F \subset C$ be given by LAV. That is, $|F| \leqslant n$ and there is an embedding

$$g\colon \sum_{k\in\omega} \mathcal{L}_k \;\to \sum_{j\in\omega, \mathrm{rem}(j,n+1)\in F} \mathcal{L}_j,$$

where $\mathrm{rem}(j, n+1)$ is the remainder of the division of j by $n+1$. We claim that this is impossible. For each k, let x_k be an element of $\mathcal{L}_k$ (say, the $\leqslant_{\mathbb{N}}$-least), and let $j_k \in \{j \in \mathbb{N} : \mathrm{rem}(j, n+1) \in F\}$ be such that $g(x_k) \in \mathcal{L}_{j_k}$. Using the fact that for every k, $\mathcal{L}_k + \mathbf{1} \not\preccurlyeq \mathcal{L}_k$, it is not hard to prove by induction on k that, for every $k \in \mathbb{N}$, $j_k \geqslant k$. Let p be an element of $C \setminus F$. Since $p = \mathrm{rem}(p, n+1) \notin F$, we have that $j_p > p$ and hence $\mathcal{L}_p \preccurlyeq \sum_{k\in\omega,k>p} \mathcal{L}_i$. Then, by Corollary 6.2.11, there exists $j > p$ such that $\mathcal{L}_p \preccurlyeq \mathcal{L}_j$, contradicting the choice of the sequence $\{\mathcal{L}_0, \mathcal{L}_1, ...\}$. $\square$

For the reverse implication, we observe that the proof of LAV in [Lav73] does not trivially go through in RCA_0+FRA. For instance, it is not known whether the following theorem [Lav73, Theorem 1.3] follows from FRA.

Theorem 6.7.3. *(Kruskal [Kru60]) If $\mathcal{Q}$ is a well-quasiordering, then $(\mathcal{FT})_{\mathcal{Q}}$ is well-quasiordered under $\leqslant_1$ and $\leqslant_m$.*

Here, $(\mathcal{FT})_{\mathcal{Q}}$ is the set of finite trees with labels in $\mathcal{Q}$. So, the elements of $(\mathcal{FT})_{\mathcal{Q}}$ are pairs $\langle T, l_T\rangle$, that we denote just by T, where T is an index for a finite subtree of $\mathrm{Seq}_{\mathbb{N}}$, and $l_T\colon T \to \mathcal{Q}$. Given $T, S \in (\mathcal{FT})_{\mathcal{Q}}$ with labeling functions $l_T\colon T \to \mathcal{Q}$ and $l_S\colon S \to \mathcal{Q}$, we let $T \leqslant_1 S$ if there exists a map $g\colon T \to S$ such that for every $\sigma, \tau \in T$,

1. if $\sigma \subsetneq \tau$ then $g(\sigma) \subsetneq g(\tau)$,
2. $l_T(\sigma) \leqslant_{\mathcal{Q}} l_S(g(\sigma))$, and
3. $g(\sigma \cap \tau) = g(\sigma) \cap g(\tau)$, where $\sigma \cap \tau$ is the greatest lower bound of σ and τ.

We let $T \leqslant_m S$ if there exists a map $g\colon T \to S$, satisfying the first two conditions above, but not necessarily the third one. So, for instance, to have $T \leqslant_m S$, g does not need to be one-to-one.

The fact that $\langle (\mathcal{FT})_{\mathcal{Q}}, \leqslant_1\rangle$ is well-quasiordered when $\mathcal{Q}$ is, is known as *Kruskal's theorem*, and it is one of the earliest results in wqo-theory. As we mentioned before, Friedman proved that it cannot be proved in ATR_0 (see [Sim85]), and Rathjen and Weiermann [RW93] found its exact proof theoretic strength. The relation between Kruskal's theorem and FRA is unknown.

But Kruskal's theorem is not used in the proof of LAV. What is used is that $\langle (\mathcal{FT})_{\mathcal{Q}}, \leqslant_m\rangle$ is well-quasiordered, which is an immediate corollary of Kruskal's theorem because $T \leqslant_1 S$ implies $T \leqslant_m S$. It might be the case that this weaker version of Kruskal's theorem does follow from ATR_0. Actually all we would need to prove is that the instances of the weaker Kruskal's theorem used in the proof of LAV follow from RCA_0+FRA.

Chapter 7
Indecomposable linear orderings and Theories of Hyperarithmetic Analysis.

7.1 Introduction

This paper is part of an ongoing project of analyzing the subsystems of second order arithmetic. This program is called Reverse Mathematics, and its main theme is the following: Given a theorem of ordinary mathematics, determine the weakest natural subsystem of second order arithmetic in which the theorem is provable. (The basic reference on Reverse Mathematics is Simpson's book [Sim99].) Surprisingly, it often happens that this question has a precise answer, and moreover, it is usually the case that the answer is one of five specific systems. This systems are RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0, listed in increasing order of proof-theoretic strength. (See [Sim99, p. 32]. We will describe the systems we will use in Subsection 7.1.6 below.) The system RCA_0, of Recursive Comprehension, is usually used as a base system; when we say that for some particular theorems the question above has a specific answer, we mean that, if RCA_0 is assumed, it can be proved that the theorem is equivalent to one of those five systems. RCA_0 resembles Computable Mathematics in the sense that, when working in RCA_0, all the sets we can assume exists are the ones that are computable from the ones we already know exist. It can be proved that the ω-models of RCA_0 are exactly the ones whose second order part is closed under Turing reduction and disjoint union, where the *disjoint union of two sets* $X, Y \subseteq \omega$ is the set $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$. The models of second order arithmetic whose first order part is the standard one $(\omega, 0, 1, +, \times)$, are called ω-models. We will identify these models with their second order parts. The system of Arithmetic Comprehension, ACA_0, has a similar behavior, but with respect to arithmetic reducibility. The ω-models of ACA_0 are exactly the ones whose second order part is closed under arithmetic reduction and disjoint union. As are the classes of recursive sets and of arithmetic sets, the class of hyperarithmetic sets is a very natural one and enjoys many closure properties. This is the class that will concern us in this paper. For more information on hyperarithmetic reductions, see Subsection 7.1.5 below.

We say that an ω-model is *hyperarithmetically closed* if it is closed under disjoint union and for every set $X, Y \subseteq \omega$, if X is hyperarithmetically reducible to Y and Y is in the model, then X is in the model too.

Definition 7.1.1. A system of axioms of second order arithmetic T is a *Theory of hyperarithmetic analysis* if

- it holds in $HYP(\mathrm{Y})$ for every $Y \subseteq \omega$, where $HYP(\mathrm{Y})$ is the ω-model consisting of the sets hyperarithmetic in Y ; and
- all its ω-models are hyperarithmetically closed.

Note that this is equivalent to say that every for every set $Y \subseteq \omega$, $HYP(\mathrm{Y})$ is the minimum ω-model of T which contains Y, and that every ω-model of T is closed under disjoint unions.

In [Ste78, Section 5], Steel defines "theories of hyperarithmetic analysis" as the ones which have $HYP{=}HYP(\emptyset)$ as their minimum ω-model. People were interested in these theories because they characterize the class HYP. Our definition is a relativized version of the previous one, and it characterizes not only HYP, but also the relation of hyperarithmetic reduction: When T is a theory of hyperarithmetic analysis, a set X is hyperarithmetically reducible to a set Y if and only if every ω-model of T which contains Y, also contains X.

The bad news is that there is no theory whose ω-models are exactly the ones that are hyperarithmetically closed. This follows from a more general result of Van Wesep [Van77, 2.2.2]: For every theory T whose ω-models are all hyperarithmetically closed, there is another theory T' whose models are all also hyperarithmetically closed and which has more ω-models than T does. So, there might not be a natural theory of hyperarithmetic analysis. Indeed, there are many. Examples of known theories of hyperarithmetic analysis are the following schemes: Σ^1_1-dependent choice (Σ^1_1-DC_0), Σ^1_1-choice (Σ^1_1-AC_0), Δ^1_1-comprehension (Δ^1_1-CA_0), and weak-Σ^1_1-choice (weak-Σ^1_1-AC_0). The unrelativezed versions of these results were proved by Harrison [Har68], Kreisel [Kre62], [Kle59] and [Sim99, Theorem VIII.4.16]. (See Subsection 7.1.6 below for definitions of these statements.) As listed, these statements go from strongest to weakest, they all imply ACA_0, and, except for Σ^1_1-DC_0, they are implied by ATR_0 (see [Sim99, VIII.3 and VIII.4]). Moreover, the implications Σ^1_1-$\mathsf{DC}_0$$\Rightarrow$$\Sigma^1_1$-$\mathsf{AC}_0$, Σ^1_1-$\mathsf{AC}_0$$\Rightarrow$$\Delta^1_1$-$\mathsf{CA}_0$, and Δ^1_1-$\mathsf{CA}_0$$\Rightarrow$weak-$\Sigma^1_1$-$\mathsf{AC}_0$ can not be reversed as proved by Friedman [Fri67], Steel [Ste78] and van Wesep [Van77], respectively.

We say that a sentence S is a *sentence of hyperarithmetic analysis* if $\mathsf{RCA}_0{+}\mathsf{S}$ is a theory of hyperarithmetic analysis. In [Fri75, Section II], Friedman mentions two sentences related to hyperarithmetic analysis. These sentences, ABW (arithmetic Bolzano-Weierstrass) and SL (sequential limit systems), use the concept of arithmetic set of reals, which is not used outside logic. Another previously known sentence of hyperarithmetic analysis is Game-AC studied by Van Wesep [Van77]. He studied it in a more general context than second order arithmetic. But if we restrict it to second order arithmetic, it essentially says that if we have a sequence of open games such that player $\mathbb{I}$ has a winning strategy in each of them, then there exists a sequence of strategies for all of them. He proved that, when restricted to second order arithmetic, Game-AC is equivalent to Σ^1_1-AC_0. (An open game is a game like the ones we describe in Subsection 7.1.2 below, with the difference that it might last for ω many steps, and if it does, player $\mathbb{I}$ is the winner.)

We will introduce five new statements of hyperarithmetic analysis: $\mathsf{G1}$, $\mathsf{G2}$, $\mathsf{G3}$, $\mathsf{G4}$ and JI. The first four statements are related to finitely terminating games. These are perfect-information games between two players, where at each turn a player might have infinitely many possible moves but every run of the game ends in finitely many steps. All these four statements are easily stated, and obviously true. So obviously true that they would not even be considered theorems, although they are proof-theoretically strong. The same happens with the statement Game-AC. The reason is that all these statements about games have the form either of comprehension axioms or of choice axioms. The last statement JI is the weakest statement of hyperarithmetic analysis we study and has to do with the iteration of the Turing jump along ordinals. We use Steel's method of forcing with tagged trees to prove that it is strictly weaker than the other statements. This is the same method used by Steel [Ste78] and by Van Wesep [Van77] to prove that the implications Σ^1_1-$\mathsf{AC}_0$$\Rightarrow$$\Delta^1_1$-$\mathsf{CA}_0$, and Δ^1_1-$\mathsf{CA}_0$$\Rightarrow$weak-$\Sigma^1_1$-$\mathsf{AC}_0$ cannot be reversed.

However, to the author's knowledge, no previously published mathematical theorem, which does not mention concepts from logic, has been proved a statement of hyperarithmetic analysis. In this paper we present an example of such a theorem. This theorem, that we call INDEC, was first proved by Pierre Jullien in his Ph.D. thesis [Jul69, Theorem IV.3.3]. INDEC is published in English in, for example, [Fra00, 6.3.4(3)] and [Ros82, Lemma 10.3]. Not only we prove that INDEC is a statement of hyperarithmetic analysis, but also that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and implies ACA_0. Note that since HYP is the minimum ω-model of INDEC, neither ACA_0, not ${\mathsf{ACA}_0}^+$ can imply it.

Another interesting fact about INDEC is that is incomparable over ACA_0 to other natural statements of mathematics. This is probably the first example of previously published purely mathematical statements which are incomparable and are between ACA_0 and ATR_0. The statements we have in mind are the following: The existence of elementary equivalence invariants for Boolean Algebras, and Ramsey Theorem. The former statement was studied by Shore [Sho04]. He first analyzed how to work with the statement in second order arithmetic and then proved that it is equivalent to ${\mathsf{ACA}_0}^+$ over RCA_0. (${\mathsf{ACA}_0}^+$ is equivalent to ACA_0 plus the sentence $\forall X(X^{(\omega}$ exists), where $X^{(\omega)}$ is the ωth Turing jump of X.) The latter statement, Ramsey's Theorem, has been extensively studied in the context of reverse mathematics (see [Sim99, III.7], [CJS01], or [Mil04, Chapter 7]). It is known that it slightly stronger than ACA_0. The reason why these statements are incomparable with INDEC is the following one. Barwise and Schlipf [BS75] proved that Σ^1_1-AC_0 (and hence also Δ^1_1-CA_0 and INDEC) is conservative over ACA_0 for Π^1_2 formulas. In other words, any Π^1_2 formula sentence which is provable in Σ^1_1-AC_0 is already provable in ACA_0. Then, since ${\mathsf{ACA}_0}^+$ can be axiomatized by a Π^1_2 sentece over ACA_0, and is strictly stonger than ACA_0, it is not implied by Σ^1_1-AC_0, and hence it is not implied by RCA_0+INDEC either. The same argument is true about Ramsey's theorem.

We now formally introduce all these statements of hyperarithmetic analysis.

7.1.1 Indecomposability Statement

We start with the most natural of all these statements. As we said in the introduction, INDEC is due to Jullien [Jul69].

Definition 7.1.2. Given a linear ordering $\mathcal{A} = \langle A, \leqslant \rangle$, a *cut in* $\mathcal{A}$ is a pair of sets $\langle L, R \rangle$ such that $L = A \smallsetminus R$ is an initial segment of $\mathcal{A}$. We say that $\mathcal{A}$ is *indecomposable* if for every cut $\langle L, R \rangle$, $\mathcal{A}$ embeds either into L or into R. (Here we are thinking of L and R as sub-orderings of $\mathcal{A}$.) We say that $\mathcal{A}$ is *indecomposable to the right* if for every cut $\langle L, R \rangle$ with $R \neq \emptyset$, we have that $\mathcal{A}$ embeds in R. Analogously we define *indecomposable to the left*. A linear ordering is *scattered* if η, the order type of the rational numbers, does not embed in it.

Statement 7.1.3. We let INDEC be the statement

> Every scattered indecomposable linear ordering is either
> indecomposable to the right or indecomposable to the left.

Indecomposable linear orderings are very useful when studying properties of linear orderings. Every scattered linear ordering can be written as a finite sum of indecomposable linear orderings, so they are in some sense the building blocks for the class of scattered linear orderings. Countable indecomposable linear orderings can be written as ω- or ω^*-sums of smaller indecomposable linear orderings. These facts are due to Laver [Lav71]; see also [Ros82, Chapter 10]. In the same paper, Laver proved Fraïssé's conjecture which says that there is no infinite descending sequence or infinite antichain in the quasi-ordering formed by the countable linear orderings ordered by embeddablity. These structure theorems together allow us to prove properties about linear orderings by transfinite induction. For example, Jullien [Jul69, Chapter V] used these structure theorems for scattered linear orderings to classify the countable extendible linear orderings. (A linear ordering is *extendible* if every countable partial ordering which does not embed it has a linearization which does not embed it either.) We used them to prove that every hyperarithmetic linear ordering is equimorphic to a recursive one in Chapter 9, and to analyze the proof theoretic strength of Jullien's theorem and Fraïssé's conjecture in Chapter 6.

We prove in section 7.2 that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and that it implies ACA_0. The former proof is not very complicated. The latter one is more interesting and has some ideas that will be used in Section 7.3 to prove that INDEC is a theory of hyperarithmetic analysis. To prove that the ω-models of INDEC are hyperarithmetically closed, we start by considering an ω-model $\mathcal{M}$ of INDEC. Of course, we think of $\mathcal{M}$ as set of subsets of ω. Then, we prove that for every computable increasing sequence of ordinals $\{\alpha_n\}_{n\in\omega}$, converging to an computable ordinal α, we have that if $(\forall n)0^{(\alpha_n)} \in \mathcal{M}$, then $0^{(\alpha)} \in \mathcal{M}$. To prove this we use Ash and Knight's machinery to construct a specific linear ordering such that when we apply INDEC to it, we can deduce that $0^{(\alpha)} \in \mathcal{M}$. Then we relativize and use effective transfinite induction to prove that for every set $X \in \mathcal{M}$ and every X-computable ordinal $\alpha \in \mathcal{M}$, $X^{(\alpha)} \in \mathcal{M}$. When we refer to Ash and Knight's machinery we refer the results that Ash and Knight derived from Ash's $0^{(\alpha)}$-priority arguments (see [AK00]).

7.1.2 Game statements

Before introducing the game statements, let us quickly review our notation for trees. We write $\mathbb{N}^{<\omega}$ for the set of finite strings of natural numbers, and $2^{<\omega}$ for the set of finite strings of zeros and ones, ordered by inclusion. A *tree* is a downward closed subset of $\mathbb{N}^{<\omega}$ and a *binary tree* is a downward subset of $2^{<\omega}$. Given a tree T and $\sigma \in T$, we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$, where $\sigma^\frown\tau$ is the string obtained by concatenating σ and τ. We use $\emptyset$ for the empty string. Given a string σ, we let $|\sigma|$ be its length, $\sigma \restriction n$ be the initial substring of σ of length n, and $\sigma^- = \sigma \restriction (|\sigma| - 1)$.

Definition 7.1.4. To each well founded tree T, we associate a game $G(T)$ which is played as follows. Player I starts by playing a number $a_0 \in \mathbb{N}$ such that $\langle a_0 \rangle \in T$. Then player II plays $a_1 \in \mathbb{N}$ such that $\langle a_0, a_1 \rangle \in T$, and then player I plays $a_2 \in \mathbb{N}$ such that $\langle a_0, a_1, a_2 \rangle \in T$. They continue like this until they get stuck. The first one who cannot play *looses*. Equivalently, the first one that reaches an end node of T *wins*. We call the sequence $\langle a_0, ..., a_k \rangle$ obtained at the end of the game, a *run* of the game, and any sequence obtained any time along the game, a *partial run*. Note that since T is well founded, the game cannot last forever. We call the games which are of the form $G(T)$ *finitely terminating games*.

Remark 7.1.5. Finitely terminating games are in one to one correspondence with clopen games. A *clopen game* is played over the full tree $\mathbb{N}^{<\omega}$ and runs of the game go for ω many steps. At the end of time, the players are left with an infinite sequence $X \in \mathbb{N}^\omega$, and player I *wins* if that sequence belongs to a previously chosen clopen set $\mathcal{A} \subset \mathbb{N}^\omega$. Otherwise II wins. This defines the game $G(\mathcal{A})$. A *clopen* set is a set which is closed and open. Every clopen set $\mathcal{A}$ is determined by a well founded tree $T_\mathcal{A}$ and a subset A of the set of end nodes of T. It is determined in the sense that $X \in \mathcal{A}$ if and only if the initial segment of X which is an end node of T belongs to A. It is not hard to see that for every clopen game $G(\mathcal{A})$, using $T_\mathcal{A}$ and A, one can construct a well-founded tree T such that $G(\mathcal{A})$ and $G(T)$ are in some sense equivalent. Also, given a well-founded tree T, it is not hard to construct a clopen set $\mathcal{A}$ that will induce an equivalent game.

Definition 7.1.6. Let $T_{\mathrm{I}} = \{\sigma \in T : |\sigma| \text{ is even}\}$ and $T_{\mathrm{II}} = \{\sigma \in T : |\sigma| \text{ is odd}\}$. So, T_{I} is the set of partial runs σ of $G(T)$ such that if σ has been played so far in a game, then it is I's turn to play. Similarly with T_{II}. Let P be either I or II. A *strategy for P* in a tree game $G(T)$ is a function $s \colon T_P \to \mathbb{N}$. We say that a partial run $\sigma \in T$ *follows a strategy s* if for every $\tau \subset \sigma$, $\tau \in T_P \Rightarrow \sigma(|\tau|) = s(\tau)$. A strategy s for P is a *winning strategy* if for every run σ of T which follows s, $\sigma \notin T_P$. In other words, s is a winning strategy for P if whenever P plays following s, he is ensured to win despite what the other player plays. A game $G(T)$ is *determined* if there is a winning strategy for one of the two players. We say that a game is *completely determined* if there is a map $d \colon T \to \{\mathtt{W}, \mathtt{L}\}$ such that for every $\sigma \in T$, if $d(s) = \mathtt{W}$, then I has a winning strategy in the game T_σ, and if $d(s) = \mathtt{L}$, then II has a winning strategy in the game $G(T_\sigma)$. We call such a d, a *winning function* for

$G(T)$. We cal a tree T *determined (completely determined)* if $G(T)$ is determined (completely determined).

It is clear that I and II cannot both have winning strategies, so winning functions, if exists, have to be unique. (This can be proved in RCA_0.) On the other hand, winning strategies do not need to be unique. This is because the value of a strategy at a node which does not follow it is not relevant at all.

Theorem 7.1.7. *The following are equivalent over RCA_0.*

1. *ATR_0;*
2. *Every finitely terminating game is determined;*
3. *Every finitely terminating game is completely determined.*

PROOF: The equivalence between (1) and (2) is proved in Steel's thesis [Ste76]. (See [Sim99, Theorem V.8.7]). The fact that (1) implies (3) follows from the uniformity in the proof of (1)⇒(3). It is clear that (3) implies (2). □

Now, we introduce four statements about finitely terminating games that we will later prove are statements of hyperarithmetic analysis. But, first, we need the following definition.

Definition 7.1.8. Given a sequence $\{T_n : n \in \mathbb{N}\}$, we let $\sum_n T_n$ be the tree S such that for each $n \in \mathbb{N}$, $S_{\langle n\rangle} = T_n$. So, we can think of the game $G(\sum_n T_n)$ as a game in which player I starts by choosing a game from $\{G(T_n) : n \in \mathbb{N}\}$, and then players I and II play it starting with player II. Whoever wins the chosen game, *wins* $G(\sum_n T_n)$.

Given a game G, let G^* be the game that is played exactly as G but players I and II are interchanged. So, for instance, if $G = G(T)$, we can assume that $G^* = G(T^*)$, where $T^*_n = \{0^\frown\sigma : \sigma \in T\}$.

Statement 7.1.9.

- $\mathsf{G1}$: Given a sequence $\{T_n : n \in \mathbb{N}\}$ of completely determined trees, there exists a set X such that $n \in X$ iff I has a winning strategy for $G(T_n)$.
- $\mathsf{G2}$: Given a sequence $\{T_n : n \in \mathbb{N}\}$ of completely determined trees, $\sum_n T_n$ is also completely determined.
- $\mathsf{G3}$: Given a sequence $\{T_n : n \in \mathbb{N}\}$ of determined trees, there exists a set X such that $n \in X$ iff I has a winning strategy for T_n.
- $\mathsf{G4}$: Given a sequence $\{T_n : n \in \mathbb{N}\}$ of determined trees, $\sum_n T_n$ is also determined.

Remark 7.1.10. The statement $\mathsf{G4}$ is equivalent to a clopen-game version of the statement Game-AC mentioned in the introduction.

7.1.3 The Jump Iteration statement

This statement will be useful when trying to prove that a sentence is one of hyperarithmetic analysis.

Statement 7.1.11. The *Jump Iteration* statement, JI, is the following:

> For every set X and every ordinal α,
> if, for every $\beta < \alpha$, $X^{(\beta)}$ exists, then $X^{(\alpha)}$ exists.

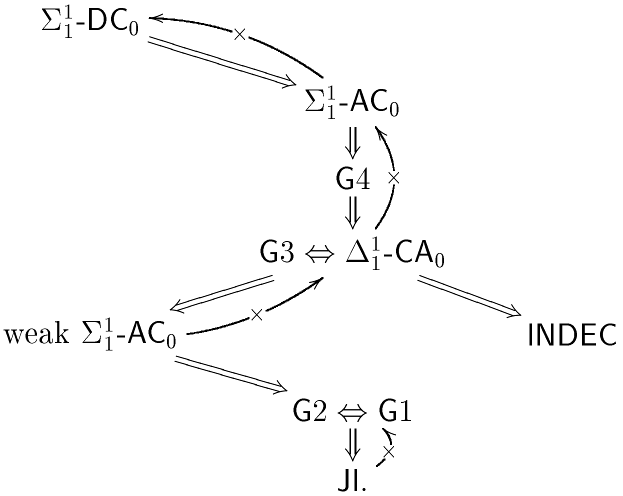
Let us prove that the ω-models of $\mathsf{RCA}_0+\mathsf{JI}$ are hyperarithmetically closed. Consider a ω-model $\mathcal{M}$ of $\mathsf{RCA}_0+\mathsf{JI}$. If it were not hyperarithmetically closed, there would be a set $X \in \mathcal{M}$ and a least X-computable ordinal α such that $X^{(\alpha)} \notin \mathcal{M}$. But this contradicts JI. So JI is a sentence of hyperarithmetic analysis.

A similar argument is used to prove that the ω-models of INDEC are hyperarithmetically closed in Section 7.3. We believe that by formalizing the ideas in that Section one can prove that $\mathsf{INDEC}\Rightarrow \mathsf{JI}$.

7.1.4 Summary or results

The following theorem contains all the implications we know how to prove between the different statements of hyperarithmetic analysis that we study in this paper.

Theorem 7.1.12. *All the theories and statements mentioned in the diagram below are ones of hyperarithmetic analysis. The implications and the non-implications in the diagram hold over RCA_0. Moreover, all the non-implications are witnessed by ω-models.*



Since all the statements in the diagram follow from Σ^1_1-DC_0, for every $Y \subseteq \omega$, $HYP(Y)$ is a model of them [Sim99, Theorem VIII.4.16]. We will prove that the ω-models of INDEC are hyperarithmetically closed in Section 7.3. Since all the other statements imply JI, every ω-model of each of them is hyperarithmetically closed.

The part of the diagram which only mentions Σ^1_1-DC_0, Σ^1_1-AC_0, Δ^1_1-CA_0 and weak-Σ^1_1-AC_0, was already mentioned in the introduction. That Δ^1_1-CA_0 implies INDEC will be proved in Section 7.2. All the other implications are proved in Section 7.4. The fact that JI does not imply $\mathsf{G1}$ is proved in Section 7.5.

Many arrows are missing from the diagram. For instance, we do not know whether $\mathsf{G4}$ is strictly in between Σ^1_1-AC_0 and Δ^1_1-CA_0, or is equivalent to one of them. We would also like to know more about how INDEC relates to the other statements. We conjecture that it implies $\mathsf{G1}$, but we have not even proved that it implies JI. Another interesting question is whether $\mathsf{G1}$ and $\mathsf{G2}$ are equivalent to weak-Σ^1_1-AC_0.

7.1.5 Hyperarithmetic Theory

Standard references for Hyperarithmecity Theory are [AK00] and [Sac90a].

Let $\mathcal{L} = \langle L, \leqslant_L \rangle$ be a presentation of a linear ordering (i.e., $\mathcal{L}$ is a linear ordering whose domain L is a subset of ω) which has a least element 0. Given $X, Y \subseteq \omega$, we say that Y is an *$H(X,\mathcal{L})$-set* if $Y^{[0]} = X$ and for every $l \in L \smallsetminus \{0\}$

$$Y^{[l]} = \bigoplus_{k <_L l} (Y^{[k]})'$$

where $Y^{[j]} = \{n : \langle j, n \rangle \in Y\}$ and $\bigoplus_{k\in A} B_k = \{\langle k, n\rangle : k \in A, n \in B_k\}$. When $\mathcal{L}$ is an ordinal it is not hard to prove by transfinite induction that there exists a unique $H(X,\mathcal{L})$-set. We denote that set by $X^{(\mathcal{L})}$. But if we consider another isomorphic presentation of $\mathcal{L}$, even the Turing degree of $X^{(\mathcal{L})}$ may change. Although, there are some cases when we know it does not change. Given an countable ordinal α and a set X, we say that α is an X-computable ordinal if there is a presentation of α recursive in X. When α is an X-computable ordinal, all the $H(X,\mathcal{L})$-sets, where $\mathcal{L}$ is an X-computable presentations of α, are Turing equivalent; this result is due to Spector [Spe55]. The least non-X-computable ordinal is denoted by ω_1^X. Note that the set of X-computable ordinals is closed downward. We use ω_1^{CK} to denote $\omega_1^{\emptyset}$, where CK stands for Church-Kleene.

Theorem 7.1.13. *[Kle55, Ash86] Given sets $X, Y \subseteq \omega$, the following are equivalent*

1. $(\exists \alpha < \omega_1^Y) X \leqslant_T Y^{(\alpha)}$, *where $\leqslant_T$ means "is computable in".*
2. $X \in \Delta^1_1(Y)$, *that is, there exists Σ^1_1 formulas ψ and φ such that $(\forall n) n \in X \Leftrightarrow \psi(n, Y) \Leftrightarrow \neg\varphi(n, Y)$.*
3. *There is a Y-computable infinitary formula φ such that $X = \{n : \varphi(n)\}$.*

If $\omega_1^Y = \omega_1^{CK}$ we also have:

4. there is a computable infinitary formula φ such that $X = \{n : \varphi(n, Y)\}$.

(A computable infinitary formula is a formula where infinite disjunctions and infinite conjunctions are allowed, so long as they are taken over computably enumerable sets of computable infinitary formulas. See [AK00, Chapter 7] for more information on these formulas.)

Definition 7.1.14. When sets $X, Y \subseteq \omega$ satisfy any of the first three condition in the theorem above we say that X *is hyperarithmetically reducible to* Y and write $X \leqslant_H Y$. We let $HYP(Y) = \{X \subseteq \omega : X \leqslant_H Y\}$ and $HYP = HYP(\emptyset)$.

7.1.6 Subsystems of second order arithmetic

We only review the subsystems of second order arithmetic that we will be using. We refer the reader to [Sim99] for more information. We use the same notation as in [Sim99], and, for instance, we use capital letters for set variables and lower case letters for number variables.

As our basic system we will use RCA_0. It consists of the axioms for semi-rings plus Δ^0_1-comprehension and Σ^0_1-induction. ACA_0 consists of RCA_0 plus the axiom scheme of arithmetic comprehension. Δ^1_1-CA_0 also includes Δ^1_1-comprehension. It is known that Δ^1_1-CA_0 is a theory of hyperarithmetic analysis.

Σ^1_1-dependent choice is the following scheme, where φ ranges over the Σ^1_1 formulas:

$$\forall Y \exists Z(\varphi(Y, Z)) \Rightarrow \exists X \forall n(\varphi(X^{[n]}, X^{[n+1]})).$$

Σ^1_1-choice is the following scheme, where φ ranges over the Σ^1_1 formulas:

$$\forall n \exists X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})).$$

The scheme weak-Σ^1_1-choice is the following, where φ ranges over the arithmetic formulas:

$$\forall n \exists! X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})),$$

where $\exists!$ stands for “there exists a unique”. Together with RCA_0 they form the systems Σ^1_1-DC_0, Σ^1_1-AC_0 and weak-Σ^1_1-AC_0 respectively.

7.1.7 Linear orderings

We use $\mathbb{N}$ for the set of all the natural numbers and ω for the linear ordering $\omega = \langle \mathbb{N}, \leqslant_{\mathbb{N}} \rangle$. (We also use ω, at the meta-level, as the standard first order model of arithmetic.)

A linear ordering $\mathcal{A} = \langle A, \leqslant_{\mathcal{A}} \rangle$ is said to be *recursive* if both A and $\leqslant_{\mathcal{A}}$ are recursive. Some recursive linear orderings we will be dealing with are: $\mathbf{1}$, the linear ordering with one element; $\mathbf{m}$, the linear ordering with m elements; ω, the ordering of the natural numbers; ζ, the ordering of the integers; η, the ordering of the rationals; ω^n, the ordering of the n-tuples of natural numbers ordered lexicographically; and ω^{n*}, the reverse linear ordering of ω^n.

Now we define some recursive operations on linear orderings. Let $\mathcal{A} = \langle A, \leqslant_{\mathcal{A}} \rangle$ be a linear ordering. Given $a \in A$, define $\mathcal{A}_{(<a)}$ to be the restriction of $\mathcal{A}$ to $\{x \in A : x <_{\mathcal{A}} a\}$. Analogously define $\mathcal{A}_{(\leqslant a)}$, $\mathcal{A}_{(>a)}$, and $\mathcal{A}_{(\geqslant a)}$. Given $a, b \in A$, let $[a, b]_{\mathcal{A}}$ be the restriction of $\mathcal{A}$ to $\{x \in A : a \leqslant_{\mathcal{A}} x \leqslant_{\mathcal{A}} v\}$. Let $(a, b)_{\mathcal{A}}$ be the restriction of $\mathcal{A}$ to $\{x \in A : a <_{\mathcal{A}} x <_{\mathcal{A}} v\}$. The *reverse* linear ordering of $\mathcal{A} = \langle A, \leqslant_{\mathcal{A}} \rangle$ is $\mathcal{A}^* = \langle A, \geqslant_{\mathcal{A}} \rangle$. Let $\mathcal{B} = \langle B, \leqslant_{\mathcal{B}} \rangle$ be another linear ordering. The *product*, $\mathcal{A} \cdot \mathcal{B}$, of two linear orderings $\mathcal{A}$ and $\mathcal{B}$ is obtained by substituting a copy of $\mathcal{A}$ for each element of $\mathcal{B}$. That is: $\mathcal{A} \cdot \mathcal{B} = \langle A \times B, \leqslant_{\mathcal{A} \cdot \mathcal{B}} \rangle$ where $\langle x, y \rangle \leqslant_{\mathcal{A} \cdot \mathcal{B}} \langle x', y' \rangle$ iff $y <_{\mathcal{B}} y'$ or $y = y'$ and $x \leqslant_{\mathcal{A}} x'$. The *sum*, $\sum_{i \in \mathcal{A}} \mathcal{B}_i$, of a set of linear orderings $\{\mathcal{B}_i\}_{i \in A}$ indexed by another linear ordering $\mathcal{A}$, is constructed by substituting a copy of $\mathcal{B}_i$ for each element $i \in A$. So, for example, $\mathcal{A} \cdot \mathcal{B} = \sum_{i \in \mathcal{B}} \mathcal{A}$. When $\mathcal{A} = \mathbf{m}$, we sometimes write $\mathcal{B}_0 + ... + \mathcal{B}_{m-1}$ or $\sum_{i=0}^{m-1} \mathcal{B}_i$ instead of $\sum_{i \in \mathbf{m}} \mathcal{B}_i$. Let $\mathcal{A}^{\mathcal{B}}$ be the linear ordering whose domain consist of finite stings $\sigma = \langle \langle a_0, b_0 \rangle, ..., \langle a_k, b_k \rangle \rangle \in (\mathcal{A} \times \mathcal{B})^{<\omega}$ such that $b_0 >_{\mathcal{B}} b_1 >_{\mathcal{B}} ... >_{\mathcal{B}} b_k$. Given $\sigma, \tau \in \mathcal{A}^{\mathcal{B}}$, let $\sigma \leqslant_{\mathcal{A}^{\mathcal{B}}} \tau$ if either $\sigma \subseteq \tau$ or for the least i such that $\sigma(i) \neq \tau(i)$ we have $\sigma(i) \leqslant_{\mathcal{A} \cdot \mathcal{B}} \tau(i)$. Observe that $\mathcal{A}^{(\mathcal{B}+\mathcal{C})} \cong \mathcal{A}^B \cdot \mathcal{A}^C$. Other recursive linear orderings that we will use are $\omega^\omega = \sum_{n \in \omega} \omega^n$ and $\omega^{\omega *} = (\omega^\omega)^*$.

If $\mathcal{A}$ can be embedded in $\mathcal{B}$, we write $\mathcal{A} \preccurlyeq \mathcal{B}$. $\mathcal{A}$ is *scattered* if $\eta \not\preccurlyeq \mathcal{A}$.

Lemma 7.1.15. *(RCA_0) If $\mathcal{Z}$ is a scattered linear ordering and $\{\mathcal{B}_z : z \in \mathcal{Z}\}$ is a family of scattered linear orderings, then $\sum_{z \in \mathcal{Z}} \mathcal{B}_z$ is also scattered. In particular, the product of scattered linear orderings is scattered.*

Proof: Suppose that f is an embedding $\eta \hookrightarrow \sum_{z \in \mathcal{Z}} \mathcal{B}_z$. If for every $q \in \eta$, $f(q)$ belongs to a different summand $\mathcal{B}_{z_q}$, then the map $q \mapsto z_q$ would be an embedding of η into $\mathcal{Z}$. So, there has to be a pair $p, q \in \eta$ and a $z \in \mathcal{Z}$, such that both $f(p)$ and $f(q)$ are in $\mathcal{B}_z$. But then $\eta \preccurlyeq [p, q]_\eta \preccurlyeq \mathcal{B}_z$. □

Trees will be an important tool in this paper. We started introducing basic notation for trees at the beginning of Subsection 7.1.2. We can linearly order the nodes of a tree in various ways. One is the *Kleene-Brouwer* ordering of $\mathbb{N}^{<\omega}$ defined as follows:

$$\sigma \leqslant_{KB} \tau \Leftrightarrow \sigma \supseteq \tau \vee \exists i(\sigma(i) \leqslant \tau(i) \;\&\; \forall j < i(\sigma(j) = \tau(j))).$$

Given a tree $T \subseteq \mathbb{N}^{<\omega}$, we let $KB(T)$ be the Kleene-Brouwer ordering restricted to T. ACA_0 can prove that if a tree T is well-founded, then $KB(T)$ is well-well-ordered [Sim99, Lemma V.1.3] (see [Hir94] for the reversal). Even though RCA_0 cannot prove this, it can prove the following.

Lemma 7.1.16. *(RCA_0) If T is well founded, then $KB(T)$ is scattered.*

Proof: Suppose that f is an embedding of η into $KB(T)$. By recursion, we construct two sequences $\langle p_n : n \in \mathbb{N} \rangle$, and $\langle q_n : n \in \mathbb{N} \rangle$ of elements of η such that for each n, $p_n \leqslant_\eta p_{n+1} <_\eta q_{n+1} \leqslant_\eta q_n$, $|f(p_n)| \geqslant n$ and $f(p_n) \upharpoonright n = f(q_n) \upharpoonright n$. Just define p_{n+1} and q_{n+1} as the least pair (in some enumeration of η^2) which satisfies the conditions above. Such a pair has to exit because f restricted to $(p_n, q_n)_\eta \cong \eta$ is a map into $KB(T_{f(p_n)\restriction n}) \cong \sum_{m\in\omega} KB(T_{(f(p_n)\restriction n)^\frown m}) + \mathbf{1}$. Finally, $\cup_{n\in\mathbb{N}} f(p_n)\restriction n$ is a path though T contradicting its well-foundedness. $\square$

On $2^{<\omega}$ we also have the *Left-to-right ordering*, $\leqslant_{LR}$. It coincides with the Kleene-Brouwer on incompatible stings, but when $\sigma \subset \tau$ we let $\sigma \leqslant_{LR} \tau$ if $\tau(|\sigma|) = 1$ and $\sigma \geqslant_{LR} \tau$ if $\tau(|\sigma|) = 0$. Observe that $\langle 2^{<\omega}, \leqslant_{LR}\rangle$ has order type η.

Lemma 7.1.17. *(RCA_0) If $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$, then $\eta \preccurlyeq \mathcal{A}$.*

PROOF: Assume $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. Observe that then $\mathcal{A} + \mathbf{1} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. So, there exist two embeddings $f_0, f_1 \colon \mathcal{A} \hookrightarrow \mathcal{A}$ and an $a \in A$ such that $\forall x, y \in A(f_0(x) <_A a <_A f_1(y))$. Now, given $\sigma \in 2^{<\omega}$ define

$$f(\sigma) = f_{\sigma(0)}(f_{\sigma(1)}(...(f_{\sigma(|\sigma|-1)}(a))...)).$$

f is an embedding of $\langle 2^{<\omega}, \leqslant_{LR}\rangle \cong \eta$ into $\mathcal{A}$. $\square$

7.2 Between ACA_0 and Δ^1_1-CA_0

In this section we prove that INDEC implies ACA_0 and is implied by Δ^1_1-CA_0 over RCA_0. From the latter of these implications we get that $HYP(Y)$ is a model of INDEC for every $Y \subseteq \omega$. That ω-models of INDEC are closed under hyperarithmetic reduction will be proved in the next section.

Definition 7.2.1. We say that a linear ordering $\mathcal{A}$ is *weakly indecomposable* if for every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$.

Note that an indecomposable linear ordering is weakly indecomposable. Also note that, by Lemma 7.1.17, if $\mathcal{A}$ is scattered then for no $a \in \mathcal{A}$ do we have both $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$.

Theorem 7.2.2. *The following are equivalent over RCA_0, and they are both implied by Δ^1_1-CA_0.*

1. *INDEC*
2. *If $\mathcal{A}$ is a scattered, weakly indecomposable linear ordering, then there exists a cut $\langle L, R\rangle$ of A such that*

$$L = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}\} \text{ and } R = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}\} \tag{7.2.1}$$

We call the cut $\langle L, R\rangle$ satisfying (7.2.1), the *middle cut* of $\mathcal{A}$.

Proof: We first prove that (1) and (2) are equivalent.

To prove (2) from INDEC, consider a scattered, weakly indecomposable linear ordering $\mathcal{A}$. If $\mathcal{A}$ is indecomposable, then, by INDEC, it is either indecomposable to the right or to the left. In the former case we would have that $\langle L, R\rangle = \langle A, \emptyset\rangle$ satisfies (7.2.1) and in the latter case $\langle L, R\rangle = \langle \emptyset, A\rangle$ satisfies (7.2.1). In both cases a cut $\langle L, R\rangle$ as in (7.2.1) exists. Suppose now that $\mathcal{A}$ is not indecomposable and let $\langle L, R\rangle$ be a cut such that neither $\mathcal{A} \preccurlyeq L$ nor $\mathcal{A} \preccurlyeq R$. We claim that $\langle L, R\rangle$ is as in (7.2.1). We have that for every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. If $a \in L$, then, since $\mathcal{A} \not\preccurlyeq L$, $\mathcal{A} \not\preccurlyeq \mathcal{A}_{(\leqslant a)}$, and hence $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. On the other hand, if $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$, then a cannot be in R, because we would have that $\mathcal{A} \preccurlyeq R$. Therefore $L = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}\}$. Analogously $R = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}\}$.

Let us now prove that (2) implies INDEC. Let $\mathcal{A}$ be a scattered indecomposable linear ordering. By (2), a cut $\langle L, R\rangle$ of $\mathcal{A}$ as in (7.2.1) exists. Since $\mathcal{A}$ is indecomposable, either $\mathcal{A} \preccurlyeq L$ or $\mathcal{A} \preccurlyeq R$. Without loss of generality, assume that $\mathcal{A} \preccurlyeq R$. If $L = \emptyset$ and $R = A$, then $\mathcal{A}$ is indecomposable to the left. Suppose, then, that $L \neq \emptyset$. Then, $1 + \mathcal{A} \preccurlyeq 1 + R \preccurlyeq \mathcal{A} \preccurlyeq R$. So, there exists an $a \in R$ such that $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. But then $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$, and by Lemma 7.1.17, $\eta \preccurlyeq \mathcal{A}$, contradicting the hypothesis on $\mathcal{A}$.

Finally, we prove that Δ^1_1-$\mathsf{CA}$$\Rightarrow$(2). Let $\mathcal{A}$ be a scattered, weakly indecomposable linear ordering. For every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$, and since $\mathcal{A}$ is scattered, it cannot be that both $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. So we have that $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)} \Leftrightarrow \mathcal{A} \not\preccurlyeq \mathcal{A}_{(>a)}$. Since $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$ are Σ^1_1 formulas, Δ^1_1-CA_0 implies that L and R as in (7.2.1) exist. $\square$

Now we turn to proving that INDEC implies ACA_0 over RCA_0. Some ideas from the proof will be used in the next section when we prove that every ω-model of INDEC is hyperarithmetically closed. The idea of the proof of ACA_0 is to construct a recursive copy $\mathcal{C}$ of $\omega^\omega + \omega^{\omega*}$ such that its middle cut computes $0'$. We also need $\mathcal{C}$ to be recursively weakly indecomposable. We say that $\mathcal{C}$ is *recursively weakly indecomposable* if for every $c \in \mathcal{C}$, there is a recursive embedding of $\mathcal{C}$ into either $\mathcal{C}_{(\leqslant c)}$ or $\mathcal{C}_{(>c)}$.

Lemma 7.2.3. *For every $n \in \mathbb{N}$, ω^n is* recursively indecomposable to the right. *That is for every $c \in \omega^n$, there is a recursive embedding of ω^n into $\omega^n_{(>c)}$. Moreover, an index for the embedding can be found uniformly in c. Furthermore, RCA_0 proves that for every n, ω^n is indecomposable to the right.*

Proof: The proof is not hard. Just consider embeddings of the form
$\langle x_0, ..., x_{n-1}\rangle \mapsto \langle x_0, ..., x_{n-1} + k\rangle$. $\square$

Theorem 7.2.4. *(RCA_0) INDEC implies ACA_0.*

Proof: We will prove that INDEC implies that $K = 0'$ exists. Then, by relativizing the proof as usual, we can get that for every set X, X' exists, and hence ACA_0 holds.

We start by constructing a linear ordering $\mathcal{Z}$ such that

- For every $s \in \mathcal{Z}$ there exists $n_s \in \mathbb{N}$ such that either $\mathcal{Z}_{(<s)}$ or $\mathcal{Z}_{(>s)}$ has n_s many elements. In the former case we say that s is on the *left side*. Otherwise s is on the *right side*.

- If the set $R_{\mathcal{Z}} = \{s \in \mathcal{Z} : s \text{ is on the right side of } \mathcal{Z}\}$ exists, it computes $0'$.

Let $\{k_0, k_1, ...\}$ be a recursive enumeration of K. For each s let $K_s = \{k_0, ..., k_s\}$ and $\sigma_s = K_s \restriction k_s + 1$. Consider the following ordering of $\mathbb{N}$.

$$s <_{B} t \Leftrightarrow \sigma_s <_{KB} \sigma_t,$$

where $<_{KB}$ is the Kleene-Brouwer ordering of $2^{<\omega}$. Let $\mathcal{Z} = \langle \mathbb{N}, \leqslant_{B} \rangle$. For each s we have that either $\forall t > s(k_t > k_s)$ (in other words, s is a *true stage*), or there exists a $t > s$ such that $k_t < k_s$. In the latter case we have that $\forall t' \geqslant t(s <_{B} t')$, and hence s is on the left side of $\mathcal{Z}$, and $n_s = |\{t' < t : t' <_{B} s\}|$. In the former case we have that s is on the right side and $n_s = |\{t' < s : s <_{B} t'\}|$. Observe that ACA_0 can prove that $\mathcal{Z}$ is isomorphic $\omega + \omega^*$ but RCA_0 cannot, since $R_{\mathcal{Z}} = \{s : s \text{ is on the right side}\}$ is the set of true stages of the enumeration of K, and hence from $R_{\mathcal{Z}}$ we can compute K.

One idea to prove that K exists would be to use 7.2.2(2) to show that the middle cut of $\mathcal{Z}$, $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}} \rangle$, has to exist. But RCA_0 cannot prove that $\mathcal{Z}$ is weakly indecomposable. (Because if s is the greatest element of $\mathcal{Z}$ and if there exists an embedding $f \colon \mathcal{Z} \to \mathcal{Z}_{(<x)}$, then we would have that $R_{\mathcal{Z}}$ is Σ^0_1: $t \in R_{\mathcal{Z}} \Leftrightarrow \exists n(f^n(s) <_{B} t)$. But we already know that $R_{\mathcal{Z}}$ is Π^0_1, so by Δ^0_1-CA it would exists, which we cannot prove in RCA_0.) So we need to consider a more complicated linear ordering.

We construct a uniformly recursive sequence of linear orderings $\{\mathcal{P}_s\}_{s \in \mathbb{N}}$ such that

$$\mathcal{P}_s \cong \begin{cases} \omega^{n_s} & \text{if } s \text{ is on the left side} \\ \omega^{n_s *} & \text{if } s \text{ is on the right side.} \end{cases},$$

To construct $\mathcal{P}_s$ recursively, uniformly in s, start by assuming that s is a true stage and enumerating ω^{n*} where $n = |\{t' < s : s <_{B} t'\}|$. If at any stage $t > s$ we discover that s is not a true stage (that is, we discover that $k_t < k_s$), we change our mind and we start constructing ω^{n_s} instead. (Note that by stage t we have enumerated only finitely many elements of ω^n.) Now define

$$\mathcal{C} = \sum_{s \in \mathcal{Z}} \mathcal{P}_s.$$

Observe that $\mathcal{C}$ is isomorphic to

$$(1 + \omega + \omega^2 + ...) + (... + \omega^{2*} + \omega^* + 1) \cong \omega^\omega + \omega^{\omega *}$$

but again, we need ACA_0 to prove it. Furthermore, if the middle cut $\langle L_{\mathcal{C}}, R_{\mathcal{C}} \rangle$ existed, where $R_{\mathcal{C}} = \{y \in \mathcal{P}_s : s \text{ on the right side}\}$, then K would exist too.

Each $\mathcal{P}_s$ is scattered; this can be proven from the fact that either $\mathcal{P}_s$ or $\mathcal{P}_s^*$ is well ordered which is provable in RCA_0. So, by Lemma 7.1.15, $\mathcal{C}$ is scattered. Now we prove that $\mathcal{C}$ is weakly indecomposable. Consider $y \in \mathcal{C}$. First suppose that $y \in \mathcal{P}_s$ and s on the right side of $\mathcal{Z}$. So s is the n_sth true stage. We will construct an embedding f of $\mathcal{C}$ into $\mathcal{C}_{(<y)}$. Let $\tau = \langle s_1, ..., s_{n_s} \rangle$ be the tuple consisting of the first n_s true stages, where $s = s_{n_s}$. Using τ as a parameter we will construct the desired embedding recursively. Let $\mathcal{D}$ be the sum of the $\mathcal{P}_s$, for s not in τ. Note that there are recursive isomorphisms $\mathcal{C} \cong \mathcal{D} + \sum_{i=n_s}^{1} \omega^{i*} \cong \mathcal{D} + \omega^{n_s*}$. (Assume, to simplify notation, that $\mathcal{C} = \mathcal{D} + \omega^{n_s*}$.) By Lemma 7.2.3 there is an embedding of ω^{n_s*} into $\omega^{n_s*}_{(>y)}$. Using this embedding we can construct an embedding $\mathcal{C} \to \mathcal{C}_{(<y)}$ by leaving the elements of $\mathcal{D}$ fixed. Now suppose that $s \in L_{\mathcal{Z}}$ and hence it is not a true stage. RCA_0 can prove the existence of a sequence $\tau = \langle s_1, ..., s_{n_s} \rangle$ with $s = s_{n_s}$, such that for all t' not in the sequence we have $s_1 <_B s_2 <_B ... <_B s_{n_s} <_B t'$. (We just have to find some stage t such that $\forall t' \geqslant t(s <_B t')$ and then analyze $<_B$ restricted to $\{t' : t' < t\}$.) As we did in the previous case we can find an embedding of $\mathcal{C}$ into $\mathcal{C}_{(>y)}$.

Since $\mathcal{C}$ is scattered and weakly indecomposable, by 7.2.2(2), the middle cut $\langle L_{\mathcal{C}}, R_{\mathcal{C}} \rangle$ exists. Therefore, $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}} \rangle$ and $0'$ exist too. $\square$

7.3 Models of INDEC

In this section we prove that INDEC is a theory of hyperarithmetic analysis. We already know that $HYP(Y) \models \mathsf{INDEC}$ for every $Y \subseteq \omega$. This is because we know that Δ^1_1-CA_0 implies INDEC, and that for every Y, $HYP(Y) \models \Delta^1_1$-$\mathsf{CA}_0$.

Theorem 7.3.1. *The ω-models of INDEC are closed under hyperarithmetic reducibility.*

Let $\mathcal{M}$ be an ω-model of INDEC. To prove that $\mathcal{M}$ is closed under hyperarithmetic reducibility we have to prove that for every $X \in \mathcal{M}$ and any X-recursive ordinal α, $X^{(\alpha)} \in \mathcal{M}$. We will actually prove that for every $\alpha < \omega_1^{CK}$, $0^{(\alpha)} \in \mathcal{M}$. Then, a relativization of the proof will give the desired result. Since INDEC implies ACA_0, we have that, if for some recursive α, $0^{(\alpha)} \in \mathcal{M}$, then $0^{(\alpha+1)} \in \mathcal{M}$ too. We will prove that if $\{\alpha_n\}_{n\in\mathbb{N}}$ is a recursive increasing sequence of recursive ordinals with limit α and $0^{(\alpha_n)} \in \mathcal{M}$ for every n, then $0^{(\alpha)} \in \mathcal{M}$ too. This implies, using transfinite induction, that for every $\alpha < \omega_1^{CK}$, $0^{(\alpha)} \in \mathcal{M}$. Fix such a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$.

We will construct a recursive scattered linear ordering $\mathcal{Y}$ (of the form $\zeta^{\alpha\cdot\omega} \cdot \mathcal{Z}$ for some other recursive linear ordering $\mathcal{Z}$), and a recursive linear ordering $\mathcal{C}$ such that

C1. $\mathcal{C} \cong \mathcal{Y} \cdot (\omega^\omega + \omega^{\omega*})$;

C2. The cut $\langle L_{\mathcal{C}}, R_{\mathcal{C}} \rangle$ of order type $\langle \mathcal{Y} \cdot \omega^\omega, \mathcal{Y} \cdot \omega^{\omega*} \rangle$ has Turing degree $0^{(\alpha)}$;

C3. For each $n \in \mathbb{N}$ there exist an $m_n \in \mathbb{N}$, a recursive linear ordering $\mathcal{D}_n$ and an isomorphism $f_n \leqslant_T 0^{(\alpha_{m_n})}$,

$$f_n \colon \mathcal{C} \to \mathcal{Y} \cdot \omega^n + \mathcal{D}_n + \mathcal{Y} \cdot \omega^{n*}.$$

Let us first see what can we do once we have constructed such a linear ordering $\mathcal{C}$. First, we note that $\mathcal{C}$ is weakly indecomposable inside $\mathcal{M}$: Consider $a \in \mathcal{C}$ and, without loss of generality, suppose that $a \in L_{\mathcal{C}}$. Then, for some n, a belongs to the initial segment of $\mathcal{C}$ of order type $\mathcal{Y} \cdot \omega^n$. By (C3), this initial segment is isomorphic to the canonical recursive presentation of $\mathcal{Y} \cdot \omega^n$ via an isomorphism which is recursive in $0^{(\alpha_{m_n})}$, and hence is inside $\mathcal{M}$. Since ω^n is recursively indecomposable to the right, we can use this isomorphism to construct an embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{(>a)}$ that is inside $\mathcal{M}$. Since $\mathcal{C}$ is the product of scattered linear orderings, it is scattered, and hence it is scattered inside $\mathcal{M}$. Now, since $\mathcal{M} \models \mathsf{INDEC}$, the middle cut of $\mathcal{C}$, which is $\langle L_{\mathcal{C}}, R_{\mathcal{C}} \rangle$, belongs to $\mathcal{M}$, and therefore $0^{(\alpha)} \in \mathcal{M}$.

7.3.1 The construction

In this subsection we will construct $\mathcal{C}$ and prove it is as desired. We will use Lemma 7.3.5 below, which we will not prove until the next subsection.

We start by constructing a linear ordering $\mathcal{Z}$ which has a cut, $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}} \rangle$, of Turing degree $0^{(\alpha)}$. Then, we will construct $\mathcal{C}$ as a $\mathcal{Z}$-linear ordering (see Definition 7.3.3 below). Essentially, $\mathcal{C}$ is a recursive $\mathcal{Z}$-linear orderings if it can be written as a recursive sum of the form $\mathcal{C} = \sum_{x \in \mathcal{Z}} P_x(\mathcal{C})$, where $P_x(\mathcal{C})$ are uniformly recursive linear orderings. Another notion that we introduce in this section is the notion of a T-sequence. We will use T-sequences to organize the construction.

Lemma 7.3.2. *There exist a recursive linear ordering $\mathcal{Z}$ and a cut $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}} \rangle$ in it such that:*

Z1. $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}} \rangle \equiv_T 0^{(\alpha)}$;

Z2. *$\mathcal{Z}$ is scattered;*

Z3. *There are recursive function $\psi, \varphi \colon \mathcal{Z} \to \mathbb{N}$ such for every $x \in \mathcal{Z}$,* $x \in L^{\mathcal{Z}} \Leftrightarrow \psi(x) \in 0^{(\alpha_{\varphi(x)})}$.

Proof: Let $\langle S_n : n \in \mathbb{N} \rangle$ be a recursive sequence of trees such that S_n has a unique path which is Turing equivalent to $0^{(\alpha_n)}$ uniformly in n. The fact that such a sequence exists is known. The reader can find a proof in Shore [Sho93, Theorem 2.3] that each such tree S_n exists, then, observing that Shore's proof is uniform, we get our sequence of S_n's. Consider $S = \{\sigma \in \mathbb{N}^{<\omega} : \forall \langle i,j \rangle < |\sigma| (\langle \sigma(i,0), ..., \sigma(i,j) \rangle \in S_n)\}$. Clearly S has a unique path, Y, which is Turing equivalent to $0^{(\alpha)}$. Let $\mathcal{Z} = KB(S)$. Let $L_{\mathcal{Z}} = \{x \in S : x \leqslant_{KB(S)} Y\}$ and $R_{\mathcal{Z}} = \mathcal{Z} \smallsetminus L_{\mathcal{Z}}$. Clearly $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}} \rangle \equiv_T 0^{(\alpha)}$. Given $\sigma \in S$, let $\varphi(\sigma) = \max\{i : \langle i,j \rangle < |\sigma|\} + 1$. Then,

$0^{(\alpha_{\varphi(\sigma)}-1)}$ can compute a string $\tau = Y \upharpoonright |\sigma|$, and then $\sigma \in L_{\mathcal{Z}} \Leftrightarrow \sigma <_{KB(S)} \tau$. Let $\psi(\sigma)$ be such that $\sigma \in L_{\mathcal{Z}} \Leftrightarrow \psi(x) \in 0^{(\alpha_{\varphi(x)})}$.

Note that $\mathcal{Z}$ is scattered: Otherwise we could find two incomparable strings σ_1 and $\sigma_2 \in S$ such that η embeds in both $KB(S_{\sigma_1})$ and $KB(S_{\sigma_2})$. But then, neither S_{σ_1} nor S_{σ_2} would be well-founded, and S would have at least two paths. $\square$

Definition 7.3.3. Given a linear ordering $\mathcal{Z}$, a *$\mathcal{Z}$-linear ordering* is a first order structure $\langle \mathcal{B}, \{P_x : x \in \mathcal{Z}\}\rangle$, where $\mathcal{B}$ is a linear ordering and the P_x are unary relation such that

- $\forall a \in \mathcal{B} \exists! x \in \mathcal{Z}(P_x(a))$,
- $\forall x \in \mathcal{Z} \exists a \in \mathcal{B}(P_x(a))$, and
- $\forall x, y \in \mathcal{Z} \forall a, b \in \mathcal{B}(P_x(a) \ \& \ P_y(b) \ \& \ x \leqslant_{\mathcal{Z}} y \Rightarrow a \leqslant_{\mathcal{B}} b)$.

We can think of a $\mathcal{Z}$-linear ordering as a linear ordering $\mathcal{B}$, together with an order-preserving, onto map $p_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{Z}$ (defined by $p_{\mathcal{B}}(b) = x \Leftrightarrow P_x(b)$). We write $P_x(\mathcal{B})$ for the sub-ordering of $\mathcal{B}$ with domain $\{b \in \mathcal{B} : P_x(b)\}$. Note that $\mathcal{B} = \sum_{x \in \mathcal{Z}} P_x(\mathcal{B})$.

If $\mathcal{B}$ is a $\mathcal{Z}$-linear ordering, and $\mathcal{X}$ is any linear ordering, we let $\mathcal{X} \cdot \mathcal{B}$ be the $\mathcal{Z}$-linear ordering which has $\mathcal{X} \cdot \mathcal{B}$ as its underlying linear ordering, and for each $z \in \mathcal{Z}$, $x \in \mathcal{X}$ and $a \in \mathcal{B}$, we let $\langle x, a\rangle \in P_x(\mathcal{X} \cdot \mathcal{B}) \Leftrightarrow a \in P_x(\mathcal{B})$.

If $a \in \mathcal{Z}$, $\mathcal{A}$ is a $\mathcal{Z}_{(\leqslant a)}$-linear ordering and $\mathcal{B}$ is a $\mathcal{Z}_{(\geqslant a)}$-linear ordering, note that we can put a $\mathcal{Z}$-linear ordering structure on $\mathcal{A} + \mathcal{B}$.

The following lemma will be the main tool in the construction of $\mathcal{C}$.

Lemma 7.3.4. *Given a computable sequence of ordinals $\{\beta_n\}_{n \in \mathbb{N}}$, a computable function ψ, a computable sequence of linear orderings $\{\mathcal{Z}_n : n \in \mathbb{N}\}$, and two computable sequences $\{\mathcal{A}_n : n \in \mathbb{N}\}$ and $\{\mathcal{B}_n : n \in \mathbb{N}\}$ where $\mathcal{A}_n$ and $\mathcal{B}_n$ are $\mathcal{Z}_n$-linear orderings, we can recursively construct a sequence $\{\mathcal{D}_n : n \in \mathbb{N}\}$ such that*

$$\mathcal{D}_n \cong \begin{cases} \zeta^{\beta_n + 1} \cdot \mathcal{A}_n & \text{if } \psi(n) \in 0^{(\beta_n)} \\ \zeta^{\beta_n + 1} \cdot \mathcal{B}_n & \text{if } \psi(n) \notin 0^{(\beta_n)}, \end{cases}$$

via an isomorphism recursive in $0^{(\beta_n)}$. Moreover, we can get an index for $\{\mathcal{D}_n : n \in \mathbb{N}\}$ recursively from indices for $\{\beta_n\}_{n \in \mathbb{N}}$, ψ, $\{\mathcal{A}_n : n \in \mathbb{N}\}$ and $\{\mathcal{B}_n : n \in \mathbb{N}\}$.

The proof of this lemma makes use of Lemma 7.3.5, whose proof we defer to the next subsection. Lemma 7.3.5 is really a corollary of the work of Ash and Knight. It can be proved using the ideas of the proof of [AJK90, Lemma 4.4]. Instead we prove it using [AK00, Theorem 18.9] and the results in [Ash91, §4] to verify the hypothesis of [AK00, Theorem 18.9] for this particular case.

Lemma 7.3.5. *Given a computable sequence of ordinals $\{\beta_n\}_{n \in \mathbb{N}}$, a computable function ψ and two recursive sequences of recursive linear orderings $\{\mathcal{A}_n : n \in \mathbb{N}\}$*

and $\{\mathcal{B}_n : n \in \mathbb{N}\}$, there is a recursive sequences of recursive linear orderings $\{\mathcal{D}_n : n \in \mathbb{N}\}$ such that

$$\mathcal{D}_n \cong \begin{cases} \zeta^{\beta_n+1} \cdot \mathcal{A}_n & \text{if } \psi(n) \in 0^{(\beta_n)} \\ \zeta^{\beta_n+1} \cdot \mathcal{B}_n & \text{if } \psi(n) \notin 0^{(\beta_n)}, \end{cases}$$

via an isomorphism recursive in $0^{(\beta_n)}$. Moreover, we can get an index for $\{\mathcal{D}_n : n \in \mathbb{N}\}$ recursively from indices for $\{\beta_n\}_{n\in\mathbb{N}}$, ψ, $\{\mathcal{A}_n : n \in \mathbb{N}\}$ and $\{\mathcal{B}_n : n \in \mathbb{N}\}$.

Proof of Lemma 7.3.4 using 7.3.5: For each $n \in \mathbb{N}$ and $x \in \mathcal{Z}_n$, let $\mathcal{A}_{n,x} = P_x(\mathcal{A}_n) = \{a \in \mathcal{A}_n : P_x(a)\}$ and $\mathcal{B}_{n,x} = P_x(\mathcal{B}_n)$. Think of $\mathcal{A}_{n,x}$ and $\mathcal{B}_{n,x}$ as linear orderings. We use Lemma 7.3.5 to construct linear orderings $\{\mathcal{D}_{n,x} : n \in \mathbb{N}, x \in \mathcal{Z}\}$ such that

$$\mathcal{D}_{n,x} \cong \begin{cases} \zeta^{\beta_n+1} \cdot \mathcal{A}_{n,x} & \text{if } \psi(n) \in 0^{(\beta_n)} \\ \zeta^{\beta_n+1} \cdot \mathcal{B}_{n,x} & \text{if } \psi(n) \notin 0^{(\beta_n)}, \end{cases}$$

via an isomorphism recursive in $0^{(\beta_n)}$. Then, we just let $\mathcal{D}_n = \sum_{x\in\mathcal{Z}_n} \mathcal{D}_{n,x}$ and $P_x(\mathcal{D}_n) = \mathcal{D}_{n,x}$. □

The idea of the construction of $\mathcal{C}$ is as follows. (We will explain it in more detail later.) Fix $a \in \mathcal{Z}$ and let $m = \varphi(a)$. Let $\mathcal{Y} = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z}$, $\mathcal{Z}_0 = \mathcal{Z}_{(\leqslant a)}$, $\mathcal{Z}_1 = \mathcal{Z}_{(\geqslant a)}$. Suppose that $\mathcal{Z}$ has first and last elements a_0 and a_1. Suppose that (using the recursion theorem) we have already constructed a $\mathcal{Z}_0$-linear ordering $\mathcal{C}_0$ and a $\mathcal{Z}_1$-linear ordering $\mathcal{C}_1$. Moreover, suppose that we know that if $a \in R_{\mathcal{Z}}$, then $\mathcal{C}_0$ satisfies conditions (C1), (C2$'$) and (C3), where (C2$'$) is

> $\langle L_{\mathcal{C}}, R_{\mathcal{C}}\rangle$, where $L_{\mathcal{C}} = p_{\mathcal{C}}^{-1}(L_{\mathcal{Z}})$ and $R_{\mathcal{C}} = p_{\mathcal{C}}^{-1}(R_{\mathcal{Z}})$, is the middle cut of $\mathcal{C}$,

and if $a \in L_{\mathcal{Z}}$, the same happens for $\mathcal{C}_1$ instead of $\mathcal{C}_0$.

We want to, uniformly from $\mathcal{C}_0$ and $\mathcal{C}_1$, construct a $\mathcal{Z}$ linear ordering $\mathcal{C}$ which also satisfies conditions (C1)-(C3). We define two $\mathcal{Z}$-linear orderings $\mathcal{A}$ and $\mathcal{B}$ as follows. Let

$$\mathcal{A} = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega + \mathcal{C}_0 + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*.$$

We need to define a $\mathcal{Z}$-linear ordering structure on $\mathcal{A}$. To do this, think of the first summand as a $\{a_0\}$-linear ordering (so $p_{\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega}$ is the constant function equal to a_0), the second summand as a $\mathcal{Z}_0$-linear ordering and the third summand as a $\mathcal{Z}_1$-linear ordering. (We define the $\mathcal{Z}_1$-linear ordering structure of $\zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*$ arbitrarily. For example, given $\langle z, y, v\rangle \in \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*$, let

$$p_{\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega^*}(\langle z, y, v\rangle) = \begin{cases} a & \text{if } v <_{\omega^*} 0 \vee (v = 0 \,\&\, y \leqslant_{\mathcal{Z}} a) \\ y & \text{if } v = 0 \,\&\, a \leqslant_{\mathcal{Z}} y.) \end{cases}$$

Let

$$\mathcal{B} = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega + \mathcal{C}_1 + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*.$$

To define a $\mathcal{Z}$-linear ordering structure on $\mathcal{B}$ we think of the first summand as a $\{\mathcal{Z}_0\}$-linear ordering, the second summand as a $\mathcal{Z}_1$-linear ordering and the third summand as a $\{a_1\}$-linear ordering. Again, the $\mathcal{Z}_0$-linear ordering structure of $\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega$ is defined arbitrarily.

Now, using Lemma 7.3.4, we construct a recursive $\mathcal{Z}$-linear ordering $\mathcal{C}$ such that

$$\mathcal{C} \cong \begin{cases} \zeta^{\alpha_m+1}\cdot\mathcal{A} & \text{if } a\in R_{\mathcal{Z}} \\ \zeta^{\alpha_m+1}\cdot\mathcal{B} & \text{if } a\in L_{\mathcal{Z}}, \end{cases}$$

and the isomorphism is recursive in $0^{(\alpha_m)}$. (Recall that $a\in L_{\mathcal{Z}} \Leftrightarrow \psi(a)\in 0^{(\alpha_m)}$.) It is not hard to see that $\mathcal{C}$ satisfies conditions (C1), (C2′) and (C3).

Now, we construct a tree that we will use to organize the construction of $\mathcal{C}$.

Lemma 7.3.6. *There is a recursive linear ordering $\mathcal{Z}$ satisfying the conditions of Lemma 7.3.2, and a recursive binary tree T such that $\mathbf{1}+LR(T)+\mathbf{1}\cong\mathcal{Z}$ via a recursive isomorphism $\sigma\mapsto a_\sigma$. Moreover, T has an path X such that $L_{\mathcal{Z}}=\{a_\sigma\in T:\sigma<_{LR(T)}X\}$ and $R_{\mathcal{Z}}=\{a_\sigma\in T: X<_{LR(T)}\sigma\}$. We can also assume that for each $\sigma\in T$, either both, $\sigma^\frown 0$ and $\sigma^\frown 1$, belong to T or both do not. Furthermore, there is a recursive family $\{\mathcal{Z}_\sigma:\sigma\in T\}$ of closed segments of $\mathcal{Z}$ such that $\mathcal{Z}_\emptyset=\mathcal{Z}$, $\mathcal{Z}_{\sigma^\frown 0}=\mathcal{Z}_{\sigma(\leqslant a_\sigma)}$ and $\mathcal{Z}_{\sigma^\frown 1}=\mathcal{Z}_{\sigma(\geqslant a_\sigma)}$ whenever $\sigma^\frown 0$ and $\sigma^\frown 1\in T$.*

Proof: Let $\mathcal{Z}^0$ be a linear ordering as in Lemma 7.3.2. Let $\mathcal{Z}=\mathbf{1}+\zeta\cdot\mathcal{Z}^0+1$. Note that $\mathcal{Z}$ still satisfies the condition of Lemma 7.3.2, and that in $\mathcal{Z}$ we can recursively decide whether two elements are separated by finitely many elements, and if so, we can compute how many elements there are in between.

We define T, the map $\sigma\mapsto a_\sigma$, and the family $\{\mathcal{Z}_\sigma:\sigma\in T\}$ simultaneously by induction. Along the induction we will preserve the property that for every $\sigma\in T$, $\mathcal{Z}_\sigma$, if it is finite, has an odd number of elements and at least three. Let $\mathcal{Z}_\emptyset=\mathcal{Z}$. Suppose now that $\sigma\in T$, and that we have already defined $\mathcal{Z}_\sigma$. If $\mathcal{Z}_\sigma$ has only three elements, then leave $\sigma^\frown 0$ and $\sigma^\frown 1$ outside of T and let a_σ be the middle element of $\mathcal{Z}_\sigma$. If $\mathcal{Z}_\sigma$ has at least five elements, enumerate $\sigma^\frown 0$ and $\sigma^\frown 1$ into T, and let a_σ be the $\leqslant_{\mathbb{N}}$-least element of $\mathcal{Z}_\sigma$ such that $\mathcal{Z}_{\sigma^\frown 0}=\mathcal{Z}_{\sigma(\leqslant a_\sigma)}$ and $\mathcal{Z}_{\sigma^\frown 1}=\mathcal{Z}_{\sigma(\geqslant a_\sigma)}$ do not have an even number of elements, and at least three. (Recall that the domain of $\mathcal{Z}$ is a subset of $\mathbb{N}$.) It is not hard to see that T, the map $\sigma\mapsto a_\sigma$, and the family $\{\mathcal{Z}_\sigma:\sigma\in T\}$ are as desired.

Let X be the leftmost path of $\{\sigma\in 2^{<\omega}:\exists\tau\in 2^{<\omega}(a_\tau\in R_{\mathcal{Z}}\ \&\ \tau\leqslant_{LR}\sigma)\}$. $\square$

From now on, fix T and $\mathcal{Z}$ as in the lemma above, and we identify $\mathcal{Z}$ with $\mathbf{1}+LR(T)+\mathbf{1}$.

Definition 7.3.7. A *T-sequence* is a family of structures $\langle\mathcal{D}_\sigma:\sigma\in T\rangle$ such that $\mathcal{D}_\sigma$ is a $\mathcal{Z}_\sigma$-linear ordering.

We will construct a recursive functional, $\mathcal{E}$, that given (an index for) a T-sequence returns (an index for) another T-sequence. Then, we will use the recursion theorem to obtain a fixed point of this operator. Since we will be using the recursion theorem, we will want, not only that $\mathcal{E}$ maps T-sequences to T-sequences in a certain way, but also that $\mathcal{E}$ maps indices which do not correspond to T-sequences to indices which do code T-sequences. This way we ensure that any fixed point of $\mathcal{E}$ is an index of a T-sequence. For this purpose we prove the following lemma.

Lemma 7.3.8. *There is a recursive function that, given an index e, returns an index for a $\mathcal{Z}$-linear ordering, $\mathcal{B}_e$, such that if e codes a $\mathcal{Z}$-linear ordering $\mathcal{A}_e$, then $\mathcal{B}_e = \zeta^3 \cdot \mathcal{A}_e$.*

Proof: Being an index for a $\mathcal{Z}$-linear ordering is an arithmetic property, say Π^0_k. Then, applying Lemma 7.3.5, we get a family of linear orderings $\{\mathcal{D}_{e,x} : e \in \mathbb{N}, x \in \mathcal{Z}\}$ such that

$$\mathcal{D}_{e,x} = \begin{cases} \zeta^{k+1} \cdot P_x(\mathcal{A}_e) & \text{if } e \text{ codes a } \mathcal{Z}\text{-linear ordering } \mathcal{A}_e; \\ \zeta^{k+1} & \text{if } e \text{ is not the code of a } \mathcal{Z}\text{-linear ordering.} \end{cases}$$

Let $\mathcal{D}_e = \sum_{x \in \mathcal{Z}} \mathcal{D}_{e,x}$ and $P_x(\mathcal{D}_e) = \mathcal{D}_{e,x}$. Actually, if for example we code $\mathcal{Z}$-linear orderings by a linear ordering and a function p onto $\mathcal{Z}$, we get that being an index for a $\mathcal{Z}$-linear ordering is a Π^0_2 property. So, we could make $k = 2$, although this is not relevant for our purposes. □

We have now introduced all the ingredients of the construction.

Construction of $\mathcal{C}$: We construct a recursive operator $\mathcal{E}$ that, given (an index d for) a T-sequence $\bar{\mathcal{D}} = \{\bar{\mathcal{D}}_\sigma : \sigma \in T\}$, returns (an index $\mathcal{E}(d)$ for) a T-sequence $\mathcal{E}(\bar{\mathcal{D}})$. (We are abusing notation here when writing $\mathcal{E}(\bar{\mathcal{D}})$ instead of $\mathcal{E}(d)$.)

We actually want $\mathcal{E}(d)$ to be an index for a T-sequence even when d is not an index for a T-sequence. We start by constructing a T-sequence $\mathcal{D} = \{\mathcal{D}_\sigma : \sigma \in T\}$, such that if d is actually an index for a T-sequence $\bar{\mathcal{D}}$, then $\mathcal{D}_\sigma = \zeta^3 \cdot \bar{\mathcal{D}}_\sigma$ for all $\sigma \in T$. For each d and σ we can uniformly compute an index d_σ such that if d is an index for a T-sequence $\bar{\mathcal{D}}$ and $\sigma \in T$, then d_σ is an index for the $\mathcal{Z}_\sigma$-linear ordering $\bar{\mathcal{D}}_\sigma$. Now, use Lemma 7.3.8 to construct a T-sequence $\mathcal{D} = \{\mathcal{D}_\sigma : \sigma \in T\}$ such that, for each $\sigma \in T$, if d_σ is actually coding a $\mathcal{Z}_\sigma$-linear ordering $\bar{\mathcal{D}}_\sigma$, then $\mathcal{D}_\sigma = \zeta^3 \cdot \bar{\mathcal{D}}_\sigma$.

Now we define two T-sequences $\mathcal{A} = \{\mathcal{A}_\sigma : \sigma \in T\}$ and $\mathcal{B} = \{\mathcal{B}_\sigma : \sigma \in T\}$. If σ is an end node of T, let $\mathcal{A}_\sigma = \zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{|\sigma|}$ and $\mathcal{B}_\sigma = \zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{|\sigma|*}$. Define a $\mathcal{Z}_\sigma$-linear ordering structure on $\mathcal{A}_\sigma$ and $\mathcal{B}_\sigma$ arbitrarily. Suppose now that σ is not an end node of T. Let $n = |\sigma|$, a_0 be the least element of $\mathcal{Z}_\sigma$ and a_1 be the greatest one. So $\mathcal{Z}_{\sigma^\frown 0} = [a_0, a_\sigma]_\mathcal{Z}$ and $\mathcal{Z}_{\sigma^\frown 1} = [a_\sigma, a_1]_\mathcal{Z}$. Let

$$\mathcal{A}_\sigma = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \mathcal{D}_{\sigma^\frown 0} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*}.$$

We need to define a $\mathcal{Z}_\sigma$-linear ordering structure on $\mathcal{A}_\sigma$. To do this, think of the first summand as a $\{a_0\}$-linear ordering, the second summand is a $\mathcal{Z}_{\sigma^\frown 0}$-linear ordering and the third summand as a $\mathcal{Z}_{\sigma^\frown 1}$-linear ordering. (We define the $\mathcal{Z}_{\sigma^\frown 1}$-linear ordering structure of $\zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{n*}$ arbitrarily.) Let

$$\mathcal{B}_\sigma = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \mathcal{D}_{\sigma^\frown 1} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*}$$

where the fist summand is a $\mathcal{Z}_{\sigma^\frown 0}$-linear ordering, the second summand is a $\mathcal{Z}_{\sigma^\frown 1}$-linear ordering and the third summand is a $\{a_1\}$-linear ordering. Again, the $\{\mathcal{Z}_{\sigma^\frown 0}\}$-linear ordering structure of $\zeta^\alpha \cdot \mathcal{Z} \cdot \omega^n$ is defined arbitrarily.

Last, using Lemma 7.3.4, we construct a T-sequence $\mathcal{E}(\bar{\mathcal{D}})$ such that for each $\sigma \in T$,

$$\mathcal{E}(\bar{\mathcal{D}})_\sigma \cong \begin{cases} \zeta^{\alpha_{\varphi(\sigma)}+1} \cdot \mathcal{A}_\sigma & \text{if } \sigma \in R_{\mathcal{Z}} \\ \zeta^{\alpha_{\varphi(\sigma)}+1} \cdot \mathcal{B}_\sigma & \text{if } \sigma \in L_{\mathcal{Z}}, \end{cases}$$

and the isomorphism is recursive in $0^{(\alpha_{\varphi(\sigma)})}$. (Recall φ is a recursive function such that $\sigma \in L^{\mathcal{Z}} \Leftrightarrow \psi(x) \in 0^{(\alpha_{\varphi(\sigma)})}$.)

By the recursion theorem there is an index c such that $\{c\} = \{\mathcal{E}(c)\}$, where $\{e\}$ is the eth Turing function. Since $\mathcal{E}$ always returns indices for T-sequences, c is the index of a T-sequence $\bar{\mathcal{C}}$. Let $\mathcal{C} = \bar{\mathcal{C}}_\emptyset$. $\Diamond$

We claim that $\mathcal{C}$ is the desired $\mathcal{Z}$-linear ordering.

Lemma 7.3.9. *$\mathcal{C}$ satisfies conditions (C1)-(C3)*

PROOF: For each n let $m_n = \varphi(X \restriction n)$. We start by observing that for every n, there is an isomorphism

$$f_n \colon \bar{\mathcal{C}}_{X\restriction n} \to \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \zeta^{\alpha_{m_n}+3} \cdot \bar{\mathcal{C}}_{X\restriction n+1} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*},$$

which is recursive in $0^{(\alpha_{m_n})}$. This, by induction on n, implies that for each n there is an isomorphism $g_n \leqslant_T 0^{(\alpha_{m_n})}$,

$$g_n \colon \mathcal{C} \to \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \zeta^{\alpha_{m_0}+3+\cdots+\alpha_{m_n}+3} \cdot \bar{\mathcal{C}}_{X\restriction n+1} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*}.$$

(We have used that $\mathbf{1}+\omega+\cdots+\omega^n$ is recursively isomorphic to ω^n, and $\zeta^{\alpha_i+3}\cdot\zeta^{\alpha\cdot\omega} \cong \zeta^{\alpha_i+3+\alpha\cdot\omega} \cong \zeta^{\alpha\cdot\omega}$.) Condition (C3) follows. Moreover, we can construct the maps g_n such that if $n_1 > n_0$, then g_{n_0} and g_{n_1} coincide on the initial segment of the form $\zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n_0}$, and on the final segment of the form $\zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{n_0 *}$. Therefore, putting all these isomorphisms together, we get an isomorphism $g \leqslant_T 0^{(\alpha)}$,

$$g \colon \mathcal{C} \to \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^\omega + \mathcal{D} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{\omega*},$$

for some possibly empty linear ordering $\mathcal{D}$. Also observe that for every $b \in \zeta^\alpha\cdot\mathcal{Z}\cdot\omega^\omega$, $p(g^{-1}(b)) \in L_{\mathcal{Z}}$ and for every $b \in \zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{\omega*}$, $p(g^{-1}(b)) \in R_{\mathcal{Z}}$. Therefore, if $b \in \mathcal{D}$, $p(g^{-1}(b))$ is either the last element of $L_{\mathcal{Z}}$ of the first element of $R_{\mathcal{Z}}$. But $L_{\mathcal{Z}}$ has no last element and $R_{\mathcal{Z}}$ has no first element (because otherwise $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}}\rangle$ would be recursive), so $\mathcal{D}$ has to be empty. Conditions (C1) and (C2′) follow. Condition (C2) easily follows from (C2′). $\square$

7.3.2 Pairs of computable structures.

In this subsection we explain how the results in [AK00, Chapter 18] and [Ash91, §4] imply Lemma 7.3.5.

We start by defining the back-and-forth relations and the notion of α-friendliness. See [AK00, Sections 15.1 and 15.2] for more information on these concepts.

Definition 7.3.10. Let K be a class of structures for a fixed language. For each ordinal α, we define the *standard back-and-forth relation* $\leqslant_\alpha$ on pairs $(A, \bar{a})$, where $A \in K$ and $\bar{a}$ is a tuple in A. Let $\bar{a}$ in A and $\bar{b}$ in B be tuples of the same length. Then,

1. $(A, \bar{a}) \leqslant_1 (B, \bar{b})$ if and only if all Σ_1 formulas true of $\bar{b}$ in B are true of $\bar{a}$ in A.

2. For $\alpha > 1$, $(A, \bar{a}) \leqslant_n (B, \bar{b})$ if and only if for each $\bar{d}$ in B, and each $\beta < \alpha$, there exists a $\bar{c}$ in A with $|\bar{c}| = |\bar{d}|$ such that $(B, \bar{b}, \bar{d}) \leqslant_\beta (A, \bar{a}, \bar{c})$.

This definition can be extended to tuples of different length, but we are only interested in pairs of tuples of the same length. We may write $A \leqslant_n B$ instead of $(A, \emptyset) \leqslant_n (B, \emptyset)$.

A pair of structures $\{A_0, A_1\}$ is *α-friendly* if the structures A_i are computable, and for $\beta < \alpha$, the standard back-and-forth relations $\leqslant_\beta$ on pairs $(A_i, \bar{a})$ with $\bar{a} \in A_i \in \{A_0, A_1\}$, are r.e. uniformly in β. That is, we can recursively enumerate all the triples $\langle\langle i, \bar{a}\rangle, \langle j, \bar{b}\rangle, \beta\rangle$ with $\beta < \alpha$, $\bar{a} \in \mathcal{A}_i$ and $\bar{b} \in \mathcal{A}_j$ such that $(A_i, \bar{a}) \leqslant_\beta (A_j, \bar{b})$.

One observation that might give the reader some intuition about the back-and-forth relation is that $(A, \bar{a}) \leqslant_n (B, \bar{b})$ if and only if all the Π_n infinitary formulas true of $\bar{a}$ in A are true of $\bar{b}$ in B [AK00, Proposition 15.1].

Given this definition we can state the main theorem on pairs of computable structures that we will be using.

Theorem 7.3.11. *(Essentially [AK00, 18.9]) For each n, let $\mathcal{A}_n$ and $\mathcal{B}_n$ be structures such that $\mathcal{B}_n \leqslant_{\alpha_n} \mathcal{A}_n$ and $\{\mathcal{A}_n, \mathcal{B}_n\}$ is α_n-friendly, uniformly in n. Let S be a $\Pi^0_{(\alpha_n)}$ set. In other words, let S be such that that is there exists a computable function f such that $n \in S \Leftrightarrow f(n) \notin 0^{(\alpha_n)}$. Then, there is a uniformly computable sequence $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ such that*

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}_n & \text{if } n \in S \\ \mathcal{B}_n & \text{otherwise.} \end{cases}$$

Moreover, the isomorphisms above are recursive in $0^{(\alpha_n)}$ and an index for $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ can be obtained uniformly from indices for S, $\{\alpha_n : n \in \mathbb{N}\}$, $\{\mathcal{A}_n : n \in \mathbb{N}\}$, $\{\mathcal{B}_n : n \in \mathbb{N}\}$ and the back-and-forth relations.

Proof: The first part of the theorem (before the “Moreover”) is exactly [AK00, Theorem 18.9]. The rest follows from the proof of [AK00, Theorem 18.9]. In the proof of [AK00, Theorem 18.9], for each n, a complicated apparatus, that outputs a computable structure $\mathcal{C}_n$ as desired, is constructed. It is constructed uniformly in n, and indices for $\mathcal{A}_n$, $\mathcal{B}_n$ and the back-and-forth relations between them. This apparatus is what they call an α_n-system (defined in [AK00, Chapter 14]), which is a $\Delta^0_{\alpha_n}$-priority construction. From the fact that the construction of each of these apparatuses is uniform in $\mathcal{A}_n$, $\mathcal{B}_n$ and the back-and-forth relations between them, we get that we can even get the sequence $\{\mathcal{C}_n\}$ uniformly from indices for S, $\{\alpha_n : n \in \mathbb{N}\}$, $\{\mathcal{A}_n : n \in \mathbb{N}\}$, $\{\mathcal{B}_n : n \in \mathbb{N}\}$ and the back-and-forth relations.

The isomorphism between $\mathcal{C}_n$ and either $\mathcal{A}_n$ or $\mathcal{B}_n$ is $\Delta^0_{\alpha_n}$ because it can be computed from a run of the α_n-system. See the proof of [AK00, 18.6]. □

The only structures we will be dealing with are linear orderings. The following two lemmas give us a way of computing the back-and-forth relations on linear orderings without having to refer to the definition given above.

Lemma 7.3.12. *[AK00, 15.7] Suppose that $\mathcal{A}$ and $\mathcal{B}$ are linear orderings. Let $\bar{a} = \langle a_0, ..., a_{n-1}\rangle$ and $\bar{b} = \langle b_0, ..., b_{n-1}\rangle$ be increasing tuples form $\mathcal{A}$, $\mathcal{B}$ respectively. For each $i \leqslant n$ let $\mathcal{A}_i$ be the interval $(a_{i-1}, a_i)_{\mathcal{A}}$ (of course, $\mathcal{A}_0 = \mathcal{A}_{(<a_0)}$ and $\mathcal{A}_n = \mathcal{A}_{(>a_{n-1})}$). Define $\mathcal{B}_i$ analogously. Then $(\mathcal{A}, \bar{a}) \leqslant_\beta (\mathcal{B}, \bar{b})$ if and only if for all $0 \leqslant i \leqslant n$, $\mathcal{A}_i \leqslant_\beta \mathcal{B}_i$.*

Lemma 7.3.13. *[AK00, 15.8] Suppose that $\mathcal{A}$ and $\mathcal{B}$ are linear orderings. Then $\mathcal{A} \leqslant_1 \mathcal{B}$ if and only if $\mathcal{A}$ is infinite or at least as large as $\mathcal{B}$. For $\beta > 1$, $\mathcal{A} \leqslant_\beta \mathcal{B}$ if and only if, for any $1 \leqslant \gamma < \beta$ and any finite partition of $\mathcal{B}$ into intervals $\mathcal{B}_1, ..., \mathcal{B}_k$, with end points in $\mathcal{B}$, there is a corresponding partition of A into intervals $\mathcal{A}_1, ..., \mathcal{A}_k$, such that for all $i < n$, $\mathcal{B}_i \leqslant_\gamma \mathcal{A}_i$.*

Now we use the results in [Ash91, §4] to prove that we can apply Theorem 7.3.11 to get Lemma 7.3.5.

Notation 7.3.14. Let $\xi_\beta = \sum_{\gamma<\beta} \zeta^\gamma \cdot \omega$ and $\nu_\beta = \xi_\beta + \xi_\beta^*$. Observe that $\zeta^\beta = \xi_\beta^* + \xi_\beta$, that every final segment of ζ^β with first element in ζ^β has order type ξ_β, and that every segment of ζ^β with both endpoints in ζ^β has order type

$$\xi_\gamma + \zeta^\gamma \cdot n + \xi_\gamma^* = \nu_\gamma \cdot (n+1),$$

for some $\gamma < \beta$ and $n \in \mathbb{N}$.

Lemma 7.3.15. *(Essentially [Ash91, Proposition 4.8]) Let α, β and γ be ordinals.*

1. *$\nu_\beta \cdot n \leqslant_{2\gamma} \nu_\alpha \cdot m$ if and only if either $\langle \beta, n\rangle = \langle \alpha, m\rangle$, or $\alpha, \beta \geqslant \gamma$.*

2. *$\nu_\beta \cdot n \leqslant_{2\gamma+1} \nu_\alpha \cdot m$ if and only if either $\langle \beta, n\rangle = \langle \alpha, m\rangle$, $\alpha \geqslant \gamma$ & $\beta > \gamma$, or $\alpha = \beta = \gamma$ & $n \geqslant m$.*

Proof: The proof is technical, but not complicated. It is by induction on γ and makes heavy use of Lemma 7.3.13. We only prove part (1) to illustrate the ideas.

Suppose that $\alpha, \beta \geqslant \gamma$; we want to show that $\nu_\beta \cdot n \leqslant_{2\gamma} \nu_\alpha \cdot m$. Let $\delta < \gamma$. Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be a partition of $\nu_\alpha \cdot m$ into intervals with endpoints in $\nu_\alpha \cdot m$. Then, there exists ordinals $\alpha_i : i = 1, ..., k$ and numbers $m_1, ..., m_k$ such that for each i, $\mathcal{A}_i = \nu_{\alpha_1} \cdot m_1$. Necessarily $\max \alpha_i = \alpha$ and $\sum_{i:\alpha_i=\alpha} m_i = m$. For i with $\alpha_i < \delta$, let $\beta_i = \alpha_i$ and $n_i = m_i$. For the first i such that $\alpha_i = \alpha$, let $\beta_i = \beta$ and $n_i = n$. For all the other i, let $\beta_i = \delta$ and $n_i = 1$. Note that for every $i = 0, ..., k$, $\nu_{\alpha_i} \cdot m_i \leqslant_{2\delta+1} \nu_{\beta_i} \cdot n_i$ and that $\sum_{i \leqslant k} \nu_i \cdot n_i \cong \nu_\beta \cdot n$. For $i = 1, .., k$, let $\mathcal{B}_i = \nu_{\beta_i} \cdot n_i$. By Lemma 7.3.13, this shows that $\nu_\beta \cdot n \leqslant_{2\gamma} \nu_\alpha \cdot m$.

Now assume that $\nu_\beta \cdot n \leqslant_{2\gamma} \nu_\alpha \cdot m$; we want to show that if $\langle \beta, n \rangle \neq \langle \alpha, m \rangle$, then $\alpha, \beta \geqslant \gamma$. If either α or β is equal to δ and $\delta + 1 < \gamma$, then, by inductive hypothesis, $\nu_\beta \cdot n \nleqslant_{2\delta+2} \nu_\alpha \cdot m$, so $\nu_\beta \cdot n \nleqslant_{2\gamma} \nu_\alpha \cdot m$. If $\gamma = \alpha = \beta + 1$, then, again by inductive hypothesis, $\nu_\beta \cdot n \nleqslant_{2\beta+1} \nu_\alpha \cdot m$, so $\nu_\beta \cdot n \nleqslant_{2\gamma} \nu_\alpha \cdot m$. If $\gamma = \beta = \alpha + 1$, then $\nu_\alpha \cdot m \nleqslant_{2\alpha+1} \nu_\beta \cdot n$, so $\nu_\beta \cdot n \nleqslant_{2\gamma} \nu_\alpha \cdot m$. □

Lemma 7.3.16. *For any linear orderings $\mathcal{B}$ and $\mathcal{D}$ and an ordinal α, $\zeta^{\alpha+1} \cdot \mathcal{B}$ and $\zeta^{\alpha+1} \cdot \mathcal{D}$ are α-back-and-forth equivalent and α-friendly.*

Proof: The proof is by induction on α. Let $\beta < \alpha$. Let $\langle z_0, d_0 \rangle, ..., \langle z_{k-1}, d_{k-1} \rangle \in \zeta^\alpha \cdot (\zeta \cdot \mathcal{D})$ be any ordered tuple. Let $c_0, ..., c_{k-1} \in \zeta \cdot \mathcal{B}$ be such that $d_i < d_j \Leftrightarrow c_i < c_j$. Observe that for each i, the interval $[\langle z_i, d_i \rangle, \langle z_{i+1}, d_{i+1} \rangle]_{\zeta^{\alpha+1} \cdot \mathcal{D}}$ is β-back and forth equivalent to $[\langle z_i, c_i \rangle, \langle z_{i+1}, c_{i+1} \rangle]_{\zeta^{\alpha+1} \cdot \mathcal{C}}$.

They are α-friendly because any closed interval interval of $\zeta^{\alpha+1} \cdot \mathcal{B}$ is either isomorphic to $\nu_\beta \cdot m$ for some $\beta \leqslant \alpha$, or of the form $\xi_{\alpha+1} + \xi^*_{\alpha+1} \cong \nu_{\alpha+1}$, or of the form $\xi_{\alpha+1} + \zeta^{\alpha+1} \cdot \mathcal{C} + \xi^*_{\alpha+1}$ for some $\mathcal{C}$, which is α-back and forth equivalent to $\xi_{\alpha+1} + \zeta^{\alpha+1} + \xi^*_{\alpha+1} \cong \nu_{\alpha+1} \cdot 2$. In all these cases we know how to compute the β-back-and-forth relations recursively. □

Lemma 7.3.5 now follows from Theorem 7.3.11 and the lemma above. This finishes the proof of Theorem 7.3.1.

Remark 7.3.17. The result of Lemma 7.3.16 is not sharp. Possibly, one could prove that α, $\zeta^{\alpha+1} \cdot \mathcal{B}$ and $\zeta^{\alpha+1} \cdot \mathcal{D}$ are β-back-and-forth equivalent for some $\beta > \alpha$. Therefore, Lemma 7.3.5 is not sharp either. But this is not relevant for our results.

7.4 The Game Statements

In this section we prove all the implications in Theorem 7.1.12 that have to do with game statements. All these statements were introduced in Subsection 7.1.2. We work in RCA_0.

Σ^1_1-$\mathsf{AC}_0 \Rightarrow$ $\mathsf{G4}$: Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of determined trees. If for some n, II has a winning strategy for $G(T_n)$, then then I has the following winning strategy in $\sum_n T_n$: Start by playing n, and then use II's winning strategy in $G(T_n)$.

(Recall that I is the second player in $G(T_n)$.) Suppose now that for every n, I has a winning strategy in $G(T_n)$. We will show that then, II has a winning strategy in $\sum_n T_n$. Using Σ^1_1-AC_0, let $\langle s_n : n \in \mathbb{N}\rangle$ be such that s_n is a winning strategy for I in $G(T_n)$. Now, if I starts playing n, II continues following s_n in $G(T_n)$ and wins.

$\mathsf{G4} \Rightarrow \mathsf{G3}$: Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of determined trees. Now, for each n, consider the game $\bar{G}_n = G_n + G^*_n$. Where $G_n = G(T_n)$ and G^*_n is in Definition 7.1.8. So $\bar{G}_n$ is the game in which player I starts by choosing whether to play G_n or G^*_n. Since G_n is determined for each n, I has a winning strategy for $\bar{G}_n$ in which he starts by choosing whichever of G_n, or G^*_n has a winning strategy for II. Therefore, by $\mathsf{G4}$, II has a winning strategy s for $\sum_n \bar{G}_n$. Let X be the set of n such that if I starts playing n in $\sum_n \bar{G}_n$, then II, following s, starts playing $\bar{G}_n$ by choosing G^*_n. So, X is the set of n such that II has a winning strategy in G^*_n, or equivalently, such that I has a winning strategy in G_n.

Δ^1_1-$\mathsf{CA}_0 \Rightarrow \mathsf{G3}$: Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of determined trees. Since there exists a winning strategy for I in $G(T_n)$ if and only if there is no winning strategy for II in $G(T_n)$. By Δ^1_1 we can define a set X such that $n \in X$ if and only if there exists a winning strategy for I. $\mathsf{G3}$ follows.

$\mathsf{G3} \Rightarrow \Delta^1_1$-$\mathsf{CA}_0$: Let φ and ψ be Σ^1_1 formulas such that $\forall n(\psi(n) \Leftrightarrow \neg\varphi(n))$. We want to show that there exists a set X such that $\forall n(n \in X \Leftrightarrow \psi(n))$ By [Sim99, Theorem V.1.7′], there exist a sequences of trees $\{T_n : n \in \mathbb{N}\}$ and $\{S_n : n \in \mathbb{N}\}$ such that for every n,

$$S_n \text{ has a path} \Leftrightarrow \psi(n) \Leftrightarrow \neg\varphi(n) \Leftrightarrow T_n \text{ is well-founded.}$$

Consider the game, $\bar{G}_n$, in which players I and II alternatively play numbers $a_0, a_1, \ldots$. Player I, when playing a_{2i}, has to make sure that $\langle a_0, a_2, \ldots, a_{2i}\rangle \in S_n$ because otherwise he looses. Player II, when playing a_{2i+1}, has to make sure that $\langle a_1, a_3, \ldots, a_{2i+1}\rangle \in T_n$; otherwise he looses.

Suppose that T_n is well-founded and S_n is not. Let X be a path through S_n. Notice that if I plays $X(i)$ in his ith move, he will surely win. Analogously, if S_n is well-founded and T_n is not, II has a winning strategy in the game $\bar{G}_n$. So, we have that for each n, $\bar{G}_n$ is determined. By $\mathsf{G3}$, there exists a set X such that

$$n \in X \Leftrightarrow \text{I has a winning strategy in } \bar{G}_n \Leftrightarrow S_n \text{ has a path} \Leftrightarrow \psi(n).$$

Δ^1_1-$\mathsf{CA}_0 \Rightarrow$weak-$\Sigma^1_1$-$\mathsf{AC}_0$ is not hard to prove. See for example [Sim99].

weak-Σ^1_1-$\mathsf{AC}_0 \Rightarrow \mathsf{G1}$: Let T be a well founded tree. We first show that d is the unique winning function of $G(T)$ if and only if

$$\forall \sigma \in T(d(\sigma) = \mathtt{W} \Leftrightarrow \exists n(\sigma^\frown n \in T \;\&\; d(\sigma^\frown n) = \mathtt{L})). \tag{7.4.1}$$

It is clear that if d is a winning function, then (7.4.1) holds. Suppose now that $d\colon T \to \{\mathtt{W}, \mathtt{L}\}$ satisfies (7.4.1). For each $\sigma \in T$ with $d(\sigma) = \mathtt{W}$ we have to define a winning strategy s_σ for I in $G(T_\sigma)$, and if $d(\sigma) = \mathtt{L}$ we have to define a winning strategy s_σ for II in $G(T_\sigma)$.

If $d(\sigma) = \mathtt{W}$ and $\tau \in T_{\sigma,\mathrm{I}}$, we let $s_\sigma(\tau)$ be the least $n \in \mathbb{N}$ such that $\tau^\frown n \in T_\sigma$ and $d(\sigma^\frown\tau^\frown n) = \mathtt{L}$. We claim that s_σ is a winning strategy for I in $G(T_\sigma)$. Observe that one can easily prove by induction that if $\tau \in T_\sigma$ is a partial run of $G(T_\sigma)$ following s_σ, then, if $\tau \in T_{\sigma,\mathrm{I}}$, $d(\sigma^\frown\tau) = \mathtt{W}$ and, if $\tau \in T_{\sigma,\mathrm{II}}$, $d(\sigma^\frown\tau) = \mathtt{L}$. Since for every end node τ of T_σ, $d(\sigma^\frown\tau) = \mathtt{L}$, we have that if I follows s_σ, he surely wins.

If $d(\sigma) = \mathtt{L}$ and $\tau \in T_{\sigma,\mathrm{II}}$, we let $s_\sigma(\tau)$ be the least $n \in \mathbb{N}$ such that $\tau^\frown n \in T_\sigma$ and $d(\sigma^\frown\tau^\frown n) = \mathtt{L}$. An argument like the one above shows that s_σ is a winning strategy for II in $G(T_\sigma)$.

Now, in order to prove $\mathsf{G1}$, consider a family of completely determined trees $\{T_n : n \in \mathbb{N}\}$. We want to show that there exists a set X such that $n \in X$ if and only if I wins $G(T_n)$. For each n, there is a unique function d_n such that (7.4.1) holds. So, by weak-Σ^1_1-AC_0, the sequence $\langle d_n : n \in \mathbb{N}\rangle$ exists. Let $X = \{n : d_n(\emptyset) = \mathtt{W}\}$.

$\mathsf{G2}\Rightarrow\mathsf{G1}$: If d is the winning function of $\sum_n T_n$ given by $\mathsf{G2}$, let $X = \{n : d(\langle n\rangle) = \mathtt{W}\}$.

$\mathsf{G1}\Rightarrow\mathsf{G2}$: Suppose we are given a sequence $\{T_n : n \in \mathbb{N}\}$ of completely determined trees. Consider the family $\{T_{n,\sigma} : n \in \mathbb{N}, \sigma \in T_n\}$, where $T_{n,\sigma}$ is the tree $\{\tau : \sigma^\frown\tau \in T_n\}$. By $\mathsf{G1}$, there is a set X such that $\forall n \forall \sigma \in T_n(\langle n,\sigma\rangle \in X \Leftrightarrow$ I has a winning strategy in $G(T_{n,\sigma}))$. Let $\langle d_n : n \in \mathbb{N}\rangle$ be such that for all $n \in \mathbb{N}$ and $\sigma \in T_n$, $d_n(\sigma) = \mathtt{W} \Leftrightarrow \langle n,\sigma\rangle \in X$ and $d_n(\sigma) = \mathtt{L}$ otherwise. Note that each d_n is a winning function for $G(T_n)$.

$\mathsf{G1}\Rightarrow\mathsf{JI}$. Let α be an ordinal and suppose that for every $\beta < \alpha$, $0^{(\beta)}$ exists. By recursive transfinite induction, we construct a family of tree games $\{G_{\beta,n} : \beta < \alpha, n \in \mathbb{N}\}$, such that $n \in 0^{(\beta)} \Leftrightarrow$ I has a winning strategy in $G_{\beta,n}$. For $\beta = 0$ and any n, let $T_{0,n} = \emptyset$. In the game $G(\emptyset)$ player I starts loosing, and II always wins. If β is a limit ordinal, then $n \in 0^{(\beta)} \Leftrightarrow n = \langle m_n, \gamma_n\rangle$ & $\gamma_n < \beta$ & $m_n \in 0^{(\gamma_n)}$. So, let $T_{\beta,n} = T_{\gamma_n,m_n}$, if $n = \langle m_n,\gamma_n\rangle$ & $\gamma_n < \beta$, and let $T_{\beta,n} = \emptyset$ otherwise. If $\beta = \gamma + 1$, then there exists a recursive function f such that $\forall n(n \in 0^{(\beta)} \Leftrightarrow \exists s(f(n,s) \notin 0^{(\gamma)}))$. Let $T_{\beta,n} = \sum_{s\in\mathbb{N}} T_{\gamma,f(n,s)}$. Then, I has a winning strategy in $T_{\beta,n}$ if and only if, for some s, II has a winning strategy in $T_{\gamma,f(n,s)}$, which happens if and only if $n \in 0^{(\beta)}$. Moreover, we claim that, using our assumption that for every $\beta < \alpha$, $0^{(\beta)}$ exists, we can prove that each game $T_{\beta,n}$ is completely determined: By recursive transfinite induction we define $0^{(\beta)}$-recursive indices for winning functions $d_{\beta,n}$ for $T_{\beta,n}$. Of course, for $\beta = 0$, $d_{0,n}$ is the empty function. When β is a limit ordinal we just let $d_{\beta,n} = d_{\gamma_n,m_n}$, if $n = \langle m_n,\gamma_n\rangle$ & $\gamma_n < \beta$, and let $d_{\beta,n} = \emptyset$ otherwise. When $\beta = \gamma+1$, we let $d_{\beta,n}(\emptyset) = \mathtt{W} \Leftrightarrow \exists s(f(n,s) \notin 0^{(\gamma)})$ and let $d_{\beta,n}(\langle s\rangle^\frown\sigma) = d_{\gamma,f(n,s)}(\sigma)$.

So, we have a family $\{G_{\beta,n} : \beta < \alpha, n \in \mathbb{N}\}$ of completely determined games such that I wins $G_{\beta,n}$ if and only if $n \in 0^{(\beta)}$. By $\mathsf{G1}$, there exits a set X such that $\langle \beta, n\rangle \in X \Leftrightarrow$ I wins $G(T_{\beta,n})$. This X is $0^{(\alpha)}$.

7.5 JI does not imply G1

This section is dedicated to prove the following theorem.

Theorem 7.5.1. *There is an ω-model of JI which is not a model of G1. Therefore, RCA+JI does not imply G1, and hence does not imply weak-Σ^1_1-AC_0 either.*

We will define a sequence $\{\langle T_i^G, d_i^G, h_i^G\rangle : i \in \omega\}$ in a generic way. Then, we will let $\mathcal{M}_\infty$ be the least ω-model closed under hyperarithmetic reduction, which contains the sequence $\{T_i^G : i \in \omega\}$ and each of the functions d_i^G. We will prove that, in $\mathcal{M}_\infty$, each T_i^G is a well-founded tree and d_i^G is a winning function for it. Even though $\mathcal{M}_\infty$ contains all the functions d_i^G, we will prove that it does not contain the sequence $\{d_i^G : i \in \omega\}$. Moreover, we will prove that it does not contain the set $\{n : d_n^G(\emptyset) = \mathtt{W}\}$. This will imply that $\mathsf{G1}$ does not hold in $\mathcal{M}_\infty$. To show that JI holds in $\mathcal{M}_\infty$ we will show that for every $X \in \mathcal{M}_\infty$ and ordinal α, $X^{(\alpha)} \in \mathcal{M}_\infty$ if and only if $\alpha < \omega_1^{CK}$. This will easily imply JI.

The functions h_i^G are going to be a kind of rank functions on T_i^G that we will specify later. We use them to ensure that the trees T_i^G look well-founded in $\mathcal{M}_\infty$, and to prove properties about the forcing notion.

7.5.1 Ranked games

Given a tree T, a *game rank* for T is a pair of functions $d\colon T \to \{\mathtt{L}, \mathtt{W}\}$ and $h\colon T \to \omega_1$ such that

1. If $\sigma \in T$ and $d(\sigma) = \mathtt{L}$, then for every immediate successor τ of σ in T, $d(\tau) = \mathtt{W}$ and $h(\sigma) = \sup\{h(\tau) + 1 : \tau \in T \;\&\; \tau^- = \sigma\}$.

2. If $\sigma \in T$ and $d(\sigma) = \mathtt{W}$, then for some immediate successor τ of σ in T, $d(\tau) = \mathtt{L}$ and $h(\sigma) = \min\{h(\tau) : d(\tau) = \mathtt{L} \;\&\; \tau^- = \sigma\}$.

Observe, that d is a winning function for $G(T)$, even when T is not well founded. By this we mean that if $d(\sigma) = \mathtt{W}$, then player I has a strategy in $G(T_\sigma)$ that will lead him to win in finitely many steps. This is because when player I moves, he always has the option to move to a node labeled $\mathtt{L}$ without increasing the ordinal label. On the other hand, player II is always forced to play to a node labeled $\mathtt{W}$ and with a strictly smaller ordinal label.

But, if a tree T has a game rank function, it is not necessarily well-founded. For example, consider the tree $T = \{0^n : n \in \omega\} \cup \{0^n{}^\frown 1 : n \in \omega\}$. A game rank function for T is defined as follows. Let $d(0^n{}^\frown 1) = \mathtt{L}$, $h(0^n{}^\frown 1) = 0$, $d(0^n) = \mathtt{W}$, $h(0^n) = 0$. The following condition guaranties that T is well-founded.

Definition 7.5.2. We say that a game rank d, h on T is *uniform* if whenever $\sigma \in T$, $d(\sigma) = \mathtt{W}$ and τ is an immediate successor of σ we have that if $d(\tau) = \mathtt{L}$, $h(\tau) = h(\sigma)$, and if $d(\tau) = \mathtt{W}$, $h(\tau) < h(\sigma)$.

Note that not every well-founded tree has a uniform game rank.

7.5.2 The forcing notion

Let $\bar{\xi}$ be a recursive ordering of order type $\omega_1^{CK} \cdot (1 + \eta)$ (i.e., a Harrison linear ordering [Har68]), and let $\xi = \bar{\xi} \cup \{\infty\}$, where ∞ is a new symbol grater than all the elements of $\bar{\xi}$. We let $\infty < \infty$. We intend the functions d_i^G, h_i^G mentioned above, to act like uniform game ranks on the trees T_i^G. They will not be actual game ranks because the image of h_i^G will not be an ordinal, but ξ. The advantage of using the Harrison ordering, instead of ω_1^{CK} as Steel does in [Ste78], is that the forcing notion is then computable.

Definition 7.5.3. We let $\mathbb{P}$ be the forcing notion which consist of conditions p of the form $\langle \langle T_i^p, d_i^p, h_i^p \rangle : i < n^p \rangle$, where

1. $n^p \in \omega$ and each T_i^p is a finite subtree of $\omega^{<\omega}$;
2. $d_i^p \colon T_i^p \to \{\mathtt{L}, \mathtt{W}\}$ and $h_i^p \colon T_i^p \to \xi$;
3. If $\sigma \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{L}$ then $d_i^p(\sigma) = \mathtt{W}$, and $h_i^p(\sigma) < h_i^p(\sigma^-)$;
4. If $\sigma \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{W}$ then, if $d_i^p(\sigma) = \mathtt{L}$, $h_i^p(\sigma) = h_i^p(\sigma^-)$, and if $d_i^p(\sigma) = \mathtt{W}$, $h_i^p(\sigma) < h_i^p(\sigma^-)$;
5. $h_i^p(\emptyset) = \infty$.

We use T^p to denote $\{\langle i, \sigma \rangle : i < n^p, \sigma \in T_i^p\}$ and d^p and h^p to denote the partial functions defined by $d^p(\langle i, \sigma \rangle) = d_i^p(\sigma)$ and $h^p(\langle i, \sigma \rangle) = h_i^p(\sigma)$. Given $p, q \in \mathbb{P}$, we let $q \leqslant_{\mathbb{P}} p$ if, $n^q \geqslant n^p$, $T^q \supseteq T^p$, $d^q \supseteq d^p$ and $h^q \supseteq h^p$ as functions. Let G be a *hyperarithmetically generic* filter. That is, G is a filter and meets every hyperarithmetic dense subset of $\mathbb{P}$. We define $\langle \langle T_i^G, d_i^G, h_i^G \rangle : i \in \omega \rangle$ in the obvious way.

Given $F \subset_f \omega$, we let $G_F = \langle T_i^G : i \in \omega \rangle \oplus \bigoplus_{j \in F} d_j^G$, and let $\mathcal{M}_F$ be the set of all sets which are hyperarithmetic in G_F. (By $F \subset_f \omega$ we mean that F is a finite subset of ω.) Let $\mathcal{M}_\infty = \bigcup_{F \subset_f \omega} \mathcal{M}_F$.

Note that being hyperarithmetically generic over $\mathbb{P}$ is a Σ_1^1 condition: It can easily be written as a formula φ of the form $(\forall X \leqslant_H \emptyset)\psi$, where ψ is arithmetic. The Spector-Gandy Theorem [Spe60, Gan60] says that every such φ is equivalent to a Σ_1^1 formula. So, we can take G generic such that $\omega_1^G = \omega_1^{CK}$. This is because of the Gandy low basis theorem [Sac90a, Corollary III.1.5] which says that every non-empty Σ_1^1 class has a hyperarithmetically low member. (A set $Y \subseteq \omega$ is *hyperarithmetically low* if $\omega_1^Y = \omega_1^{CK}$.) Fix such a G. Therefore, every set $X \in \mathcal{M}_\infty$

is computable from $G_F^{(\alpha)}$ for some $F \subset_f \omega$ and $\alpha < \omega_1^{CK}$, and hence is of the form $\{x : \psi(x, G_F)\}$ for some computable infinitary formula ψ. See [AK00, Chapter 7] for a definition of computable infinitary formulas.

We shall prove that $\mathcal{M}_\infty \models \mathsf{JI}\ \&\ \neg\mathsf{G1}$.

7.5.3 The forcing relation

Both Steel [Ste78] and Van Wesep [Van77] used a ramified language as a forcing language, when they worked with tagged trees forcing. Instead, we use computable infinitary formulas in the language of first-order arithmetic augmented with a unary relation symbol $\cdot \in T$ and binary relation symbols $d_i(\cdot) = \cdot$, for each $i \in \omega$. For $F \subset_f \omega$, we denote the set of formulas which do not mention d_i for $i \notin F$ by $\mathcal{L}^F$. We let $\mathcal{L}^\infty = \bigcup_{F \subset_f \omega} \mathcal{L}^F$. We associate to each formula of $\mathcal{L}^\infty$ a *rank* $\alpha < \omega_1^{CK}$ defined by transfinite induction as follows: if φ is an atomic formula of arithmetic, then $\mathrm{rk}(\varphi) = 0$, $\mathrm{rk}(x \in T) = 1$, $\mathrm{rk}(d(x) = \mathtt{L}) = \mathrm{rk}(d(x) = \mathtt{W}) = 2$, $\mathrm{rk}(\forall x \psi(x)) = \mathrm{rk}(\neg\psi) = \mathrm{rk}(\psi) + 1$ and $\mathrm{rk}(\bigwedge_{i\in\omega} \psi_i) = \sup\{\mathrm{rk}(\psi_i) + 1 : i \in \omega\}$. (The motivation for the base case in the definition of rk is just to prove Lemma 7.5.8.)

Definition 7.5.4. The forcing relation for formulas of $\mathcal{L}^\infty$ is defined as usual:

1. $p \Vdash \psi \Leftrightarrow \psi$ when ψ is a quantifier free formula of arithmetic;
2. $p \Vdash \langle i, \sigma\rangle \in T$ if either $|\sigma| < 2$, or $\sigma^{--} \in T_i^p$ and $h_i^p(\sigma^{--}) \geqslant 1$;
3. $p \Vdash d_i(\sigma) = \mathtt{L}$ if one of the following holds:
   - $\sigma \in T_i^p$ and $d_i^p(\sigma) = \mathtt{L}$,
   - $\sigma^- \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{W}$ and $h_i^p(\sigma^-) = 0$,
   - $\sigma^{--} \in T_i^p$, $d_i^p(\sigma^{--}) = \mathtt{L}$ and $h_i^p(\sigma^{--}) = 1$;
4. $p \Vdash d_i(\sigma) = \mathtt{W}$ if one of the following holds:
   - $\sigma \in T_i^p$ and $d_i^p(\sigma) = \mathtt{W}$,
   - $\sigma^- \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{L}$ and $h_i^p(\sigma^-) > 0$;
5. $p \Vdash \forall x \psi(x)$ if for all n, $p \Vdash \psi(n)$;
6. $p \Vdash \bigwedge_{i\in\omega} \psi_i$ if for every i, $p \Vdash \psi_i$;
7. $p \Vdash \neg\psi$ if for every $q \leqslant_{\mathbb{P}} p$, $q \nVdash \psi$.

It can be proved by induction on the formulas that $p \Vdash \psi$ if and only if whenever G is a hyperarithmetically generic filter, $p \in G$ and $\mathcal{M}_\infty$ is the model defined from G, we have that $\mathcal{M}_\infty \models \psi$. This property is what motivated the definition of $p \Vdash d_i(\sigma) = \mathtt{L}$ and $p \Vdash d_i(\sigma) = \mathtt{W}$.

Observe that for a formula ψ of rank α, $0^{(\alpha)}$ can decide whether $p \Vdash \psi$ uniformly in ψ, p and α. This can be easily proved by transfinite induction. (Actually, less than $0^{(\alpha)}$ is required.)

We are now ready to prove that $\mathcal{M}_\infty \models \mathsf{JI}$.

Lemma 7.5.5. *Let $\alpha \in \mathcal{M}_\infty$ be a linear ordering and $X \in \mathcal{M}_\infty$ be an $H(\emptyset, \alpha)$-set. Then α and is a well ordering and $\alpha < \omega_1^{CK}$.*

(See definition of $H(\emptyset,\alpha)$-set in Subsection 7.1.5.)

Proof: Since $X \in \mathcal{M}_\infty$, there exist F and $\beta < \omega_1^{CK}$ such that $X \leqslant_T G_F^{(\beta)}$. Suppose toward a contradiction that α is not a well ordering. Then, there is a decreasing sequence $a_0 > a_1 > a_2 > \ldots$ of elements of α, and we have that for every k, $X^{[a_k]} \geqslant_T (X^{[a_{k+1}]})'$. Then, by [Sac90a, Lemma III.3.3], we have that for every recursive ordinal γ, $0^{(\gamma)} \leqslant_T X \leqslant_T G_F^{(\beta)}$ uniformly. So, there is a computable infinitary $\Sigma^0_{\beta+1}$ formula φ such that for every $\delta < \omega_1^{CK}$, $\{n : \varphi(G,\delta,n)\} = 0^{(\delta)}$. Let γ be the rank of the formula φ. We will get a contradiction by proving that $0^{(\gamma)} \geqslant_T 0^{(\gamma+1)}$. Let $p \Vdash \{n : \varphi(G,\gamma+1,n)\} = 0^{(\gamma+1)}$. Now, given n, recursively in $0^{(\gamma)}$ find $q \leqslant_{\mathbb{P}} p$ which decides $\varphi(G,\gamma+1,n)$. Then $n \in 0^{(\gamma+1)}$ if and only if $q \Vdash \varphi(G,\gamma+1,n)$.

If we had $\alpha \geqslant \omega_1^{CK}$, we would also have that for every recursive ordinal γ, $0^{(\gamma)} \leqslant_T X \leqslant_T G_F^{(\beta)}$ uniformly, and we would get a contradiction the same way. $\square$

It follows from the lemma above that for $\alpha, X \in \mathcal{M}_\infty$, $X^{(\alpha)} \in \mathcal{M}_\infty$ if and only if $\alpha < \omega_1^{CK}$.

Lemma 7.5.6. *$\mathcal{M}_\infty$ satisfies JI.*

Proof: Let X and α be such that, in $\mathcal{M}_\infty$, α is an ordinal and $\forall \beta < \alpha$, $X^{(\beta)} \in \mathcal{M}_\infty$. In particular, we have that for all $\beta < \alpha$, $0^{(\beta)} \in \mathcal{M}_\infty$, and hence, by the previous lemma, $\alpha < \omega_1^{CK}$. It then follows that $X^{(\alpha)} \in \mathcal{M}_\infty$. $\square$

7.5.4 Retaggings

The goal of this subsection is to prove that $\mathsf{G1}$ does not hold in $\mathcal{M}_\infty$. We need to show that in $\mathcal{M}_\infty$ all the trees T_i^G are well-founded and completely determined by d_i^G, but that the set $\{n : d_n^G(\emptyset) = \mathtt{W}\}$ is not in $\mathcal{M}_\infty$.

The next definition and lemma are key when forcing with tagged trees.

Definition 7.5.7. Let $p, p^* \in \mathbb{P}$, $F \subset_f \omega$ and $\alpha \in \omega_1^{CK}$. We say that p^* is an *α-F-absolute retagging* of p, and we write $\mathrm{Ret}(\alpha, F; p, p^*)$, if

1. $n^p = n^{p^*}$, $T^p = T^{p^*}$ and for $i \in F$, $d_i^p = d_i^{p^*}$;
2. for all $i < n^p$ and $\sigma \in T_i^p$, if $h_i^p(\sigma) < \alpha$, then $h_i^{p^*}(\sigma) = h_i^p(\sigma)$ and $d_i^{p^*}(\sigma) = d_i^p(\sigma)$; and

3. if $h_i^p(\sigma) \geqslant \alpha$, then $h_i^{p^*}(\sigma) \geqslant \alpha$.

Lemma 7.5.8. *Let ψ be a formula in $\mathcal{L}^F$ of rank less than or equal to α and let $p, p^* \in \mathbb{P}$ be α-F-*absolute retaggings*. Then, $p^* \Vdash \psi$ if and only if $p \Vdash \psi$.*

Proof: The proof is by transfinite induction on α. All the cases are trivial except for $\psi = \neg\varphi$. Suppose that $p^* \Vdash \neg\varphi$; we want to show that $p \Vdash \neg\varphi$. Consider $q \leqslant_{\mathbb{P}} p$; we need to show that $q \nVdash \varphi$. Let $\beta < \alpha$ be the rank of φ. We claim that that there is a $q^* \leqslant_{\mathbb{P}} p^*$ which is a β-F-absolute retagging of q. From the claim we would get what we want because, since $p^* \Vdash \neg\varphi$, we have that $q^* \nVdash \varphi$, and hence $q \nVdash \varphi$.

Let us now prove the claim. Note that we can assume that $T^q \setminus T^p$ has only one element $\langle j, \sigma\rangle$; we can then prove our claim for a general q by induction on $|T^q \setminus T^p|$. Let $T^{q^*} = T^q = T^{p^*} \cup \{\langle j, \sigma\rangle\}$ and for $\tau \in T^{p^*}$, let $h_i^{q^*}(\tau) = h_i^{p^*}(\tau)$ and $d_i^{q^*}(\tau) = d_i^{p^*}(\tau)$. Now, if $\sigma = \emptyset$, let $d_j^{q^*}(\sigma) = d_j^q(\sigma)$ and $h_j^{q^*}(\sigma) = h_j^q(\sigma) = \infty$. Suppose now that $\sigma \neq \emptyset$ and let $\tau = \sigma^-$. There are two cases. The first case is $h_j^q(\sigma) < \alpha$, where we need to define $h_j^{q^*}(\sigma) = h_j^q(\sigma)$ and $d_j^{q^*}(\sigma) = d_j^q(\sigma)$. We need to verify that this definition is consistent. To do this we have to look at all the possible values of $h_j^{p^*}(\tau)$ and $d_j^{p^*}(\tau)$. All the possibilities are easy to analyze. The second case is $h_j^q(\sigma) > \alpha$. In this case we only need to worry to define $h_j^{q^*}(\sigma) \geqslant \beta$ and $d_j^{q^*}(\sigma)$ to be consistent with $h_j^{p^*}(\tau)$ and $d_j^{p^*}(\tau)$, which is not hard to do. □

Lemma 7.5.9. *The trees T_i^G, $i \in \omega$, have no infinite paths in $\mathcal{M}_\infty$.*

Proof: Suppose, toward a contradiction, that $X \in \mathcal{M}_\infty$ is a path through T_i^G. The sequence $\{h_i^G(X \restriction n) : n \in \omega\}$ is a descending sequence in ξ, and therefore, for every n, $h_i^G(X \restriction n) > \omega_1^{CK}$. There are some $F \subset_f \omega$ and formula $\varphi \in \mathcal{L}^F$ such that $(\forall n, m)(X(n) = m \Leftrightarrow \varphi(n, m))$. Let $\psi(k)$ be the formula that says that $\{\langle n, m\rangle : \varphi(n, m, G_F)\}$ is a path through T_i^G and that $\varphi(0, k)$ (i.e., the path starts with $\langle k\rangle$). Let α be the rank of $\psi(k)$ and let $p \in G$ force $\psi(k)$ for some $k \in \omega$. It is not hard to prove that there exists q such that $\mathrm{Ret}(\alpha, F; p, q)$ and $h_i^q(\langle k\rangle) < \omega_1^{CK}$, using the fact that $h_i^p(\langle k\rangle) > \omega_1^{CK} > \alpha$. Then, by the previous lemma, we have that $q \Vdash \psi$, which is impossible because, since $h_i^q(\langle k\rangle) < \omega_1^{CK}$, there cannot be any path though T_i starting with $\langle k\rangle$ in any model defined from a generic extension of q. □

Corollary 7.5.10. *In $\mathcal{M}_\infty$, $\{T_i^G : i \in \omega\}$ is a sequence of completely determined well-founded trees.*

Proof: From the definition of $\mathbb{P}$ and the fact that G is generic, we get that for every i, d_i^G satisfies (7.4.1) and hence is a winning function for $G(T_i^G)$. □

Lemma 7.5.11. *In $\mathcal{M}_\infty$, there is no set X such that $n \in X$ if and only if* I *has a wining strategy in the game determined by T_n.*

Proof: If such a set X existed in $\mathcal{M}_\infty$, there would be a formula $\varphi(n) \in \mathcal{L}^F$, for some $F \subset_f \omega$, such that $(\forall n)\varphi(n) \Leftrightarrow d_n(\emptyset) = \mathtt{W}$. Let $\alpha = \mathrm{rk}((\forall n)\varphi(n) \Leftrightarrow d_n(\emptyset) = \mathtt{W})$, and let $p \in G$ be such that $p \Vdash (\forall n)\varphi(n) \Leftrightarrow d_n(\emptyset) = \mathtt{W}$ and for some $i \in \omega \setminus F$, $d_i^p(\emptyset) = \mathtt{L}$ and $p \Vdash \neg\varphi(i)$. Such p has to exists by the genericity of G. We will get a contradiction by proving that there exists q such that $\mathrm{Ret}(\alpha, F; p, q)$ and $d_i^q(\emptyset) = \mathtt{W}$. Let q be such that $T^q = T^p$, $h^q = h^p$, and except at $\langle i, \emptyset\rangle$, $d^q = d^p$. Let $d_i^q(\emptyset) = \mathtt{W}$. Since $h_i^q(\emptyset) = \infty$, we have that $\mathrm{Ret}(\alpha, F; p, q)$. To show that $q \in \mathbb{P}$, observe that for all immediate successors σ of $\emptyset$ in T^q, $d_i^q(\sigma) = \mathtt{W}$, so condition 7.5.3(4) is satisfied just because ∞ is grater than any other element of ξ. $\square$

Theorem 7.5.1 now follows.

Chapter 8
Ranked Structures and Arithmetic Transfinite Recursion (*with Noam Greenberg*).

8.1 Introduction

Classification of mathematical objects is often achieved by finding *invariants* for a class of objects - a method of representing the equivalence classes of some notion of sameness (such as isomorphism, elementary equivalence, bi-embeddability) by simple objects (such as natural numbers or ordinals). A related logical issue is the question of complexity: if the invariants exist, how complicated must they be; when does complexity of the class make the existence of invariants impossible; and how much information is implied by the statement that invariants of certain type exist. To mention a far from exhaustive list of examples: in descriptive set theory, Hjorth and Kechris ([HK95]) investigated the complexity of the existence of Ulm-type classification (and of the invariants themselves) in terms of the Borel and projective hierarchy; see also Camerlo and Gao ([CG01]) and Gao ([Gao04]). In computability theory, complexity of index-sets of isomorphism relations on structures have been studied, among others, by Goncharov and Knight ([GK02b]) and Calvert ([Cal04]); index-sets for elementary equivalence are considered as well (Selivanov [Sel91]); and in reverse mathematics, the proof-theoretic strength of the statement of existence of invariants was studied by Shore ([Sho]).

For some classes, closely connected to invariants is the notion of rank. For example, the Cantor-Bendixon rank of a countable compact metric space is obtained by an iterated process of weeding out isolated points. This rank, then, together with the number of points left at the last step, constitutes an invariant for the homeomorphism relation. Similar processes of iterating some derivative can be used to classify well-founded trees, superatomic Boolean algebras, and reduced Abelian p-groups. In this paper we investigate the proof-theoretic strength of various statements directly relating to the existence of invariants and ranks on these classes. We do it from the viewpoint of Reverse Mathematics; we refer the reader to [Sim99] for more information about the program of Reverse Mathematics. We assume that the reader is familiar with at least the introductory chapter of [Sim99].

It turns out that in some sense, the structures under consideration effectively code the ordinals which are their ranks. Thus, the study of these structures is closely related to two issues: the translation processes between these classes, which reduce statements about one class to another (and in particular, to ordinals); and the strength of related questions for the class of ordinals. The corresponding subsystem of second-order arithmetic is ATR_0, the system which allows us to iterate arithmetic comprehension along ordinals. To quote Simpson ([Sim99, Page 176]):

> ... ATR_0 is the weakest set of axioms which permits the development of a decent theory of countable ordinals.

Our general aim is to demonstrate that a similar statement can be made for well-founded trees, superatomic Boolean algebras, etc.; general statements about these classes will be shown to be equivalent to ATR_0, thereby implying the necessary use of ordinal ranks in the investigation of these classes. Our work continues investigations of ordinals (Friedman and Hirst [FH90], see [Hir] for a survey), of reduced Abelian p-groups (see Simpson [Sim99] and Friedman, Simpson and Smith [FSS83]), of countable compact metric spaces (see Friedman [Fria] and Frideman and Hirst [FH91]), and of well-founded directed graphs (Hirst [Hir00]).

As we mentioned, key tools for establishing our results are reductions between various classes of objects. These reductions are an interesting object of study in their own right. Indeed we have two points of view: classical - we investigate when there are continuous (or even computable) reductions of one class to another; and proof-theoretic - we ask in what system can one show that these reductions indeed preserve notions such as isomorphism and embeddability.

8.1.1 Reverse mathematics

In this paper we only use common subsystems of second-order arithmetic. The base theory we use will usually be RCA_0 - the system that consists of the semi-ring axioms, Δ^0_1 comprehension and Σ^0_1 induction. We often use the stronger system ACA_0 which adds comprehension for arithmetic formulas. We note that over RCA_0, ACA_0 is equivalent to the existence of the range of any one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$.

The focus, though, is Friedman's even stronger system ATR_0 which enables us to iterate arithmetical comprehension along any well-ordering. As mentioned above, this is the system which is both sufficient and necessary for a theory of ordinals in second-order arithmetic. For example, comparability of well-orderings is equivalent to ATR_0 over RCA_0.

For recursion-theoretic intuition, we mention that ATR_0 is equivalent to the statement that for every $X \subseteq \mathbb{N}$ and every ordinal α, $X^{(\alpha)}$, the α^{th} iterate of the Turing jump of X, exists.

8.1.2 The classes

We discuss the various classes of objects with which we deal only briefly in this introduction, as greater detail will be given at the beginning of each section. The common feature of these classes is that they form the "well-founded part" of a larger class which is simply (arithmetically) definable (whereas the classes themselves are usually Π^1_1). [All structures are naturally coded as subsets of $\mathbb{N}$ and so the classes can be considered as sets of reals.] The fact that the larger class has both well-founded and ill-founded elements will usually imply large complexity: the isomorphism relation will be Σ^1_1-complete, and natural statements about the class will require Π^1_1-comprehension. On the other hand, when we focus our attention on the well-founded part, the hyperarithmetic hierarchy (and ATR_0) suffice.

When we discuss each class in detail, we specify a notion of isomorphism $\cong$ and a notion of embedding $\preccurlyeq$; we also define the notion of rank and describe which structures are ranked.

- The class of ordinals, that is, well-orderings of natural numbers (which we denote by $\mathcal{O}n$), is of course a sub-class of the class of linear orderings. For embedding we use weak embeddings (one-to-one, order preserving maps).
- We let $\mathcal{WFT}$ denote the class of well-founded trees, a sub-class of the class of trees of height $\leqslant \omega$. As the tree structure we take not only the partial ordering but also the predecessor relation; we thus may assume that all trees are trees of finite sequences of natural numbers (with the extension relation). The notion of embedding only requires preservation of strict order, so an embedding isn't necessarily one-to-one.
- $\mathcal{SABA}$ denotes the class of superatomic Boolean algebras (a sub-class of the class of Boolean algebras). As far as we know, this class has not been discussed in the setting of reverse mathematics, and so we give a detailed treatment of various definitions and their proof-theoretic content.
- Fixing a prime number p, we let $\mathcal{R}$-p-$\mathcal{G}$ denote the class of reduced Abelian p-groups, a sub-class of the class of all p-groups.
- On the analytic side, we let $\mathcal{CCS}$ denote the class of compact, very countable topological spaces. *Very countable* means Hausdorff, countable and second countable. It turns out that each countable, compact Hausdorff space is second countable, but in the setting of second-order arithmetic, we can only treat very countable spaces as reals (so this last statement is not expressible in this setting). In fact, we show that the compact spaces are all metrizable, and so the class coincides with countable, compact metric spaces. However, all properties we discuss are purely topological and so we pick the topological presentation. To be strict, the class of compact spaces does not consist of all "well-founded" very countable spaces; the latter class (the class of *scattered* spaces) is larger, but ill behaved, so we restrict ourselves to the compact case. As isomorphisms we take homeomorphisms, and as embeddings we take one-to-one, continuous and open maps.

8.1.3 The statements

We now discuss the various statements we analyze. Let $\mathcal{X}$ be a class of structures as above, equipped with a notion of isomorphism $\cong$, a notion of embeddability $\preccurlyeq$, and a subclass of ranked structures.

Rank

For each of the classes we study, there is a notion of derivative which is analogous to the Cantor-Bendixon operation of removing isolated points (such as removing leaves from trees or eliminating atoms in Boolean algebras by means of a quotient). Iterating the derivative yields a rank and an invariant, which has the expected properties (for example, it characterizes the isomorphism relation and is well-behaved with regards to the embeddability relation). When dealing with the class $\mathcal{X}$, we define this rank formally and thus the class of structures in $\mathcal{X}$ which are ranked. We thus define the following statement:

Statement 8.1.1. $\mathsf{RK}(\mathcal{X})$ Every structure in $\mathcal{X}$ is ranked.

We remark that we often show directly that $\mathsf{RK}(\mathcal{X})$ implies other statements $\varphi(\mathcal{X})$ (without appealing to ATR_0).

Implications of invariants

Suppose that an invariant for isomorphism for the class $\mathcal{X}$ exists. Now as this is a third-order statement, we follow Shore ([Sho]) and discuss a statement which is immediately implied by this existence. Suppose that a sequence $\{A_n\}_{n\in\mathbb{N}}$ of structures in $\mathcal{X}$ is given; if each A_n is uniformly assigned a simple object which characterized its isomorphism type, then we could uniformly decide which pairs (A_n, A_m) are isomorphic. We thus define:

Statement 8.1.2. $\exists$-$\mathsf{ISO}(\mathcal{X})$ If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{X}$, then the set $\{(n,m) : A_n \cong A_m\}$ exists.

Suppose that the invariant is even stronger; that the simpler objects assigned are quasi-ordered and that the invariant preserves the notion of embeddability. Then, as above, we could decide the embeddability relation. Thus we define:

Statement 8.1.3. $\exists$-$\mathsf{EMB}(\mathcal{X})$ If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{X}$, then the set $\{(n,m) : A_n \preccurlyeq A_m\}$ exists.

Natural statements

We define simple statements which are elementary in the analysis of the class $\mathcal{X}$. Relating to Simpson's words, we consider these statements (when true) necessary for the study of $\mathcal{X}$.

Statement 8.1.4. $\mathsf{COMP}(\mathcal{X})$ For every A and B in $\mathcal{X}$, either $A \preccurlyeq B$ or $B \preccurlyeq A$.

Statement 8.1.5. $\mathsf{EQU{=}ISO}(\mathcal{X})$ For every A and B in $\mathcal{X}$, if $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \cong B$.

The structure of the embeddability relation

It turns out that for the classes that we study, the embeddability relation is well-founded; moreover, it forms a *well-quasi ordering*: whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{X}$, there are some $n < m$ such that $A_n \preccurlyeq A_m$. This fact can be added to our list.

[Another familiar definition for the notion of well-quasi-orderings is a quasi-ordering which has no infinite descending sequences and no infinite antichains. However, this equivalence uses Ramsey's theorem for pairs, and so cannot be carried out in our base theory RCA_0. In fact, the equivalence uses the existence of the embeddability relation (to which Ramsey's theorem is applied) and so by our results the standard proof uses ATR_0. (See [CMS04] for a comparison of the different definitions of well-quasi-orderings from the viewpoint of reverse mathematics.)]

Statement 8.1.6. $\mathsf{WQO}(\mathcal{X})$ The class $\mathcal{X}$, quasi-ordered by $\preccurlyeq$, forms a well-quasi ordering.

8.1.4 Reductions

As we mentioned, reductions between classes of structures provide means of proving equivalences to ATR_0 be means of reducing statements from class to class. However, these reductions are interesting in their own right. It turns out many of the classes in question are as equivalent as they can be.

We consider two kinds of reductions. One should perhaps be called "effective Wadge" reducibility. The classes we consider are complicated in the sense that membership is often Π^1_1-complete. Recall that each class $\mathcal{X}$ we consider is the well-founded part of a simpler class $\mathcal{Y}$ which is arithmetic. "Effective Wadge" reducibility is the analogue of many-one reducibility in the context of sets of reals.

Definition 8.1.7. For a pair of classes $\mathcal{X}_1, \mathcal{X}_2$ which are subclasses of simpler classes $\mathcal{Y}_1$ and $\mathcal{Y}_2$, we say that $\mathcal{X}_1$ *is EW-reducible to* $\mathcal{X}_2$ *within* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ (and write $\mathcal{X}_1 \leqslant_{EW} \mathcal{X}_2$) if there is some computable functional $\Phi\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ such that $\Phi^{-1}\mathcal{X}_2 = \mathcal{X}_1$.

The idea is that the question of membership in $\mathcal{X}_1$ is effectively reduced to an oracle which gives us membership for $\mathcal{X}_2$.

Another notion of reducibility is closer to the notion of Borel reducibility for Borel equivalence relations, which is extensively investigated by descriptive set-theorists (see, for instance, [HK01]). Here we consider not the elements of classes $\mathcal{X}_1$ and $\mathcal{X}_2$ but rather the collection of isomorphism types of these classes, and we look for an embedding of one class into the other which is induced by a computable transformation. The structures we work with may have domain which is a proper subset of ω. If we would like to factor out the influence of the complexity of the domain, we arrive at the following definition made by Calvert, Cummins, Knight and S. Miller ([CCKM]), which is in fact stronger than a mere embedding induced by a computable function:

Definition 8.1.8. A *computable transformation* of a class of structures $\mathcal{X}_1$ to another class $\mathcal{X}_2$ is a function $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ for which there is some recursively enumerable collection Φ such that for all $A \in \mathcal{X}_1$, for every finite collection of statements b in the language of $\mathcal{X}_2$, $b \subseteq D(f(A))$ (the atomic diagram of $f(A)$) iff there is some finite collection $a \subseteq D(A)$ such that $(a, b) \in \Phi$. A computable transformation f is an *embedding* if f preserves $\cong$ and $\ncong$. We write $\mathcal{X}_1 \leqslant_c \mathcal{X}_2$ if there is a computable embedding of $\mathcal{X}_1$ into $\mathcal{X}_2$.

Another way to think of computable transformations is as functionals which from any enumeration of $D(A)$ produce, uniformly, an enumeration of $D(f(A))$.

Computable embeddings (unlike Turing embeddings) preserve the substructure relation, hence preserve embeddability.

In some cases, we cannot have computable reductions: for example, from well-founded trees to ordinals - simply because $\mathsf{EQU}{=}\mathsf{ISO}(\mathcal{WFT})$ fails. If instead of isomorphism classes we consider equimorphism (bi-embeddability) classes, we get a slightly different notion of reduction. This reduction is not just a one-to-one map of $\mathcal{X}_1$-equimorphism types into $\mathcal{X}_2$-equimorphism types; it preserves the partial ordering on these equivalence classes induced by embeddability.

Definition 8.1.9. For classes of structures $\mathcal{X}_1$ and $\mathcal{X}_2$, $\mathcal{X}_1$ is *equicomputably reducible* to $\mathcal{X}_2$ (we write $\mathcal{X}_1 \leqslant_{ec} \mathcal{X}_2$) if there is a computable transformation $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ which preserves both $\preccurlyeq$ and $\not\preccurlyeq$.

We introduce notation which indicates that two reductions are induced by the same function. For example, $\mathcal{X}_1 \leqslant_{EW,c} \mathcal{X}_2$ if there is some computable transformation f which is both an EW- and a c-reduction of $\mathcal{X}_1$ to $\mathcal{X}_2$.

8.1.5 Results

We first consider the proof-theoretic strength of the various statements we discussed earlier.

Theorem 8.1.10.

	RK	$\exists$-ISO	$\exists$-EMB	COMP	$\mathsf{EQU}{=}\mathsf{ISO}$	WQO
$\mathcal{O}n$	N/A	✓	✓	✓	✓	✓
$\mathcal{WFT}$	✓	✓	✓	✓	F	✓
$\mathcal{SABA}$	✓	✓	✓	✓	✓	✓
$\mathcal{R}$-p-$\mathcal{G}$	✓	✓	✓	F	F	✓
$\mathcal{CCS}$	✓	✓	✓	✓	✓	✓

For a statement φ and class $\mathcal{X}$, a ✓ indicates that $\varphi(\mathcal{X})$ is equivalent to ATR_0 over RCA_0. A square labelled by "F" indicates that $\varphi(\mathcal{X})$ is false. A square labelled by "N/A" *indicates that $\varphi(\mathcal{X})$ is meaningless.*

Of course, not all of these results are new. Friedman and Hirst ([FH90]) showed that both $\mathsf{COMP}(\mathcal{O}n)$ and $\mathsf{EQU{=}ISO}(\mathcal{O}n)$ are equivalent to ATR_0 over RCA_0. Shore ([Sho93]) showed that $\mathsf{WQO}(\mathcal{O}n)$ is equivalent to ATR_0 over RCA_0. Hirst ([Hir00]) showed that ATR_0 implies $\mathsf{RK}(\mathcal{WFT})$ (actually he proved that every well-founded directed graph is ranked, which implies the result for trees.) Friedman, Simpson and Smith showed that ATR_0 is equivalent to $\mathsf{RK}(\mathcal{R}\text{-p-}\mathcal{G})$ over RCA_0 (see [Sim99, Theorem V.7.3]). In [Frib], Friedman shows that over ACA_0, $\mathsf{WQO}(\mathcal{R}\text{-p-}\mathcal{G})$ is equivalent to ATR_0, and leaves open the question of whether the equivalence can be proved over RCA_0. Shore and Solomon (unpublished) proved that $\exists\text{-}\mathsf{ISO}(\mathcal{R}\text{-p-}\mathcal{G})$ is equivalent to ATR_0 over RCA_0. That $\mathsf{COMP}(\mathcal{CCS})$ is equivalent to ATR_0 (over ACA_0) can be deduced from results in either [FH91] or [Fria].

Next, we turn to reducibilities. The classes we deal with are all highly equivalent. In second order arithmetic we often find that ATR_0 shows the existence of reductions between the classes; in fact, usually what we really use is the fact that structures are ranked. As this comes for free when the original class is the class of ordinals we can usually show reductions from ordinals to other classes in weaker systems. However, we do not have reversals to ATR_0 from the statements asserting the existence of computable reductions starting from other classes. In particular it is interesting to know if for a class $\mathcal{X}$, the existence of a computable reduction from $\mathcal{X}$ to the ordinals is as strong as ATR_0, as we can think of that statement as another way to say that invariants for $\mathcal{X}$ exist.

Theorem 8.1.11.

1. *Let* $\mathcal{X} \in \{\mathcal{WFT}, \mathcal{SABA}, \mathcal{R}\text{-p-}\mathcal{G}, \mathcal{CCS}\}$. *Then in* RCA_0 *we can show that* $\mathcal{O}n \leqslant_{EW,c} \mathcal{X}$. *In* ACA_0 *we can show that* $\mathcal{O}n \leqslant_{EW,c,ec} \mathcal{X}$.
2. ATR_0 *implies the following:* $\mathcal{WFT} \leqslant_{EW,ec} \mathcal{O}n$, $\mathcal{SABA} \leqslant_{EW,c,ec} \mathcal{O}n$ *and* $\mathcal{R}\text{-p-}\mathcal{G} \leqslant_{EW} \mathcal{O}n$.

We remark that EW-equivalence of all of our classes follows immediately from the fact that $\mathcal{O}n$ is EW-reducible to every other class. For $\mathcal{O}n$ is Π^1_1-complete, and these reductions show that each of our classes is Π^1_1-complete, hence all EW-equivalent. The extra information here is that these reductions can be made by computable transformations (rather than merely Turing reductions), and furthermore these transformations often preserve isomorphism, non-isomorphism, etc.

Note that we don't have reductions from $\mathcal{CCS}$ to other classes. Turing reductions can be found, but computable transformations have not been found yet.

We also note that $\mathcal{R}\text{-p-}\mathcal{G} \nleqslant_{ec} \mathcal{O}n$. This is because $\preccurlyeq$ is not a total relation on $\mathcal{R}\text{-p-}\mathcal{G}$.

Proofs of the various parts of the theorems appear in the relevant sections.

8.1.6 More Results

The last section of this paper is not about Reverse Mathematics as are the previous ones. Rather, it is about a property shared by all the classes of structures we study.

Clifford Spector proved the following well known classical theorem in Computable Mathematics.

Theorem 8.1.12. *[Spe55] Every hyperarithmetic well ordering is isomorphic to a recursive one.*

This result was later extended in Chapter 9 as follows.

Theorem 8.1.13. *Every hyperarithmetic linear ordering is equimorphic to a recursive one.*

Note that Theorem 8.1.13 extends Spector's theorem because if a linear ordering is equimorphic to an ordinal, it is actually isomorphic to it.

As for the connection to ATR_0, this result can be extended to classes of structures studied in this paper. For example, Ash and Knight mention the following:

Theorem 8.1.14. *[AK00] Every hyperarithmetic superatomic Boolean algebra is isomorphic to a recursive one.*

The two following theorems are straightforward, the third less so. We give proofs for all in section 8.7.

Theorem 8.1.15.

1. *Every hyperarithmetic tree is equimorphic with a recursive one.*
2. *Every hyperarithmetic Boolean algebra is equimorphic with a recursive one.*

Theorem 8.1.16. *Every hyperarithmetic compact metric space is isomorphic to a computable one.*

Theorem 8.1.17. *Every hyperarithmetic Abelian p-group is equimorphic with a recursive one.*

We refer the reader to [AK00, Chapter 5] or to [Sac90b] for background on hyperarithmetic theory.

8.2 Ordinals

A survey of the theory of ordinals in reverse mathematics can be found in [Hir]. We follow his notation and definitions. As our notion of embedding we take $\leqslant_w$, an order-preserving injection.

We first show below (proposition 8.2.1) that the statement $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ is equivalent to ATR_0 over ACA_0; we then mention some facts about the Kleene-Brouwer ordering of a tree - these will be useful also in later sections). Using this results we show (8.2.6) that $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ is equivalent to ATR_0 over ACA_0. Finally (8.2.7) we reduce the base to RCA_0 for both statements.

Equivalences of other statements about ordinals to ATR_0 are not new; references were made in the introduction.

8.2.1 Equivalence over ACA_0

Proposition 8.2.1 (ACA_0). *$\exists$-$\mathsf{ISO}(\mathcal{O}n)$ is equivalent to ATR_0.*

PROOF: First assume ATR_0 and let $\{\alpha_n : n \in \mathbb{N}\}$ be a sequence of ordinals. For all $i < j$, there is a unique comparison map between α_i and α_j. This shows that $\{(i,j) \,:\, \alpha_i \equiv \alpha_j\}$ is Δ^1_1-definable. By Δ^1_1-comprehension, which holds in ATR_0 ([Sim99, Lemma VIII.4.1]), this set exists. Thus $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ holds.

Suppose now that $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ holds. We will prove $\mathsf{COMP}(\mathcal{O}n)$, which implies ATR_0. Let α and β be ordinals. Let

$$F = \{(x,y) \in \alpha \times \beta \,:\, \alpha \restriction x \cong \beta \restriction y\}$$

(where $\alpha \restriction x$ is the induced ordering from α on the collection of α-predecessors of x). This set exists by $\exists$-$\mathsf{ISO}(\mathcal{O}n)$. We claim that F itself is a comparison map between α and β.

Recall that no ordinal can be isomorphic to any of its proper initial segments. It follows that F is a one-to-one function on its domain. Further, we observe that if $(x,y),(x',y') \in F$ then $x < x'$ (in α) if and only if $y < y'$ (in β). For if not, say $x < x'$ and $y > y'$, we compose the isomorphisms $\alpha \restriction x' \to \beta \restriction y'$ and $\beta \restriction y \to \alpha \restriction x$ to get an isomorphism between $\alpha \restriction x'$ and an initial segment of $\alpha \restriction x$. Also not hard to prove is that $\operatorname{dom} F$ and $\operatorname{range} F$ are initial segments of α and β (this is where we use ACA_0).

F is an isomorphism between $\operatorname{dom} F$ and $\operatorname{range} F$. If they are both proper initial segments of α and β respectively, let $\alpha \restriction x = \operatorname{dom} F$, $\beta \restriction y = \operatorname{range} F$. Then F witnesses that $(x,y) \in F$ for a contradiction. $\square$

We could prove that $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ is equivalent to ATR_0 over ACA_0 using a similar argument. Instead we give a different proof.

Remark 8.2.2. In the following, we use effective (Δ^0_1)-transfinite recursion in RCA_0. The proof that it works is the classical one (using the recursion (fixed-point) theorem). Also, we may perform the recursion along any well-founded relation (not necessarily linear).

Remark 8.2.3. Recall that in RCA_0, for every ordinal α one can construct the linear ordering ω^α. In fact, for any linear ordering L, one can construct ω^L in an analogous fashion. However, one needs ACA_0 to show that if α is an ordinal then so is ω^α. See Hirst [Hir94].

Recall the following: a *tree* is a downwards closed subset of $\mathbb{N}^{<\mathbb{N}}$; a tree is well-founded if it does not have an infinite path (all common definitions coincide in RCA_0). For linear orderings X and Y, $T(X,Y)$ denotes the *tree of double descent* for X which consists of the descending sequences in the partial ordering $X \times Y$. For a tree T, $\mathrm{KB}(T)$ denotes the Kleene-Brouwer ordering on T ([Sim99, Section V.1]); $X * Y = \mathrm{KB}(T(X,Y))$. In RCA_0 we know that if either X or Y are well-founded then so is $T(X,Y)$. In ACA_0 we know that T is well-founded iff $\mathrm{KB}(T)$ is.

Lemma 8.2.4 (RCA_0). *Let α be an ordinal and L be a linear ordering. Then there is an embedding of $\alpha * L$ into $\omega^\alpha + 1$.*

PROOF: Let $T = T(\alpha, L)$; we know that T is well-founded. By effective transfinite recursion on T we construct, for every $\sigma \in T$ with last element (β, l), a recursive function $i_\sigma \colon T_\sigma \to \omega^\beta + 1$ (if $\sigma = \langle\rangle$ then $\beta = \alpha$), where $T_\sigma = \{\tau \in T : \sigma \subseteq \tau\}$. Given $i_{\sigma^\frown x}$ for every $x < \omega$ (such that $\sigma^\frown x \in T$), we construct i_σ by pasting these $i_{\sigma^\frown x}$s linearly and placing σ at the end. In detail: Let $S = \{x \in \mathbb{N} : \sigma^\frown x \in T\}$. For $x \in S$, let $\beta_x = (x)_0$. We have $i_\sigma \colon T_{\sigma^\frown x} \to \omega^{\beta_x}$. For $x \in S$ let

$$\gamma_x = \sum_{y <_{\mathbb{N}} x, y \in S} (\omega^{\beta_y} + 1)$$

which is smaller than ω^β; for $\tau \in T_{\sigma^\frown x}$ let $i_\sigma(\tau) = \gamma_x + i_{\sigma^\frown x}(\tau)$. Finally let $i_\sigma(\sigma) = \omega^\beta$.

Now by Π^0_1-transfinite induction on T, which holds in RCA_0 ([Hir]), we can show that for all $\sigma \in T$, for all $\tau_0, \tau_1 \in T_\sigma$, $i_\sigma(\tau_0) < i_\sigma(\tau_1)$ iff $\tau_0 <_{\mathrm{KB}} \tau_1$. □

Remark 8.2.5 *(ACA_0).* For the next proof, we need the fact that if α is an ordinal and L is a non-well-founded linear ordering, then α embeds into $\alpha * L$ ([Sim99, Lemma V.6.5]). The embedding is obtained by considering $T(\alpha)$, the tree of (single) descent of elements of α. We first embed α into $\mathrm{KB}(T(\alpha))$ by taking $\beta < \alpha$ to the KB-least $\sigma \in T(\alpha)$ whose last element is β. Next we embed $T(\alpha)$ into $T(\alpha, L)$ by fixing a descending sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ of L and taking $\langle \beta_1, \dots, \beta_n \rangle$ to $\langle (\beta_1, x_1), \dots, (\beta_n, x_n) \rangle$. To see that this embedding induces an embedding of $\mathrm{KB}(T(\alpha))$ into $\mathrm{KB}(T(\alpha, L))$ we note that if $a <_{\mathbb{N}} b$ then for all x, $(a,x) <_{\mathbb{N}} (b,x)$.

Proposition 8.2.6 (ACA_0). *$\exists$-$\mathsf{EMB}(\mathcal{O}n)$ is equivalent to ATR_0.*

PROOF: We show ATR_0 by showing the equivalent principle of Σ^1_1-separation ([Sim99, Theorem V.5.1]). Suppose that φ, ψ are Σ^1_1 formulas which define disjoint classes of natural numbers. From φ and ψ we can manufacture sequences $\langle X_n \rangle$ and $\langle Y_n \rangle$ of linear orderings such that for all n, X_n is a well-ordering iff $\neg\varphi(n)$ and Y_n is a well-ordering iff $\neg\psi(n)$. Let

$$\alpha_n = (\omega^{X_n} + 2) * Y_n$$

and

$$\beta_n = X_n * (\omega^{Y_n} + 2).$$

Note that at least one of X_n and Y_n are well-founded (and X_n is well-founded implies ω^{X_n} well-founded) thus both α_n and β_n are indeed ordinals.

Suppose that X_n is well-founded and that Y_n is not. We claim that β_n embeds into α_n but α_n does not embed into β_n. By Lemma 8.2.4, $\beta_n \preccurlyeq \omega^{X_n} + 1$ and by Remark 8.2.5, $\omega^{X_n} + 2 \preccurlyeq \alpha_n$. So $\beta_n \preccurlyeq \alpha_n$, but we cannot have $\alpha_n \preccurlyeq \beta_n$ or we would have $\omega^{X_n} + 2 \preccurlyeq \omega^{X_n} + 1$ which is impossible.

We can thus let $A = \{n : \beta_n \preccurlyeq \alpha_n\}$. By $\exists$-$\mathsf{EMB}(\mathcal{O}n)$, A exists. If $\psi(n)$ holds then X_n is an ordinal and Y_n is not, and so $n \in A$. If $\varphi(n)$ holds then by a similar argument we get $\beta_n \not\preccurlyeq \alpha_n$ so $n \notin A$, as required. □

8.2.2 Proofs of arithmetic comprehension

Proposition 8.2.7 (RCA_0). *Both $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ and $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ imply ACA_0.*

PROOF: Let φ be a Σ^0_1 formula. For each n, construct an ordinal α_n by letting $\alpha_n \cong 3$ if $\neg\varphi(n)$ and $\alpha_n \cong 17$ if $\varphi(n)$.

Now
$$\{n : \varphi(n)\} = \{n : \alpha_n \cong 17\} = \{n : \alpha_n \not\preccurlyeq 5\}.$$
$\exists$-$\mathsf{ISO}(\mathcal{O}n)$ implies the second set exists; $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ implies that the third set exists. □

8.3 Well-founded trees

We denote the class of well-founded trees by $\mathcal{WFT}$. If T, S are trees then $T \preccurlyeq S$ if there is some $f : T \to S$ which preserves strict inclusion. Note that f does not need to preserve non-inclusion, in fact f may be not injective.

The layout of this section is fairly straightforward. The standard rank of a well-founded tree is defined in the language of second-order arithmetic; we mention that Hirst showed that ATR_0 implies that every well-founded tree is ranked. We then show how to get the other (true) statements from $\mathsf{RK}(\mathcal{WFT})$, except for $\exists$-$\mathsf{ISO}(\mathcal{WFT})$, which follows directly from ATR_0.

We then define the reduction $L \mapsto T(L)$ which maps ordinals to well-founded trees, and use this reduction to get reversals. To get the reduction from trees to ordinals, we need the notion of a *fat tree* which we discuss in subsection 8.3.2. In the last subsection we derive ACA_0 from the statements for which the previous reversals required this comprehension.

Notation 8.3.1. Let T be a tree and $\sigma \in T$. Then $T[\sigma] = \{\tau \in T : \tau \not\subset \sigma\}$, $T - \sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $\sigma^\frown T = \{\sigma^\frown\tau : \tau \in T\}$.

8.3.1 Ranked Trees

Definition 8.3.2. Let T be a tree. A node $\tau \in T$ is an *immediate successor* of a node σ if $\sigma \subseteq \tau$ and $|\tau| = |\sigma| + 1$. A function $\mathrm{rk}: T \to \alpha$ for some ordinal α is *a rank function* for T if for every $\sigma \in T$, $\mathrm{rk}(\sigma) = \sup\{\mathrm{rk}(\tau) + 1 :$

τ is an immediate successor of σ on $T\}$, and further $\alpha = \mathrm{rk}(\langle\rangle) + 1$. We say that a tree T is *ranked* if a rank function of T exists.

Lemma 8.3.3 (RCA_0). *Let $f : T \to \alpha$ be a rank function on a tree T. Then* $\operatorname{range} f = \alpha$.

PROOF: Let T be a well-founded tree and $\mathrm{rk}\colon T \to \alpha$ a rank function on it. Suppose, toward a contradiction, that there is a $\gamma < \alpha$ not in the range of rk. We prove by Π^0_1-transfinite induction that every β such that $\gamma < \beta < \alpha$ is not in the range of rk. This will contradict that $\alpha = \mathrm{rk}(\langle\rangle) + 1$.

Suppose that every β' between γ and β is not in the range of rk. Then for no node σ can we have $\mathrm{rk}(\sigma) = \sup\{\mathrm{rk}(\tau){+}1 : \tau$ is an immediate successor of σ on $T\}$. □

Let T be a tree and rk be a rank function on T. Let $\sigma \in T$. By Π^0_1-induction on $|\tau|$ we can show in RCA_0 that if $\sigma \subsetneq \tau$ then $\mathrm{rk}(\tau) < \mathrm{rk}(\sigma)$. It follows that

$$\mathrm{rk}(\sigma) = \sup\{\mathrm{rk}(\tau) + 1 \,:\, \tau \in T, \sigma \subsetneq \tau\}.$$

Another immediate corollary is:

Lemma 8.3.4 (RCA_0). *Every ranked tree is well-founded.*

(As an infinite path through T would give rise to a descending sequence in T's rank.)

The following two propositions are proved in [Hir00] for well-founded directed graphs, a class which essentially contains the class of trees.

Proposition 8.3.5 (RCA_0). *Let T be a tree and let $f_1\colon T \to \alpha_1$ and $f_2\colon T \to \alpha_2$ be rank functions. Then there is a bijection $g\colon \alpha_1 \to \alpha_2$ such that $f_2 = g \circ f_1$.*

Thus ranks are unique up to isomorphism; if T is ranked by a function $f\colon T \to \alpha$ then we let $\mathrm{rk}(T) = \alpha - 1 = f(\langle\rangle)$. (Note though that most set theory texts let $\mathrm{rk}(T) = \alpha$).

Proposition 8.3.6 (ATR_0). *Every well founded tree is ranked.*

Implications of rank

Lemma 8.3.7 (RCA_0). *Suppose that S and T are ranked trees and that* $\mathrm{rk}(S) \preccurlyeq \mathrm{rk}(T)$. *Then $S \preccurlyeq T$.*

PROOF: Let $g\colon \mathrm{rk}(S) \to \mathrm{rk}(T)$ be an embedding of ordinals. For each $\sigma \in S$, we define $f(\sigma) \in T$ by induction on $|\sigma|$. Along the construction we make sure at every step that for every $\sigma \in S$, $g(\mathrm{rk}_S(\sigma)) \leqslant \mathrm{rk}_T(f(\sigma))$. Let $f(\langle\rangle) = \langle\rangle$. Suppose we have defined $f(\sigma)$ and we want to define $f(\tau)$ where τ is an immediate successor of σ on S. Since $g(\mathrm{rk}_S(\tau)) < g(\mathrm{rk}_S(\sigma)) \leqslant \mathrm{rk}_T(f(\sigma))$, there exists $\pi \supsetneq f(\sigma)$ with $\mathrm{rk}_T(\pi) \geqslant g(\mathrm{rk}_S(\tau))$. Let $f(\tau)$ be the $<_{\mathbb{N}}$-least such π. □

Lemma 8.3.8 (ACA_0). *Let S and T be ranked trees and assume that $S \preccurlyeq T$. Then $\mathrm{rk}(S) \preccurlyeq \mathrm{rk}(T)$.*

Proof: Suppose first that there is an embedding $f\colon S \to T$; we want to construct an embedding $g\colon \mathrm{rk}(S) \to \mathrm{rk}(T)$. Given $\alpha < \mathrm{rk}(S)$, let $g(\alpha) = \min(\mathrm{rk}_T(f(\sigma)) : \sigma \in S \;\&\; \mathrm{rk}_S(\sigma) = \alpha)$. We claim that g is an embedding of $\mathrm{rk}(S)$ into $\mathrm{rk}(T)$. Consider $\alpha_0 < \alpha_1 < \mathrm{rk}(S)$. Let $\sigma \in S$ be such that $\mathrm{rk}_S(\sigma) = \alpha_1$ and $g(\alpha_1) = \mathrm{rk}_T(f(\sigma))$. Let $\tau \supsetneq \sigma$ be such that $\mathrm{rk}_S(\tau) = \alpha_0$. Such τ exists by Lemma 8.3.3 applied to S_σ. Since $f(\tau) \supsetneq f(\sigma)$, $g(\alpha_1) = \mathrm{rk}_T(f(\sigma)) > \mathrm{rk}_T(f(\tau)) \geqslant g(\alpha_0)$. □

Corollary 8.3.9 (ACA_0). *If every well-founded tree is ranked then $\exists$-$\mathsf{EMB}(\mathcal{WFT})$ holds.*

Proof: Let $\langle T_n\rangle$ be a sequence of well-founded trees. Let $T = \bigoplus T_n$; this is the tree obtained by placing a common root below all of the T_ns: $T = \{\langle\rangle\} \cup \bigcup_n \langle n\rangle^\frown T_n$. The tree T is well-founded and so has a rank function rk_T.

For $n, m \in \mathbb{N}$, $\mathrm{rk}(T_n) \preccurlyeq \mathrm{rk}(T_m)$ iff $\mathrm{rk}_T(\langle n\rangle) \leqslant \mathrm{rk}_T(\langle m\rangle)$. This is because $\mathrm{rk}_T \restriction T_n$ is a rank function for T_n (of course we mean $\langle n\rangle^\frown T_n$); and because for $\beta, \gamma < \mathrm{rk}(T)$, $\beta \preccurlyeq \gamma$ iff $\beta \leqslant \gamma$. It follows that $T_n \preccurlyeq T_m$ iff $\mathrm{rk}_T(\langle m\rangle) \leqslant \mathrm{rk}_T(\langle n\rangle)$ so the set $\{(n,m) : T_n \preccurlyeq T_m\}$ exists. □

Corollary 8.3.10 (RCA_0). *If every well-founded tree is ranked then $\mathsf{COMP}(\mathcal{WFT})$ holds.*

Proof: Let T, S be well-founded trees; let rk^* be a rank function on $T \oplus S$ (as before this is $\{\langle\rangle\} \cup 0^\frown T \cup 1^\frown S$). Let $\alpha = \mathrm{rk}^*(\langle 0\rangle)$ and $\beta = \mathrm{rk}^*(\langle 1\rangle)$. Now, since $\alpha, \beta < \mathrm{rk}(T \oplus S)$, either $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$; suppose the former. Then $\mathrm{rk}(T) \preccurlyeq \mathrm{rk}(S)$. It follows that $T \preccurlyeq S$. □

Corollary 8.3.11 (RCA_0). *If every well-founded tree is ranked then $\mathsf{WQO}(\mathcal{WFT})$ holds.*

Proof: Let $\langle T_n\rangle$ be a sequence of well-founded trees. Let $T = \bigoplus T_n$ and let rk_T be a rank function on T. Now $\langle \mathrm{rk}_T(\langle n\rangle)\rangle_{n\in\mathbb{N}}$ cannot be strictly decreasing. It follows that for some $n \leqslant m$ we have $\mathrm{rk}_T\langle n\rangle \leqslant \mathrm{rk}_T\langle m\rangle$ so $\mathrm{rk}(T_n) \preccurlyeq \mathrm{rk}(T_m)$ so $T_n \preccurlyeq T_m$. □

We have no direct argument to get $\exists$-$\mathsf{ISO}(\mathcal{WFT})$ from $\mathsf{RK}(\mathcal{WFT})$. Rather, we give an argument from ATR_0.

Proposition 8.3.12 (ATR_0). $\exists$-$\mathsf{ISO}(\mathcal{WFT})$ *holds.*

Proof: We prove that given two recursive trees T and S, both of rank α, we can decide whether $T \cong S$ recursively uniformly in $0^{(3\alpha+3)}$, which exists by ATR_0. We do it by effective transfinite induction. For each i let $T_i = T - \langle i\rangle, S_i = S - \langle i\rangle$. Observe that $T \cong S$ if and only if for every i, the number of trees T_j such that $T_j \cong T_i$ is equal to the number of trees S_j such that $S_j \cong T_i$ (this number is possibly infinite). We can check whether $T_i \cong T_j$ and whether $T_i \cong S_j$ recursively uniformly in $0^{(3\alpha)}$, so we can check whether the sentence above holds recursively in $0^{(3\alpha+3)}$. □

A reversal

We will later get all reversals by translating ordinals into trees. However, we also have one direct reversal akin to the proof for ordinals (proposition 8.2.6); it is simpler. For any tree T, temporarily let $1+T = \{\langle\rangle\} \cup \langle 0\rangle^\frown T$. As for ordinals, if T is well-founded then we cannot have $1+T \preccurlyeq T$; for iterating the embedding on $\langle\rangle$ would yield a path in T.

Proposition 8.3.13 (ACA_0). *$\exists$-$\mathit{EMB}(\mathcal{WFT})$ implies ATR_0.*

PROOF: We show Σ^1_1-separation. Suppose that φ, ψ are Σ^1_1 formulas which define disjoint classes. From φ and ψ we can manufacture sequences $\langle T_n\rangle$ and $\langle S_n\rangle$ of trees such that for all n, T_n is a well-founded iff $\neg\varphi(n)$ and S_n is well-founded iff $\neg\psi(n)$.

Consider $A_n = T_n \times (1+S_n)$ and $B_n = (1+T_n) \times S_n$. Both A_n and B_n are well-founded for all n. Suppose that T_n is well founded and that S_n is not. We always have $T_n \preccurlyeq 1+T_n$; since S_n is not well-founded we have $1+S_n \preccurlyeq S_n$ (map everything onto an infinite path). Thus $A_n \preccurlyeq B_n$.

On the other hand, again since S_n is not well-founded, there is an embedding of $1+T_n$ into B_n (again use an infinite path for the second coordinate). By omitting the second coordinate, we have $A_n \preccurlyeq T_n$. It follows that we cannot have $B_n \preccurlyeq A_n$, or we would have $1+T_n \preccurlyeq T_n$. We can thus again let the separator be $\{n : B_n \not\preccurlyeq A_n\}$. □

8.3.2 Reductions

From Ordinals to Trees

Definition 8.3.14 (RCA_0). Given a linear ordering L, let $T(L)$ be the tree of L-decreasing sequences of elements of L.

It is easy to show in RCA_0 that for a linear ordering L, $T(L)$ is well-founded iff L is, so we get an EW-reduction. In RCA_0 we can show that $L \mapsto T(L)$ is a computable transformation.

Lemma 8.3.15 (RCA_0). *For every α, $T(\alpha)$ is ranked and has rank α.*

PROOF: For every nonzero $\sigma \in T(\alpha)$ let $\mathrm{rk}(\sigma)$ be the last element of σ, and let $\mathrm{rk}(\langle\rangle) = \alpha$. rk is indeed a rank function because for all $\sigma \in T$ of rank β, the set of ranks of immediate successors of σ is exactly all $\gamma < \beta$. □

The next corollary follows from 8.3.5; the one after it follows from 8.3.7 and 8.3.8.

Corollary 8.3.16 (RCA_0). *For all ordinals α and β, $\alpha \cong \beta$ iff $T(\alpha) \cong T(\beta)$.*

Corollary 8.3.17 (ACA_0). *Let α and β be ordinals. Then $\alpha \preccurlyeq \beta$ iff $T(\alpha) \preccurlyeq T(\beta)$.*

Proposition 8.3.18 (RCA_0). *$\exists$-ISO($\mathcal{WFT}$) implies ATR$_0$.*

PROOF: Corollary 8.3.16 shows that $\exists$-ISO($\mathcal{WFT}$) implies $\exists$-ISO($\mathcal{O}n$). $\square$

Proposition 8.3.19 (RCA_0). *RK($\mathcal{WFT}$) implies ATR$_0$.*

PROOF: We show that $\exists$-ISO($\mathcal{O}n$) holds. Let $\langle\alpha_n\rangle$ be a sequence of ordinals; consider $T = \bigoplus T(\alpha_n)$. Let rk be a rank function on T. Then $\alpha_n \cong \alpha_m$ iff $T(\alpha_n) \cong T(\alpha_m)$ iff $\mathrm{rk}(\langle n\rangle) = \mathrm{rk}(\langle m\rangle)$. $\square$

Proposition 8.3.20 (ACA_0). *$\exists$-EMB($\mathcal{WFT}$), COMP($\mathcal{WFT}$) and WQO($\mathcal{WFT}$) imply ATR$_0$.*

PROOF: It follows from the previous lemma that each of these three statements imply the corresponding ones for ordinals, and hence ATR_0. $\square$

From trees to ordinals

We describe fat trees.

Definition 8.3.21. Given a tree T, let T^∞ be the tree consisting of sequences of the form $\langle(\sigma_0, n_0), \dots, (\sigma_k, n_k)\rangle$ where $\langle\rangle \neq \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_k \in T$ and $n_i \in \mathbb{N}$.

Of course, in RCA_0, T is well-founded iff T^∞ is.

Let T be a tree; for $p \in T^\infty$ ending with the pair (τ, n) we let $i(p) = \tau$. We also let $i(\langle\rangle) = \langle\rangle$.

Lemma 8.3.22 (RCA_0). *Let T be a tree. Then $(T^\infty)^\infty \cong T^\infty$.*

PROOF: We define a map f from T^∞ to $(T^\infty)^\infty$ by induction. At every step we make sure that $i(p) = i(i(f(p)))$. Let $f(\langle\rangle) = \langle\rangle$. Suppose we have defined $f(q) = \bar\sigma \in (T^\infty)^\infty$ and we want to define $f(q^\frown\langle\sigma, n\rangle)$ for some $q^\frown\langle\sigma, n\rangle \in T^\infty$. Let $B_{q,\sigma} = \{\langle\sigma, m\rangle \in T \times \mathbb{N} : q^\frown\langle\sigma, m\rangle \in T^\infty\}$. Let $A_{\bar\sigma,\sigma} = \{\langle p, m\rangle \in T^\infty \times \mathbb{N} : i(p) = \sigma \ \&\ \bar\sigma^\frown\langle p, m\rangle \in (T^\infty)^\infty\}$. Both $\mathcal{B}_{q,\sigma}$ and $A_{\bar\sigma,\sigma}$ are infinite because $\sigma \supset i(p) = i(i(\bar\sigma))$, so we can find a bijection $g_{q,\sigma}$ between them. Let $f(q^\frown\langle\sigma, n\rangle) = \bar\sigma^\frown g_{q,\sigma}(\langle\sigma, n\rangle)$. $\square$

Definition 8.3.23. A tree T is called *fat* if $T^\infty \cong T$.

Of course, a tree is fat iff it is isomorphic to T^∞ for some tree T. Note that for all T and $p \in T^\infty$ we have $(T \upharpoonright i(p))^\infty \cong T^\infty \upharpoonright p$. It follows that if T is fat and $\sigma \in T$ then $T \upharpoonright \sigma$ is fat.

Also, if T is fat, then for all $\sigma \in T$ and for all successors τ of σ in T, there are infinitely many τ' which are immediate successors of σ and such that $T \upharpoonright \tau \cong T \upharpoonright \tau'$. (Work in T^∞; consider all τ' such that $i(\tau') = i(\tau)$.)

Lemma 8.3.24 (RCA_0). *Let T be a tree. T is ranked iff T^∞ is ranked and in that case,* $\mathrm{rk}(T) = \mathrm{rk}(T^\infty)$.

Proof: Suppose that T is ranked. For all $p \in T^\infty$, let $\mathrm{rk}^\infty(p) = \mathrm{rk}_T(i(p))$. It is immediate that rk^∞ is a rank function on T^∞ and that $\mathrm{rk}(T) = \mathrm{rk}(T^\infty)$.

Suppose that T^∞ is ranked by a rank function rk_{T^∞}. We note that for all $p, q \in T^\infty$, if $i(p) = i(q)$ then $\mathrm{rk}_{T^\infty}(p) = \mathrm{rk}_{T^\infty}(q)$. This is because $T^\infty \restriction p \cong T^\infty \restriction q$ as they are both isomorphic to $(T \restriction i(p))^\infty$. It follows that we can define rk_T on T by letting $\mathrm{rk}_T(\sigma) = \mathrm{rk}_{T^\infty}(p)$ where p is any such that $i(p) = \sigma$. It is clear that rk_T is order-inversing. If $i(p) = \sigma$ then the collection of $i(q)$s of immediate successors q of p is exactly the collection of all extensions of σ in T. It follows that rk_T is indeed a rank function on T. $\square$

Together with lemmas 8.3.7 and 8.3.8, we get:

Corollary 8.3.25 (ACA_0). *Let S and T be ranked trees. Then $S \preccurlyeq T$ iff $S^\infty \preccurlyeq T^\infty$.*

In fact, EQU=ISO holds for fat trees:

Lemma 8.3.26 (RCA_0). *Let T be a ranked fat tree. Then for every $\sigma \in T$ and every $\gamma < \mathrm{rk}_T(\sigma)$ there are infinitely many immediate successors τ of σ such that $\mathrm{rk}(\tau) = \gamma$.*

(So a ranked fat tree is saturated within its rank.)

Proof: Work with T^∞. Let $p \in T^\infty$ have rank β; we know that $\mathrm{rk}_T(i(p)) = \mathrm{rk}_{T^\infty}(p)$. Let $\gamma < \beta$. As the rank function on $T \restriction i(p)$ is onto $\beta + 1$, there is some extension τ of σ such that $\mathrm{rk}_T(\tau) = \gamma$. Then for every $n \in \mathbb{N}$, the immediate extension q of p which is determined by adding (τ, n) as last pair has rank γ. $\square$

Corollary 8.3.27 (RCA_0). *Suppose that T and S are ranked fat trees. If $\mathrm{rk}(T) \cong \mathrm{rk}(S)$ then $T \cong S$.*

Proof: We define $f\colon T \to S$ by induction on the levels of T. Suppose that $f(\sigma)$ is defined. For each $\gamma < \mathrm{rk}(\sigma)$, consider the set $A_{\sigma,\gamma}$ which consists of those immediate extensions of σ of rank γ; similarly $B_{\sigma,\gamma}$ consists of those immediate extension of $f(\sigma)$ of rank γ. For all $\gamma < \mathrm{rk}(\sigma)$, $A_{\sigma,\gamma}$ and $B_{\sigma,\gamma}$ are infinite; we can thus build bijections between them and thus extend f. $\square$

Next we go from fat trees to ordinals. We need the following because $T \to \mathrm{rk}(T)$ is far from computable.

Lemma 8.3.28 (RCA_0). *Let T be a ranked fat tree. Then $\mathrm{KB}(T)$ is isomorphic to $\omega^{\mathrm{rk}(T)} + 1$.*

Proof: We can mimic the construction in the proof of Lemma 8.2.4: by effective transfinite recursion we construct maps from $T \restriction \sigma$ to $\omega^{\mathrm{rk}(\sigma)} + 1$ which preserve $<_{\mathrm{KB}}$. We then use Π^0_1-transfinite induction on T to show that each such map is onto its range. In fact, we can directly compute the final embedding: For any $\sigma \in T$, let P_σ be the collection of those $\tau \in T$ which lie lexicographically to the left of σ but such that $\tau \restriction |\tau| - 1 \subseteq \sigma$. This is in fact finite. Order P_σ by $<_{\mathrm{KB}}$ (which is the same as the lexicographic ordering on P_σ) as $\langle \tau_0, \dots, \tau_k \rangle$. We let $f(\sigma) = \omega^{\mathrm{rk}(\tau_0)} + 1 + \omega^{\mathrm{rk}(\tau_1)} + 1 + \cdots + \omega^{\mathrm{rk}(\tau_k)} + 1 + \omega^{\mathrm{rk}(\sigma)}$. $\square$

8.3.3 Proofs of arithmetic comprehension

Proposition 8.3.29 (RCA_0). *$\exists$-$\mathsf{EMB}(\mathcal{WFT})$ implies ACA_0.*

PROOF: Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. We construct a sequence of trees $\langle T_n \rangle_{n\in\mathbb{N}}$. We have $\langle\rangle \in T_n$ for all n; further, we put $\langle x \rangle$ in T_n if $f(x) = n$. Let $T = \{\langle\rangle\}$. Let $A = \{n \in \mathbb{N} : T_n \preccurlyeq T\}$. We observe that $\mathbb{N} \smallsetminus A = \operatorname{range} f$. □

Proposition 8.3.30 (RCA_0). *$\mathsf{COMP}(\mathcal{WFT})$ implies ACA_0.*

PROOF: This is similar to ideas of Friedman's [Frib] for p-groups. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. We construct two trees by defining their leaves. Let the leaves of T be $\{\langle 0\rangle^\frown\langle n\rangle^n : n \in \mathbb{N}\}$; let the leaves of S be $\{\langle nn\rangle^\frown\langle m\rangle^m : n = f(m)\} \cup \{\langle nn\rangle : n \notin \operatorname{range} f\}$.

T does not embed in S: say $g\colon T \to S$ is an embedding. Let $\sigma = g(\langle 0\rangle)$. There is some n such that $\langle n\rangle \subseteq \sigma$. If $n \notin \operatorname{range} f$ then $g(\langle 022\rangle)$ cannot be defined. If $n = f(m)$ then $g(\langle 0\rangle^\frown\langle m+2\rangle^{m+2})$ cannot be defined.

Thus let $g\colon S \to T$ be an embedding. If $f(m) = n$ then $g(\langle nn\rangle)$ must extend $\langle 0k\rangle$ for some k, and in fact we must have $k > m$ because in this case $g(\langle nn\rangle^\frown\langle m\rangle^m)$ must be defined. Thus $n \in \operatorname{range} f$ iff for the unique k such that $\langle 0k\rangle \subseteq g(\langle nn\rangle)$, $n \in \operatorname{range} f \restriction k$. □

WQO

We will first prove that $\mathsf{WQO}(\mathcal{WFT})$ implies ACA_0 using RCA_2, and then show that over RCA_0, $\mathsf{WQO}(\mathcal{WFT})$ implies RCA_2 (recall that RCA_2 is RCA_0 together with Σ^0_2-induction.) Our proofs of $\mathsf{WQO}(\mathcal{X}) \Rightarrow \mathsf{ACA}_0$ in this and later sections are motivated by Shore's technique of proving $\mathsf{WQO}(\mathcal{O}n) \Rightarrow \mathsf{ACA}_0$ [Sho93].

Proposition 8.3.31 (RCA_2). *$\mathsf{WQO}(\mathcal{WFT})$ implies ACA_0.*

PROOF: Let $\langle k_s\rangle$ be an effective enumeration of $0'$. $s \in \mathbb{N}$ is a *true stage* of this enumeration if for all $t > s$, $k_t > k_s$. $s \in \mathbb{N}$ *appears to be a true stage at stage* $t > s$ if for all $r \in (s, t]$ we have $k_r > k_s$.

Let T consist of all sequences $\sigma^\frown\langle t\rangle$ where $t \in \mathbb{N}$ and σ is an increasing enumeration of the stages which appear to be true at t. T is indeed a tree because if $t_1 < t_2$ and t_1 appears to be true at t_2, then for all $s < t_1$, s appears to be true at t_1 iff it appears to be true at t_2. We let $\max\sigma$ denote the last element of a sequence $\sigma \in T$.

Assume for contradiction that $0'$ does not exist. Then T is well-founded: If f is an infinite path in T then every $s \in \operatorname{range} f$ is a true stage, since it appears to be true at unboundedly many later stages.

For $n \in \mathbb{N}$, let $T_n = \{\tau - \tau \restriction n : \tau \in T\}$ (i.e. the elements of T_n are the final segments of sequences in T, the first n elements removed.) Each T_n is a well-founded tree because T is.

For $\sigma \in T_n$, we say that σ is *true* if $\max\sigma$ is a true stage.

By assumption, there are some $n < m$ and an embedding $g\colon T_n \to T_m$. We claim that the image, under g, of a true sequence, is also true. This is because if $\tau \in T_m$ is not true, then $T_n[\tau]$ is finite. On the other hand, for any number r, there is a true string $\sigma \in T_n$ of length $> r$ (this requires Σ^0_2-induction). And of course, the true strings on T_n are linearly ordered. Thus if σ is true then $g(\sigma)$ can never be off a true string, for in that case g would be "stuck".

The second point is that if $\sigma \in T_n$ is true, then $\max g(\sigma) > \max\sigma$; this is because $|g(\sigma)| \geqslant |\sigma|$ and the fact that in T_m we "chopped off" more of the beginning of each string. This shows that given any true stage, we can manufacture a bigger true stage; iterating, we compute $0'$. □

To derive $I\Sigma_2$, we use ideas of Shore ([Sho93, Theorem 3.1]). Let $\psi(x) = \exists u \forall v \phi(x,u,v)$ be a Σ^0_2 formula, and fix $n \in \mathbb{N}$. We let $Z = \{x < n : \psi(x)\}$ and for $p \in \mathbb{N}$, we let $Z_p = \{x < n : \exists u \leqslant p\ \forall v \phi(x,u,v)\}$. Obviously if $p < q$ then $Z_p \subseteq Z_q \subseteq Z$. Each Z_p exists (by bounded Σ^0_1-comprehension, which holds in RCA_0. See [Sim99, Definition II.3.8 and Theorem II.3.9]), so if for some p we have $Z_p = Z$ then Z exists. This is enough to get induction on ψ up to n.

Lemma 8.3.32 (RCA_0). *There is a sequence $\langle\alpha_p\rangle_{p\in\mathbb{N}}$ of ordinals such that for all p, if $Z_p \neq Z$ then for all $q > p$, $\alpha_q + 1 \preccurlyeq \alpha_p$.*

Proof: In Shore's construction, each α_i (M_i for Shore), is defined as a sum $\sum_{j<n} N_{i,j}$, where $N_{i,n-j}$ is isomorphic to ω^{3j+1} if $\neg\psi(n-j)$, and $\omega^{3j}\cdot u_j - i + \omega^{3j-1}$ if $\psi(n-j)$ and u_j is the least witness, i.e, the least number u such that $\forall v \phi(n-j,u,v)$. (If there is a witness, a least one exists by Σ^0_1-induction.) Of course, if $u_j < i$ we let $u_j - i = 0$. Now, if $Z_p \subsetneq Z$, there exists $0 < j \leqslant n$ such that $\psi(n-j)$ but $\neg\exists u \leqslant p \forall v \phi(n-j,u,v)$, or in other words, $u_j > p$. Then, $\sum_{k=n-j,\dots,n-1} N_{q,k} + 1 \preccurlyeq \omega^{3j}\cdot u_j - q + \omega^{3j-1}\cdot 2 \preccurlyeq \omega^{3j}\cdot u_j - p \preccurlyeq N_{p,n-j} \preccurlyeq \sum_{k=n-j,\dots,n-1} N_{p,k}$. Then, since for all $k < n-j$, $N_{q,k} \preccurlyeq N_{p,k}$, we have that $\alpha_q + 1 \preccurlyeq \alpha_p$. □

Proposition 8.3.33 (RCA_0). *WQO($\mathcal{WFT}$) implies Σ^0_2 induction.*

Proof: Let ψ, Z and Z_p be as above, and let $\langle\alpha_p\rangle$ be the sequence given by Lemma 8.3.32

By $\mathsf{WQO}(\mathcal{WFT})$, there exist $p < q$ such that $T(\alpha_p) \preccurlyeq T(\alpha_q)$. In RCA_0 we cannot deduce that $\alpha_p \preccurlyeq \alpha_q$, but we can deduce that $\alpha_q + 1 \not\preccurlyeq \alpha_p$. This is because otherwise, we would have $1 + T(\alpha_q) \preccurlyeq T(\alpha_q + 1) \preccurlyeq T(\alpha_q)$, contradicting the well-foundedness of $T(\alpha_q)$. Thus $Z_p = Z$ and we're done. □

8.4 Superatomic Boolean algebras

Superatomic Boolean algebras have not, as far as we can tell, been studied in the context of reverse mathematics. This is why we first discuss various possible definitions for this class and see how they relate (for some of the equivalences we seem to require ACA_0). We then follow the plan that was executed in the last section.

8.4.1 Definitions

Boolean Algebras

In this subsection, unless we mention otherwise, we work in RCA_0.

Definition 8.4.1. A *Boolean algebra* is a set A endowed with two binary operations $\wedge$ and $\cup$, a unary operation $\neg$ and a distinguished element 0_A, which satisfies the familiar axioms of Boolean algebras. (For basic properties, see [Kop89].)

The partial ordering on A given by $x \cup y = y$ (equivalently $x \wedge y = x$) also exists; $\cup$ and $\wedge$ are indeed the least upper bound and greatest lower bound, and so a Boolean algebra is indeed a complemented distributive lattice. We use $\triangle$ to denote the symmetric difference operation: $x \triangle y = (x - y) \cup (y - x)$, where $x - y = x \cap \neg y$.

The notions of an ideal of a Boolean algebra, a homomorphism of Boolean algebras, products (and infinite sums) of Boolean algebras and subalgebras can be copied from the algebra textbooks and carried out in RCA_0. We can also formalize the notion of the quotient algebra:

Definition 8.4.2. Let I be an ideal of a Boolean algebra A. For $a, b \in A$, let $a =_I b$ if $a \triangle b \in I$. Let B be the collection of $a \in A$ which are the $<_{\mathbb{N}}$-least elements of their $=_I$-equivalence class. B exists as it is Δ^0_0. For $a, b \in B$, let $a \cup_B b = c$ if $a \cup_A b =_I c$, and similarly for $\wedge, \neg, 0$. Again these operations exist, and the resulting structure is a Boolean algebra; this is the *quotient algebra A/I*.

A particular example is the free Boolean algebra. Let V be a set. We let $\mathtt{Prop}(V)$ be the collection of all propositional formulas with variables in V (this can be effectively coded). We let $\Leftrightarrow$ denote logical equivalence on propositional formulas (it exists as it is computable by using truth tables). We let $\mathbb{B}(V) = \mathtt{Prop}(V)/\Leftrightarrow$; this is a Boolean algebra which we call the *free Boolean algebra over V*.

In particular, we fix some infinite set of variables V^* and let $\mathtt{Prop} = \mathtt{Prop}(V^*)$. For $\varphi(\bar{x}) \in \mathtt{Prop}$, a Boolean algebra A and $\bar{a} \in A$, $\varphi^A(\bar{a})$ is well-defined (by induction on φ). If A is a Boolean algebra and $X \subseteq A$ then we let

$$\langle X \rangle_A = \{\varphi^A(\bar{a}) : \varphi \in \mathtt{Prop}, \bar{a} \in X\}.$$

In ACA_0 we can show that $\langle X \rangle_A$ (which we call the *subalgebra of A generated by X*) indeed exists, and we can also show that $\langle X \rangle_A$ is the inclusion-wise smallest subalgebra of A containing X. In fact the existence of subalgebras generated by sets is equivalent to ACA_0. However, we note that if X is finite, then in RCA_0 we can show that $\langle X \rangle_A$ exists, as there are only $2^{|X|}$ many propositional formulas with variables in X.

Superatomicity in RCA_0

We turn to describe superatomic algebras. We need some definitions.

Definition 8.4.3. Let A be a Boolean algebra. An element $x \in A$ is an *atom* if $x > 0$ but there is no $y \in A$, $0 < y < x$. A Boolean algebra is *atomless* if it has no atoms.

For example, if V is infinite then $\mathbb{B}(V)$ is atomless.

If A is a Boolean algebra then we let $A^+ = A \setminus \{0_A\}$.

Definition 8.4.4. An embedding of the full binary tree into a Boolean algebra A is a map $f\colon 2^{<\mathbb{N}} \to A^+$ such that for all $\sigma, \tau \in 2^{<\mathbb{N}}$, $\sigma \subseteq \tau$ implies $f(\tau) \leqslant f(\sigma)$ and $\sigma \perp \tau$ implies $f(\sigma) \wedge f(\tau) = 0$.

Lemma 8.4.5 (RCA_0). *If A is atomless then there is an embedding of the full binary tree into A.*

Proof: For $\sigma \in 2^{<\mathbb{N}}$, we define $f(\sigma)$ by induction on $|\sigma|$. Let $s(\langle\rangle) = 1_A$. Say that $f(\sigma)$ is defined. It is not an atom; so we can let $f(\sigma^\frown 0)$ be the $<_{\mathbb{N}}$-least $y \in A$ such that $0 < y < f(\sigma)$, and let $f(\sigma^\frown 1) = f(\sigma) - f(\sigma^\frown 0)$. $\square$

Definition 8.4.6. Let A be a Boolean algebra. A set $X \subseteq A$ is *free* if for all $\bar{a} \in X$ and $\varphi(\bar{x}) \in \texttt{Prop}$, if $\varphi^A(\bar{a}) = 0$ then φ is logically false.

For example, for any set V, V is free in $\mathbb{B}(V)$.

Lemma 8.4.7 (RCA_0). *Let A be a Boolean algebra. Then there is an embedding of the full binary tree into A iff there is some infinite free set $X \subseteq A$.*

Proof: In one direction, suppose that $X \subseteq A$ is infinite and free; let $g\colon \mathbb{N} \to X$ be a one-to-one enumeration of X. For any $a \in A$, let $a^0 = a$ and $a^1 = \neg a$. We define $f\colon 2^{<\mathbb{N}} \to A^+$ by letting $f(\sigma) = \bigwedge_{n<|\sigma|} g(n)^{\sigma(n)}$. That f preserves extension and incompatibility is immediate; the point is that for all σ, $f(\sigma) > 0$. This follows from freeness; $f(\sigma) \neq 0$ because $\bigwedge_{n<\sigma} x_n^{\sigma(n)}$ is not logically false.

In the other direction, let $f\colon 2^{<\mathbb{N}} \to A^+$ be an embedding of the full binary tree into A. Let $g(n) = \bigvee_{\sigma\in 2^n} f(\sigma^\frown 0)$ (so $g(0) = f(0)$). We first show that g has "free range": for all distinct $n_0, \dots, n_k \in \mathbb{N}$ and for all $\varphi(x_0, \dots, x_k) \in \texttt{Prop}$, if $\varphi^A(f(n_0), \dots, f(n_k)) = 0_A$ then φ is logically false. Let $n > 1$ and let $\varphi(x_0, \dots, x_{n-1})$ be a propositional sentence. For $\sigma \in 2^n$, let $\varphi_\sigma = \bigwedge_{k<n} x_k^{\sigma(k)}$. We have $\varphi_\sigma^A(g(0), \dots, g(n-1)) = f(\sigma)$. For some $F \subseteq 2^n$, φ is equivalent to $\bigvee_{\sigma\in F} \varphi_\sigma$. If φ is not logically false then $F \neq 0$; take some $\sigma \in F$. We have $\varphi^A(g(0), \dots, g(n-1)) \geqslant \varphi_\sigma^A(g(0), \dots, g(n-1)) > 0$.

Now g is one-to-one; it follows that its range contains an infinite set X; X must be free. $\square$

It is clear that if A has an atomless subalgebra then there is an embedding of the full binary tree into A. In fact these notions are equivalent.

Lemma 8.4.8 (RCA_0). *Suppose that a Boolean algebra A contains an infinite free set. Then A has an atomless subalgebra.*

PROOF: Let $X \subseteq A$ be an infinite free set. We construct $B \subseteq A$ as an increasing union of finite subalgebras $\langle B_n \rangle$ which we construct by induction. Suppose that we have $B_n = \langle x_0, \ldots, x_{n-1} \rangle_A$ where $x_i \in X$. Let m_n be the $<_{\mathbb{N}}$-least natural number which is $<_{\mathbb{N}}$-greater than all $y \in B_n$. Let x_n be the $<_{\mathbb{N}}$-least element of X such that $\langle x_0, \ldots, x_n \rangle_A \cap \{0, \ldots, m_n - 1\} = B_n$; one exists by freeness: if $y, z \in X$, and y, z and the x_i's are distinct then $\langle B_n \cup \{y\} \rangle_A \cap \langle B_n \cup \{z\} \rangle_A = B_n$, so the possibilities below m_n soon exhaust themselves.

Now $B = \cup B_n$ is isomorphic to the free Boolean algebra on infinitely many elements, and so is atomless. □

We thus make the following definition:

Definition 8.4.9. A Boolean algebra is *superatomic* if it has no atomless subalgebra.

Let $\mathcal{SABA}$ denote the class of superatomic Boolean algebras.

Superatomicity in ACA_0

Lemma 8.4.10 (RCA_0). *A superatomic Boolean algebra has no atomless quotient.*

PROOF: Suppose that a Boolean algebra A has an atomless quotient $B = A/I$. Let $X \subseteq B$ be infinite and free. Recall that we designed our subalgebras such that as sets, $B \subseteq A$. So we have $X \subseteq A$ and X is free in A. □

We work toward the converse. Recall that a set $X \subseteq A$ is *dense* if for all nonzero $a \in A$ there is some nonzero $b \in X$ such that $b \leqslant a$. If I is an ideal of A, B is a subalgebra of A and $I \cap B = \{0\}$ then the quotient map $\varphi \colon A \to A/I$ is one-to-one on B, and we identify B with its image φ"B (which exists under ACA_0).

Lemma 8.4.11 (ACA_0). *If B is a subalgebra of a Boolean algebra A then there is an ideal I of A such that $I \cap B = \{0\}$ and B is dense in A/I.*

PROOF: Let B be a subalgebra of A. Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be an enumeration of the elements of A. We define a sequence of ideals I_n of A such that for all n, $I_n \cap B = \{0\}$. We let $I_0 = \{0_A\}$. At stage n, given I_n, we ask if a_n bounds some element of B^+ modulo I_n. If so, we let $I_{n+1} = I_n$; Otherwise, we let I_{n+1} be the ideal generated by I_n together with a_n.

Now the sequence $\langle I_n \rangle_{n \in \mathbb{N}}$ exists because each I_n is finitely generated and we can keep track of the finite sets of generators. Thus we can let $I = \cup_n I_n$. By the construction, $I \cap B = \{0\}$. Also, B is dense in A/I, because each element at its turn is either discovered to bound some element of B^+ modulo an ideal contained in I, or is thrown into I. □

Corollary 8.4.12 (ACA_0). *A Boolean algebra is superatomic iff it has no atomless quotient.*

Proof: Suppose that A is not superatomic; it has some atomless subalgebra B. Let I be given by Lemma 8.4.11. Then A/I is atomless, because B is a dense, atomless subalgebra of A/I. □

Question 8.4.13. Does the statement of corollary 8.4.12 imply ACA_0?

Remark 8.4.14. Lemma 8.4.11 is equivalent to ACA_0 over RCA_0.

8.4.2 Ranked Boolean algebras

In this section we define rank functions on superatomic Boolean algebras; this follows Simpson [Sim99, Section V.7], where reduced Abelian p-groups and their Ulm resolutions are discussed.

Let B be a Boolean algebra. The *Cantor-Bendixon derivative* of B is the quotient B/I, where I is the ideal generated by the atoms of B. I is called the *Cantor-Bendixon ideal* of B.

Let α be an ordinal. A *partial resolution* of B along α is a sequence of ideals $\langle I_\beta \rangle_{\beta<\alpha}$ such that $I_0 = \{0_B\}$, if $\beta + 1 < \alpha$ then $I_{\beta+1}$ is the (pullback to B) of the CB-ideal of B/I_β and for limit $\beta < \alpha$, $I_\beta = \cup_{\gamma<\beta} I_\gamma$. A *resolution* of B is a partial resolution of B along some ordinal $\alpha + 1$ such that $I_\alpha = B$ and for all $\beta < \alpha$ $I_\beta \neq B$.

Suppose that $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ is a resolution of a Boolean algebra B. We define associated rank and degree functions. For $x \in B$, $\mathrm{rk}(x)$ is the unique $\beta < \alpha$ such that $x \in I_{\beta+1} \setminus I_\beta$; and $\deg(x) = n$ if in $B/I_{\mathrm{rk}(x)}$, x is the join of n many atoms. We let $\mathrm{inv}(x) = (\mathrm{rk}(x), \deg(x))$; we let $\mathrm{rk}(B) = \mathrm{rk}(1_B)$, $\deg(B) = \deg(1_B)$ and $\mathrm{inv}(B) = \mathrm{inv}(1_B)$.

Definition 8.4.15. A Boolean algebra is *ranked* if it has some resolution such that the associated invariant function exists.

Lemma 8.4.16 (RCA_0). *Let B be a superatomic Boolean algebra. Let α, α' be ordinals and suppose that $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ and $\langle I'_\beta \rangle_{\beta \leqslant \alpha'}$ are two resolutions of B. Further assume that the associated invariant functions* inv *and* inv′ *exist. Then $\alpha \cong \alpha'$, and the isomorphism commutes with* rk, rk′ *(i.e. if $f\colon \alpha \to \alpha'$ is the isomorphism then $f \circ \mathrm{rk} = \mathrm{rk}'$).*

Proof: We remark that for all $\beta < \alpha, \beta' < \alpha'$, if $I_\beta \subseteq I'_{\beta'}$ then $I_{\beta+1} \subseteq I'_{\beta'+1}$. For take some atom x of B/I_β. For all $z \leqslant x$, either $z \in I_\beta$ or $z =_{I_\beta} x$. It follows that for all $z \leqslant x$, either $z \in I'_{\beta'}$ or $z =_{I'_{\beta'}} x$. It follows that $x \in I'_{\beta'}$ or x is an atom of $B/I'_{\beta'}$. Thus every finite join of atoms of B/I_β is in $I'_{\beta'}$ or is a finite join of atoms of $B/I'_{\beta'}$.

Of course, rk and rk$'$ are symmetric here. Thus, if $I_\beta = I'_{\beta'}$ then $I_{\beta+1} = I'_{\beta'+1}$.

Using Π^0_1-transfinite induction on $\beta < \alpha$ we show that for all $x \in B$, if $\mathrm{rk}(x) = \beta$ then $I_\beta = I'_{\mathrm{rk}'(x)}$.

Suppose that the claim is verified up to β; let $x \in B$ be such that $\mathrm{rk}(x) = \beta$. Let $\beta' = \mathrm{rk}'(x)$. We need to see that $I_\beta = I'_{\beta'}$.

Let $y \in I_\beta$; let $\gamma = \mathrm{rk}(y)$. Now $\gamma < \beta$ so $x \notin I_{\gamma+1}$. Let $\gamma' = \mathrm{rk}'(y)$. By induction, $I_\gamma = I'_{\gamma'}$ and so $I_{\gamma+1} = I'_{\gamma'+1}$ and so $x \notin I'_{\gamma'+1}$; as $x \in I_{\beta'+1}$ it follows that $\gamma' < \beta'$. Thus $I_\beta \subseteq I'_{\beta'}$.

Next, we note that for no $\gamma' < \beta'$ can we have $I_\beta \subseteq I'_{\gamma'}$; for then we would have $I_{\beta+1} \subseteq I'_{\gamma'+1} \subseteq I'_{\beta'}$, contrary to $\mathrm{rk}'(x) = \beta'$.

Let $\gamma' < \beta'$. Then $I_\beta \nsubseteq I'_{\gamma'}$; so there is some $y \in I_\beta$ such that $\mathrm{rk}'(y) \geqslant \gamma'$. Let $\delta = \mathrm{rk}(y)$; $\delta < \beta$ so by induction, $I_{\delta+1} = I_{\mathrm{rk}'(y)+1}$. Thus $I_{\gamma'+1} \subseteq I_\beta$. As $I'_{\beta'} = \cup_{\gamma'<\beta'} I'_{\gamma'+1}$, we have $I'_{\beta'} \subseteq I_\beta$ as required.

Now we can define a function $f\colon \alpha \to \alpha'$ by letting $f(\beta) = \beta'$ if for some (all) x such that $\mathrm{rk}(x) = \beta$ we have $\mathrm{rk}'(x) = \beta'$. The function f exists as it is Δ^0_1-definable. We show that f is an isomorphism. Of course, $\mathrm{dom}\, f = \alpha$ as the sequence of ideals $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ is strictly increasing. By the same argument, $\mathrm{range}\, f = \alpha'$. f is order preserving: say $\gamma < \beta < \alpha$. Then $I_\beta = I'_{f(\beta)}$ and $I_\gamma = I'_{f(\gamma)}$. Also, $I_\gamma \subsetneq I_\beta$. It follows that $I'_{f(\gamma)} \subsetneq I'_{f(\beta)}$ and so $f(\gamma) < f(\beta)$. □

Corollary 8.4.17 (RCA_0). *Suppose that A, B are ranked Boolean algebras and that $f\colon A \to B$ is an isomorphism. Let $\mathrm{rk}_A\colon A \to \alpha$ and $\mathrm{rk}_B\colon B \to \alpha'$ be the rank functions. Then $\alpha \cong \alpha'$ and the isomorphism g commutes with $\mathrm{rk}_A, \mathrm{rk}_B, f$ (i.e. $\mathrm{rk}_B \circ f = g \circ \mathrm{rk}_A$).*

Lemma 8.4.18 (RCA_0). *A ranked Boolean algebra is superatomic.*

Proof: Let B be a ranked Boolean algebra of rank α. Suppose toward a contradiction that B is not superatomic. Then, there is an embedding f of the full binary tree into B. By recursion we construct a decreasing sequence $\langle \sigma_n \rangle_{n\in\mathbb{N}} \subseteq 2^{<\omega}$ such that $\langle \mathrm{inv}(f(\sigma_n)) \rangle_{n\in\mathbb{N}}$ is a descending sequence in $\alpha \times \omega$, getting a contradiction. Let $\sigma_0 = \langle\rangle$. Given σ_n, either $\mathrm{inv}(f(\sigma_n{}^\frown 0)) < \mathrm{inv}(f(\sigma_n))$ or $\mathrm{inv}(f(\sigma_n{}^\frown 1)) < \mathrm{inv}(f(\sigma_n))$; for if $\mathrm{inv}(f(\sigma_n)) = \mathrm{inv}(f(\sigma_n{}^\frown 0)) = \mathrm{inv}(f(\sigma_n{}^\frown 1)) = (\beta, n)$, then necessarily $f(\sigma_n) = f(\sigma_n{}^\frown 0) = f(\sigma_n{}^\frown 1)$ in B/I_β, contradicting $f(\sigma_n{}^\frown 0) \wedge f(\sigma_n{}^\frown 1) = 0_B$. Let σ_{n+1} one of $\sigma_n{}^\frown 0$ or $\sigma_n{}^\frown 1$ which has smaller invariant. □

Lemma 8.4.19 (ATR_0). *Every superatomic Boolean algebra is ranked.*

Proof: Let B be a Boolean algebra, and assume that there is no ordinal α such that a full iteration of the derivative of B along $\alpha + 1$ exists.

Let $\varphi(L, \langle I_a \rangle_{a\in L})$ say that L is a linear ordering, that for all $a \in L$, I_a is a proper ideal of B, and that if $a <_L b$ then the (pullback to B of) the CB-ideal of B/I_a is contained in I_b.

For every ordinal α, by arithmetic transfinite recursion we can construct an iteration $\langle I_\beta \rangle_{\beta<\alpha}$ of the derivative of B along α. Then $\varphi(\alpha, \langle I_\beta \rangle_{\beta<\alpha})$ holds.

Since the collection of ordinals is not Σ^1_1-definable (this is provable in ACA_0; see [Sim99, V.1.9]), there is some linear ordering L which is not well-founded and such that there is a sequence $\langle I_a \rangle_{a\in L}$ such that $\varphi(L, \langle I_a \rangle_{a\in L})$ holds.

Let $a_0 >_L a_1 >_L a_2 >_L \dots$ be an infinite descending sequence in L. Let $I = \bigcap_{n\in\mathbb{N}} I_{a_n}$. Then B/I is atomless (showing that B is not superatomic). For if $y \in B \setminus I$ then for some n, $y \notin I_{a_n}$. If in B/I, y is an atom, then y is an atom in $B/I_{a_{n+1}}$, in which case we would have $y \in I_{a_n}$.

Thus if B is superatomic then there is a full iteration $\langle I_\beta \rangle_{\beta\leqslant\alpha}$ along some ordinal $\alpha+1$. For every $\beta < \alpha$, The collection of atoms of B/I_α exists. Given any $b \in B$, we can find the unique $\beta < \alpha$ such that $b \in I_{\beta+1} \setminus I_\beta$; and then find the finite set F of atoms of B/I_β such that $b = \cup F$. Then the invariant of b in B is $(\beta, |F|)$. □

Implications of rank

The following is a converse to corollary 8.4.17.

Lemma 8.4.20 (RCA_0). *Suppose that A, B are ranked Boolean algebras which have the same CB invariant. Then $A \cong B$.*

Proof: This is a back-and-forth construction; we define $f\colon A \to B$. Let $f(0_A) = 0_B$ and $f(1_A) = 1_B$.

Let $a_0 \in A$. We look for $b_0 \in B$ such that $\mathrm{inv}_A(a_0) = \mathrm{inv}_B(b_0)$ and $\mathrm{inv}_A(\neg a_0) = \mathrm{inv}_B(\neg b_0)$. Why does such exist? We first note that $\mathrm{inv}_A(a_0) \leqslant_{\mathrm{lex}} \mathrm{inv}(A)$ and that for all pairs $(\gamma, n) \leqslant_{\mathrm{lex}} \mathrm{inv}(B)$ there is some $c \in B$ such that $\mathrm{inv}_B(c) = (\gamma, n)$. Next we note that for all $(\gamma, n) <_{\mathrm{lex}} \mathrm{inv}(A)$ there is a unique $(\beta, m) \leqslant_{\mathrm{lex}} \mathrm{inv}(A)$ such that for all $a \in A$ such that $\mathrm{inv}_A(a) = (\gamma, n)$ we have $\mathrm{inv}_A(\neg a) = (\beta, m)$. By replacing a_0 by its complement, if necessary, we can assume that $\mathrm{inv}_A(a_0) <_{\mathrm{lex}} \mathrm{inv}(A)$. Thus we can pick any $b_0 \in B$ such that $\mathrm{inv}_B(b_0) = \mathrm{inv}_A(a_0)$. We let $f(a_0) = b_0$ and $f(\neg a_0) = \neg b_0$.

We now repeat the process **back**ward, in the other direction, in the Boolean algebras $B(\leqslant b_0)$ and $B(\leqslant \neg b_0)$; we pick new elements, find their equivalents in $A(\leqslant a_0)$ and $A(\leqslant \neg a_0)$ as above, and extend f to be defined on the subalgebra of A generated by all the elements picked so far. We then repeat inside the four new smaller algebras we got, and so **forth**. □

Lemma 8.4.21 (RCA_0). *A (finite or infinite) direct sum of superatomic Boolean algebras is superatomic.*

Proof: Let A and B be Boolean algebras, and suppose that $f\colon 2^{<\mathbb{N}} \to A \times B$ is an embedding of the full binary tree into $A \times B$; write $f(\sigma) = (a_\sigma, b_\sigma)$. There are two possibilities. Suppose that there is some σ such that $a_\sigma = 0_A$. Then we can define $g\colon 2^{<\mathbb{N}} \to B$ by letting $g(\tau) = b_{\sigma^\frown\tau}$. For all τ, $g(\tau) > 0$, as $f(\sigma^\frown\tau) > 0$. If $\tau_1 \perp \tau_2$ then $\sigma^\frown\tau_1 \perp \sigma^\frown\tau_2$ so $f(\sigma^\frown\tau_0) \perp f(\sigma^\frown\tau_1)$ so $g(\tau_0) \perp g(\tau_1)$. Thus B is not superatomic. Similarly, if for some σ we have $b_\sigma = 0$ then A is not superatomic.

Otherwise, we can let $g(\sigma) = a_\sigma$. By assumption, $g(\sigma) > 0$ for all σ; and is $\tau_0 \perp \tau_1$ then $f(\tau_0) \perp f(\tau_1)$ which implies $g(\tau_0) \perp g(\tau_1)$. Then A (and B) are not superatomic.

Let $\langle B_n \rangle_{n\in\mathbb{N}}$ be a sequence of Boolean algebras, and let $B = \bigoplus_{n\in\mathbb{N}} B_n$. Recall that the elements of B are those sequences of $\prod_n B_n$ which have either almost all elements 1 or almost all elements 0. Suppose that $f\colon 2^{<\mathbb{N}} \to B$ is an embedding of the full binary tree into B. Then for either $f(0)$ or $f(1)$, almost all elements are 0, which essentially means that either $f(0)$ or $f(1)$ (and all extensions) lie in some finite product. Thus some finite product of the B_ns is not superatomic, so some B_n is not superatomic. □

Corollary 8.4.22 (RCA_0). *Suppose that every superatomic Boolean algebra is ranked. Then* $\exists$-$\mathsf{ISO}(\mathcal{SABA})$ *holds.*

Proof: Let $\langle B_n \rangle$ be a sequence of superatomic Boolean algebras. Let $B = \bigoplus_n B_n$. Then B is superatomic; let inv_B be the invariant function for B. We claim that $\mathrm{inv}_B \restriction B_n$ is the invariant function for B_n (where we identify $b \in B_n$ with the sequence containing b and otherwise only 0s): by Π^0_1-transfinite induction on $\beta < \mathrm{rk}(B)$ we can see that $I_\beta(B_n) = I_\beta(B) \cap B_n$. For the successor step note that B_n is an initial segment of B; at limit stages take unions.

By corollary 8.4.17 and Lemma 8.4.20, $B_n \cong B_m$ iff $\mathrm{inv}_B(B_n) = \mathrm{inv}_B(B_m)$. This is equality of elements of $\mathrm{inv}(B)$ rather than merely isomorphism of ordinals; we use the fact that if $\beta, \gamma < \alpha$ then $\beta \cong \gamma$ iff $\beta = \gamma$. Thus $\{(n,m) : B_n \cong B_m\}$ exists. □

If we care for only one direction, we have the following.

Lemma 8.4.23 (RCA_0). *Suppose that A, B are ranked Boolean algebras and that* $\mathrm{inv}(A) \leqslant_{\mathrm{lex}} \mathrm{inv}(B)$*. Then there is an embedding of A into B.*

The proof is similar to the proof of Lemma 8.4.20, without going back; f, instead of preserving the invariant, simply does not decrease it (lexicographically).

Corollary 8.4.24 (RCA_0). *Assume that every superatomic Boolean algebra is ranked. Then* $\mathsf{COMP}(\mathcal{SABA})$ *holds.*

Proof: Let A, B be superatomic Boolean algebras; get an invariant inv on $A \times B$. We know that either $\mathrm{inv}(1_A) \leqslant_{\mathrm{lex}} \mathrm{inv}(1_B)$ or vice-versa; thus $A \preccurlyeq B$ or $B \preccurlyeq A$. □

Corollary 8.4.25 (RCA_0). *Assume that every superatomic Boolean algebra is ranked. Then* $\mathsf{WQO}(\mathcal{SABA})$ *holds.*

Proof: Let $\langle B_n \rangle$ be a sequence of superatomic Boolean algebras. Let $B = \bigoplus_n B_n$ and let inv be an invariant on B. Since $\langle \mathrm{inv}(1_{B_n}) \rangle$ cannot be a strictly $<_{\mathrm{lex}}$-decreasing sequence, we must have some $n < m$ such that $\mathrm{inv}(1_{B_n}) \leqslant_{\mathrm{lex}} \mathrm{inv}(1_{B_m})$. It follows that $B_n \preccurlyeq B_m$. □

The analog of corollary 8.4.17 (i.e. the converse of Lemma 8.4.23) seems to require ACA_0.

Lemma 8.4.26 (ACA_0). *Let A, B be ranked Boolean algebras such that A embeds into B. Then* $\mathrm{rk}(A) \preccurlyeq \mathrm{rk}(B)$.

Proof: Let $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ be the CB resolution for A and let $\langle J_\beta \rangle_{\beta \leqslant \alpha'}$ be the CB resolution for B. Let $f \colon A \to B$ be an embedding. Define $g(\beta) = \beta'$ if β' is the least ordinal below α' such that there is some $x \in A$ such that $\mathrm{rk}(x) = \beta$ and $\mathrm{rk}(f(x)) = \beta'$.

Now we prove that g is order preserving. Let $\beta < \gamma < \alpha$. Take $x \in A$ such that $\mathrm{rk}(x) = \gamma$ and $g(\gamma) = \mathrm{rk}(f(x))$. $x \notin I_{\beta+1}$ so in A/I_β there are infinitely many atoms below x. By induction we can pick an infinite collection $X \subseteq A$ such that for all $y \in X$, $y \leqslant x$ and in A/I_β, y is an atom (so $\mathrm{rk}(y) = \beta$); and further, for distinct $y, y' \in X$ we have $y \wedge y' = 0$. For all $y \in X$, $f(y) \notin J_{g(\beta)}$, $f(y) \leqslant f(x)$, and the $f(y)$'s are pairwise disjoint. Also, f is one-to-one so $f``X$ is infinite. It follows that $f(x) \notin J_{g(\beta)+1}$, and hence that $g(\beta) < g(\gamma)$. □

Remark 8.4.27 *(ACA_0).* Suppose that A, B are ranked Boolean algebras and that $\mathrm{rk}(A) = \mathrm{rk}(B)$. Then $A \preccurlyeq B$ iff $\deg(A) \leqslant \deg(B)$. For if $\deg(A) \leqslant \deg(B)$ then $\mathrm{inv}(A) \leqslant_{\mathrm{lex}} \mathrm{inv}(B)$ so $A \preccurlyeq B$ (Lemma 8.4.23). Suppose that $f \colon A \to B$ is an embedding; let $\alpha = \mathrm{rk}(A)$ and $n = \deg(A)$. Let $X \subseteq A$ be a set of size n of pairwise disjoint elements of rank α. By Lemma 8.4.26, for each $x \in X$, $\mathrm{rk}_B(f(x)) = \mathrm{rk}(B(\leqslant f(x))) \geqslant \mathrm{rk}(A(\leqslant x)) = \mathrm{rk}_A(x) = \alpha$, so in B there are n pairwise disjoint elements of rank α; it follows that $\deg(B) \geqslant n$.

Corollary 8.4.28 (ACA_0). *Assume that every superatomic Boolean algebra is ranked. Then $\exists$-$\mathsf{EMB}(\mathcal{SABA})$ holds.*

Proof: The proof is similar to that of corollary 8.4.22. We are given a sequence $\langle B_n \rangle$ of superatomic Boolean algebras and get a rank on $B = \bigoplus_n B_n$. Now if $\mathrm{inv}(B_n) \leqslant_{\mathrm{lex}} \mathrm{inv}(B_m)$ then B_n embeds into B_m. On the other hand, if B_n embeds into B_m then $\mathrm{rk}(B_n) \preccurlyeq \mathrm{rk}(B_m)$ and if they are equal then $\deg(B_n) \leqslant \deg(B_m)$. However, as all of these ordinals are initial segments of $\mathrm{rk}(B)$, $\preccurlyeq$ and $\leqslant$ coincide, so if B_n embeds into B_m then $\mathrm{inv}(B_n) \leqslant_{\mathrm{lex}} \mathrm{inv}(B_m)$. The conclusion follows. □

Corollary 8.4.29 (ACA_0). *Assume that every superatomic Boolean algebra is ranked. Then EQU=$\mathsf{ISO}(\mathcal{SABA})$ holds.*

Proof: Let A, B be superatomic Boolean algebras such that $A \preccurlyeq B$ and $B \preccurlyeq A$. Again let inv be an invariant on $A \times B$. By Lemma 8.4.26 and Remark 8.4.27, we have that $\mathrm{inv}(1_A) = \mathrm{inv}(1_B)$. It follows that $A \cong B$. □

8.4.3 Reductions

Ordinals to superatomic Boolean algebras

Let L be a linear ordering. We let $\mathrm{Int}(L)$ be the Boolean algebra consisting of finite unions of half open intervals of L of the form $[a,b)$ (where we allow $b=\infty$). [In RCA_0, the elements of this Boolean algebra are coded by the finite sequences of the pairs of endpoints of these intervals; all Boolean operations exist.]

For any linear ordering ordinal L, let $\mathbb{B}(L)=\mathrm{Int}(\omega^L)$.

Lemma 8.4.30 (RCA_0). *For all α, $\mathbb{B}(\alpha)$ is ranked, and its invariant is $(\alpha,1)$.*

PROOF: The important thing to recall is that the operations of ordinal addition and subtraction ($\alpha-\beta=\gamma$ if $\beta+\gamma=\alpha$) below ω^α exist; also, taking the logarithm of base ω exists. That is, given $\gamma=\omega^{\beta_1}n_1+\omega^{\beta_2}n_2+\cdots+\omega^{\beta_k}n_k$ where $\beta_1>\beta_2>\cdots>\beta_k$ are elements of α and $n_i\in\omega$, we know that β_1 is the greatest element β of α such that $\omega^\beta\leqslant\gamma$; we in fact let $\mathrm{inv}(\gamma)=(\beta_1,n_1)$ and for every interval $[\beta,\gamma)$ (where $\beta<\gamma\leqslant\omega^\alpha$) we let $\mathrm{inv}([\beta,\gamma))=\mathrm{inv}(\gamma-\beta)$. Note that for the special case $\gamma=\infty(=\omega^\alpha)$ we always have $\omega^\alpha-\beta=\omega^\alpha$ and $\mathrm{inv}(\omega^\alpha)=(\alpha,1)$. When faced by a finite disjoint union of intervals, we take the natural "sum" of invariants, namely the invariant is (β,n) where β is the maximal rank of the intervals and n is the sum of the degrees of those intervals which have rank β.

The fact that inv is indeed the correct invariant for $\mathbb{B}(\alpha)$ follows from three facts (all of which are properties of ordinal addition up to ω^α): one, that finite unions do not increase rank; second, that the inclusion relation respects the lexicographic ordering of invariants (so in particular, I_γ, the collection of all elements of $\mathbb{B}(\alpha)$ which have rank below γ, forms an ideal); and third, that an interval of length ω^γ cannot include two disjoint subintervals of length ω^γ; this implies that the intervals of invariant $(\gamma,1)$ (which modulo I_γ have length ω^γ) are the atoms of $\mathbb{B}(\alpha)/I_\gamma$. □

Lemma 8.4.31 (RCA_0). *For all linear orderings L, L is well-founded iff $\mathbb{B}(L)$ is superatomic.*

PROOF: The direction from left to right follows from the previous lemma and from Lemma 8.4.18. For the other direction suppose that L is not well-founded and that $\{a_n\}_{n\in\mathbb{N}}$ is a descending sequence in L. We define an embedding f of the full binary tree into $\mathbb{B}(L)$. Given $\sigma\in 2^{<\mathbb{N}}$, we let $x_\sigma=\sum_{i:\sigma(i)=1}\omega^{a_{i+1}}$ and we let

$$f(\sigma)=[x_\sigma,x_\sigma+\omega^{a_{|\sigma|}}).$$

Is not hard to check that f is as wanted. □

We get (aided by 8.4.17 and 8.4.20):

Corollary 8.4.32 (RCA_0). *Let α,β be ordinals. Then $\alpha\cong\beta$ iff $\mathbb{B}(\alpha)\cong\mathbb{B}(\beta)$.*

The following is immediate;

Lemma 8.4.33 (RCA_0). *Let α, β be ordinals and suppose that $\alpha \preccurlyeq \beta$. Then $\mathbb{B}(\alpha) \preccurlyeq \mathbb{B}(\beta)$.*

By 8.4.26, we get

Corollary 8.4.34 (ACA_0). *Let α, β be ordinals. Then $\alpha \preccurlyeq \beta$ iff $\mathbb{B}(\alpha) \preccurlyeq \mathbb{B}(\beta)$.*

As a corollary we have:

Corollary 8.4.35 (RCA_0). *The statement $\exists$-$\mathsf{ISO}(\mathcal{SABA})$ implies $\exists$-$\mathsf{ISO}(\mathcal{O}n)$, and EQU=$\mathsf{ISO}(\mathcal{SABA})$ implies EQU=$\mathsf{ISO}(\mathcal{O}n)$. Therefore, both $\exists$-$\mathsf{ISO}(\mathcal{SABA})$ and EQU=$\mathsf{ISO}(\mathcal{SABA})$ are equivalent to ATR_0.*

Corollary 8.4.36 (ACA_0). *The statement $\exists$-$\mathsf{EMB}(\mathcal{SABA})$ implies $\exists$-$\mathsf{EMB}(\mathcal{O}n)$, the statement $\mathsf{COMP}(\mathcal{SABA})$ implies $\mathsf{COMP}(\mathcal{O}n)$ and the statement $\mathsf{WQO}(\mathcal{SABA})$ implies $\mathsf{WQO}(\mathcal{O}n)$. Therefore, $\exists$-$\mathsf{EMB}(\mathcal{SABA})$, $\mathsf{COMP}(\mathcal{SABA})$ and $\mathsf{WQO}(\mathcal{SABA})$ are equivalent to ATR_0 over ACA_0.*

Well-founded trees to superatomic Boolean algebras

We do not need the following for the proof of Theorem 8.1.11, but we thought to include a natural operation which takes us directly from trees to Boolean algebras.

Let T be a tree. We let $\mathbb{B}(T)$ be the *tree algebra* of T, as described in [Kop89, Section 16]. There, it is defined as the subalgebra of $\mathbb{P}(T)$ generated by the cones $\{\sigma \in T : \tau \subseteq \sigma\}$ (for all $\tau \in T$). Of course, this definition only makes sense in ACA_0, but in fact we can get the tree algebra in RCA_0. We let $\mathbb{B}_0(T)$ be a Boolean algebra generated freely over a basis $\{b_\sigma : \sigma \in T\}$; We let I be the ideal generated by the equations $b_\sigma \leqslant b_\tau$ if $\tau \subseteq \sigma$ and $b_\sigma \wedge b_\tau = 0$ if $\sigma \perp \tau$. The ideal I exists: by reducing to disjunctive normal form, it is sufficient to decide whether an element of the form $a = b_{\sigma_1} \wedge \cdots \wedge b_{\sigma_n} \wedge -b_{\tau_1} \wedge \cdots \wedge -b_{\tau_m}$ is in I. If the σ_i are not linearly ordered then $a \in I$. Otherwise, we let σ be the greatest σ_i; modulo I, $a = b_\sigma \wedge -b_{\tau_1} \wedge \cdots \wedge -b_{\tau_m}$. If for some i we have $\sigma \supset \tau_i$ then $a \in I$; otherwise $a \notin I$. Finally we let $\mathbb{B}(T) = \mathbb{B}_0(T)/I$.

Remark 8.4.37. The argument showing that I exists and some more work yield a normal form for elements of $\mathbb{B}(T)$; all the elements can be written as *disjoint* sums of elements of the form $b_\sigma - \sum_{\tau \in S} b_\tau$, where S is a finite antichain in T and for all $\tau \in S$, $\sigma \subseteq \tau$. Again, see [Kop89].

Recall that for a tree T and $\tau \in T$, $T - \tau = \{\sigma : \tau^\frown\sigma \in T\}$.

Fact 8.4.38. *Let T be a tree. For all $\tau \in T$, $\mathbb{B}(T - \tau) \cong \mathbb{B}(T)(\leqslant b_\tau)$.*

In fact, we can break up the tree algebra into pieces. Let $\tau \in T$ let $S(\tau)$ be the collection of immediate successors of τ on T. If $S(\tau)$ is infinite, then

$$\mathbb{B}(T - \tau) \cong \oplus \{\mathbb{B}(T - \sigma) : \sigma \in S(\tau)\}.$$

However, if $S(\tau)$ is finite (for example, if τ is a leaf), then $b_\tau - \sum_{\sigma\in S(\tau)}$ is an atom of $\mathbb{B}(T-\tau)$ and so we have

$$\mathbb{B}(T-\tau) \cong \{0,1\} \oplus \{\mathbb{B}(T-\sigma) \,:\, \sigma \in S(\tau)\}$$

(So if τ is a leaf then b_τ is an atom of $\mathbb{B}(T)$ and $\mathbb{B}(T-\tau) \cong \{0,1\}$.

If T is well-founded then we can build $\mathbb{B}(T)$ from the leaves up.

Lemma 8.4.39 (ACA_0). *If T is well-founded then $\mathbb{B}(T)$ is superatomic.*

PROOF: Suppose that f is an embedding of the full binary tree into $\mathbb{B}(T)$. By induction, we find some $\sigma_n \in T$ and some embedding f_n of the full binary tree into $\mathbb{B}(T \restriction \sigma_n)$. Given $f_n(\langle\rangle) = b_{\sigma_n} = 1_{\mathbb{B}(T-\sigma_n)}$, for some $i < 2$, there is some finite $S \subseteq S(\sigma_n)$ such that $f_n(i) \leqslant \sum_{\tau\in S} b_\tau$. We can thus view $\rho \mapsto f_n(i\frown\rho)$ as an embedding of the full binary tree into $\oplus\{\mathbb{B}(T-\tau) \,:\, \tau \in S\}$. By asking finitely many questions, as in the proof of Lemma 8.4.21, we find some coordinate from which we can pick a σ_{n+1} and let f_{n+1} be the adequate restriction. $\square$

It is clear that if $T \cong T'$ then $\mathbb{B}(T) \cong \mathbb{B}(T')$.

Superatomic Boolean algebras to well-founded trees

Definition 8.4.40. A *uniformly splitting tree* is a tree of the form $\{\sigma \in \omega^{<\alpha} : \forall i < |\sigma| (\sigma(i) < m_i)\}$ where $\alpha \in \mathbb{N} \cup \{\mathbb{N}\}$ and for all i, $m_i \in \mathbb{N}$ and $m_i \geqslant 2$. A *finite tree embedding* into a Boolean algebra B is a partial function $g\colon T \to B^+$ where T is a finite uniformly splitting tree and g preserves $\leqslant$, $\perp$ and 1 (i.e. $g(\langle\rangle) = 1_B$).

Fix a sequence of constants $\langle q_n\rangle_{n\geqslant 2}$ which grows sufficiently quickly so that for all $l, k < n$, $q_l + q_k \leqslant q_n$ (for example, let $q_k = 2^{2^k}$). Also, fix an infinite recursive set $\mathbb{C} = \{c_n \,:\, n \geqslant 2\}$ (say $c_n = \langle 0, n\rangle$) and fix a coding of all finite sequences and functions such that the collection of code numbers is disjoint from $\mathbb{C}$. For $m \in \mathbb{N}$, $n \geqslant 2$ we let $\varrho(n;m) = \langle c_n\rangle^m$ and let $\varrho_n = \varrho(n; q_n - 1)$.

Given a Boolean algebra B we will code all finite tree embeddings into B using a tree. Let $T \subseteq m_1 \times \cdots \times m_k$ be a finite uniformly splitting tree and let $g\colon T \to B$ be a finite tree embedding. For $l \leqslant k$ let $T_l = T \cap \omega^l$. We let the *code* for g, be the string

$$\varpi(g) = \varrho_{m_1}\frown\langle g \restriction T_1\rangle\frown\varrho_{m_2}\frown\langle g \restriction T_2\rangle\frown\cdots\frown\varrho_{m_k}\frown\langle g \restriction T_k\rangle.$$

Note that we also allow g to be defined on $T_0 = \{\langle\rangle\}$; there is a unique such g (we must have $g(\langle\rangle) = 1_B$) and in this case $\varpi(g) = \langle\rangle$. The function $g \mapsto \varpi(g)$ is one-to-one and its range is computable from B, so it exists (in RCA_0).

The tree $\mathcal{T}(B)$ consists of all of the strings $\varpi(g)\frown\varrho(n;m)$ where $n \geqslant 2$ and $m < q_n$, and g is a finite tree embedding into B. In RCA_0 we can show that $\mathcal{T}(B)$ exists. Also, $\mathcal{T}(B)$ is closed under initial segments so it is indeed a tree.

Lemma 8.4.41 (RCA_0). *B is superatomic iff $\mathcal{T}(B)$ is well-founded.*

Proof: Suppose that B is not superatomic. Then there is an embedding of the full binary tree into B, and by shifting elements around we may assume that the top of the embedding is 1_B. This embedding would yield a path in $\mathcal{T}(B)$.

On the other hand, from a path in $\mathcal{T}(B)$ we can recover an embedding of an infinite uniformly splitting tree into B. The reason is that on an infinite path, codes for functions must occur infinitely often: if f is an infinite path and $f(n)$ is a code for a function then if $c_m = f(n+1)$ we know that $f(n+q_m)$ must be a code for a function. Restricting the embedding we get to $2^{<\omega}$ we get an embedding of the full binary tree into B. $\square$

For the purpose of the following computation, we let $q_1 = 0$. Also recall that $I_0 = \{0\}$ so that $\mathrm{rk}(0_B) = -1$ but for every finite join of atoms x, $\mathrm{rk}(x) = 0$.

Lemma 8.4.42 ($\mathbf{RCA}_0$). *Suppose that B is a ranked Boolean algebra and* $\mathrm{inv}(B) = (\alpha, n)$. *Then $\mathcal{T}(B)$ is ranked and has rank $\omega(2\alpha+1)+q_n$.*

(Recall that if γ is limit and $\alpha = \gamma + k$ then $2\alpha = \gamma + 2k$.)

Proof: For a finite tree embedding g into B we let $\mathrm{inv}(g) = \min\{\mathrm{inv}(x) : x \in \operatorname{range} g\}$, where the ordering is of course the lexicographic one. For each such g, if $\mathrm{inv}(g) = (\beta, k)$ then we let $f(\varpi(g)) = \omega(2\beta+1)+q_k$.

Now we claim that f extends to a rank function on $\mathcal{T}(B)$. We first discuss how we should define f on nodes of the form $\varpi(g)^\frown\varrho_n$ for $n \geqslant 2$ and finite tree embeddings g into B. Let g be such and let $(\beta, k) = \mathrm{inv}(g)$; let $x \in \operatorname{range} g$ such that $\mathrm{inv}(x) = (\beta, k)$. Let $n \geqslant 2$. If $n > k$ then we cannot split x into n many disjoint elements of rank β; on the other hand, we can split x into n many disjoint elements, each of which have invariant below $(\beta, 0)$ but as large as we like below $(\beta, 0)$. It follows that the immediate extensions of $\varpi(g)^\frown\varrho_n$ in $\mathcal{T}(B)$ are of the form $\varpi(g')$ where $\mathrm{inv}(g') < (\beta, 0)$ (so if $\beta = 0$ there are no such g' and $\varpi(g)^\frown\varrho_n$ is terminal), and all invariants below $(\beta, 0)$ occur; then the supremum of f on the immediate extensions of $\varpi(g)^\frown\varrho_n$ is $\sup\{\omega(2\gamma+1)+l : \gamma < \beta, l \in \mathbb{N}\} = \omega(2\beta)$. We thus let $f(\varpi(g)^\frown\varrho_n) = \omega(2\beta)$ for such n.

If, however, $n \leqslant k$, then x can be split into n disjoint elements of rank β. In fact, there is then an immediate successor $\varpi(g')$ of $\varpi(g)^\frown\varrho_n$ in $\mathcal{T}(B)$ of maximal invariant (β, l) (where l is the largest possible size of a smallest set in a partition of k into n nonempty sets). We can thus let $f(\varpi(g)^\frown\varrho_n) = \omega(2\beta+1)+q_l+1$. We note that $l = \lfloor k/n \rfloor$.

Having defined f on the nodes $\varpi(g)^\frown\varrho_n$ we can extend it to nodes $\varpi(g)^\frown\varrho(n;m)$ because from $\varpi(g)^\frown\varrho(n;1)$ to $\varpi(g)^\frown\varrho_n$ there is no splitting on $\mathcal{T}(B)$. We thus have $f(\varpi(g)^\frown\varrho(n;1)) = \omega(2\beta)+q_n-2$ if $n > k$, $\omega(2\beta+1)+q_l+q_n-1$ if $n \leqslant k$ where $\lfloor k/n <\rfloor$. Since we chose the q_ks to rise quickly we have $q_l + q_n \leqslant q_k$ for $n < k$. Thus if $k \geqslant 2$ then we indeed have that $\varpi(g)^\frown\varrho(k;1)$ has maximal rank among the immediate successors of $\varpi(g)$, and indeed we assigned it rank $f(\varpi(g))-1$, so f is continuous at $\varpi(g)$. If $k = 1$ then for all n, $f(\varpi(g)^\frown\varrho(n;1)) = \omega(2\beta)+q_n-2$ and $q_n \to \infty$, so again f is continuous at $\varpi(g)$ as required. $\square$

Corollary 8.4.43 (ACA_0). *Suppose that A, B are ranked Boolean algebras. Then $A \cong B$ iff $\mathcal{T}(A)^\infty \cong \mathcal{T}(B)^\infty$ and $A \preccurlyeq B$ iff $\mathcal{T}(A)^\infty \preccurlyeq \mathcal{T}(B)^\infty$.*

Note that this yields another proof that ATR_0 implies the various statements for superatomic Boolean algebras; we deduce them from $\mathsf{RK}(\mathcal{SABA})$ and the corresponding statements for fat trees.

8.4.4 Proofs of arithmetic comprehension

Proposition 8.4.44 (RCA_0). *$\exists$-$\mathsf{EMB}(\mathcal{SABA})$ implies ACA_0.*

Proof: This is immediate (same as Proposition 8.3.29). $\square$

Proposition 8.4.45 (RCA_0). *$\mathsf{COMP}(\mathcal{SABA})$ implies ACA_0.*

Proof: Let $f\colon \mathbb{N} \to \mathbb{N}$ be a one-to-one function. For each $n \in \mathbb{N}$ let B_n be a finite boolean algebra such that if n is not in the range of f, it has only one atom, and if $n = f(m)$, it has m many atoms. Let $B = \bigoplus_{n\in\mathbb{N}} B_n$ and let B' be a ranked boolean algebra of invariant $(\omega, 2)$. First we observe that $B' \not\preccurlyeq B$: B' has two elements with meet $0_{B'}$ and infinitely many atoms below, but B does not. So, $\mathsf{COMP}(\mathcal{SABA})$ implies $B \preccurlyeq B'$. Let g be such an embedding. For at most two (actually one) values of n we may have $\mathrm{rk}(g(1_{B_n})) = 1$. For every other n we have $\mathrm{inv}(g(1_{B_n})) = (0, k_n)$. As in the proof for trees, we observe that n is in the range of f iff it is in the range of $f \restriction k_n$. $\square$

WQO

As in subsection 8.3.3, we go via RCA_2. As mentioned in that subsection, we make use of some ideas of Shore's proof that $\mathsf{WQO}(\mathcal{O}n)$ implies ACA_0 [Sho93, Theorems 2.17 and 3.1].

Proposition 8.4.46 (RCA_2). *$\mathsf{WQO}(\mathcal{SABA})$ implies ACA_0.*

Proof: Again fix an enumeration of $0'$. Let $\beta^n_t = \omega + 1$ if t is the k^{th} true stage of the enumeration for some $k > n$, and finite otherwise. [Set $1 <^* 2 <^* 3 <^* \cdots <^* 0$. At stage $s > t$, determine that $s \in \beta^n_t$ if at s, t appears to be a true stage and there are n other stages before t which also appear to be true at s. If t is not true then eventually this will be found out. If t is true then already at t we know all the true stages $< t$, so if there are fewer than n true stages $< t$, then no $s > t$ is in β^n_t.]

Let $\alpha_n = \sum_{t\in\mathbb{N}} \beta^n_t$ and let $B_n = \mathrm{Int}(\alpha_n)$. Each B_n is superatomic (to see this quickly, note that $B_n = \oplus_t \mathrm{Int}(\beta^n_t)$, and see Lemma 8.4.21). Suppose that $n < m$ and that $f\colon B_n \to B_m$ is an embedding.

By Σ^0_2-induction, there are more than n many true stages. Suppose then that t is the $(n+k)^{th}$ true stage for some $k > 0$. Thus the interval $I = \cup_{s\leqslant t}\beta^n_s$ has exactly k limit points, which in B_n means that it is the join of k many elements below each of which there are infinitely many atoms. This has to be true of $f(I)$ in B_m. Suppose that $\sup f(I) \in \beta_r^m$. Then r bounds a true stage larger than t (for example, the $(m+k)^{th}$ true stage). The stage u between t and r at which the smallest number ever to be enumerated between t and r was actually enumerated, is a true stage $> t$. This process can be iterated to get infinitely many true stages and thus $0'$. □

Now we observe that in light of Lemma 8.3.32 and the proof of proposition 8.3.33, in order to show

Proposition 8.4.47 (RCA_0). *$\mathsf{WQO}(\mathcal{SABA})$ implies $I\Sigma_2$.*

it is enough to show two things in RCA_0: first, that if α, β are ordinals and $\alpha \preccurlyeq \beta$ then $\mathbb{B}(\alpha) \preccurlyeq \mathbb{B}(\beta)$ (this is Lemma 8.4.33), and that if α is an ordinal, then $\mathbb{B}(\alpha+1)$ does not embed into $\mathbb{B}(\alpha)$. Armed with these facts, the proof follows exactly as it did for trees. The second statement can be shown using:

Fact 8.4.48 (RCA_0). *Suppose that A and B are ranked Boolean algebras with resolutions $\langle I_\beta \rangle_{\beta<\alpha}$ and $\langle J_\beta \rangle_{\beta<\alpha}$ along the* same ordinal *α (we allow for a final segment of $I_\beta = A$ or $J_\beta = B$). Suppose that $f\colon A \to B$ is an embedding. Then for all $\beta < \alpha$ and $x \in A$, if $\mathrm{rk}(x) \geqslant \beta$ then $\mathrm{rk}(f(x)) \geqslant \beta$. This is shown by Π^0_1-transfinite induction on α.*

8.5 Reduced p-groups

Fix a prime number p. A *p-group* is a group in which every element has order a power of p.

Convention 8.5.1. *From now on, all groups are Abelian.*

A group G is *divisible* if for every $a \in G$ and every $n \in \mathbb{N}$, there exists $b \in G$ such that $nb = a$.

Definition 8.5.2. A group is *reduced* if it has no divisible subgroup.

Fact 8.5.3 (ACA_0). *A p-group G is reduced iff there is no sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ of elements of G such that for all n, $pg_{n+1} = g_n$. This is because the subgroup generated by the g_ns is the direct limit of $\mathbb{Z}_{p^n}$, and in each $\mathbb{Z}_{p^n}$ one can divide by numbers not divisible by p.*

Again, reduced groups are the "well-founded part" of the collection of groups, and it takes Π^1_1-comprehension to weed out this part. A classic result is the following theorem of Friedman, Simpson and Smith:

Theorem 8.5.4 ([FSS83]). *The statement "every group is the direct product of a reduced group and a divisible groups" is equivalent to Π^1_1-CA_0 over RCA_0.*

As our notion of embedding $\preccurlyeq$ we take the usual notion of group embedding (one-to-one homomorphism).

8.5.1 Ranked p-groups

We define rank functions for reduced p-groups. Let α be an ordinal and G a p-group. A *partial Ulm resolution* of G along α is a sequence of subgroups $\langle G_\beta \rangle_{\beta<\alpha}$ such that $G_0 = G$, if $\beta+1<\alpha$ then $G_{\beta+1} = \{pg : g \in G_\beta\}$, and for limit $\lambda < \alpha$, $G_\lambda = \bigcap_{\beta<\lambda} G_\beta$ (we sometimes write $p^\beta G$ for G_β). An *Ulm resolution* of G is a partial Ulm resolution of G along some ordinal $\alpha+1$ such that $G_\alpha = \{0\}$ and for all $\beta<\alpha$, $G_\beta \neq \{0\}$. We call such an α the *length* of G.

Notation 8.5.5 *(RCA_0).* If G is a p-group then $G[p]$, its *socle*, is the subgroup consisting of elements of G of order p.

Suppose that $\langle G_\beta \rangle_{\beta\leqslant\alpha}$ is a resolution of a p-group G; we define an associated Ulm sequence. For each $\beta<\alpha$, $G_\beta[p]/G_{\beta+1}[p]$ is a vector space over $\mathbb{Z}_p$; we let $U_G(\alpha)$ be its dimension. The sequence $\langle U_G(\beta) : \beta<\alpha\rangle$ is called the *Ulm sequence* of G, and it characterizes G up to isomorphism. We also define an associated rank function: for $x \in G$, let $\mathrm{rk}_G(x)$ be the unique $\beta<\alpha$ such that $x \in G_\beta \smallsetminus G_{\beta+1}$. (In ACA_0, if a resolution exists then so does the sequence and the rank function; but not in RCA_0.)

Definition 8.5.6. A p-group G is *weakly ranked* if it has an Ulm resolution and the associated rank function exists. It is *ranked* if further, the associated Ulm sequence exists.

RCA_0 is enough to prove that reduced p-groups with the same Ulm sequence are isomorphic [Sim99, Theorem V.7.1]. Simpson [Sim99, Lemma V.7.2] uses ACA_0 to prove that Ulm sequences are unique up to isomorphisms of ordinals.

Lemma 8.5.7 (RCA_0). *Let G be a reduced p-group. Let α and α' be ordinals and suppose that $\langle G_\beta\rangle_{\beta<\alpha}$ and $\langle G'_\beta\rangle_{\beta<\alpha'}$ are two resolutions of G. Further, assume that the associated rank function* rk *and* rk′ *exist. Then $\alpha \cong \alpha'$, and the isomorphism commutes with* rk, rk′.

Proof: This is similar (but not identical) to the proof of Lemma 8.4.16. By Π^0_1-transfinite induction on $\beta<\alpha$ we show that for all x of rank β, $G_{\mathrm{rk}(x)} = G'_{\mathrm{rk}(x')}$ (in particular it follows that if $\mathrm{rk}(x)=\mathrm{rk}(y)=\beta$ then $\mathrm{rk}'(x)=\mathrm{rk}'(y)$). Suppose that the claim is verified up to β. There are two options.

First, suppose that β is a limit ordinal. Then $G_\beta = \cap_{\gamma<\beta} G_\gamma$. Let x have rank β and let $\beta' = \mathrm{rk}'(x)$. So far, we have a map $f\colon \beta \to \alpha'$ which is defined by taking $\gamma<\beta$ to the unique γ' such that for some (all) $y \in G$ of rk γ, $\mathrm{rk}'(y)=\gamma'$ (so $G_\gamma = G'_{\gamma'}$). For all $\gamma<\beta$ we thus have $x \in G'_{f(\gamma)}$ so $\operatorname{range} f \subseteq \beta'$. Of course, f is order-preserving, and since β is limit, $\operatorname{range} f$ has no last element. Suppose that f is not cofinal in β': that there is some $\delta'<\beta'$ such that for all $\gamma' \in \operatorname{range} f$, $\gamma' \leqslant \delta'$. But then we have $G_\beta = \cap_{\gamma'\in\operatorname{range} f} G'_{\gamma'} \supset G'_{\delta'}$; on the other hand, $\beta'>\delta'$ implies that there is some $y \in G'_{\delta'}$ such that $py = x$. But there is no such y in G_β. Thus $\beta' = \sup\operatorname{range} f$ is limit and $G'_{\beta'} = G_\beta$ as required.

Next, suppose that $\beta = \gamma + 1$. Take $x \in G$ of rank β; let $\beta' = \mathrm{rk}'(x)$. Since $x \in G_{\gamma+1}$, $p|x$ in G_γ so we can find $y \in G_\gamma$ such that $py = x$. We cannot have $y \in G_\beta$ (or $x \in G_{\beta+1}$); so $\mathrm{rk}(y) = \gamma$. Let $\gamma' = \mathrm{rk}'(y)$. By induction, $G_\gamma = G'_{\gamma'}$, so $G_\beta = G'_{\gamma'+1}$. But $\beta' = \mathrm{rk}'(x) = \gamma' + 1$. Why is that? Otherwise, we have $\mathrm{rk}'(x) > \gamma' + 1$ so $x \in G'_{\gamma'+2}$; thus there is some $y \in G'_{\gamma'+1}$ such that $py = x$; and so there is some $z \in G'_{\gamma'}$ such that $pz = y$. But then $z \in G_\gamma$ and so $x \in G_{\gamma+2}$ which is false. Thus $G_\beta = G'_{\beta'}$ as required.

When we are done with the induction we define $f \colon \alpha \to \alpha'$ as we did in limit stages and get the desired isomorphism. □

Lemma 8.5.8 (RCA_0). *Every ranked p-group is reduced.*

Proof: Suppose G is a p-group which is not reduced. Let H be a divisible subgroup of G and let $x_0 \in H$. By primitive recursion construct a sequence $\langle x_i : i \in \mathbb{N}\rangle$ such that for each i, $px_{i+1} = x_i$. Note now that for each i, $\mathrm{rk}(x_{i+1}) < \mathrm{rk}(x_i)$, because if $x_{i+1} \in G_\beta$, then $x_i \in G_{\beta+1}$. So, we have a contradiction because the length of G is well-founded. □

The statement that every reduced p-group is weakly ranked is equivalent to ATR_0 over RCA_0 [Sim99, Theorem V.7.3]. It follows that the statement that every reduced p-group is ranked is also equivalent to ATR_0 over RCA_0, because as mentioned earlier, ACA_0 is enough to compute the Ulm sequence.

Given a group G, a set $A \subseteq G \setminus \{0\}$ is *independent* if for any $a_1, \dots, a_k \in A$, if $n_1a_1 + \dots + n_ka_k = 0$, then $n_1a_1 = \dots = n_ka_k = 0$. Let G be a p-group. Then if $a_1, \dots, a_k \in G$ are independent and the order of a_i is p^{n_i} then $p^{n_1-1}a_1, \dots, p^{n_k-1}a_k$ is an independent subset of $G[p]$, which is a vector space. Hence all maximal independent sets in G have the same cardinality which we denote by $r(G)$, the (for our purposes unfortunately named) *rank* of G; $r(G) = r(G[p])$.

Lemma 8.5.9 (ACA_0). *Suppose that G, G' are ranked p-groups (with resolutions $\langle G_\beta\rangle_{\beta\leqslant\alpha}$, $\langle G'_{\beta'}\rangle_{\beta'\leqslant\alpha'}$). Suppose that $f \colon G \to G'$ is an embedding. Then there is some embedding $h \colon \alpha \to \alpha'$ such that for all $\beta < \alpha$, $r(G_\beta) \leqslant r(G'_{h(\beta)})$.*

(This extends the discussion about embeddings in [Frib].)

Proof: For $\beta < \alpha$, we let $h(\beta)$ be the maximal $\beta' < \alpha$ such that $f``G_\beta \subseteq G'_{\beta'}$. h is strictly order-preserving because for all $\beta < \alpha$, $G_{\beta+1} = pG_\beta \subseteq G'_{h(\beta)+1}$. Now $h \restriction G_\beta[p]$ is a vector space embedding of $G_\beta[p]$ into $G'_{h(\beta')}[p]$ so the rank cannot decrease. □

Remark 8.5.10. Suppose that $\langle G_\beta\rangle_{\beta\leqslant\alpha}$ is the Ulm resolution of a p-group G. Then for $\beta < \gamma \leqslant \alpha$ we have $r(G_\beta) - r(G_\gamma) = \sum_{\delta\in[\beta,\gamma)} U_G(\delta)$. Thus, if $\alpha = \epsilon + n$ where ϵ is limit, then for all $k < n$, $r(G_{\epsilon+k}) = U_G(\epsilon + k) + \dots + U_G(\epsilon + n - 1)$. If $\beta < \epsilon$ then there are infinitely many $\delta \in [\beta, \epsilon)$ such that $U_G(\delta) > 0$ and so $r(G_\beta) = \omega$. (See Barwise and Eklof [BE71].)

We, in fact, have a converse for Lemma 8.5.9.

Lemma 8.5.11 (ACA_0). *Suppose that G, G' are ranked p-groups (with resolutions $\langle G_\beta \rangle_{\beta \leqslant \alpha}$, $\langle G'_{\beta'} \rangle_{\beta' \leqslant \alpha'}$). Suppose that there is an embedding $h \colon \alpha \to \alpha'$ such that for all $\beta < \alpha$, $r(G_\beta) \leqslant r(G'_{h(\beta)})$. Then $G \preccurlyeq G'$.*

PROOF: See [BE71, Corollary 5.4] and [Frib]. □

As before, we need to see how ranks correspond to direct sums.

Lemma 8.5.12 (RCA_0). *Suppose that $\langle H_n \rangle$ is a sequence of reduced p-groups. Then $G = \bigoplus_n H_n$ is reduced. If $\langle G_\beta \rangle_{\beta \leqslant \alpha}$ is the Ulm resolution for G, then for all $n \in \mathbb{N}$, $\langle G_\beta \cap H_n \rangle_{\beta \leqslant \alpha}$ is an Ulm resolution of H_n (we may need to trim the end of the sequence though).* □

Proposition 8.5.13 (ACA_0). $\mathsf{RK}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$ *implies* $\exists\text{-}\mathsf{EMB}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$.

PROOF: This is similar to what we did before. Let $\langle H_n \rangle$ be a sequence of reduced p-groups. Let $G = \bigoplus_n H_n$ and get a resolution of G; we get the induced resolutions of the H_ns uniformly. Again comparability of the ordinals in question is equivalent to their position in the length of G; we can check ranks of tail-ends to see if the condition in lemmas 8.5.9, 8.5.11 holds. □

Similarly,

Proposition 8.5.14 (ACA_0). $\mathsf{RK}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$ *implies* $\exists\text{-}\mathsf{ISO}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$.

PROOF: Note that we need ACA_0 to check not only equality of lengths but also of the Ulm function along the length. □

8.5.2 Reductions

Ordinals to groups

Definition 8.5.15. Given a tree T we let $\mathcal{G}(T)$ be the (Abelian) group generated freely by the elements of T, modulo the relations $\langle\rangle = 0$ and $p\tau = \sigma$ whenever τ is an immediate successor of σ on T. Given an ordinal α we let $\mathcal{G}(\alpha) = \mathcal{G}(T(\alpha))$.

Despite the presentation as generators / relations, the group $\mathcal{G}(T)$ is computable from T. For more information on the reduction $\mathcal{G}(T)$, as for example how to compute its Ulm Sequence, see [Bar95].

The following is proved in [Frib]:

Lemma 8.5.16 (RCA_0). *For any tree T, T is well-founded iff $\mathcal{G}(T)$ is reduced.*

Also,

Lemma 8.5.17 (RCA_0). *For any ordinal α, $\mathcal{G}(\alpha)$ is a weakly ranked p-group of length α.*

As usual, in ACA_0 we can also prove that $\mathcal{G}(\alpha)$ is ranked.

Proof, following [Frib]: Given $\beta < \alpha$, let G_β be the set of elements of $\mathcal{G}(\alpha)$ of the form $n_0\sigma_0 + \cdots + n_k\sigma_k$ such that for all $i \leqslant k$, the last element of σ_i is at least β. We claim that $\langle G_\beta : \beta < \alpha\rangle$ is an Ulm resolution for $\mathcal{G}(\alpha)$. Clearly for λ limit $G_\lambda = \cap_{\beta<\lambda} G_\beta$. Now consider $x = n_0\sigma_0 + \cdots + n_k\sigma_k$. If $x \in G_{\beta+1}$, then $y = n_0\sigma_0{}^\frown\beta + \cdots + n_k\sigma_k{}^\frown\beta \in G_\beta$ and $py = x$. Conversely, if $x \notin G_{\beta+1}$, then for some σ_i, the last element of σ_i is at most β. So, there is no $y \in G_\beta$ such that $py = x$. Thus $\langle G_\beta\rangle_{\beta\leqslant\alpha}$ is indeed an Ulm resolution of $G(\alpha)$. It is also easy to find the rank of any element. $\square$

Lemma 8.5.18 (RCA_0). *Given ordinals α and β we have that $\alpha \cong \beta \Leftrightarrow \mathcal{G}(\alpha) \cong \mathcal{G}(\beta)$.*

Proof: Clearly if $\alpha \cong \beta$ then $\mathcal{G}(\alpha) \cong \mathcal{G}(\beta)$. Suppose now that $\mathcal{G}(\alpha) \cong \mathcal{G}(\beta)$. By the previous lemma $\mathcal{G}(\alpha)$ and $\mathcal{G}(\beta)$ are both weakly ranked and have lengths α and β. By Lemma 8.5.7, we have that $\alpha \cong \beta$. $\square$

Corollary 8.5.19 (RCA_0). *$\exists$-$\mathsf{ISO}(\mathcal{R}$-p-$\mathcal{G})$ implies ATR_0. Therefore, $\mathsf{RK}(\mathcal{R}$-p-$\mathcal{G})$ and $\exists$-$\mathsf{ISO}(\mathcal{R}$-p-$\mathcal{G})$ are equivalent to ATR_0.*

Lemma 8.5.20 (ACA_0). *Given ordinals α and β we have that $\alpha \preccurlyeq \beta \Leftrightarrow \mathcal{G}(\alpha) \preccurlyeq \mathcal{G}(\beta)$.*

Proof: If $\mathcal{G}(\alpha) \preccurlyeq \mathcal{G}(\beta)$ then by Lemma 8.5.11, $\alpha \preccurlyeq \beta$. If $\alpha \preccurlyeq \beta$ then we can directly construct an embedding of $\mathcal{G}(\alpha)$ into $\mathcal{G}(\beta)$. $\square$

Corollary 8.5.21 (ACA_0). *$\exists$-$\mathsf{EMB}(\mathcal{R}$-p-$\mathcal{G})$ implies ATR_0.*

Groups to trees

Given a p-group G, we let $T(G)$ (essentially) consist of the elements of G: we declare that 0_G corresponds to $\langle\rangle$, and $x \in G$ of order p^n is identified with the sequence $\langle p^{n-1}x, \dots, px, x\rangle$. It is immediate that G is reduced iff $T(G)$ is well-founded; if G is ranked then so is $T(G)$ (the rank of every nonzero x on $T(G)$ is its rank in G, the rank of the tree is the length of G). As we noticed, however, because of the intricate structure of the equimorphism classes of reduced p-groups, this operation can preserve neither non-isomorphism nor non-embedding.

8.5.3 Proofs of arithmetic comprehension

Proposition 8.5.22 (RCA_0). *$\exists$-EMB($\mathcal{R}$-p-$\mathcal{G}$) implies ACA_0, and hence it is equivalent to ATR_0.*

PROOF: Let φ be a Σ^0_1 formula. For each n, construct a p-group G_n by letting $G_n = \mathbb{Z}_p$ if $\neg\varphi(n)$ and $G_n = \mathbb{Z}_{p^2}$ if $\varphi(n)$. Then $\{n : \varphi(n)\} = \{n : \mathbb{Z}_{p^2} \preccurlyeq G_n\}$. $\square$

We next show that $\mathsf{WQO}(\mathcal{R}\text{-p-}\mathcal{G})$ implies ACA_0. As before we go through Σ^0_2-induction. Let T and the sequence $\langle T_n\rangle$ be as in subsection 8.3.3, and let $G_n = \mathcal{G}(T_n)$. Again assuming that $0'$ does not exist, by the results of Friedman quoted earlier (Lemma 8.5.16), each G_n is reduced.

Suppose $n < m$ and that $g\colon G_n \to G_m$ is an embedding. Suppose that $\sigma \in T_n$ is a true string. Considered as an element of G_n, we write $g(\sigma)$ in normal form as $\sum_{i<k} m_i\sigma_i$, where $\sigma_i \in T_m$ and $m_i \in \mathbb{Z}_p$. We claim that every σ_i is true. Suppose that some σ_j is not true; let r be the height of the (finite) tree $T_m[\sigma_j]$. By Σ^0_2-induction, there is some true $\tau \supset \sigma$ on T_n which is sufficiently long so that $p^s\tau = \sigma$, where $s > r - |\sigma_j|$. Then $p^s g(\tau) = g(\sigma)$. Writing $g(\tau)$ in normal form as $\sum_{i<k'} n_i\tau_i$ and multiplying by p^s, we get

$$g(\sigma) = \sum_{i<k} m_i\,\sigma_i = \sum_{i<k'} n_i\,\tau_i',$$

where τ_i' is τ_i with the last s bits chopped off. Thus the set of the τ_i's equals the set of the σ_is, which shows that some τ_i is an extension of σ_j of length $> r$. This is impossible.

By the same kind of calculation, we see that if $\sigma_0 \subsetneq \sigma_1 \in T_n$ are true, then each τ appearing in the normal form of $g(\sigma_0)$ is properly extended by some τ' which appears in $g(\sigma_1)$. This shows that if $\sigma \in T_n$ is true then via g we can obtain some true $\tau \in T_m$ of length at least $|\sigma|$. This allows us to iterate and get $0'$.

Next, we see that Σ^0_2-induction follows from $\mathsf{WQO}(\mathcal{R}\text{-p-}\mathcal{G})$. This follows the proof of Lemma 8.3.33. As for Boolean algebras, all we really need is that in RCA_0:

1. If α, β are ordinals and $\alpha \preccurlyeq \beta$ then $G(\alpha) \preccurlyeq G(\beta)$.

2. For any ordinal α, $G(\alpha + 1)$ does not embed into $G(\alpha)$.

As for Boolean algebras (subsection 8.4.4), the first follows from a direct construction, and the second follows from an analogue of fact 8.4.48 (with the same proof). Thus:

Proposition 8.5.23 (RCA_0). *$\mathsf{WQO}(\mathcal{R}\text{-p-}\mathcal{G})$ implies ACA_0.*

8.6 Scattered and compact spaces

In this section we introduce the class of very countable topological spaces. Unfortunately, the well founded part (the collection of *scattered* spaces) of our class is not very-well behaved. We thus leave open the analysis of the class of scattered spaces and concentrate on compact spaces, which turn out to be metrizable. This allows us to refer to a rich body of research on metric spaces in reverse mathematics.

Remark 8.6.1. We do not give a reduction from topological spaces to other classes. In fact, we do not know whether such computable embeddings exist. There are Turing reductions from compact spaces to well-founded trees and ordinals which preserve embedding, non-embedding, isomorphism and non-isomorphism; however, they make use of a particular listing of the points of the space or of its basic open sets.

The most natural reduction from compact spaces is to the class of superatomic Boolean algebras - Stone duality. All facts about Stone duality can, in fact, be proved in ATR_0 (including the fact that the corresponding Boolean algebra is countable); however, this is not a continuous operation, so we do not consider it in this paper.

8.6.1 Definitions

Definition 8.6.2. A *very countable topological space* is a set $X \subseteq \mathbb{N}$, equipped with a (countable) collection of subsets $\mathcal{O}_X$ which are a basis for a topology on X (i.e., for every finite subset F of $\mathcal{O}_X$ and every $x \in \bigcap F$, there exists $U \in \mathcal{O}_X$ such that $x \in U \subseteq \bigcap F$).

We assume all usual topological notions (see, for example, [Mun00]). So, for instance, $V \subseteq X$ is an *open set* (or an *$\mathcal{O}_X$-open set*) if $\forall x \in V \exists U \in \mathcal{O}_X (x \in V \subseteq U)$. Two topologies $\mathcal{O}_X$ and $\mathcal{O}'_X$ on a same set X are *equivalent* if the $\mathcal{O}_X$-opens sets are exactly the $\mathcal{O}'_X$-open sets. Note that, up to equivalence, we can assume that $\mathcal{O}_X$ is always closed under finite intersections, since closing up under finite intersections is a computable operation.

As *isomorphism* we use homeomorphisms. A one-to-one map $f \colon X \to Y$ is *bi-continuous* if it is continuous, and its inverse is continuous as well (for formalization in RCA_0, we do not assume that a map necessarily has a range; the notion of continuity still makes sense, because the range is definable.) An *embedding* of a space X into a space Y is a one-to-one, bi-continuous function. An embedding $f \colon X \to Y$ is a homeomorphism of X onto its perhaps non-existent range. We note that the standard definition of the subspace topology makes sense in this setting.

All spaces we deal with are very countable, and so we drop this prefix. Also, unless otherwise stated, all spaces are *Hausdorff*. (That is, for every $x, y \in X$, there exists disjoint $U, V \in \mathcal{O}_X$ such that $x \in U$, $y \in V$.) In Hausdorff spaces we can use familiar notions such as converging sequences and limit points. So, when we say *space* we mean very countable Hausdorff topological space.

We note that very countable topological spaces are just countable, second countable spaces. The reader familiar with these concepts should note that when a space is countable, the notions of first countable (N1) and second countable (N2) coincide. We deal with very countable topological spaces because they are the ones that can be easily encoded in Second Order Arithmetic as a set of natural numbers (rather than a class of reals).

Example 8.6.3 *(RCA_0).* Let L be a linear ordering. There is a natural topology on L, the *order topology*, which makes L a very countable space: $\mathcal{O}_L$ is the set of open intervals of L, defined by endpoints in $A \cup \{-\infty, \infty\}$.

We observe that if B is a sub-ordering of A, then the order topology on B is equivalent to the subspace topology.

A "well-founded" topological space is called *scattered*.

Definition 8.6.4. Let X be a space. We say that $y \in Y$ is *isolated* if $\{y\} \in \mathcal{O}_Y$. A set $Y \subseteq X$ is *dense in itself* if as a subspace, Y has no isolated points. A space is *scattered* if it contains no subset which is dense in itself.

8.6.2 Metrizable spaces

A *metric space* consists of a set $M \subseteq \mathbb{N}$ and a sequence $\langle r(a,b)\rangle_{a,b\in M}$ of real numbers (real numbers in the sense of [Sim99, Chapter II], quickly converging Cauchy sequences of rationals,) satisfying the classical properties of a metric.

Recall that a metric on a set M, induces a topology on it: $\mathcal{O}_M = \{B_{x,r} : x \in M, r \in \mathcal{Q}\}$. (Of course, we can then close $\mathcal{O}_M$ under finite intersections.) A topological space is *metrizable* if there is a metric on X consistent with its topology. That is, a metric such that the topology it induces is equivalent to the topology on X.

Example 8.6.5 *(RCA_0).* If α is an ordinal, then there is a canonical embedding of α into the interval $(0,1)$ - see [FH91]. This embedding is bi-continuous with respect to the order topology on α, which shows that α, as a topological space, is metrizable.

A topological space X is *normal* if it is Hausdorff, and for every disjoint closed sets $C, D \subseteq X$, there exists disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. X is *regular* if the condition above holds when C is a singleton.

Lemma 8.6.6. *A space X is regular iff for all $x \in X$ and all open neighborhoods U of x, there is some neighborhood V of X such that $\bar{V} \subseteq U$.*

The proof in [Mun00, Theorem 31.1] goes through in ACA_0.

We note that being an open subset of X is an arithmetic property. Similarly, the relation $x \in \bar{A}$ is arithmetic, as the closure of $A \subseteq X$ is the collection of points which do not have basic open neighborhoods disjoint from A. As a corollary of the previous lemma, we notice that regularity is arithmetically definable, as a space X is regular iff for all $x \in X$ and all *basic* open neighborhoods U of x there is some *basic* neighborhood V of x such that $\bar{V} \subseteq U$. Further, we note that if X is regular, then the function taking some $x \in X$ and a basic neighborhood U of x to the least (according to some fixed enumeration) basic neighborhood V of x such that $\bar{V} \subseteq U$ is arithmetically definable. It is not "topological" as it depends on the enumeration of $\mathcal{O}_X$ - so does not respect homeomorphism. Of course, we view the structure X as equipped with some enumeration of $\mathcal{O}_X$, so the function is indeed definable from X.

Similarly, the function taking some closed $A \subseteq X$ and $x \in X \setminus A$ to the least basic neighborhood of x disjoint from B is arithmetically definable.

Lemma 8.6.7 (ACA_0). *Every regular space is normal.*

PROOF: Essentially, the proof is the standard one, given for example in [Mun00, Theorem 32.1]. All we need is to note that when, for each $x \in A$, we choose a basic open neighborhood of x whose closure is disjoint from B, we can do that arithmetically, and so the resulting cover exists. □

The proof in fact yields more: if X is regular, then there is an arithmetically definable function which takes disjoint, closed $A, B \subseteq X$ to a pair of open subsets of X which separate A and B. Or, equivalently, given some closed A and open $B \supset A$ we can get in an arithmetical way some open $C \supset A$ such that $\bar{C} \subseteq B$.

We need to refine even further. Let X be a normal space. Fix some enumeration $\langle U_n \rangle_{n \in \mathbb{N}}$ of $\mathcal{O}_X$. Suppose that $A \subseteq X$ is open. An *open presentation* of A is a set $N \subseteq \mathbb{N}$ such that $A = \cup_{n \in N} U_n$. A *closed presentation* of a closed set $B \subseteq X$ is an open presentation of $X \setminus B$. Among all open presentations of some open $A \subseteq X$ there is one maximal; it is of course $\{n : U_n \subseteq A\}$. Given some open set $A \subseteq X$, ACA_0 ensures that some presentation of A, in fact its maximal one, exists.

ACA_0 ensures the existence of sets such as $\{(x, n) : x \in \bar{U}_n\}$, $\{(n, m) : \bar{U}_n \subseteq U_m\}$ and functions such as the one taking n to the maximal closed presentation of $\bar{U}_n$ and (n, m) to the maximal closed presentation of $U_n \cap U_m$. When we fix these sets and functions as *oracle*, we can, by the proof of Lemma 8.6.7, *effectively* construct, given some closed presentations of disjoint $A, B \subseteq X$, open sets U, V separating A and B; further, we can effectively construct open presentations of U and V. This allows us to iterate the process of finding separators, which shows that the proof of Urysohn's Lemma goes through in ACA_0:

Lemma 8.6.8 (ACA_0). *(Urysohn's Lemma [Mun00, Theorem 33.1]) If X is a normal space and A, B are disjoint closed subsets of X, then there is some continuous $f\colon X \to [0, 1]$ such that $A \subseteq f^{-1}\{0\}$ and $B \subseteq f^{-1}\{1\}$.*

Remark 8.6.9. We may assume that range $f \subseteq \mathcal{Q}$. For given f, we can use the standard "forth" argument to get an order-preserving, hence continuous, $g\colon$ range $f \to \mathcal{Q}$; ACA_0 ensures that $<\restriction$ range f exists.

We also remark that f is obtained effectively given the discussed oracle, uniformly in (closed presentations of) A and B. This uniformity allows us to see that if X is regular then there is a countable collection of functions $f\colon X \to [0,1]$ such that for any $x \in X$ and any neighborhood U of x, some f in the collection is positive at x and vanishes outside U. The rest of the proof of the Urysohn metrization theorem goes through in ACA_0:

Theorem 8.6.10 (ACA_0). *(Urysohn metrization theorem [Mun00, Theorem 34.1]) Let X be a space. The following are equivalent.*

1. *X is regular.*
2. *X is normal.*
3. *X is metrizable.*

Moreover, given a sequence of regular topological spaces, there is a sequence of metrics for them.

Example 8.6.11 *(ACA_0).* The order topology of every linear ordering is regular and hence metrizable.

8.6.3 Compact spaces

Definition 8.6.12. A space X is *compact* if every open covering of X which consists of basic open sets, contains a finite sub-covering. That is, if for every sequence of basic open sets $\langle U_n \rangle_{n\in\mathbb{N}}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} U_n$, there exists a finite $F \subseteq \mathbb{N}$ such that $X \subseteq \bigcup_{n\in F} U_n$.

We note that when working in ZFC, the definition of compactness requires consideration of uncountable open coverings. But if a space is countable, then every open covering has a countable sub-covering. Further, if a space is very countable, then from an arbitrary countable open covering, we can find a refinement which is both countable and consists of basic open sets. So, when dealing with very countable spaces, the definition of compactness given above is equivalent to the usual one.

Definition 8.6.13. A space X is *sequentially compact* if every infinite sequence of elements of X has a converging subsequence.

The trick of having every infinite Σ^0_1 class containing an infinite set yields the following.

Lemma 8.6.14 (RCA_0). *A space X is sequentially compact iff every infinite subset of X has a limit point in X.*

The standard proofs of equivalence, formalized, yield the following:

Lemma 8.6.15 (RCA_0). *Every sequentially compact space is compact.*

Lemma 8.6.16 (ACA_0). *Every compact space is sequentially compact.*

We observe that when working in ZFC with countable, compact, Hausdorff spaces, the condition of very countability comes for free.

Observation 8.6.17. (ZFC) Every countable, compact, Hausdroff topological space is second countable, and hence very countable.

Proof: Suppose that X is not second countable. Then, it is not first countable either. That means that there exists an $x \in X$ which has no countable basis of open neighborhoods, i.e., such that for every decreasing sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of open neighborhoods of x, there exists an open neighborhood V of x such that for no n, $U_n \subseteq V$.

Let $\langle x_n \rangle_{n \in \omega}$ be an enumeration of $X \setminus \{x\}$. We will construct a subsequence $\langle x_{n_k} \rangle_{k \in \omega}$ with no converging subsequence, contradicting the compactness of X. First, using the fact that X is Hausdorff construct two sequences of open sets $\langle U_n \rangle_{n \in \omega}$ and $\langle V_n \rangle_{n \in \omega}$ such that for every n, U_n and V_n are disjoint neighborhoods of x and x_n respectively, $U_{n+1} \subsetneq U_n$. Now there is an open set W containing x such that for no n, $U_n \subseteq W$. Define $\langle x_{n_k} \rangle_{k \in \omega}$ as follows: Let $x_{n_0} \in U_0 \setminus W$; given x_{n_k}, let $m > n_k$ be such that $x_{n_k} \notin U_m$ and let $x_{n_{k+1}} \in U_m \setminus W$. Now, x is not a limit of any subsequence of $\langle x_{n_k} \rangle_{k \in \omega}$ because W is a neighborhood of x which contains no point in that sequence. Also, any point x_n is not a limit of any subsequence of $\langle x_{n_k} \rangle_{k \in \omega}$ because V_n is a neighborhood of x_n which contains no point x_{n_k} for $n_k > n$. □

Observation 8.6.18. The above proof can be carried through in a subsystem of second order arithmetic with sufficiently much choice, provided that it is meaningful: that is, when the given space is a definable class.

We now see that compact spaces are nice and well-founded. For the first, the standard proof will do.

Lemma 8.6.19 (ACA_0). *Every compact space is normal.*

Lemma 8.6.20 (ACA_0). *Every compact space is scattered.*

Proof: Suppose that X is a topological space and suppose that $Y \subseteq X$ is dense in itself. We will construct a sequence $\langle y_n \rangle_{n \in \omega} \subseteq Y$ with no convergent subsequence. Let $\langle x_n \rangle_{n \in \omega}$ be an enumeration of X. Let y_0 be such that there exits disjoint basic open sets U_0 and V_0 around y_0 and x_0 respectively. Suppose we have already defined y_n and U_n, and $y_n \in U_n$. Let $y_{n+1} \in U_n$ be any point different from x_{n+1}, which exists because Y is dense in itself. Let $U_{n+1} \subseteq U_n$ and V_{n+1} be basic open neighborhoods of y_{n+1} and x_{n+1} respectively. Now, since for every n, $x_n \in V_n$ and $\forall m \geqslant n \ (y_m \notin V_n)$, $\langle y_n \rangle$ has no convergent subsequence. □

We shall need to following basic facts:

1. If X is compact and $Y \subseteq X$ is closed, then Y is compact (in the subspace topology).
2. If $Y \subseteq X$ is compact and X is Hausdorff then Y is closed in X.

Standard proof go through in ACA_0, so we get (remembering that all spaces are Hausdorff):

Fact 8.6.21 (ACA_0). *If X is compact and $f\colon X \to Y$ is one-to-one and continuous, then f is closed (so f is an embedding).*

Spaces of well-orderings

In the following, by *complete* spaces we of course mean perhaps uncountable, definable spaces.

Lemma 8.6.22 (ATR_0). *[FH91] Every countable, closed, totally bounded subset of a complete separable metric space is homeomorphic to the canonical metric space of some well-ordered set.*

Lemma 8.6.23 (ACA_0). *Let X be a topological space. The following are equivalent.*

1. *X is compact.*
2. *X is homeomorphic to a countable, closed, totally bounded subset of a complete, separable metric space.*

Proof: If X is compact, then we can consider it as a metric space. We let $\bar{X}$ be the completion of X (see [Sim99, Section II.5]). This is a complete, separable metric space. Since X is compact, it is a closed in $\bar{X}$ and is also totally bounded.

For the other direction see [Mun00, Theorem 45.1]. □

Thus every compact space is homeomorphic to the order topology of some ordinal.

Lemma 8.6.24 (RCA_0). *Let α be an ordinal. Then $\alpha + 1$ is a compact space.*

This can be derived in ACA_0 from theorems 2.2 and 2.3 of [FH91].

Proof: Let $\langle U_n \rangle$ be a sequence of basic open sets which cover $\alpha + 1$. The point is that from n we can get the pair (a_n, b_n) defining U_n. Thus, if $\langle U_n \rangle$ does not have a finite sub-cover, we can inductively choose a descending sequence $\langle c_k \rangle$ in α as follows: together with $\langle c_k \rangle$, we find a sequence $\langle U_{n_k} \rangle$ such that for each k, $(c_k, \infty) \subseteq \cup_{l \leqslant k} U_{n_l}$. Given c_k, we let $U_{n_{k+1}}$ be some open set on the list which contains c_k; and we let $c_{k+1} = a_{n_{k+1}}$. □

We can now define the reduction from ordinals to compact spaces:

Definition 8.6.25. Let α be an ordinal and $n \in \mathbb{N}$. The space $\mathbb{C}(\alpha, n)$ is the topological space given by the order topology on $\omega^\alpha \cdot n + 1$. We let $\mathbb{C}(\alpha) = \mathbb{C}(\alpha, 1)$.

8.6.4 Ranked spaces

The *Cantor-Bendixon derivative* X' of a space X is the collection of limit points of X (i.e., X with its isolated points removed.) Let α be an ordinal; a *partial Cantor-Bendixon resolution* of X along α is a sequence of subspaces $\langle X_\beta \rangle_{\beta<\alpha}$ such that $X_0 = X$, if $\beta+1 < \alpha$ then $X_{\beta+1} = X'_\beta$, and for limit $\lambda < \alpha$, $X_\lambda = \bigcap_{\beta<\lambda} X_\beta$. A *Cantor-Bendixon resolution* of X is a partial Cantor-Bendixon resolution of X along some ordinal $\alpha+1$ such that $X_\alpha = \emptyset$ but for $\beta < \alpha$, $X_\beta \neq \emptyset$. If there is a resolution of X along $\alpha+1$ then we let $\mathrm{rk}(X) = \alpha$ (which is also called the *length* of X). If α is a successor ordinal, then the *degree* of X is the number of points in $X_{\alpha-1}$; if α is a limit ordinal we let $\deg(X) = 0$. We define $\mathrm{inv}(X) = (\mathrm{rk}(X), \deg(X))$. We also define an associated rank function; for $x \in X$, we let $\mathrm{rk}(x)$ be the unique $\beta < \alpha$ such that $x \in X_\beta \smallsetminus X_{\beta+1}$.

Definition 8.6.26. A space X is *ranked* if it has a Cantor-Bendixon resolution and the associated rank function exists.

As usual, in RCA_0 we can show that any two rankings of a ranked space X are isomorphic. See, for example, 8.4.16.

Lemma 8.6.27 (ATR_0). *Every scattered space is ranked.*

PROOF: Friedman [Fria] proved this result for countable metric spaces. The same proof works for topological spaces. $\square$

Lemma 8.6.28 (RCA_0). *Every ranked space is scattered.*

PROOF: Suppose that $Y \subseteq X$ is dense in itself. Let $\langle X_\beta \rangle_{\beta<\alpha}$ be any partial resolution of X. Then by Π^0_1-transfinite induction on $\beta < \alpha$ we can show that $Y \subseteq X_\beta$ for every $\beta < \alpha$. Thus $\langle X_\beta \rangle$ cannot be a full resolution of X. $\square$

Lemma 8.6.29 (RCA_0). *Let α be and ordinal and $n \in \mathbb{N}$. Then $\mathbb{C}(\alpha, n)$ is scattered, indeed it is ranked, and its invariant is $(\alpha+1, n)$.*

Of course, in RCA_0, $\mathbb{C}(\alpha, n)$ is compact.

PROOF: Let $X = \mathbb{C}(\alpha, n)$. Given $x = \sum_{i<k} \omega^{\beta_i} \cdot n_i \in X$, we let $\mathrm{rk}(x) = \min\{\beta_i : i < k\}$. For $\beta \leqslant \alpha+1$, let $X_\beta = \{x \in X : \mathrm{rk}(x) \geqslant \beta\}$. We claim that $\langle X_\beta : \beta \leqslant \alpha+1 \rangle$ is a Cantor-Bendixon resolution of X and rk is a rank function for X. It is clear that when γ is a limit ordinal, $X_\gamma = \bigcap_{\delta<\gamma} X_\delta$. We then have to prove that for every $\gamma \leqslant \alpha$, $X_{\gamma+1} = X'_\gamma$. This follows from the fact that $X_\gamma = \{\omega^\gamma \cdot \delta : \delta \leqslant \omega^{\alpha'} \cdot n\}$, and $X_{\gamma+1} = \{\omega^\gamma \cdot \delta : \delta \leqslant \omega^{\alpha'} \cdot n$ and δ is a limit ordinal$\}$, where α' is such that $\gamma + \alpha' = \alpha$ (note that we do not assume that we can regard α' as an initial segment of α).

It follows that $\mathrm{rk}(X) = \alpha+1$ and since $X_\alpha = \{\omega^\alpha \cdot i : 0 < i \leqslant n\}$, $\deg(X) = n$. $\square$

The importance of the Cantor-Bendixon invariant is that it classifies compact spaces up to isomorphism, and is also compatible with the embedding relation. For the case of countable metric spaces, Friedman essentially proved the following lemma.

Lemma 8.6.30 (ACA_0). *([Fria]) Let X and Y be ranked countable metric spaces with invariants $\langle \alpha, n\rangle$ and $\langle \beta, m\rangle$ respectively. Then, there is an one-to-one, continuous function $f\colon X \to Y$ if and only if $\langle \alpha, n\rangle \leqslant_{\text{lex}} \langle \beta, m\rangle$.*

Recalling fact 8.6.21, we get:

Corollary 8.6.31 (ACA_0). *Let X and Y be ranked compact spaces. Then $X \preccurlyeq Y$ iff $\operatorname{inv}(X) \leqslant_{\text{lex}} \operatorname{inv}(Y)$.*

As in earlier sections, we will want to get uniform rankings of a sequence of spaces.

Lemma 8.6.32 (ACA_0). *Assume that every compact space is ranked. Let $\langle X_n\rangle$ be a sequence of compact spaces. Then there is an ordinal α and a sequence of functions $f_n\colon X_n \to \alpha$ such that each f_n is a rank function for X_n.*

Proof: Let Y be the disjoint union of the X_ns and let X be a simple version of the 1-point compactification of Y: The basic open subsets of X are the basic open subsets of Y, together with the sets $X \setminus X_n$ for $n \in \mathbb{N}$. It is straightforward to check that X is compact and that each X_n is an open subset of X. Thus, a ranking of X gives uniform rankings of all the X_ns (see Friedman, [Fria], to see that if $A \subseteq B$ is open in B and $\langle B_\gamma\rangle$ is a resolution of B, then $\langle A \cap B_\gamma\rangle$ is a resolution of A.) $\square$

Corollary 8.6.33 (ACA_0). *$\mathsf{RK}(\mathcal{CCS})$ implies $\exists$-$\mathsf{EMB}(\mathcal{CCS})$, $\mathsf{COMP}(\mathcal{CCS})$ and $\mathsf{WQO}(\mathcal{CCS})$.*

Corollary 8.6.34 (ATR_0). *Every compact space X is homeomorphic to $\mathbb{C}(\alpha, n)$, where $(\alpha+1, n) = \operatorname{inv}(X)$.*

Proof: By lemmas 8.6.22 and 8.6.23, X is homeomorphic to the canonical metric space of some ordinal β. (Note that β has to be a successor ordinal, because otherwise X would not be compact.) Using the Cantor normal form, write β as:

$$\beta = \omega^{\alpha_0} \cdot n_0 + \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k,$$

where $\alpha_0 > \alpha_1 > \ldots > \alpha_k$. We can write β as

$$(\omega^{\alpha_0} \cdot n_0 + 1) + (\omega^{\alpha_1} \cdot n_1 + 1) + \ldots + (\omega^{\alpha_{k'}} \cdot n'_{k'} + 1).$$

It is not hard to prove that β is homeomorphic to

$$(\omega^{\alpha_{k'}} \cdot n'_{k'} + 1) + \ldots + (\omega^{\alpha_1} \cdot n_1 + 1) + (\omega^{\alpha_0} \cdot n_0 + 1),$$

which, as an ordinal, is isomorphic to $\omega^{\alpha_0} \cdot n_0 + 1$. Finally, $(\alpha+1, n) = \operatorname{inv}(X) = \operatorname{inv}(\omega^{\alpha_0} \cdot n_0 + 1) = (\alpha_0 + 1, n_0)$, so $\alpha_0 = \alpha$ and $n_0 = n$. $\square$

Corollary 8.6.35 (ATR_0). *Let X an Y be compact spaces with invariants $\langle \alpha, n\rangle$ and $\langle \beta, m\rangle$ respectively. Then, X and Y are homeomorphic if and only if $\langle \alpha, n\rangle = \langle \beta, m\rangle$.*

Corollary 8.6.36. *ATR_0 implies the statements $\exists$-$\mathsf{ISO}(\mathcal{CCS})$ and $\mathsf{EQU}{=}\mathsf{ISO}(\mathcal{CCS})$.*

8.6.5 Reversals

To get reversals, we need to apply some of the aforementioned results in weaker systems. The next lemma follows from Lemma 8.6.29.

Lemma 8.6.37 (RCA_0). *Let α and β be ordinals. Then $\alpha \cong \beta$ iff $\mathbb{C}(\alpha)$ and $\mathbb{C}(\beta)$ are homeomorphic.*

Corollary 8.6.38 (RCA_0). *$\exists$-$\mathsf{ISO}(\mathcal{CCS})$ is equivalent to ATR_0.*

The next lemma follows from Lemma 8.6.29 and corollary 8.6.31.

Lemma 8.6.39 (ACA_0). *Let α and β be ordinals. Then, $\mathbb{C}(\alpha)$ embeds in $\mathbb{C}(\beta)$ if and only if $\alpha \preccurlyeq \beta$.*

Corollary 8.6.40 (ACA_0). *Each of $\exists$-$\mathsf{EMB}(\mathcal{CCS})$, $\mathsf{COMP}(\mathcal{CCS})$, $\mathsf{EQU}{=}\mathsf{ISO}(\mathcal{CCS})$, and $\mathsf{WQO}(\mathcal{CCS})$, is equivalent to ATR_0.*

To get the EW-reduction, we first observe the following. Suppose that L is an ill-founded linear ordering; let $\langle a_n\rangle$ be an infinite descending sequence in L. We can construct a copy of $1+\mathcal{Q}$ inside ω^L by considering all elements of the form $\omega^{a_1}n_1+\omega^{a_2}n_2+\ldots\omega^{a_k}n_k$ for $k, n_l \in \mathbb{N}$. Thus ω^L is not scattered. As a conclusion, we see that for all linear orderings L, if L is a well-ordering then $\mathbb{C}(L)$ is compact, and if L is ill-founded, then $\mathbb{C}(L)$ is not scattered, and so not compact. All this can be done in RCA_0.

8.6.6 Proofs of arithmetic comprehension

Proposition 8.6.41 (RCA_0). *$\exists$-$\mathsf{EMB}(\mathcal{CCS})$ implies ACA_0.*

Proof: See Proposition 8.3.29 or 8.5.22. □

Proposition 8.6.42 (RCA_0). *$\mathsf{RK}(\mathcal{CCS})$ implies ACA_0.*

Proof: This is immediate, given Lemma 8.6.32. We construct a sequence $\langle X_n\rangle$ of compact spaces and points $x_n \in X_n$, such that if $n \in 0'$ then x_n is a limit point in X_n, and if $n \notin 0'$ then $X_n = \{x_n\}$. Then from a uniform ranking of the X_ns we can uniformly get the rank of each x_n (in X_n) and thus get $0'$. □

Proposition 8.6.43 (RCA_0). *$\mathsf{COMP}(\mathcal{CCS})$ implies ACA_0.*

PROOF: We again show that $0'$ exists by showing how to enumerate infinitely many true stages.

As in the proof of proposition 8.4.46, we construct a sequence of ordinals $\langle \alpha_s \rangle_{s\in\mathbb{N}}$ such that if s is a true stage, then α_s is a canonical copy of $\omega+1$, and otherwise α_s is a copy of some $n<\omega$ (where we can tell which is the last element and what is the place of the other elements). We let $\alpha = \sum_{s\in\mathbb{N}} \alpha_s$.

It is easy to see that α is an ordinal (from a decreasing sequence in α we can construct either a decreasing sequence of $s \in \mathbb{N}$ or a decreasing sequence in some α_s). The limit points of α are exactly those last elements of α_s where s is a true stage.

Let $X = \alpha \cdot 2 + 1$, with the order topology. Let Y be a canonical copy of ω^2+1. There cannot be an embedding of X into Y. This is because for any embedding, the image of a limit point is a limit point and the image of a limit of limit points is also a limit of limit points; of which X has two but Y only one.

By $\mathsf{COMP}(\mathcal{CCS})$, there is an embedding g of Y into X. Write $X = \alpha_0 + \alpha_1 + 1$ (α_i is a copy of α), and let A be the class of limit points of X. For some $i<2$, $B_i = A \cap g^{-1}\alpha_i$ is infinite. For such i, $g \restriction B_i$ allows us to enumerate infinitely many limit points of α, and so infinitely many true stages. $\square$

Proposition 8.6.44 (RCA_0). *$\mathsf{EQU{=}ISO}(\mathcal{CCS})$ implies ATR_0.*

PROOF: Let α and β be ordinals; we will prove that they are comparable. Note that $\delta = \alpha+\beta+\alpha+...$ and $\gamma = \beta+\alpha+\beta+\alpha+...$ are equimorphic via continuous embeddings. So, $\omega^\delta+1$ and $\omega^\gamma+1$ are also equimorphic by continuous embeddings, and hence equimorphic as compact spaces. By $\mathsf{EQU{=}ISO}(\mathcal{CCS})$ we have that they are homeomorphic. Then, by Lemma 8.6.37, δ and γ are isomorphic. It follows that α and β are comparable. $\square$

Proposition 8.6.45 (RCA_0). *$\mathsf{WQO}(\mathcal{CCS})$ implies ACA_0.*

PROOF: As usual, we first work over RCA_2.

We define β^n_t and α_n in a similar way to what is done in the proof of proposition 8.4.46; in this case, $\beta^n_t = \omega+1$ exactly when t is the k^{th} true stage for some $k>2n$; $\alpha_n = \sum_t \beta^n_t$. Let $X_n = \alpha_n + 1$ with the order topology; every X_n is compact. Suppose that $n<m$ and that f is an embedding of X_n into X_m.

Let t_k be the $(2n+k)^{th}$ true stage. For all t, let $a_t = \max \beta^n_t$ and let $b_t = \max \beta^m_t$. For all $k>0$, a_{t_k} is a limit point of X_n. Consider $f(a_{t_1})$ and $f(a_{t_2})$. At least one of them is in α_m (that is, it is not the last limit point we added to make X_m compact), and in fact, it has to be b_{t_k} for some $k>2$, say, for example, $k=7$. From t_7 we can find $t_3, t_4, \ldots, t_6$. From $f(a_{t_1}), f(a_{t_2}), \ldots, f(a_{t_7})$, at least six are in α_m and are b_{t_k}s for distinct $k>2$; thus at least one must be b_{t_k} for $k>7$. Now the process repeats to get all t_k.

The rest (getting $I\Sigma_2$ from $\mathsf{WQO}(\mathcal{CCS})$) is identical to the proof in all the previous sections. That is, we verify in RCA_0 that if $\alpha \preccurlyeq \beta$ then $\mathbb{C}(\alpha) \preccurlyeq \mathbb{C}(\beta)$, and that the analog of fact 8.4.48 holds for compact spaces, and so that $\mathbb{C}(\alpha+1)$ does not embed into $\mathbb{C}(\alpha)$. $\square$

8.7 Up to equimorphism, hyperarithmetic is recursive

In this section we prove the theorems in subsection 8.1.6. As we mentioned, the first two are not difficult. They rely on the fact, already mentioned in [AK00] for Boolean algebras and groups, that if X is a hyperarithmetic structure in the well-founded part of any of the classes we considered, its rank is computable. This is simple; we give a general proof: for each class $\mathcal{X}$ we considered, we showed that ATR_0 suffices to prove that each well-founded structure is ranked. By [Sim99, Corollary VII.2.12], we know that there is a β-model M of ATR_0 which consists of hyper-low sets. Each hyperarithmetic structure X in $\mathcal{X}$ is in M; a rank for X, in the sense of M, exists in M; and since M is Σ^1_1-correct, this rank is really an ordinal.

Since the invariant of a compact space determines its isomorphism type, we get Theorem 8.1.16 immediately. This also yields the result 8.1.14 for superatomic Boolean algebras mentioned in [AK00]. We also know that the rank of a well-founded tree determines its equimorphism type, so we get that every hyperarithmetic well-founded tree is equimorphic with a recursive one.

Theorem 8.1.15 follows, because we know that if B is a Boolean algebra which is not superatomic, then it contains a copy of the atomless Boolean algebra, into which every countable Boolean algebra can be embedded. And if T is an ill-founded tree, then every countable tree can be embedded into T.

We turn to the third theorem.

Proof of Theorem 8.1.17:: Let G be a hyperarithmetic p-group. If G is reduced, then (see [AK00, Theorem 8.17]) it has some length $\alpha + n < \omega_1^{CK}$, where α is a limit ordinal and $n < \omega$. By [Bar95, Proposition 4.3] and [BE71, Theorem 4.1], there is a recursive group H of length $\alpha+n$ such that for all $\beta < \alpha$, $U_H(\beta) = \infty$ and for all $m \leqslant n$, $U_H(\alpha + m) = U_G(\alpha + m)$. From Lemma 8.5.11 and Remark 8.5.10 we obtain that H and G are equimorphic.

Suppose now that G is not reduced. It can be written as a sum $G_d + G_r$, where G_d is divisible and G_r is reduced (see [Kap69, Theorem 3]). Every countable divisible p-group is of the form $\mathbb{Z}(p^\infty)^m$, for some $m \leqslant \omega$ (see [Kap69, Theorem 4]), and hence has a recursive copy. Again by [AK00, Theorem 8.17], G_r has some length $\alpha \leqslant \omega_1^{CK}$. If G_r has length $\alpha < \omega_1^{CK}$, by the previous argument it is equimorphic to a recursive group, and hence $G = G_d + G_r$ is too. Suppose now that $\alpha = \omega_1^{CK}$. We claim that then, $G_d \cong \mathbb{Z}(p^\infty)^\omega$, and hence G is equimorphic to $\mathbb{Z}(p^\infty)^\omega$. (Note that any countable p-group embeds in $\mathbb{Z}(p^\infty)^\omega$.) Suppose instead, toward a contradiction, that $G_d \cong \mathbb{Z}(p^\infty)^n$ for some $n < \omega$. Note that from Remark 8.5.10 we get that for all $\beta < \omega_1^{CK}$, $r(p^\beta G) = \infty$. Consider the partial ordering P whose elements are $n+1$ tuples of independent elements of G, and such that $\langle x_0, \dots, x_n \rangle \leqslant \langle x'_0, \dots, x'_n \rangle$ iff there exists some $k \in \mathbb{N}$ such that for every $i \leqslant n$ $p^k x_i = x'_i$. We claim that P is well founded and has rank $\leqslant \omega_1^{CK}$. This will be a contradiction because P is hyperarithmetic. We prove this by defining a rank function on P. Given $\langle x_0, \dots, x_n \rangle \in T$, first write each x_i as $y_i + z_i$ where

$y_i \in G_r$ and $z_i \in G_d$, and then let $g(\langle x_0,\dots,x_n\rangle) = \min\{\mathrm{rk}_{G_r}(y_i) : i \leqslant n\}$, where $\mathrm{rk}_{G_r}(0) = \infty$. We claim that $g\colon P \to \omega_1^{CK}$ is a rank function. First, we observe that for no $\bar{x} \in P$, $g(\bar{x}) = \infty$: If $g(\langle x_0,\dots,x_n\rangle) = 0$ then for all $i \leqslant n$, $y_i = 0$, and hence $x_i = z_i \in G_d$. but this cannot be the case because, since $r(G_d) = n$, $\{x_0,\dots,x_n\}$ cannot be an independent set. Second, we observe that if $\langle x_0,\dots,x_n\rangle < \langle x'_0,\dots,x'_n\rangle$, then $g(\langle x_0,\dots,x_n\rangle) < g(\langle x'_0,\dots,x'_n\rangle)$: This is because if $g(\langle x_0,\dots,x_n\rangle) = \mathrm{rk}_{G_r}(x_{i_0})$, then $g(\langle x'_0,\dots,x'_n\rangle) \leqslant \mathrm{rk}_{G_r}(x'_{i_0}) < \mathrm{rk}_{G_r}(x_{i_0})$. Last, we show that if $\beta < g(\langle x_0,\dots,x_n\rangle)$, there exists some $\langle x'_0,\dots,x'_n\rangle < \langle x_0,\dots,x_n\rangle$ such that $g(\langle x'_0,\dots,x'_n\rangle) \geqslant \beta$. By definition of rk_{G_r}, for each $i \leqslant n$ there exists x'_i such that $x'_i p = x_i$ and $\mathrm{rk}_{G_r}(x'_i) \geqslant \beta$. Clearly $\langle x'_0,\dots,x'_n\rangle < \langle x_0,\dots,x_n\rangle$ and $g(\langle x'_0,\dots,x'_n\rangle) \geqslant \beta$. We still have to prove that $\langle x'_0,\dots,x'_n\rangle \in P$. Suppose that $\sum_{i\leqslant n} m_i x'_i = 0$. Then $\sum_{i\leqslant n} m_i x_i = p\sum_{i\leqslant n} m_i x'_i = 0$, and hence $m_i x_i = 0$ for every i. This implies that $p|m_i$ for every i. Then $\sum_{i\leqslant n}(m_i/p)x_i = \sum_{i\leqslant n} m_i x'_i = 0$, and hence $m_i x'_i = (m_i/p)x_i = 0$ for every i. We have proved that $\langle x'_0,\dots,x'_n\rangle$ is an independent set and hence belongs to P. The fact that P has rank ω_1^{CK} follows from the fact that for all $\beta < \omega_1^{CK}$, $r(p^\beta G) = \infty$. $\square$

Part III

Computable Mathematics

Chapter 9
Up to equimorphism, hyperarithmetic is recursive.

Published in the Journal of Symbolic Logic, volume 70 (2), year 2005, pages 360-378.

9.1 Introduction

Clifford Spector proved the following well known classical theorem in Computable Mathematics.

Theorem 9.1.1. *[Spe55] Every hyperarithmetic well ordering is isomorphic to a recursive one.*

Recall that a set is hyperarithmetic if and only if it is Δ^1_1. Then, for instance, every arithmetic set is hyperarithmetic.

The direct generalization of Theorem 9.1.1 to the class of linear orderings does not hold. It is not the case that every linear ordering with a hyperarithmetic presentation is isomorphic to a recursive one. Feiner constructed in [Fei67] and [Fei70] (see also [Dow98, Theorem 2.5]) a Π^0_1 subset of $\mathcal{Q}$ that, as a linear ordering, is not isomorphic to a computable one. Other examples were given later. It follows from the work of Lerman [Ler81] that for every Turing degree $\mathbf{a}$ such that $\mathbf{a}'' >_T 0''$ there is a linear ordering of degree $\mathbf{a}$ without a recursive copy. This result was later extended, first to any non-recursive recursively enumerable degree $\mathbf{a}$ by Jockusch and Soare [JS91], then to any non-recursive Δ^0_2 degree $\mathbf{a}$ by Downey [Dow98] and Seetapun (unpublished), and finally to any non-recursive degree $\mathbf{a}$ by Knight [AK00]. Many other results have been proved about presentations of linear orderings; we refer the reader to [Dow98] for a survey on the effective mathematics of linear orderings.

But there are other ways in which we can generalize Theorem 9.1.1. We say that two linear orderings are *equimorphic* if each one can be embedded into the other one. Observe that if a linear ordering $\mathcal{L}$ is equimorphic to an ordinal α, then $\mathcal{L}$ and α are actually isomorphic. (It is clear that two equimorphic well orderings are isomorphic. Note that $\mathcal{L}$ has to be a well ordering because since ω^* does not embed in α and $\mathcal{L}$ embeds in α, ω^* does not embed in $\mathcal{L}$ either (where ω^* is the order type of the negative integers).) So, actually, we can state Theorem 9.1.1 as "every hyperarithmetic well ordering is equimorphic to a recursive linear ordering." The main theorem of this paper is the following generalization of Theorem 9.1.1.

Theorem 9.1.2. *Every hyperarithmetic linear ordering is equimorphic to a recursive one.*

Many properties of linear orderings are invariant under equimorphisms. An interesting example is extendibility. A linear ordering $\mathcal{L}$ is *extendible* if every partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a *linearization* (i.e.: a linear extension) which does not embed $\mathcal{L}$ either. The notion of *weakly extendible* is defined similarly but only considering countable partial orderings $\mathcal{P}$. It is not hard to see that these notions depend only on the equimorphism type of the linear ordering $\mathcal{L}$. Classifications of extendible and weakly extendible linear orderings have been given by Bonnet and Pouzet [BP82], and Jullien [Jul69]. See Chapter 6 and [DHLS03] for an analysis of Jullien's theorem and of the extendibility of certain linear orderings form the viewpoint of Computable Mathematics and Reverse Mathematics.

Three other properties that are invariant under equimorphisms, and which will be very important in this paper, are being scattered, being indecomposable and having a certain Hausdorff rank. A linear ordering is *scattered* if it does not contain a copy η, the order type of the rationals. Then, for a countable linear ordering, being scattered is equivalent to not being equimorphic to η. We say that a linear ordering $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have that $\mathcal{L}$ can be embedded in either $\mathcal{A}$ or $\mathcal{B}$. It is not hard to prove that a linear ordering equimorphic to an indecomposable one is also indecomposable. The *Hausdorff rank* of a scattered linear ordering is the least ordinal α such that only finitely many point are left after α iterations of the operation of collapsing points of $\mathcal{L}$ which have only finitely many points in between (see Definition 9.2.1 below). We will prove that a scattered linear ordering has Hausdorff rank less than ω_1^{CK} (the first non-recursive ordinal) if and only if it is equimorphic to a recursive linear ordering.

Contrary to the case of countable well orderings, the partial ordering, $\mathbb{L}$, of countable linear orderings modulo equimorphism ordered by embeddablity is not a well understood structure. (See [Ros82, § 10.2] for more information on $\mathbb{L}$.) Note that $\mathbb{L}$ has the equimorphism type of η as its top element. (An *equimorphism type* is an equivalence class for the equimorphism relation.) Let $\widehat{\mathbb{L}}$ be obtained by removing the equimorphism type of η from $\mathbb{L}$. So, $\widehat{\mathbb{L}}$ consists of the equimorphism types of scattered linear orderings. Roland Fraïssé conjectured in [Fra48] that $\widehat{\mathbb{L}}$ is well founded and that every element has only countably many elements below it. Later, the statement that says that $\mathbb{L}$ is a well partial ordering became known as Fraïssé's conjecture. (A partial ordering is a *well partial ordering* if it contains no infinite descending sequence and no infinite antichain. See Definition 9.2.10 below.) All these statements were proved by Richard Laver, twenty three years later, in [Lav71] using Nash-Williams's complicated notion of better quasiordering [NW68]. As a corollary of our construction, we prove that for every $\alpha < \omega_1^{CK}$, $\mathbb{L}_\alpha$, the subordering of $\mathbb{L}$ containing the the equimorphism types of linear orderings of Hausdorff rank less than α, is recursively presentable. This result might be useful when studying Fraïssé's conjecture from the viewpoint of Reverse Mathematics. Logicians have been interested in Fraïssé's conjecture because of the complexity of its proof. Some results have been proved about its proof theoretic strength: Shore [Sho93] proved that it implies ATR_0, and we proved in Chapter 6 that it is equivalent to Jullien's theorem, to the finite decomposability of scattered linear orderings and to the statement that says that the class of signed trees is well quasiordered. But, its exact proof theoretic strength is still unknown. It has been conjectured by Clote [Clo90], Simpson [Sim99, Remark X.3.31] and Marcone [Mar] that it is equivalent to ATR_0 over RCA_0. It would be interesting, and maybe useful when studying Fraïssé's conjecture, to know what the rank of $\mathbb{L}_\alpha$, as a well founded partial ordering, is for a given α.

Outline

In Section 9.2 we present the most important ideas in the proof of our main result, Theorem 9.1.2. In Section 9.3 we introduce and study the structure of signed forests. Signed forests extend the notion of signed trees which was introduced in Chapter 6. The use of signed trees is very helpful when studying the structure of indecomposable linear orderings up to equimorphisms. In Section 9.4 we formally describe the construction, already mentioned in Section 9.2, but this time using the results of Section 9.3.

Basic Notions

An *embedding* between linear orderings $\mathcal{L}$ and $\mathcal{Q}$ is a one-to-one, order preserving map $f\colon L \to Q$. If this is the case, we write $f\colon \mathcal{L} \hookrightarrow \mathcal{Q}$, and we write $\mathcal{L} \preccurlyeq \mathcal{Q}$ to mean that $\mathcal{L}$ *embeds* in $\mathcal{Q}$. $\mathcal{L}$ and $\mathcal{Q}$ are *equimorphic* if $\mathcal{L} \preccurlyeq \mathcal{Q}$ and $\mathcal{Q} \preccurlyeq \mathcal{L}$, in which case we write $\mathcal{L} \sim \mathcal{Q}$.

A *presentation* of a linear ordering $\mathcal{L}$ is another linear ordering $\mathcal{A} = \langle A, \leqslant_A \rangle$ isomorphic to $\mathcal{L}$ such that $A \subseteq \omega$. The *Turing degree* of a presentation $\mathcal{A}$ is the join of the degrees of A and $\leqslant_A$.

Given a function $f\colon X \to Y$ and $Z \subseteq X$, we let $f[Z] = \{f(z) : z \in Z\}$.

Given two linear orderings $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} + \mathcal{B}$ is obtained by considering the disjoint union of $\mathcal{A}$ and $\mathcal{B}$ and letting all the elements of $\mathcal{B}$ be bigger than the ones in $\mathcal{A}$. This can be generalized to infinite sums of the form $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 + \ldots$, in an obvious way.

9.2 General ideas of the Proof

In this section we start proving Theorem 9.1.2. The first easy observation is that if a linear ordering has a subset isomorphic to η, then it is equimorphic to η, which has a recursive presentation. So we can restrict our attention to *scattered* linear orderings.

The second step is to analyze the Hausdorff rank of hyperarithmetic scattered linear orderings.

Definition 9.2.1. Let $\mathcal{L} = \langle L, \leqslant \rangle$ be a scattered linear ordering. For each ordinal α we define an equivalence relation $\approx_\alpha$ on L by transfinite recursion. Let $\approx_0$ be the identity relation. If α is a limit ordinal, let $x \approx_\alpha y$ if $x \approx_\beta y$ for some $\beta < \alpha$.

If $\alpha = \beta + 1$, let $x \approx_\alpha y$ if there are only finitely many different $\approx_\beta$-equivalence classes between x and y. In other words

$$x \approx_\alpha y \Leftrightarrow \exists n \exists x_1, ..., x_n \forall z (x < z < y \Rightarrow \exists i < n (z \approx_\beta x_i)).$$

We define the *Hausdorff rank* of $\mathcal{L}$, $\mathrm{rk}_H(\mathcal{L})$, to be the least α such that $\approx_\alpha$ has only finitely many equivalence classes if such an α exists, and we let $\mathrm{rk}_H(\mathcal{L}) = \infty$ otherwise.

It can be proved by transfinite induction that if $f\colon \mathcal{L}_0 \hookrightarrow \mathcal{L}_1$, and $f(x) \approx_\alpha f(y)$ then $x \approx_\alpha y$. Therefore, $\mathcal{L}_0 \preccurlyeq \mathcal{L}_1$ implies that $\mathrm{rk}_H(\mathcal{L}_0) \leqslant \mathrm{rk}_H(\mathcal{L}_1)$, and hence, the Hausdorff rank is preserved under equimorphisms. Also note that $\mathrm{rk}_H(\eta) = \infty$ and hence $\mathrm{rk}_H(\mathcal{L}) = \infty$ for every non-scattered $\mathcal{L}$. It is also known that if $\mathcal{L}$ is scattered, then $\mathrm{rk}_H(\mathcal{L}) < \infty$. (This follows from the relativized version Lemma 9.2.2 below.) See [Ros82, Chapter 5] for more background on Hausdorff rank.

In the following lemma we prove that when $\mathcal{L}$ is hyperarithmetic and scattered, its Hausdorff rank cannot be arbitrarily high. This is the only place in the paper where we use hyperarithmecity. All we use is that if a set is Σ^1_1 in a hyperarithmetic set, then it is Σ^1_1 and hence it cannot be the set of indices for recursive well orderings which is Π^1_1 complete. See [Sac90a] or [AK00] for information on the hyperarithmetic hierarchy.

Lemma 9.2.2. *If $\mathcal{L}$ is a hyperarithmetic scattered linear ordering, then* $\mathrm{rk}_H(\mathcal{L}) < \omega_1^{CK}$.

The proof of this lemma is somewhat similar to the proof of [Clo89, Lemma 13], where Clote proved that Hausdorff's theorem holds in ATR_0. The basic idea of both proofs is the use of pseudohierarchies.

Proof: Assume that $\mathcal{L}$ is hyperarithmetic and $\mathrm{rk}_H(\mathcal{L}) \geqslant \omega_1^{CK}$. We will show that then, there is an embedding of η into $\mathcal{L}$. Given a linear ordering $\mathcal{A} = \langle A, \leqslant \rangle$, and a family $E = \{\simeq_a : a \in A\}$ of equivalence relations on L, let $\phi(\mathcal{A}, E)$ be the hyperarithmetic formula that says:

- For every $a \in A$ there is a pair of non-$\simeq_a$-equivalent elements, and
- for every $a \in A$, if $x \not\simeq_a y$, then, for every $b < a$ there are infinitely many elements of L between x and y which are mutually non-$\simeq_b$-equivalent.

Observe that if $\alpha < \mathrm{rk}_H(\mathcal{L})$, then $E = \{\approx_\beta : \beta < \alpha\}$ satisfies $\phi(\alpha, E)$. Then, for every recursive well ordering α, $\exists E(\phi(\alpha, E))$. The formula $\exists E(\phi(x, E))$ is Σ^1_1. Then, since the set of recursive well orderings cannot be defined by a Σ^1_1 formula, there is a recursive non-well-ordered linear ordering $\mathcal{A}$ such that $\exists E(\phi(\mathcal{A}, E))$. Let $E = \{\simeq_a : a \in A\}$ and $\{a_i\}_{i \in \mathbb{N}}$ be a descending sequence in $\mathcal{A}$. Let x_0 and x_1 be two elements of L such that $x_0 \not\simeq_{a_0} x_1$. Since there are infinitely many $\simeq_{a_1}$-equivalence classes between x_0 and x_1, there is an $x_{1/2} \in L$ such that $x_0 < x_{1/2} < x_1$ and $x_0 \not\simeq_{a_1} x_{1/2} \not\simeq_{a_1} x_1$. In the same way we define $x_{1/4}$ and $x_{3/4}$ such that

$$x_0 < x_{1/4} < x_{1/2} < x_{3/4} < x_1$$

and

$$x_0 \not\simeq_{a_2} x_{1/4} \not\simeq_{a_2} x_{1/2} \not\simeq_{a_2} x_{3/4} \not\simeq_{a_2} x_1.$$

Continue in this way to define an embedding of the dyadic rationals into $\mathcal{L}$. $\square$

From the lemma we have just proved, we get that Theorem 9.1.2 will follow from the following theorem.

Theorem 9.2.3. *A scattered linear ordering has Hausdorff rank less than ω_1^{CK} if and only if it is equimorphic to a recursive linear ordering.*

Since equimorphism preserves Hausdorff rank, the direction from right to left follows from the lemma above.

Richard Laver [Lav71] proved that every scattered linear ordering is a finite sum of indecomposable linear orderings. Thus, it would be enough to prove Theorem 9.2.3 for indecomposable linear orderings. Dealing with equimorphism classes of indecomposable linear orderings might be complicated, so we will work with signed trees instead. Signed trees were introduced in Chapter 6 to represent indecomposable linear orderings up to equimorphism.

Definition 9.2.4. A *signed tree* is pair $\langle T, s_T\rangle$, where T is a *well founded subtree* of $\omega^{<\omega}$ (i.e.: a downwards closed subset of $\omega^{<\omega}$ with no infinite paths) and s_T is a map, called *sign function*, from T to $\{+,-\}$. We will usually write T instead of $\langle T, s_T\rangle$. A *homomorphism* from a signed tree T to another signed tree $\check{T}$ is a map $f\colon T \to \check{T}$ such that

- for all $\sigma \subset \tau \in T$ we have that $f(\sigma) \subset f(\tau)$ and
- for all $\sigma \in T$, $s_{\check{T}}(f(\sigma)) = s_T(\sigma)$.

(Here $\subset$ is the strict inclusion of strings.) We define a binary relation $\preccurlyeq$ on the class of signed trees. We let $T \preccurlyeq \check{T}$ if there exists a homomorphism $f\colon T \to \check{T}$. We say that T and $\check{T}$ are *equimorphic*, and write $T \sim \check{T}$, if $T \preccurlyeq \check{T}$ and $\check{T} \preccurlyeq T$.

Remark 9.2.5. For f to be a homomorphism, we do not require that $\sigma|\tau$ implies $f(\sigma)|f(\tau)$.

Notation 9.2.6. For $\sigma \in T$, we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $s_{T_\sigma}(\tau) = s_T(\sigma^\frown\tau)$. For $n \in \omega$ with $\langle n\rangle \in T$, we let $T_n = T_{\langle n\rangle}$.

We associate to each signed tree T, a linear ordering $\operatorname{lin}(T)$.

Definition 9.2.7. The definition of $\mathrm{lin}(T)$ is by effective transfinite induction. If $T = \{\emptyset\}$, we let $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$ depending on whether $s_T(\emptyset) = +$ or $s_T(\emptyset) = -$. Now suppose $T \supsetneq \{\emptyset\}$. If $s_T(\emptyset) = +$, we want $\mathrm{lin}(T)$ to be an ω sum of copies of $\mathrm{lin}(T_0)$, $\mathrm{lin}(T_1)$,..., where each $\mathrm{lin}(T_i)$ appears infinitely often in the sum. So, we let

$$\mathrm{lin}(T) = \mathrm{lin}(T_0) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1)) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1) + \mathrm{lin}(T_2)) + ...$$

If $s_T(\emptyset) = -$, we let

$$\mathrm{lin}(T) = ... + (\mathrm{lin}(T_2) + \mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + (\mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + \mathrm{lin}(T_0).$$

We say that a linear ordering, $\mathcal{L}$, is *h-indecomposable* if it is of the form $\mathrm{lin}(T)$ for some signed tree T.

It was proved in Chapter 6 that every indecomposable linear ordering is equimorphic either to **1** or to an h-indecomposable linear ordering. (Note that in Chapter 6, **1** is considered an h-indecomposable linear ordering.) It was also proved in Chapter 6 that given signed trees T and $\check{T}$, $T \preccurlyeq \check{T}$ if and only if $\mathrm{lin}(T) \preccurlyeq \mathrm{lin}(\check{T})$, and hence $T \sim \check{T}$ if and only if $\mathrm{lin}(T) \sim \mathrm{lin}(\check{T})$. The ranks of T and of $\mathrm{lin}(T)$ are very closely related too. We define $\mathrm{rk}(T)$ to be the rank of the well founded partial ordering $\langle T, \supseteq\rangle$. On a well founded partial ordering $\mathcal{P} = \langle P, \leqslant\rangle$, the *rank* function is defined as usual:

$$\mathrm{rk}(\mathcal{P}, x) = \sup\{\mathrm{rk}(\mathcal{P}, y) + 1 : y \in P, y < x\}$$

and $\mathrm{rk}(\mathcal{P}) = \sup\{\mathrm{rk}(\mathcal{P}, x) + 1 : x \in P\}$.

Remark 9.2.8. Observe that if $T \preccurlyeq S$, then $\mathrm{rk}(T) \leqslant \mathrm{rk}(S)$. To prove this first let f is a homomorphism $f\colon T \to S$. Then, by transfinite induction on $\mathrm{rk}(T, x)$, prove that for every $x \in T$, $\mathrm{rk}(T, x) \leqslant \mathrm{rk}(S, f(x))$.

Lemma 9.2.9. *Let T be a signed tree. Then, if T has finite rank, then* $\mathrm{rk}(T) = \mathrm{rk}_H(\mathrm{lin}(T))$. *If T has infinite rank, then* $\mathrm{rk}(T) = \mathrm{rk}_H(\mathrm{lin}(T)) + 1$.

Proof: The proof is by transfinite induction on the rank of T. If $T = \{\emptyset\}$, then $\mathrm{rk}(T) = 1$ and, since either $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$, $\mathrm{rk}_H(\mathrm{lin}(T)) = 1$ too. For the inductive step it is enough to prove that for any linear orderings $\mathcal{L}_0$, $\mathcal{L}_1$,... we have that

$$\mathrm{rk}_H(\mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + ...) = \sup\{\mathrm{rk}_H(\mathcal{L}_i) + 1 : i \in \omega\}. \quad (9.2.1)$$

Let $\mathcal{L} = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + ...$ and $\alpha = \sup\{\mathrm{rk}_H(\mathcal{L}_i) + 1 : i \in \omega\}$. First observe that $\mathrm{rk}_H(\mathcal{L}_i + \mathcal{L}_i + \mathcal{L}_i + ...) = \mathrm{rk}_H(\mathcal{L}_i) + 1$, and since $\mathcal{L}_i + \mathcal{L}_i + ... \preccurlyeq \mathcal{L}$ we have that $\mathrm{rk}_H(\mathcal{L}) \geqslant \mathrm{rk}_H(\mathcal{L}_i) + 1$ for every i, and hence $\mathrm{rk}_H(\mathcal{L}) \geqslant \alpha$. On the other hand, if we let $\alpha_i = \max\{\mathrm{rk}_H(\mathcal{L}_j) : j \leqslant i\}$, then the initial segment of $\mathcal{L}$,

$$\mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + ... + (\mathcal{L}_0 + \mathcal{L}_1 + ... + \mathcal{L}_i)$$

has only finitely many $\approx_{\alpha_i}$-equivalence classes, and hence it has only one $\approx_{\alpha_i+1}$-equivalence class. Therefore, every pair of elements of $\mathcal{L}$ is $\approx_\alpha$-equivalent, and hence $\mathrm{rk}_H(\mathcal{L}) \leqslant \alpha$.

Now that we have proved (9.2.1), the induction step is straightforward in both, the finite and the infinite case. The discrepancy between the finite and the infinite case is due to the following fact: when $\mathrm{rk}(T, \emptyset) = \omega$, we have that $\mathrm{rk}(T) = \omega + 1$ and

$$\mathrm{rk}_H(\mathrm{lin}(T)) = \max\{\mathrm{rk}_H(\mathrm{lin}(T_i)) + 1 : \langle i \rangle \in T\} = \max\{\mathrm{rk}(T_i) + 1 : \langle i \rangle \in T\} = \omega.$$

$\square$

Therefore, since the functional lin is recursive, it is enough to show that every signed tree of rank less than ω_1^{CK} is equimorphic to a recursive one. This will follow from the following proposition that we will prove in section 9.4.

Proposition 9.4.1: *For every recursive ordinal α there is a recursive partial ordering $\langle A_\alpha, \preccurlyeq_\alpha \rangle$ and a recursive function t_α that assigns to each element of A_α a recursive signed tree of rank at most α such that*

- *for every signed tree T of rank at most α there is an $x \in A_\alpha$ with $t_\alpha(x) \sim T$, and*
- *for $x, y \in A_\alpha$, $x \preccurlyeq_\alpha y$ if and only if $t_\alpha(x) \preccurlyeq t_\alpha(y)$.*

We start by giving the general idea of the proof of this proposition. The construction is by effective transfinite recursion. Suppose we have already defined A_β, $\preccurlyeq_\beta$ and t_β and we want to define these objects for $\alpha = \beta + 1$. Every signed tree T is determined, up to equimorphism, by $s_T(\emptyset)$ and the set of *branches* of T,

$$\mathrm{bran}(T) = \{T_i : \langle i \rangle \in T\}.$$

If T has rank α, then for every $\check{T} \in \mathrm{bran}(T)$, there is some $x \in \mathcal{A}_\beta$ such that $T \sim t_\beta(x)$. Let $\mathrm{bran}(T){\downarrow} = \{\check{T} \in t_\beta[A_\beta] : \exists i(\langle i \rangle \in T \;\&\; \check{T} \preccurlyeq T_i)\}$. Then, observe that the tree $\widehat{T}$ determined by $s_{\widehat{T}}(\emptyset) = s_T(\emptyset)$ and $\mathrm{bran}(\widehat{T}) = \mathrm{bran}(T){\downarrow}$ is equimorphic to T. Also observe that T has rank α if and only if $\sup\{\mathrm{rk}(\check{T}) : \check{T} \in \mathrm{bran}(T)\} = \beta$, or equivalently, if and only if for every $\gamma < \beta$ there is a tree $\check{T} \in \mathrm{bran}(T)$ such that $\gamma < \mathrm{rk}(\check{T})$. Therefore, to construct $A_\alpha \setminus A_\beta$, we have to consider all the trees T such that $\mathrm{bran}(T) \subseteq t_\beta[A_\beta]$ is *downwards closed* (i.e. $\mathrm{bran}(T)$ is equal to $\mathrm{bran}(T){\downarrow}$ up to equimorphism), and $\mathrm{rk}[\mathrm{bran}(T)]$ is unbounded below β. (We say that a subset $X \subseteq \beta + 1$ is *unbounded below* β if $\forall \gamma < \beta \exists \delta \in X(\delta > \gamma)$.)

Now comes one of the key ideas of the construction. We need the following definition.

Definition 9.2.10. A *quasiordering* is a pair $\mathcal{P} = \langle P, \leqslant_P \rangle$ where $\leqslant_P$ is transitive and reflexive. If $\mathcal{P}$ is also antisymmetric, then $\mathcal{P}$ is a *partial ordering*. A *well* *quasiordering* is a quasiordering $\mathcal{P}$ such that, for every sequence $\{x_i : i \in \omega\} \subseteq P$, there exists $i < j$ such that $x_i \leqslant_P x_j$. A *well partial ordering* is a well quasiordering that is also a partial ordering. A partial ordering is *well founded* if it has no infinite descending sequences. For more information on well quasiorderings see [Mil85].

Remark 9.2.11. Observe that a well quasiordering has no infinite descending sequences and no infinite antichain. Conversely, it can be proved using Ramsey's theorem that a quasiordering which has no infinite descending sequences and no infinite antichain is a well quasiordering. Also observe that if we have a quasiordering $\mathcal{P}$ and we take the quotient over the equivalence relation $x \equiv_P y \Leftrightarrow x \leqslant_P y$ & $y \leqslant_P x$, we obtain a partial ordering that we denote by $\mathcal{P}/\equiv_P$. Moreover, $\mathcal{P}$ is a well quasiordering if and only if $\mathcal{P}/\equiv_P$ is a well partial ordering, and if $\mathcal{P}$ is recursive, then so is $\mathcal{P}/\equiv_P$.

By Fraïssé's conjecture we have that, in particular, the set of indecomposable linear orderings, ordered by $\preccurlyeq$, is a well quasiordering. Then, since the operator lin preserves order, we have that the set of signed trees, ordered by $\preccurlyeq$ is well quasiordered too. Therefore $\langle A_\beta, \preccurlyeq_\beta\rangle$ is a well partial ordering, and hence it is well founded too. Given a subset F of A_β, let

$$\mathcal{I}_{A_\beta}(F) = \{x \in A_\beta : \forall y \in F(y \not\preccurlyeq_\beta x)\}.$$

Conversely, given a downwards closed subset $\mathcal{I}$ of A_β, let $F_{\mathcal{I}}$ be the set of minimal elements of $A_\beta \setminus \mathcal{I}$. Since $F_{\mathcal{I}}$ is an antichain, and $\langle A_\beta, \preccurlyeq_\beta\rangle$ is a well partial ordering, $F_{\mathcal{I}}$ is finite. Moreover, since $\langle A_\beta, \preccurlyeq_\beta\rangle$ is well founded, $\mathcal{I} = \mathcal{I}_{A_\beta}(F_{\mathcal{I}})$. We have proved the following lemma.

Lemma 9.2.12. *Let T be a signed tree. Then T has rank $\alpha = \beta + 1$ if and only if there is a finite antichain F of A_β such that* $\operatorname{bran}(T) = t_\beta[\mathcal{I}_{A_\beta}(F)]$ *and* $\operatorname{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ *is unbounded below β.*

In section 9.4 we will represent the trees of rank α by pairs $\langle *, F\rangle$, where $* \in \{+, -\}$, and $F \subseteq A_\beta$ is a finite antichain such that $\operatorname{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β. The difficulty here is that there is no obvious way of checking recursively whether $\operatorname{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β. In the next section we will analyze the structure of signed trees further and find a recursive way of doing this.

To define $\preccurlyeq_\alpha$ we will use the following lemma.

Lemma 9.2.13. *Consider $x \in A_\beta$, $*, \check{*} \in \{+, -\}$ and $F, \check{F}$ finite antichains of A_β such that both* $\operatorname{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ *and* $\operatorname{rk}[t_\beta[\mathcal{I}_{A_\beta}(\check{F})]]$ *are unbounded below β. Let $S = t_\beta(x)$, T be a signed tree with $s_T(\emptyset) = *$ and* $\operatorname{bran}(T) = t_\beta[\mathcal{I}_{A_\beta}(F)]$, *and $\check{T}$ be a signed tree with $s_{\check{T}}(\emptyset) = \check{*}$ and* $\operatorname{bran}(\check{T}) = t_\beta[\mathcal{I}_{A_\beta}(\check{F})]$. *Then:*

1. *$T \not\preccurlyeq S$.*
2. *$T \preccurlyeq \check{T}$ if and only if $* = \check{*}$ and $\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F})$.*

3. $S \preccurlyeq T$ *if and only if either* $x \in \mathcal{I}_{A_\beta}(F)$, *or* $s_S(\emptyset) = *$ *and* $\mathrm{bran}(S) \subseteq t_\beta[\mathcal{I}_{A_\beta}(F)]$.

PROOF: Part (1) is because S has rank less than or equal to β and T has rank $\alpha = \beta + 1$.

For part (2) note that, since both T and $\check{T}$ have rank α, a homomorphism between them has to map the root of T into the root of $\check{T}$, and each branch of T into a branch of $\check{T}$. Since $\mathrm{bran}(\check{T})$ is downwards close, this is equivalent to $* = \check{*}$ and $\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F})$. For part (3) observe that a homomorphism $S \to T$ either maps the root of S to the root of T, in which case $s_S(\emptyset) = *$ and $\mathrm{bran}(S) \subseteq t_\beta[\mathcal{I}_{A_\beta}(F)]$, or it maps S into a branch of T, in which case $x \in \mathcal{I}_{A_\beta}(F)$. $\square$

Remark 9.2.14. Note that whether $\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F})$ or not can be decided recursively. This is because

$$\mathcal{I}_{A_\beta}(F) \subseteq \mathcal{I}_{A_\beta}(\check{F}) \Leftrightarrow \check{F} \cap \mathcal{I}_{A_\beta}(F) = \emptyset \Leftrightarrow \forall x \in \check{F} \exists y \in F (y \preccurlyeq_\beta x).$$

9.3 Signed Forests

In this section we study *ideals* (downwards closed subsets) of the partial ordering of signed trees modulo equimorphisms. Since the class of signed trees is well quasi-ordered, every antichain is finite. So, for every ideal $\mathcal{I}$ there is a finite set $\mathrm{com}(\mathcal{I})$ such that

$$T \in \mathcal{I} \Leftrightarrow \neg(\exists \check{T} \in \mathrm{com}(\mathcal{I}))\check{T} \preccurlyeq T,$$

namely the set of minimal elements of the complement of $\mathcal{I}$. The objective of this section is to define $\mathrm{com}(\mathcal{I})$, for some ideals $\mathcal{I}$, in a recursive way. The results of this section will be used in the next one when we prove Proposition 9.4.1.

Since we will be dealing with signed trees and ideals of signed trees at the same time, we will work with the more general notion of signed forests. Before introducing signed forests we prove some properties about ranks of partial ordering that we will need later.

9.3.1 Natural sum of ordinals and ranks

Given an ordinal α we let ω^α be the linear ordering whose elements are the finite sequences $\langle \beta_0, \beta_1, ..., \beta_n \rangle$ such that $\alpha > \beta_0 \geqslant \beta_1 \geqslant ... \geqslant \beta_n \geqslant 0$. We order the elements of ω^α lexicographically; that is, $\langle \beta_0, ..., \beta_n \rangle \leqslant_{\omega^\alpha} \langle \gamma_0, ..., \gamma_m \rangle$ if either $n \leqslant m$ and for all $i \leqslant n$, $\beta_i = \gamma_i$, or, for the first i such that $\beta_i \neq \gamma_i$, we have that $\beta_i \leqslant \gamma_i$. It can be shown that ω^α is also a well ordering, and that the initial segment of ω^α up to $\langle \beta_0, ..., \beta_n \rangle$ has order type

$$\omega^{\beta_0} + \omega^{\beta_1} + ... + \omega^{\beta_n}.$$

The *Cantor normal form* of an ordinal α is a tuple $\langle\alpha_0, ..., \alpha_n\rangle$ such that $\alpha \geqslant \alpha_0 \geqslant \alpha_1 \geqslant ... \geqslant \alpha_n \geqslant 0$ and

$$\alpha \cong \omega^{\alpha_0} + ... + \omega^{\alpha_n}.$$

(See [AK00, Chapter 4] or [Ros82, Chapter 3 §4] for more information on ordinal operations and the Cantor normal form. The definition we give here of Cantor normal form is slightly different, but obviously equivalent.) Given two ordinals $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$ and $\beta = \omega^{\beta_0} + ... + \omega^{\beta_{m-1}}$, we define the *natural sum* between α and β to be

$$\alpha \oplus \beta = \omega^{\gamma_0} + \omega^{\gamma_1} + ...\omega^{\gamma_{n+m-1}},$$

where $\gamma_0, ..., \gamma_{n+m-1}$ are such that $\gamma_0 \geqslant \gamma_1 \geqslant ... \geqslant \gamma_{n+m-1}$ and there exists two disjoint subsets $\{a_0, ..., a_{n-1}\}$ and $\{b_0, ..., b_{m-1}\}$ of $\{0, ..., n + m - 1\}$ such that $\gamma_{a_i} = \alpha_i$ and $\gamma_{b_i} = \beta_i$. The natural sum, sometimes called the Hessenberg sum, was introduced in [Hes06]; see [AB99] for more information on Hessenberg based operations. Note that if we are only considering ordinals which are initial segments of ω^α for a big recursive ordinal α, then the operations $+$, $\oplus$ and taking Cantor normal forms are recursive. There are only a few properties of the natural sum that we will use:

NS1. $(\alpha \oplus \beta) + 1 = \alpha \oplus (\beta + 1) = (\alpha + 1) \oplus \beta$,

NS2. $\alpha + \beta \leqslant \alpha \oplus \beta$,

NS3. if $\alpha, \beta < \omega^\gamma$, then $\alpha \oplus \beta < \omega^\gamma$,

NS4. if $\alpha_0 \leqslant \alpha_1$ and $\beta_0 \leqslant \beta_1$, then $\alpha_0 \oplus \alpha_1 \leqslant \beta_0 \oplus \beta_1$.

The proofs of these facts are not hard. An ordinal δ is said to be *additively indecomposable* if for every $\alpha, \beta < \delta$, $\alpha \oplus \beta < \delta$. A well known fact is that for an ordinal δ the following are equivalent:

1. δ is additively indecomposable;

2. δ is indecomposable as a linear ordering;

3. $\delta = \omega^\gamma$ for some ordinal γ.

To prove that (1) implies (2) use (NS2). To prove that (2) implies (3) use transfinite induction on δ (see [Ros82, Exercise 10.4]). That (3) implies (1) follows from (NS3).

The following lemma will be very useful later. We need to define some notation first. Given a partial ordering $\mathcal{P} = \langle P, \leqslant_P\rangle$, and $x \in P$, we let $P_{(<x)} = \{y \in P : y <_P x\}$ and $\mathcal{P}_{(<x)} = \langle P_{(<x)}, \leqslant_P\rangle$. Observe that

$$\mathrm{rk}(\mathcal{P}) = \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P\}.$$

Lemma 9.3.1. *Let $\mathcal{P} = \langle P, \leqslant\rangle$ be a well founded partial ordering. Let $P_0, P_1 \subseteq P$ be such that $P_0 \cup P_1 = P$, and let $\mathcal{P}_0 = \langle P_0, \leqslant\rangle$ and $\mathcal{P}_1 = \langle P_1, \leqslant\rangle$. Then*

$$\mathrm{rk}(\mathcal{P}) \leqslant \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1).$$

If we also have that P_0 and P_1 are closed upwards, then

$$\mathrm{rk}(\mathcal{P}) = \max(\mathrm{rk}(\mathcal{P}_0), \mathrm{rk}(\mathcal{P}_1)).$$

Proof: We use transfinite induction on $\mathrm{rk}(\mathcal{P})$. For the first part we have that

$$\begin{aligned}
\mathrm{rk}(\mathcal{P}) &= \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P\} \\
&\leqslant \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) \oplus \mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \in P\} \\
&= \max(\sup\{(\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1) \oplus \mathrm{rk}(\mathcal{P}_{1(<x)}) : x \in P_0\}, \\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) \oplus (\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1) : x \in P_1\}) \\
&\leqslant \max(\sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1 : x \in P_0\} \oplus \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) : x \in P_0\}, \\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) : x \in P_1\} \oplus \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \in P_1\}) \\
&\leqslant \max(\mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1), \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1)) \\
&= \mathrm{rk}(\mathcal{P}_0) \oplus \mathrm{rk}(\mathcal{P}_1).
\end{aligned}$$

The second inequality being because of NS4. For the second part we use that if $x \notin P_0$, then $\mathcal{P}_{(<x)} = \mathcal{P}_{1(<x)}$.

$$\begin{aligned}
\mathrm{rk}(\mathcal{P}) &= \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P\} \\
&= \max(\sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \in P_0 \cap P_1\}, \\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \notin P_0\}, \sup\{\mathrm{rk}(\mathcal{P}_{(<x)}) + 1 : x \notin P_1\}) \\
&= \max(\sup\{\max(\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1, \mathrm{rk}(\mathcal{P}_{1(<x)}) + 1) : x \in P_0 \cap P_1\}, \\
&\qquad \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \notin P_0\}, \sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1 : x \notin P_1\}) \\
&= \max(\sup\{\mathrm{rk}(\mathcal{P}_{0(<x)}) + 1 : x \in P_0\}, \sup\{\mathrm{rk}(\mathcal{P}_{1(<x)}) + 1 : x \in P_1\}) \\
&= \max(\mathrm{rk}(\mathcal{P}_0), \mathrm{rk}(\mathcal{P}_1)).
\end{aligned}$$

□

9.3.2 Signed forests and signed sequences

Definition 9.3.2. A *signed forest* is a structure $\mathcal{P} = \langle P, \leqslant, s_P\rangle$ such that

1. $\langle P, \leqslant\rangle$ is a countable well founded partial ordering;
2. for every $x \in P$, $\{y \in P : y \geqslant x\}$ is finite and linearly ordered;
3. $s_P \colon P \to \{+, -\}$.

A *homomorphism* between two signed forests $\mathcal{P}_0 = \langle P_0, \leqslant, s_{P_0}\rangle$ and $\mathcal{P}_1 = \langle P_1, \leqslant, s_{P_1}\rangle$ is a map $f\colon P_0 \to P_1$ such that $x < y \Rightarrow f(x) < f(y)$ and $s_{P_0} = s_{P_1} \circ f$. We let $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ if there is a homomorphism $f\colon \mathcal{P}_0 \to \mathcal{P}_1$. We say that $\mathcal{P}_0$ and $\mathcal{P}_1$ are *equimorphic* if $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ and $\mathcal{P}_1 \preccurlyeq \mathcal{P}_0$. The *rank* of a signed forest is the rank of the underlying well founded partial ordering.

A signed tree $\langle T, s_T\rangle$ can be thought of as the signed forest $\langle T, \leqslant, s_T\rangle$, where $\leqslant$ is the reverse inclusion relation $\supseteq$. Conversely, a *rooted signed forest* (that is a signed forest which has a top element, called *root*), can be thought of as a signed tree. Given a rooted signed forest $\langle P, \leqslant, s_P\rangle$ with $P \subseteq \omega$, consider the signed tree $T \subseteq \omega^{<\omega}$, whose nodes are the sequences $\langle x_0, ..., x_m\rangle$, where $\{r < x_0 < ... < x_m\rangle = \{y \in P : y \leqslant x_m\}$ and r is the root of P, and $s_T(\langle x_0, ..., x_m\rangle) = s_P(x_m)$.

Countable ideals of signed trees can also be represented by signed forests. Given an ideal $\mathcal{I}$ of signed trees we consider the signed forest $\biguplus \mathcal{I}$ defined to be the disjoint union of the trees on $\mathcal{I}$ where elements of different trees are considered incomparable. Formally, $\biguplus \mathcal{I} = \langle \bigsqcup \mathcal{I}, \leqslant_{\mathcal{I}}, s_{\mathcal{I}}\rangle$, where $\bigsqcup \mathcal{I} = \{\langle t, T\rangle : t \in T \in \mathcal{I}\}$, $\langle t, T\rangle \leqslant_{\mathcal{I}} \langle s, S\rangle$ if and only if $S = T$ and $t \supseteq s$, and $s_{\mathcal{I}}(\langle t, T\rangle) = s_T(t)$. Observe that given two countable ideals $\mathcal{I}$ and $\check{\mathcal{I}}$ we have that $\mathcal{I} \subseteq \check{\mathcal{I}}$ if and only if $\biguplus \mathcal{I} \preccurlyeq \biguplus \check{\mathcal{I}}$, and given a signed tree T, $T \in \mathcal{I}$ if and only if $T \preccurlyeq \biguplus \mathcal{I}$ as signed forests.

Lemma 9.3.3. *Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be signed forest and suppose that both s_{P_0} and s_{P_1} are constant and equal to $* \in \{+, -\}$. Then, $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ if and only if* $\operatorname{rk}(\mathcal{P}_0) \leqslant \operatorname{rk}(\mathcal{P}_1)$.

Proof: First, suppose that $\mathcal{P}_0 \preccurlyeq \mathcal{P}_1$ and f is a homomorphism $f\colon \mathcal{P}_0 \to \mathcal{P}_1$. It can be proved by transfinite induction on $\operatorname{rk}(\mathcal{P}_0, x)$ that for every $x \in P_0$, $\operatorname{rk}(\mathcal{P}_0, x) \leqslant \operatorname{rk}(\mathcal{P}_1, f(x))$. This implies that $rk(\mathcal{P}_0) \leqslant \operatorname{rk}(\mathcal{P}_1)$.

Now suppose that $\operatorname{rk}(\mathcal{P}_0) \leqslant \operatorname{rk}(\mathcal{P}_1)$. For $x \in P_0$ we define $f(x) \in P_1$ by induction on the size of $\{y \in P_0 : y > x\}$, and we do it so that $\operatorname{rk}(\mathcal{P}_0, x) \leqslant \operatorname{rk}(\mathcal{P}_1, f(x))$. If x is a maximal element of $\mathcal{P}_0$, since $\operatorname{rk}(\mathcal{P}_0) \leqslant \operatorname{rk}(\mathcal{P}_1)$, we can define $f(x)$ to be some element of $\mathcal{P}_1$ such that $\operatorname{rk}(\mathcal{P}_0, x) \leqslant \operatorname{rk}(\mathcal{P}_1, f(x))$. Now suppose that x has an immediate successor y. Since $\operatorname{rk}(\mathcal{P}_0, x) < \operatorname{rk}(\mathcal{P}_0, y) \leqslant \operatorname{rk}(\mathcal{P}_1, f(y))$, we can define $f(x)$ to be some element of $P_{1(<f(y))}$ such that $\operatorname{rk}(\mathcal{P}_0, x) \leqslant \operatorname{rk}(\mathcal{P}_1, f(x))$. $\square$

This lemma implies that given α and $* \in \{+, -\}$, there is only one signed forest of rank α and with signed function constant equal to $*$, up to equimorphism. We now define a canonical forest in this equivalence class.

Definition 9.3.4. Given an ordinal α and $* \in \{+, -\}$, let $\operatorname{Sf}(\alpha, *)$ be the signed forest $\langle P, \leqslant, s_P\rangle$ where P is the set of non-empty strictly descending finite sequences of elements of α, $\leqslant$ is reverse inclusion on sequences, and s_P is the constant function equal to $*$. If α is a successor ordinal, say $\alpha = \beta + 1$, then consider only the sequences that start with β.

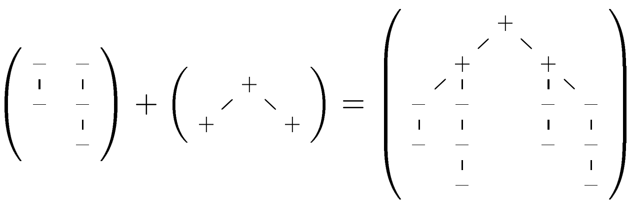
Observe that $\operatorname{rk}(\operatorname{Sf}(\alpha, *)) = \alpha$ and that if α is a successor ordinal then $\operatorname{Sf}(\alpha, *)$ is rooted, and hence a signed tree.

Lemma 9.3.5. *Let α be an indecomposable ordinal and $\mathcal{P}$ be a signed forest of rank at least α. Then, either* $\mathrm{Sf}(\alpha,+) \preccurlyeq \mathcal{P}$ *or* $\mathrm{Sf}(\alpha,-) \preccurlyeq \mathcal{P}$.

Proof: For $* \in \{+,-\}$, let $P^* = \{x \in P : s_P(x) = *\}$ and $\mathcal{P}^*$ be the induced signed forest with domain P^*. By Lemma 9.3.1, $\mathrm{rk}(\mathcal{P}) \leqslant \mathrm{rk}(\mathcal{P}^+) \oplus \mathrm{rk}(\mathcal{P}^-)$. Then, since α is additively indecomposable, either $\mathrm{rk}(\mathcal{P}^+) \geqslant \alpha$ or $\mathrm{rk}(\mathcal{P}^-) \geqslant \alpha$. From the previous lemma we get that then, either $\mathrm{Sf}(\alpha,+) \preccurlyeq \mathcal{P}^+ \preccurlyeq \mathcal{P}$ or $\mathrm{Sf}(\alpha,-) \preccurlyeq \mathcal{P}^- \preccurlyeq \mathcal{P}$. □

Definition 9.3.6. Given two signed forests $\mathcal{P}_0$ and $\mathcal{P}_1$, let $\mathcal{P}_0 + \mathcal{P}_1$ be the signed forest obtained by putting a copy of $\mathcal{P}_0$ below each minimal element of $\mathcal{P}_1$.

See the picture below for an example. In the picture the elements of the forests are marked with either a $+$ or a $-$ and the lines between them represent the order relation.



Lemma 9.3.7. *1.* $(\mathcal{P}_0 + \mathcal{P}_1) + \mathcal{P}_2 = \mathcal{P}_0 + (\mathcal{P}_1 + \mathcal{P}_2)$.

2. $\mathrm{rk}(\mathcal{P}_0 + \mathcal{P}_1) = \mathrm{rk}(\mathcal{P}_0) + \mathrm{rk}(\mathcal{P}_1)$.

3. *If $\mathcal{P}_0 \preccurlyeq \mathcal{Q}_0$ and $\mathcal{P}_1 \preccurlyeq \mathcal{Q}_1$, then $\mathcal{P}_0 + \mathcal{P}_1 \preccurlyeq \mathcal{Q}_0 + \mathcal{Q}_1$.*

4. *If $\bar{P}$ is an upwards closed subset of $\mathcal{P}$ such that for all $x \in \bar{P}$, $\mathcal{Q} \preccurlyeq P_{(<x)}$, then $\mathcal{Q} + \bar{\mathcal{P}} \preccurlyeq \mathcal{P}$.*

5. *If $\bar{P}$ is a non-empty upwards closed subset of $\mathcal{P}$ such that for all $x \notin \bar{P}$, $P_{(\leqslant x)} \preccurlyeq \mathcal{Q}$, then $\mathcal{P} \preccurlyeq \mathcal{Q} + \bar{\mathcal{P}}$.*

Proof: Part (1) is immediate. Part (2) can be easily proved by transfinite induction on $\mathrm{rk}(\mathcal{P}_1)$ using that for all $x \in P_1$, $(\mathcal{P}_0 + \mathcal{P}_1)_{(<x)} = (\mathcal{P}_0 + \mathcal{P}_{1(<x)})$. Part (3) follows from part (4). To prove part (4) construct the map $f\colon \mathcal{Q}+\bar{\mathcal{P}} \preccurlyeq \mathcal{P}$ as follows: First, for each minimal element y of $\bar{\mathcal{P}}$ let g_y be an embedding, $g_y\colon \mathcal{Q} \hookrightarrow \mathcal{P}_{(<y)}$. Then if $x \in \bar{P}$, let $f(x) = x$, and if x is in the copy of $\mathcal{Q}$ that is below some minimal element y of $\bar{\mathcal{P}}$, define $f(x)$ using g_y in the obvious way. For part (5), construct the map $f\colon \mathcal{P} \hookrightarrow \mathcal{Q} + \bar{\mathcal{P}}$ as follows: First, for each maximal element y of $P \smallsetminus \bar{P}$, let g_y be an embedding, $g_y\colon \mathcal{P}_{(\leqslant y)} \hookrightarrow \mathcal{Q}$. Then, if $x \in \bar{P}$, let $f(x) = x$, and if $x \in P \smallsetminus \bar{P}$, let y be the maximal element of $P \smallsetminus \bar{P}$ that is greater that or equal to x and let $f(x) = g_y(x)$. It is not hard to see that in both cases f is the desired embedding. □

Definition 9.3.8. A *signed sequence* is a finite sequence of the form

$$\pi = \langle\langle\alpha_0, *_0\rangle, \langle\alpha_1, *_1\rangle, ..., \langle\alpha_{n-1}, *_{n-1}\rangle\rangle,$$

where each α_i is an ordinal and $*_i \in \{+,-\}$. The rank of a signed sequence is $\mathrm{rk}(\pi) = \alpha_0 + \alpha_1 + ... + \alpha_{n-1}$. Given a signed sequence π we define a signed forest $\mathrm{Sf}(\pi)$ by induction on $|\pi|$. $\mathrm{Sf}(\langle\langle\alpha, *\rangle\rangle) = \mathrm{Sf}(\alpha, *)$ and $\mathrm{Sf}(\pi^\frown\langle\alpha, *\rangle) = \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha, *)$. Just for completeness, we let $\mathrm{rk}(\emptyset) = 0$ and let $\mathrm{Sf}(\emptyset)$ be the empty signed forest.

Observation 9.3.9. If $last(\pi) = \langle\alpha_{n-1}, *_{n-1}\rangle$ and α_{n-1} is a successor ordinal, then $\mathrm{Sf}(\pi)$ is rooted, and therefore a signed tree. Also observe that $\mathrm{rk}(\mathrm{Sf}(\pi)) = \mathrm{rk}(\pi)$.

Proposition 9.3.10. *Let $\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + ... + \omega^{\alpha_{n-1}}$ with $\alpha_0 \geqslant \alpha_1 \geqslant ... \geqslant \alpha_{n-1}$, and let $\mathcal{P}$ be a signed forest of rank $\geqslant \alpha$. Then, there exists a $\sigma = \langle *_0, ..., *_{n-1}\rangle \in \{+,-\}^n$ such that*

$$\mathrm{Sf}(\langle\langle\omega^{\alpha_0}, *_0\rangle, \langle\omega^{\alpha_1}, *_1\rangle, ..., \langle\omega^{\alpha_{n-1}}, *_{n-1}\rangle\rangle) \preccurlyeq \mathcal{P}.$$

Proof: We use induction on n. If $n = 1$, the proposition follows from Lemma 9.3.5. Suppose now we have proved the lemma for n. Let $\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + ... + \omega^{\alpha_{n-1}} + \omega^{\alpha_n}$, and $\mathcal{P}$ be a signed forest of rank $\geqslant \alpha$. For each $x \in P$ with $\mathrm{rk}(\mathcal{P}, x) \geqslant \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$ we have, by inductive hypothesis, that for some $\sigma_x = \langle *_0, ..., *_{n-1}\rangle \in \{+,-\}^n$,

$$\mathrm{Sf}(\langle\langle\omega^{\alpha_0}, *_0\rangle, \langle\omega^{\alpha_1}, *_1\rangle, ..., \langle\omega^{\alpha_{n-1}}, *_{n-1}\rangle\rangle) \preccurlyeq \mathcal{P}_{(<x)}.$$

Let $Q = \{x \in P : \mathrm{rk}(\mathcal{P}, x) \geqslant \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}\}$ and $\mathcal{Q}$ the induced signed forest with domain Q. Observe that $\mathrm{rk}(\mathcal{Q}) \geqslant \omega^{\alpha_n}$. (This is because $\forall x \in Q(\mathrm{rk}(\mathcal{P}, x) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}} + \mathrm{rk}(\mathcal{Q}, x)$, which can be easily proved by transfinite induction on $\mathrm{rk}(\mathcal{Q}, x)$.) For each $\sigma \in \{+,-\}^n$, let

$$\pi_\sigma = \langle\langle\omega^{\alpha_0}, \sigma(0)\rangle, \langle\omega^{\alpha_1}, \sigma(1)\rangle, ..., \langle\omega^{\alpha_{n-1}}, \sigma(n-1)\rangle\rangle,$$

and let Q_σ be the set of $y \in Q$ such that $\mathrm{Sf}(\pi_\sigma) \preccurlyeq \mathcal{P}_{(<y)}$. Since $Q = \cup_{\sigma\in\{+,-\}^n} Q_\sigma$, from Lemma 9.3.1, we get that

$$\bigoplus_{\sigma\in\{+,-\}^n} \mathrm{rk}(\mathcal{Q}_\sigma) \geqslant \mathrm{rk}(\mathcal{Q}) \geqslant \omega^{\alpha_n}.$$

Then, since ω^{α_n} is additively indecomposable, for some $\sigma \in \{+,-\}^n$, $\mathrm{rk}(Q_\sigma) \geqslant \omega^{\alpha_n}$, and from Lemma 9.3.5, we get that for some $* \in \{+,-\}$, $\mathrm{Sf}(\omega^{\alpha_n}, *) \preccurlyeq \mathcal{Q}_\sigma$. Thus, from Lemma 9.3.7(3) and (4), we get that

$$\mathrm{Sf}(\pi_\sigma{}^\frown\langle\omega^{\alpha_n}, *\rangle) = \mathrm{Sf}(\pi_\sigma) + \mathrm{Sf}(\omega^{\alpha_n}, *) \preccurlyeq \mathrm{Sf}(\pi_\sigma) + \mathcal{Q}_\sigma \preccurlyeq \mathcal{P}.$$

$\square$

Definition 9.3.11. Given α as in the proposition above, let com_α be the set of all signed sequences of the form

$$\langle\langle\omega^{\alpha_0}, *_0\rangle, \langle\omega^{\alpha_1}, *_1\rangle, ..., \langle\omega^{\alpha_{n-1}}, *_{n-1}\rangle\rangle.$$

The set $\mathrm{com}_{\beta+1}$ will be used later to compute the minimal elements of the complement of A_β.

Note that, assuming we could compute the Cantor normal form of α uniformly, com_α could be computed uniformly in α too.

Corollary 9.3.12. *A signed forest $\mathcal{P}$ has rank greater than or equal to α if and only if for some $\sigma \in \mathrm{com}_\alpha$, $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$.*

Proof: The implication from left to right follows immediately form Proposition 9.3.10. For the other direction, observe that if $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$ then

$$\alpha = \mathrm{rk}(\mathrm{Sf}(\sigma)) \leqslant \mathrm{rk}(\mathcal{P}).$$

□

This corollary will allow us to identify the unbounded ideals of A_β in the proof of Proposition 9.4.1.

9.3.3 The complements

To identify the unbounded ideals of A_β we will also need to be able to find, for each $\tau \in \mathrm{com}_\beta$, a finite subset $F \subseteq A_\beta$ such that

$$\{x \in A_\beta : t_\beta(x) \preccurlyeq \mathrm{Sf}(\tau)\} = \mathcal{I}_{A_\beta}(F).$$

For this purpose, for each such τ we will define $\mathrm{com}(\tau)$, a finite set of signed sequences, such that for every signed tree T,

$$T \preccurlyeq \mathrm{Sf}(\tau) \Leftrightarrow \neg(\exists \pi \in \mathrm{com}(\tau))\, \mathrm{Sf}(\pi) \preccurlyeq T.$$

In the next section we will define a function isf_β that, given a signed sequence π of rank at most β, which ends in $\langle 1, *\rangle$, returns an index in $\mathcal{A}_\beta$ for the signed tree $\mathrm{Sf}(\pi)$. So, the desired F will be $\{\mathit{isf}_\beta(\pi) : \pi \in \mathrm{com}(\tau)\ \&\ \mathrm{rk}(\pi) \leqslant \beta\}$. The definition of $\mathrm{com}(\pi)$ might seem obscure at first; it is defined the way it is just to make Proposition 9.3.14 below work.

Definition 9.3.13. Given a signed sequence π we will define $\mathrm{com}(\pi)$, a finite set of signed sequences, by induction on $|\pi|$ as follows: Let $\mathrm{com}(\emptyset) = \{\langle 1, +\rangle, \langle 1, -\rangle\}$ and

$$\begin{array}{rcl}\mathrm{com}(\pi^\frown\langle\alpha, *\rangle) & = & \{\sigma^\frown\langle 1, \bar{*}\rangle : \sigma \in \mathrm{com}(\pi),\ \mathit{last}(\sigma) = \langle 1, *\rangle\}\ \cup \\ & & \{\sigma^{-\frown}\langle\alpha, *\rangle^\frown\langle 1, *\rangle : \sigma \in \mathrm{com}(\pi),\ \mathit{last}(\sigma) = \langle 1, *\rangle\}\ \cup \\ & & \{\sigma : \sigma \in \mathrm{com}(\pi),\ \mathit{last}(\sigma) = \langle 1, \bar{*}\rangle\}.\end{array}$$

We are using the following notation: For $* \in \{+, -\}$, $\bar{*}$ is the opposite of $*$, that is $\bar{+} = -$ and $\bar{-} = +$. For a string $\sigma = \langle x_0, ..., x_{n-1}\rangle$, $\mathit{last}(\sigma) = x_{n-1}$ and $\sigma^- = \langle x_0, ..., x_{n-2}\rangle$.

Note that for all $\sigma \in \operatorname{com}(\pi)$, $last(\sigma)$ is either $\langle 1,+\rangle$ or $\langle 1,-\rangle$, and hence $\mathrm{Sf}(\sigma)$ is a signed tree.

Proposition 9.3.14. *For a signed forest $\mathcal{P}$ and a signed sequence π we have that $\mathcal{P} \not\preccurlyeq \mathrm{Sf}(\pi)$ if and only if for some $\sigma \in \operatorname{com}(\pi)$, $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$.*

Proof: We use induction on $n = |\pi|$. For $\pi = \emptyset$ the result is trivial. Now suppose we know the result for π and we want to prove it for $\pi' = \pi^\frown\langle\alpha,*\rangle$.

Let us start by proving the implication from right to left. It is enough to prove that for every $\tau \in \operatorname{com}(\pi')$, $\mathrm{Sf}(\tau) \not\preccurlyeq \mathrm{Sf}(\pi')$. There are two possible cases. First suppose that $last(\tau) = \langle 1,\bar{*}\rangle$ and either $\tau^- = \sigma \in \operatorname{com}(\pi)$ or $\tau \in \operatorname{com}(\pi)$. In any case, by induction hypothesis, $\mathrm{Sf}(\tau) \not\preccurlyeq \mathrm{Sf}(\pi)$. But, if $\mathrm{Sf}(\tau) \preccurlyeq \mathrm{Sf}(\pi') \preccurlyeq \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha,*)$, then necessarily $\mathrm{Sf}(\tau) \preccurlyeq \mathrm{Sf}(\pi)$ because the root of $\mathrm{Sf}(\tau)$ is signed $\bar{*}$. So $\mathrm{Sf}(\tau) \not\preccurlyeq \mathrm{Sf}(\pi')$. Second, suppose that $\tau = \sigma^\frown\langle\alpha,*\rangle^\frown\langle 1,*\rangle$ and $\sigma^\frown\langle 1,*\rangle \in \operatorname{com}(\pi)$. Suppose, toward a contradiction, that we have an homomorphism $f\colon \mathrm{Sf}(\tau) \to \mathrm{Sf}(\pi')$. Let $\bar{P}$ be the copy of $\mathrm{Sf}(\langle\langle\alpha,*\rangle,\langle 1,*\rangle\rangle)$ inside $\mathrm{Sf}(\tau) = \mathrm{Sf}(\sigma) + \mathrm{Sf}(\langle\langle\alpha,*\rangle,\langle 1,*\rangle\rangle)$ and $\bar{Q}$ the copy of $\mathrm{Sf}(\alpha,*)$ inside $\mathrm{Sf}(\pi') = \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha,*)$. By inductive hypothesis $\mathrm{Sf}(\sigma^\frown\langle 1,*\rangle) \not\preccurlyeq \mathrm{Sf}(\pi)$, so, for every $x \in \bar{P}$ it has to be the case that $f(x) \in \bar{Q}$ because

$$\mathrm{Sf}(\pi')_{(\leqslant f(x))} \succcurlyeq \mathrm{Sf}(\tau)_{(\leqslant x)} \succcurlyeq \mathrm{Sf}(\sigma^\frown\langle 1,*\rangle) \not\preccurlyeq \mathrm{Sf}(\pi).$$

But then $\mathrm{Sf}(\langle\langle\alpha,*\rangle,\langle 1,*\rangle\rangle) = \bar{\mathcal{P}} \preccurlyeq \bar{\mathcal{Q}} = \mathrm{Sf}(\alpha,*)$, contradicting Lemma 9.3.3.

Now we prove the other implication. Let $\mathcal{P}$ be such that $\mathcal{P} \not\preccurlyeq \mathrm{Sf}(\pi')$. Let $\bar{P} = \{x \in P : \mathcal{P}_{(\leqslant x)} \not\preccurlyeq \mathrm{Sf}(\pi)\}$. Note that $\bar{P}$ is upwards closed. By the inductive hypothesis, for each $x \in \bar{P}$ there is some $\sigma_x \in \operatorname{com}(\pi)$ such that $\mathrm{Sf}(\sigma_x) \preccurlyeq \mathcal{P}_{(\leqslant x)}$. If for some of these $x \in \bar{P}$, $last(\sigma_x) = \langle 1,\bar{*}\rangle$, then $\sigma_x \in \operatorname{com}(\pi')$ too, and we would be done. So, suppose this is not the case and that for every $x \in \bar{P}$ $last(\sigma_x) = \langle 1,*\rangle$. If some $x \in \bar{P}$ is signed $\bar{*}$, then actually $\mathrm{Sf}(\sigma_x) \preccurlyeq \mathcal{P}_{(<x)}$, since $\mathrm{Sf}(\sigma_x)$ has a top element signed $*$. But then $\mathrm{Sf}(\sigma_x{}^\frown\langle 1,\bar{*}\rangle) \preccurlyeq \mathcal{P}_{(\leqslant x)} \preccurlyeq \mathcal{P}$ and since $\sigma_x{}^\frown\langle 1,\bar{*}\rangle \in \operatorname{com}(\pi')$, we would be done. So, suppose that every $x \in \bar{P}$, $last(\sigma_x) = \langle 1,*\rangle$ and $s_P(x) = *$. We want to show that for some $\sigma \in \operatorname{com}(\pi)$ with $last(\sigma) = \langle 1,*\rangle$ we have that $\mathrm{Sf}(\sigma^{-\frown}\langle\alpha,*\rangle^\frown\langle 1,*\rangle) \preccurlyeq \mathcal{P}$. First we observe that $\mathrm{rk}(\bar{\mathcal{P}}) > \alpha$. Because otherwise, by Lemma 9.3.3, $\bar{\mathcal{P}} \preccurlyeq \mathrm{Sf}(\alpha,*)$, and then using Lemma 9.3.7(5) and (3) and the fact that $\forall x \notin \bar{P}(\mathcal{P}_{(\leqslant x)} \preccurlyeq \mathrm{Sf}(\pi))$ we would get that

$$\mathcal{P} \preccurlyeq \mathrm{Sf}(\pi) + \bar{\mathcal{P}} \preccurlyeq \mathrm{Sf}(\pi) + \mathrm{Sf}(\alpha,*) = \mathrm{Sf}(\pi').$$

For each $\sigma \in \operatorname{com}(\pi)$, let $\bar{P}_\sigma$ be the set of $x \in \bar{P}$ such that $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}_{(\leqslant x)}$. The sets $\bar{P}_\sigma$ are closed upwards and have union $\bar{P}$, so, by Lemma 9.3.1,

$$\max\{\mathrm{rk}(\mathcal{P}_\sigma) : \sigma \in \operatorname{com}(\pi)\} = \mathrm{rk}(\bar{\mathcal{P}}) \geqslant \alpha + 1.$$

Therefore, for some $\sigma \in \operatorname{com}(\pi)$, $\mathrm{rk}(\bar{\mathcal{P}}_\sigma) \geqslant \alpha + 1$, and hence, by Lemma 9.3.3, $\mathrm{Sf}(\alpha+1,*) \preccurlyeq \bar{\mathcal{P}}_\sigma$. Notice that for for all $x \in \bar{P}_\sigma$, $\mathrm{Sf}(\sigma^-) \preccurlyeq \mathcal{P}_{(<x)}$. Then, again by Lemma 9.3.7(3) and (4),

$$\mathrm{Sf}(\sigma^{-\frown}\langle\alpha,*\rangle^\frown\langle 1,*\rangle) \sim \mathrm{Sf}(\sigma^-) + \mathrm{Sf}(\alpha+1,*) \preccurlyeq \mathrm{Sf}(\sigma^-) + \bar{\mathcal{P}}_\sigma \preccurlyeq \mathcal{P}.$$

□

Definition 9.3.15. Let $\mathrm{com}_\alpha(\pi) = \{\sigma \in \mathrm{com}(\pi) : \mathrm{rk}(\sigma) \leqslant \alpha\}$.

Corollary 9.3.16. *Given $\mathcal{P}$ of rank at most α, we have that $\mathcal{P} \not\preccurlyeq \mathrm{Sf}(\pi)$ if and only if there is some $\sigma \in \mathrm{com}_\alpha(\pi)$ such that $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$.*

PROOF: It follows immediately from the proposition above and the fact that if $\mathrm{Sf}(\sigma) \preccurlyeq \mathcal{P}$, then necessarily $\mathrm{rk}(\sigma) \leqslant \mathrm{rk}(\mathcal{P}) \leqslant \alpha$. $\square$

9.4 The construction

In this section we put everything we have done together and prove Proposition 9.4.1. We have already shown in section 9.2 that Proposition 9.4.1 implies Theorems 9.2.3 and 9.1.2.

Proposition 9.4.1. *For every recursive ordinal α there is a recursive partial ordering $\langle A_\alpha, \preccurlyeq_\alpha\rangle$ and a recursive function t_α that assigns to each element of A_α a recursive signed tree of rank at most α such that*

- *for every signed tree T of rank less than or equal to α there is an $x \in A_\alpha$ with $t_\alpha(x) \sim T$, and*
- *for $x, y \in A_\alpha$, $x \preccurlyeq_\alpha y$ if and only if $t_\alpha(x) \preccurlyeq t_\alpha(y)$.*

PROOF: Let ξ be a big additively indecomposable recursive ordinal such that the operations $+$, $\oplus$, and taking Cantor normal forms of ordinals below ξ are recursive. For each $\alpha < \xi$ we will construct, uniformly in α, a recursive set A_α, a recursive partial ordering $\preccurlyeq_\alpha$ on A_α, and two recursive functions t_α and isf_α such that the following condition are satisfied.

1. t_α assigns to each $x \in A_\alpha$ a recursive signed tree of rank less than or equal to α.
2. For every recursive signed tree T of rank less than or equal to α there exists an $x \in A_\alpha$ such that $t_\alpha(x) \sim T$.
3. $x \preccurlyeq_\alpha y$ if and only if $t_\alpha(x) \preccurlyeq t_\alpha(y)$.
4. isf_α maps signed sequences π, with $\mathit{last}(\pi) = \langle 1, *\rangle$ and of rank less than or equal to α, into A_α, such that $t_\alpha(\mathit{isf}_\alpha(\pi)) \sim \mathrm{Sf}(\pi)$.
5. For $\beta < \alpha$, $A_\beta = A_\alpha \cap \omega^{[\leqslant\beta]}$, $t_\beta \subseteq t_\alpha$, $\mathit{isf}_\beta \subseteq \mathit{isf}_\alpha$, and $\preccurlyeq_\beta$ is the restriction of $\preccurlyeq_a$ to $A_\beta \times A_\beta$, where $\omega^{[\leqslant\beta]} = \{\langle\gamma, y\rangle : \gamma \leqslant \beta, y \in \omega\}$.

Observe that Condition (5) above implies that $\mathrm{rk}(t_\alpha(x)) = (x)_0$, where $(\cdot)_0$ is the projection onto the first coordinate, so for example $(\langle y, z\rangle)_0 = y$. So, we want to construct t_α and isf_α such that the following diagram commutes up to equimorphism of signed trees.

$$\begin{tikzcd}
A_\alpha \arrow[r, "t_\alpha"] \arrow[dr, "(\cdot)_0"] & \text{signed trees of rank} \leqslant \alpha \arrow[d, "\mathrm{rk}"] \\
\text{signed sequences of rank} \leqslant \alpha \arrow[u, "\mathit{isf}_\alpha"] \arrow[ur, "\mathrm{Sf}"] \arrow[r, "\mathrm{rk}"'] & \{\gamma : \gamma \leqslant \alpha\}
\end{tikzcd}$$

The construction is by effective transfinite recursion. Let A_1 consist of two incomparable elements $\langle 1, +\rangle$ and $\langle 1, -\rangle$, and for $* \in \{+, -\}$, let $t_1(\langle 1, *\rangle) = \mathrm{Sf}(1, *)$ and $\mathit{isf}_1(\langle\langle 1, *\rangle\rangle) = \langle 1, *\rangle \in A_1$.

Suppose now we have already constructed A_β, $\preccurlyeq_\beta$, t_β and isf_β for each $\beta < \alpha$ satisfying the conditions above. When α is a limit ordinal, just take A_α to be $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ and define $\preccurlyeq_\alpha$, t_α and isf_α also by taking unions. It is not hard to see that the conditions above are still satisfied. (Recall that there are no signed trees whose rank is a limit ordinal.)

Now suppose $\alpha = \beta + 1$. Let B_α the set of pairs $\langle *, F\rangle$ where $* \in \{+, -\}$ and F is a finite antichain of $\langle A_\beta, \preccurlyeq_\beta\rangle$ such that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β. By Lemma 9.2.12, for every signed tree of rank α there is some $\langle *, F\rangle \in B_\alpha$ such that $s_T(\emptyset) = *$ and $\mathrm{bran}(T){\downarrow} = t_\beta[\mathcal{I}_{A_\beta}(F)]$. Conversely, if, for a signed tree T, $s_T(\emptyset) = *$ and $\mathrm{bran}(T){\downarrow} = t_\beta[\mathcal{I}_{A_\beta}(F)]$ for some $\langle *, F\rangle \in B_\alpha$ then T has rank α. Now, observe that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β if and only if the signed forest $\biguplus t_\beta[\mathcal{I}_{A_\beta}(F)]$ has rank β. By Corollary 9.3.12, this happens if and only if there is a $\sigma \in \mathrm{com}_\beta$ such that $\mathrm{Sf}(\sigma) \preccurlyeq \biguplus t_\beta[\mathcal{I}_{A_\beta}(F)]$. By Corollary 9.3.16, $\mathrm{Sf}[\mathrm{com}_\beta(\sigma)]$ is the set of trees of rank at most β which are minimal in the complement of the ideal $\{T : \mathrm{rk}(T) \leqslant \beta \mathbin{\&} T \preccurlyeq \mathrm{Sf}(\sigma)\}$, everything up to equimorphism. Therefore $\mathrm{Sf}(\sigma) \sim \biguplus t_\beta[\mathcal{I}_{A_\beta}(\mathit{isf}_\beta[\mathrm{com}_\beta(\sigma)])]$. So, we have that $\mathrm{rk}[t_\beta[\mathcal{I}_{A_\beta}(F)]]$ is unbounded below β if and only if for some $\sigma \in \mathrm{com}_\beta$,

$$\mathcal{I}_{A_\beta}(\mathit{isf}_\beta[\mathop{\mathrm{com}}_\beta(\sigma)]) \subseteq \mathcal{I}_{A_\beta}(F),$$

By Remark 9.2.14, we can check whether $\mathcal{I}_{A_\beta}(\mathit{isf}_\beta[\mathrm{com}_\beta(\sigma)]) \subseteq \mathcal{I}_{A_\beta}(F)$ recursively. So B_α is recursive.

Let $A_\alpha = A_\beta \cup (\{\alpha\} \times B_\alpha)$. For $x \in A_\beta$, let $t_\alpha(x) = t_\beta(x)$. For $\langle *, F\rangle \in B_\alpha$, let $t_\alpha(\langle \alpha, \langle *, F\rangle\rangle)$ be the signed tree T such that $s_T(\emptyset) = *$ and $\mathrm{bran}(T) = t_\beta[\mathcal{I}_{A_\beta}(F)]$. Note that, because of what we said above about B_α, t_α satisfies conditions (1) and (2).

Now we want to define the relation $\preccurlyeq_\alpha$ on A_α. Consider $x, y \in A_\alpha$. We let $x \preccurlyeq_a y$ if and only if one of the following conditions holds

- $x, y \in A_\beta$ and $x \preccurlyeq_\beta y$;
- $x = \langle \alpha, \langle *_0, F_0\rangle\rangle$, $y = \langle \alpha, \langle *_1, F_1\rangle\rangle$, $*_0 = *_1$, and $\mathcal{I}_{A_\beta}(F_0) \subseteq \mathcal{I}_{A_\beta}(F_1)$;

- $x \in A_\beta$, $y = \langle \alpha, \langle *, F\rangle\rangle$ and
  - either $x \in \mathcal{I}_{A_\beta}(F)$,
  - or $x = \langle \gamma+1, \langle \check{*}, \check{F}\rangle\rangle$, for some $\gamma < \beta$, $* = \check{*}$ and
$$\mathcal{I}_{A_\gamma}(\check{F}) \subseteq \mathcal{I}_{A_\beta}(F).$$

Condition (3) follows from Lemma 9.2.13. Observe that

$$\mathcal{I}_{A_\gamma}(\check{F}) \subseteq \mathcal{I}_{A_\beta}(F) \Leftrightarrow F \cap \mathcal{I}_{A_\gamma}(\check{F}) = \emptyset \Leftrightarrow \forall x \in F((x)_0 \geqslant \gamma \vee \exists y \in \check{F}(y \preccurlyeq_\beta x)).$$

So $\preccurlyeq_\alpha$ is recursive.

Finally, let us define isf_α. For a signed sequence π of rank less than α, let $\mathit{isf}_\alpha(\pi) = \mathit{isf}_\beta(\pi)$. For $\pi' = \pi^\frown\langle 1, *\rangle$ of rank α, let $\mathit{isf}_\alpha(\pi') = \langle \alpha, \langle *, \mathit{isf}_\beta[\mathrm{com}_\beta(\pi)]\rangle\rangle$. So we get that $t_\alpha(\mathit{isf}_\alpha(\pi'))$ is the signed tree T such that $s_T(\emptyset) = *$ and

$$\begin{aligned}
\mathrm{bran}(T) &= t_\beta[\mathcal{I}_{A_\beta}(\mathit{isf}_\beta[\underset{\beta}{\mathrm{com}}(\pi)])] \\
&= \{\check{T} : \mathrm{rk}(\check{T}) \leqslant \beta \;\&\; \neg(\exists x \in \mathit{isf}_\beta[\underset{\beta}{\mathrm{com}}(\pi)]) t_\beta(x) \preccurlyeq \check{T}\} \\
&= \{\check{T} : \mathrm{rk}(\check{T}) \leqslant \beta \;\&\; \neg(\exists \sigma \in \underset{\beta}{\mathrm{com}}(\pi))\, \mathrm{Sf}(\sigma) \preccurlyeq \check{T}\},
\end{aligned}$$

which, by Corollary 9.3.16 is equal to $\{\check{T} : \check{T} \preccurlyeq \mathrm{Sf}(\pi)\}$. Therefore, $T \sim \mathrm{Sf}(\pi')$ and condition (4) follows. Condition (5) is immediate from the definitions. $\square$

An interesting consequence of the proof of this proposition is given in Corollary 9.4.3. The following basic observation about indecomposable linear orderings will be used in the proof of Corollary 9.4.3.

Observation 9.4.2. If $\mathcal{L}$ is indecomposable and $\mathcal{L} \preccurlyeq L_1 + ... + L_n$, then $L \preccurlyeq L_i$ for some i. This fact can easily be proved by induction on n using the definition of indecomposability.

Corollary 9.4.3. *Given $\alpha < \omega_1^{CK}$, $\mathbb{L}_\alpha$, the partial ordering of equimorphism types of linear orderings of Hausdorff rank less than α ordered by $\preccurlyeq$, is recursively presentable.*

Proof: Let A_α, $\preccurlyeq_\alpha$ and t_α be as in the proof above. Let $\hat{A}_\alpha = (A_\alpha \cup \{0\})^{<\omega} \smallsetminus \{\emptyset\}$, the set of finite, non-empty, sequences of elements of $A_\alpha \cup \{0\}$. For $x \in A_\alpha \cup \{0\}$ let

$$l(x) = \begin{cases} \mathrm{lin}(t_\alpha(x)) & \text{if } x \in \mathcal{A}_\alpha \\ \mathbf{1} & \text{if } x = 0. \end{cases}$$

Define the function $\hat{l}$ to $\hat{A}_\alpha$ by:

$$\hat{l}(\langle x_0, ..., x_{n-1}\rangle) = l(x_0) + ... + l(x_{n-1}).$$

Since every scattered linear ordering is equimorphic to a finite sum of **1**s and h-indecomposable linear orderings, for every linear ordering of Hausdorff rank less than or equal to α, there is a $\sigma \in \hat{A}_\alpha$ such that $\hat{l}(\sigma)$ is equimorphic to it. (Here we are using Lemma 9.2.9. So, if α is finite, we need to consider $A_{\alpha-1}$ instead of A_α.) We now need to compute the embeddablity relation between linear orderings. We will define a relation $\preccurlyeq$ on $\hat{A}_\alpha$ such that for $\sigma, \tau \in \hat{A}_\alpha$, $\sigma \preccurlyeq \tau \Leftrightarrow \hat{l}(\sigma) \preccurlyeq \hat{l}(\tau)$. First, suppose we are given $\sigma = \langle x_0, ..., x_n \rangle \in \hat{A}_\alpha$ and $x \in A_\alpha \cup \{0\}$, and we want to know whether $l(\sigma) \preccurlyeq \hat{l}(x)$. If $x = 0$, then $\hat{l}(\sigma) \preccurlyeq l(x) = \mathbf{1}$ if and only if $\hat{l}(\sigma) = \mathbf{1}$, or equivalently $\sigma = \langle 0 \rangle$. So, suppose that $x = \langle \beta + 1, \langle +, F \rangle \rangle$ for some $\beta < \alpha$. (The case when $x = \langle \beta + 1, \langle -, F \rangle \rangle$ is analogous.) Then

$$l(x) = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + ...,$$

where $\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, ...\} = l[\mathcal{I}_{A_\beta}(F)]]$. Suppose that $\hat{l}(\sigma) \preccurlyeq l(x)$. Then, necessarily, $l(x_n) \preccurlyeq l(x)$ and for each $i < n$, $l(x_i)$ embeds into a proper initial segment of $l(x)$. Since every proper initial segment of $l(x)$ is contained in a finite sum of linear orderings of the form $l(y)$ for $y \in \mathcal{I}_{A_\beta}(F)$, and $l(x_i)$ is indecomposable, it has to be the case that for some $y \in \mathcal{I}_{A_\beta}(F)$, $l(x_i) \preccurlyeq l(y)$, and hence that $x_i \in \mathcal{I}_{A_\beta}(F)$. Therefore, for $\hat{l}(\sigma) \preccurlyeq l(x)$ to hold we have to have that

$$x_{n-1} \preccurlyeq_\alpha x \;\&\; \forall i < n(x_i \in \mathcal{I}_{A_\beta}(F)), \tag{9.4.1}$$

which we can check recursively since

$$x_i \in \mathcal{I}_{A_\beta}(F) \Leftrightarrow \mathrm{rk}(x_i) \leqslant \beta \;\&\; \neg\exists y \in F(y \preccurlyeq_\alpha x_i).$$

Conversely, if (9.4.1) holds, then $\hat{l}(\sigma^-)$ embeds into a proper initial segment of $l(x)$ because $l(x)$ contains infinitely many segments isomorphic to $l(x_i)$ for each $i < n$. Since $l(x)$ embeds into every proper final segment of itself, (9.4.1) implies that $l(x_n)$ embeds in every proper final segment of $l(x)$. Therefore $\hat{l}(\sigma) = \hat{l}(\sigma^-) + l(x_n) \preccurlyeq l(x)$. We have shown how to check whether $\hat{l}(\sigma) \preccurlyeq l(x)$ recursively.

Now, suppose we are given $\sigma = \langle x_0, ..., x_n \rangle$ and $\tau = \langle y_0, ..., y_m \rangle \in \hat{A}_\alpha$ and we want to check whether $\hat{l}(\sigma) \preccurlyeq \hat{l}(\tau)$. Suppose that there is an embedding $g\colon \sum_{i=0}^n l(x_i) \hookrightarrow \sum_{j=0}^m l(y_j)$. Observe that, since each $l(x_i)$ is indecomposable, we can assume that for every $i \leqslant n$, $g[l(x_i)] \subseteq l(y_j)$ for some $j \leqslant m$. (This is because of the property of indecomposable linear orderings mentioned above.) Therefore, $\hat{l}(\sigma) \preccurlyeq \hat{l}(\tau)$ if and only if

$$\bigvee_{0=i_0 \leqslant ... \leqslant i_m \leqslant n} \left(\bigwedge_{k \leqslant m} \hat{l}(\langle x_{i_k}, x_{i_k+1}, ..., x_{i_{k+1}-1} \rangle) \preccurlyeq l(y_k) \right).$$

(In the formula above we are taking $i_{m+1} = n + 1$.) Now we have that the quasiordering relation $\preccurlyeq$ on $\hat{A}_\alpha$ is recursive. Hence the induced equivalence relation, $\sim$, defined by $x \sim y \Leftrightarrow x \preccurlyeq y \;\&\; y \preccurlyeq x$, is recursive, and therefore the quotient partial ordering $\langle \hat{A}_\alpha, \preccurlyeq \rangle / \sim$ is recursive too. Observe that $\langle \hat{A}_\alpha, \preccurlyeq \rangle / \sim$ is the desired partial ordering. $\square$

Chapter 10
Boolean Algebras, Tarski Invariants, and Index Sets (*with Barbara F. Csima and Richard A. Shore*).

This chapter will appear in the Notre Dame Journal of Formal Logic.

10.1 Introduction

A common theme in mathematical investigations is the classification of structures (within a specified class) and the characterization of the (sub)classes delineated. Indeed, Hodges [Hod93] offers the classification process (along with constructions of specified types of structures) as the essence of model theory. Of course, the general endeavor pervades many branches of mathematics. Our topic in this paper has its origin in such a study of the class of Boolean algebras. It begins with Tarski's classification [Tar49] of Boolean algebras into countably many classes each consisting of the models of a complete extension of the basic theory. (Of course, this classifies Boolean algebras up to elementary equivalence.) His motivation was to prove that the theory of Boolean algebras was decidable and he did this by producing a uniformly computable list of axioms for (each of) the complete extensions corresponding to his classification.

Given such a classification (or the prospect of one), one may well want to characterize membership in each subclass in some way and analyze the complexity of the classes (i.e. of membership in each). The algebraist asks for invariants corresponding to structural properties that determine membership in each class. The model theorist might ask for the (simplest) axioms that insure such membership. The descriptive set theorist or recursion theorist wants to determine the location of the classes in some standard hierarchy. The former, expresses the results as completeness properties for the classes of countable structures at levels of the Borel hierarchy. The latter, takes the lightface approach of proving completeness of the subclasses of the computable structures in the arithmetic, hyperarithmetic or analytic hierarchy. (Typically, relativization of such lightface characterizations produces the boldface Borel ones.)

For the classification of Boolean algebras up to elementary equivalence, Tarski [Tar49] (see also [Ers64], [Gon97, Ch. 2] and [Mon89, Ch. 7]) provides the structural information by describing algebraic invariants as well as axiomatizations for each class. The determination of the simplest form of such axiom systems (in the sense syntactic complexity) is given by Wasziewicz [Was74]. In this paper, we provide the recursion (and so descriptive set) theoretic characterizations of these classes as complete at specified levels of the arithmetic hierarchy and a bit more. The classes provide not only index sets complete at the Σ_n or Π_n level for each $n < \omega$ but also for level $\Pi_{\omega+1}$ (the sets co-c.e. in $0^{(\omega)}$) and even more unusually for the classes $\Sigma_n \wedge \Pi_n$ (the sets which are intersections of one in Σ_n and one in Π_n) for $n \equiv 1, 2 \pmod 4$. As a by-product of our analysis we reprove the results of

[Was74] as well.

A standard question related to classifying the complexity of membership in such subclasses is how to characterize the complexity of the isomorphism problem (when two structures are isomorphic) for structures in the class or specified subclasses. Again, there are natural descriptive set theoretic as well as recursion theoretic versions of this problem. For the class of all Boolean algebras the isomorphism problem is as complicated as possible, i.e. Σ_1^1 complete, and so one typically says that there is no way to classify all Boolean algebras up to isomorphism or provide isomorphism invariants. There is, however, an algebraically defined class of Boolean algebras, the dense Boolean algebras (see Definition 10.4.2), for which elementary equivalence is the same as isomorphism. (So model theoretically these are the saturated Boolean algebras.) We construct dense Boolean algebras as witnesses for all the hardness results for membership in each of the elementary classes. Thus we can deduce analogous results for isomorphism problems on these classes of Boolean algebras. (Some care needs to be taken as being dense is itself a complicated property.) We present the results in terms of typical *strong* index set notation, e.g. $(\Sigma_n, \Pi_n) \leqslant_{\mathtt{m}} (\mathcal{DB}_r, \mathcal{DB}_s)$ (where $\mathcal{DB}_r$ and $\mathcal{DB}_s$ are classes of dense Boolean algebras) as in Soare [Soa87, IV.3.1] and explained in Definition 10.2.9. This easily translates into the terminology proposed by Knight of the isomorphism relation being, e.g. Π_n, within some class of dense Boolean algebras. (See Definition 10.2.11 and also [Cal] for further discussion of this notion.) Thus our results also supply examples of classes complete (in a strong way) at the same syntactic levels for a collection of isomorphism problems. (Isomorphism problems at certain higher levels of the hyperartihmetical hierarchy are provided by classes of reduced Abelian p-groups as shown in Calvert [Cal].)

While all of these issues are natural in their own right, we should note that we came to the particular questions addressed here from the problem of classifying the complexity of related issues in terms of Reverse Mathematics. The question raised in [Sho04] is the proof theoretic complexity of the existence of invariants for (countable) Boolean Algebras classifying them up to elementary equivalence. Answers to such questions are often provided by index set type results. Indeed, as explained in [Sho04] it seemed plausible, because of the nature of the results and the proof theoretic issues, that one might need such results in this case. As it turned out, weaker hardness theorems for membership in some of the classes sufficed to reach the desired proof theoretic system of ACA_0^+ (corresponding to the existence of $X^{(\omega)}$ for every set X). Nonetheless, the recursion theoretic questions remained interesting. In particular, the class at level $\Pi_{\omega+1}$ plays no role in the proof theoretic analysis and we thank Jim Schmerl for raising the corresponding question.

As we were about to submit this paper for publication, we came across [Sel03], a survey of positive (i.e. computably enumerable) structures. Selivanov describes there (Theorems 4.5.5-4.5.7) a number of results on index sets for computably enumerable Boolean algebras which, along with many others, appear in [Sel90] and [Sel91]. He also states (Remark 1 following Theorem 4.5.7) that analogs of the results mentioned may be proven for the computable Boolean algebras (with the index sets one step lower in every case) by straightforwardly generalizing his proofs for the computably enumerable ones. The analogs of the results mentioned in [Sel03] and appearing in [Sel91] cover our completeness results for the finitely axiomatizable classes of Boolean algebras. Others in [Sel90, Lemma 12], if also generalized to the computable case, would cover the other cases except for the nonarithmetic class at level $\Pi_{\omega+1}$. (The explicit results of [Sel90, Lemma 12] give the strong index set form of the results corresponding to the first four lines of our table in Theorem 10.2.10. The general ones for finitely axiomatizable classes as in [Sel91, p. 168] provide completeness results but do not explicitly give the strong form of the index set results as in the fifth and sixth lines of our table.) The question corresponding to the nonarithmetic class of computably enumerable Boolean algebras is explicitly left open in [Sel90]. All of our proofs, including the nonarithmetic case, immediately supply the corresponding results for computably enumerable algebras. (The index sets are one level higher in the arithmetic hierarchy than those for computable Boolean algebras in the arithmetic cases and at the same level ($\Pi_{\omega+1}$) in the nonarithmetic one. To see this, note that one can go from computable to computably enumerable at the cost of one level in the hierarchy by simply relativizing to algebras computable in $0'$ as every Δ^0_2 (i.e. computable in $0'$) Boolean algebra is isomorphic to a uniformly constructed Σ^0_1, i.e. computably enumerable, one (essentially by [Fei67] according to [Dow97, Cor. 3.10] or explicitly by [OS89, Th. 2]). Of course, $\Pi_{\omega+1}$ relativized to $0'$ is still $\Pi_{\omega+1}$ and so the result is the same for the computably enumerable algebras as for the computable ones in this case.) Thus we also reprove some of the results of [Sel90] and [Sel91].

Our methods are quite different from Selivanov's. We use no representations as tree algebras but extensively exploit the back and forth relations and notions of k-friendliness of [AK00] to unify and simplify our analysis in the arithmetic cases. The nonarithmetic case also needs some specific constructions using interval algebras. All our results are proven for dense Boolean algebras and so also provide new results on index sets for the isomorphism problem for these algebras as mentioned above.

We provide the basic definitions for Boolean algebras needed to define our classes and state the main index set type theorems in Section 10.2. We prove the easy, quantifier counting aspect of our complexity results in Section 10.3. We define dense Boolean algebras in Section 10.4 and present some useful lemmas about them. Section 10.5 introduces the back and forth relations of Ash and Knight [AK00] and their notion of k-friendly structures. The remaining sections prove the hardness results for the various classes of Boolean algebras: Σ_n or Π_n for every $n < \omega$; $\Sigma_n \wedge \Pi_n$ for $n \equiv 1, 2 (\mathrm{mod}\, 4)$; and, finally, $\Pi_{\omega+1}$.

We refer the reader to Monk [Mon89] (especially Ch. 7) and Goncharov [Gon97] (especially Ch. 2) for general background about Boolean algebras. For recursion theory, we suggest Soare [Soa87].

For a study of the complexity of the elementary theory of Boolean Algebras in terms of space, time and number of alternations, see [Koz80].

10.2 Definitions and Theorems

We begin with some basic definitions.

Definition 10.2.1. Let B be a Boolean algebra. We use the usual notation of constants 0 and 1 and operations $\wedge$, $\vee$, and $\neg$. We define the following abbreviations. We let $x \leqslant y$ abbreviate $x \wedge y = x$; $x - y$ abbreviate $x \wedge \neg y$; and $x \triangle y$ abbreviate $(x-y) \vee (y-x)$. We say that $x \in B$ is an *atom* if $x \neq 0 \,\&\, \forall z < x (z = 0)$; x is *atomic* if for every non-zero element $z < x$, there is an atom $y \leqslant z$; x is *atomless* if it has no atoms below it.

Let $\mathcal{I}(B)$ denote the ideal of all elements x of B such that $x = y \vee z$, where y is atomic and z is atomless. Let $B^{[0]} = B$, and $B^{[n+1]} = B^{[n]}/\mathcal{I}(B^{[n]})$. We now define the *invariant* of B to be $\mathrm{inv}(B) = \langle p, q, r\rangle$, where $p \leqslant \omega$, $q \leqslant \omega$, $r \leqslant 1$, and

$$p = \begin{cases} \min\{n : B^{[n+1]} = 0\} & \text{if it exists,} \\ \omega & \text{otherwise,} \end{cases}$$

$$q = \begin{cases} \sup\{n : B^{[p]} \text{ has at least } n \text{ atoms}\} & \text{if } p < \omega, \\ 0 & \text{if } p = \omega, \end{cases}$$

$$r = \begin{cases} 1 & \text{if } p < \omega \text{ and } B^{[p]} \text{ contains an atomless element,} \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathrm{inv}(B) = \langle p, q, r\rangle$, we write $\mathrm{inv}_1(B) = p$, $\mathrm{inv}_2(B) = q$, and $\mathrm{inv}_3(B) = r$. We let $\mathtt{In}$ be the set of possible invariants. That is, $\mathtt{In}$ is the set of triplets $\langle p, q, r\rangle \in (\omega+1) \times (\omega+1) \times 2$ such that if $p = \omega$ then $q = r = 0$ and if $p < \omega$ then q and r are not both 0.

The original theorem showing that these are invariants for elementary equivalence is Tarski's:

Theorem 10.2.2 (Tarski). *[Tar49] If A and B are Boolean algebras, then* $\mathrm{inv}(A) = \mathrm{inv}(B)$ *if and only if A and B are elementarily equivalent.*

To simplify our notation we assign names to the classes (of computable algebras) corresponding to each invariant and an additional *level* value that will roughly correspond to the level of the associated index sets.

Definition 10.2.3. Given $\langle p, q, r\rangle \in \mathtt{In}$, we let $\mathcal{B}_{\langle p,q,r\rangle}$ be the set of indices of computable Boolean algebras with invariant $\langle p, q, r\rangle$. To each $x \in \mathtt{In}$ we assign a *level*, $l(x) \in \omega + 1$, as follows

$$l(x) = \begin{cases} 4p+1 & \text{if } x \in \{\langle p, q, 0\rangle : q < \omega\}, \\ 4p+2 & \text{if } x \in \{\langle p, q, 1\rangle : q < \omega\}, \\ 4p+3 & \text{if } x = \langle p, \omega, 0\rangle, \\ 4p+4 & \text{if } x = \langle p, \omega, 1\rangle, \\ \omega & \text{if } x = \langle \omega, 0, 0\rangle. \end{cases}$$

For a Boolean algebra B, we let $l(B) = l(\mathrm{inv}(B))$. Given $n \in \omega$, we let $\mathcal{B}_n$ be $\mathcal{B}_{\langle p,1,0\rangle}$ if $n = 4p+1$, $\mathcal{B}_{\langle p,0,1\rangle}$ if $n = 4p+2$, $\mathcal{B}_{\langle p,\omega,0\rangle}$ if $n = 4p+3$, and $\mathcal{B}_{\langle p,\omega,1\rangle}$ if $n = 4p+4$. We let $\mathcal{B}_{\langle p,\bar{q},r\rangle} = \cup\{\mathcal{B}_{\langle p,q',r\rangle} | q' \neq q, \omega\}$ and $\mathcal{B}_{\langle \bar{p},q,r\rangle} = \cup\{\mathcal{B}_{\langle p',q,r\rangle} | p' \neq p\}$.

We can now formulate our main results in terms of characterizing the complexity of these index sets. First, we deal with the standard levels of the arithmetic hierarchy.

Theorem 10.2.4. *For every n, $\mathcal{B}_n$ is Σ_n-complete if 4 divides n and Π_n-complete if 4 does not divide n.*

Next, we turn to completeness results that fall between some of the Σ_n and Σ_{n+1} levels.

Definition 10.2.5. A set S is in $\Sigma_n \wedge \Pi_n$ if there are $\phi \in \Sigma_n$ and $\psi \in \Pi_n$ such that $x \in S \leftrightarrow \phi(x) \,\&\, \psi(x)$.

Theorem 10.2.6. *For every $p < \omega$, and $1 < q < \omega$, $\mathcal{B}_{\langle p,q,0\rangle}$ is $\Pi_n \wedge \Sigma_n$-complete, where $n = 4p+1 = l(\langle p,q,0\rangle)$. For every $p < \omega$, and $0 < q < \omega$, $\mathcal{B}_{\langle p,q,1\rangle}$ is $\Pi_n \wedge \Sigma_n$-complete, where $n = 4p+2 = l(\langle p,q,1\rangle)$.*

Finally, we reach the level beyond the arithmetic ones.

Definition 10.2.7. A set S is $\Sigma^0_{\omega+1}$ if it is c.e. in $0^{(\omega)}$, and it is $\Pi^0_{\omega+1}$ if its complement is $\Sigma_{\omega+1}$.

(Note that we here follow the notation used in Soare [Soa87, XII.4]. In Ash-Knight [AK00], these classes are called Σ^0_ω and Π^0_ω, respectively.)

It is well known, and not hard to prove, that a set S is $\Pi_{\omega+1}$ if there is a computable f such that $n \in S \Leftrightarrow \forall j(f(n,j) \notin 0^{(j)})$. A set S is $\Sigma_{\omega+1}$ iff $\overline{S}$ is $\Pi_{\omega+1}$, that is, if there is a computable f such that $n \in S \Leftrightarrow \exists j(f(n,j) \in 0^{(j)})$.

Theorem 10.2.8. *$\mathcal{B}_{\langle\omega,0,0\rangle}$ is $\Pi_{\omega+1}$-complete.*

In fact, in every case our proofs will show more.

Definition 10.2.9. For any class Γ (of subsets of ω) and its complementary class $\breve{\Gamma} = \{\bar{S} | S \in \Gamma\}$ and for any $A, B \subseteq \omega$, $(\Gamma, \breve{\Gamma}) \leqslant_{\mathtt{m}} (A, B)$ means that for every $S \in \Gamma$ there is a computable function f such that $\forall x(x \in S \to f(x) \in A)$ and $\forall x(x \notin S \to f(x) \in B)$.

Our constructions will control the outcomes required in the proofs of hardness so as to improve the hardness conclusion. We can summarize our results as follows:

Theorem 10.2.10. *For each $x \in \mathtt{In}$, $\mathcal{B}_x$ is in Γ_x where Γ_x is specified in the second column of the table below. Moreover, $\mathcal{B}_x$ is complete for Γ_x and, indeed,* *complete in the sense of a reduction for $(\Gamma_x, \breve{\Gamma}_x)$ as given by the third column:*

x	Γ_x	$(\Gamma_x, \breve{\Gamma}_x) \leqslant_{\mathtt{m}}$
$\langle p,1,0\rangle$	Π_{4p+1}	$(\mathcal{B}_{\langle p,1,0\rangle}, \mathcal{B}_{\langle p,0,1\rangle})$
$\langle p,0,1\rangle$	Π_{4p+2}	$(\mathcal{B}_{\langle p,0,1\rangle}, \mathcal{B}_{\langle p,\omega,0\rangle})$
$\langle p,\omega,0\rangle$	Π_{4p+3}	$(\mathcal{B}_{\langle p,\omega,0\rangle}, \mathcal{B}_{\langle p,\omega,1\rangle})$
$\langle p,\omega,1\rangle$	Σ_{4p+4}	$(\mathcal{B}_{\langle p,\omega,1\rangle}, \mathcal{B}_{\langle p+1,1,0\rangle})$
$\langle p,q,0\rangle,\ 1<q<\omega$	$\Sigma_{4p+1} \wedge \Pi_{4p+1}$	$(\mathcal{B}_{\langle p,q,0\rangle}, \mathcal{B}_{\langle p,\bar{q},0\rangle})$
$\langle p,q,1\rangle,\ 0<q<\omega$	$\Sigma_{4p+2} \wedge \Pi_{4p+2}$	$(\mathcal{B}_{\langle p,\omega,1\rangle}, \mathcal{B}_{\langle p,\bar{\omega},1\rangle})$
$\langle \omega,0,0\rangle$	$\Pi_{\omega+1}$	$(\mathcal{B}_{\langle \omega,0,0\rangle}, \mathcal{B}_{\langle \bar{\omega},\omega,0\rangle})$

In addition, in every case we will also be able to restrict the sets of (indices for) Boolean algebras in the third column to the (indices for) dense ones (Definition 10.4.2) in the same classes. (When we say that $\mathcal{B}_x$ is in Γ_x for $\langle 0,1,0\rangle$ and $\langle 0,1,1\rangle$ we mean that there formulas of the form specified by Γ_x such that any Boolean algebra satisfying them is in $\mathcal{B}_x$. The issue here is that to say that a number is an index of a Boolean algebra (or even a structure at all) is already Π_2.)

As Goncharov [Gon97, 2.3.2] proves that any two countable dense Boolean algebras with the same invariant are isomorphic, we can restate some of these results in terms of the terminology introduced by Knight (see [GK02a] and [Cal, 3.1, 3.2] and the accompanying discussion) for classifying the complexity of the problem of determining if two structures are isomorphic.

Definition 10.2.11. Let Γ be a class of subsets of ω (e.g. a complexity class such as Π_n), $A \subseteq B \subseteq \omega$ (e.g. the sets of indices of some subclass and class, respectively, of structures). We say that A is Γ *complete within* B if, for any $S \in \Gamma$, there is a computable function $f : \omega \to B$ such that $\forall n(n \in S \Leftrightarrow f(n) \in A)$.

Corollary 10.2.12. *The isomorphism problem for dense Boolean algebras, i.e. the set $A = \{\langle i,j\rangle |\ i$ and j are indices of isomorphic dense computable Boolean algebras$\}$, is $\Pi_{\omega+1}$ complete within $\mathcal{DB}$, the set of indices of dense computable Boolean algebras. Indeed, for $x \in \mathtt{In}$, the finer problem of being isomorphic to the dense Boolean algebra D_x of level $l(x)$, $\{i | i$ is an index of a dense Boolean algebra isomorphic to $D_x\}$, is Γ_x complete within $\mathcal{DB}$ for Γ_x as specified in the table in Theorem 10.2.10.*

The results of Wasziewicz [Was74] are also derived along the way and slightly improved. (See Section 10.3 and the final remarks of Section 10.6 for the proofs.)

Theorem 10.2.13. *[Was74] If $x \in \mathtt{In}$ and $l(x) = n < \omega$, then the class of Boolean* *algebras B with* $\mathrm{inv}(B) = x$ *is axiomatized as follows:*

x	*Axioms*
$\langle p, 1, 0\rangle$	one $\forall_{4p+1}$
$\langle p, 0, 1\rangle$	one $\forall_{4p+2}$
$\langle p, \omega, 0\rangle$	one $\forall_{4p+3}$ and a computable set of $\exists_{4p+2}$
$\langle p, \omega, 1\rangle$	one $\exists_{4p+4}$ and a computable set of $\exists_{4p+2}$
$\langle p, q, 0\rangle,\ 1 < q < \omega$	one $\exists_{4p+1}$ and one $\forall_{4p+1}$
$\langle p, q, 1\rangle,\ 0 < q < \omega$	one $\exists_{4p+2}$ and one $\forall_{4p+2}$
$\langle \omega, 0, 0\rangle$	one $\forall_n$ for each n

By [Tar49], each class corresponds to a complete theory and so, for any $m < \omega$, if $l(B), l(B') \leqslant m$ and $B \equiv_m B'$ (i.e. they satisfy the same $\exists_m$ sentences) then $B \equiv B'$. On the other hand, if $l(B), l(B') > m$ then $B \equiv_m B'$.

Corollary 10.2.14. *[Was74] The class of Boolean algebras B with* $\mathrm{inv}(B) = x$ *are not axiomatizable by sentences in $\exists_{n-1}$ and $\forall_{n-1}$ where $n = l(x) < \omega$. The classes with invariants $\langle p, \omega, 0\rangle$ and $\langle p, \omega, 1\rangle$ are not finitely axiomatizable. The class of Boolean algebras with invariant $\langle \omega, 0, 0\rangle$ is not axiomatizable by sentences at any bounded level of the $\exists_n$ hierarchy.*

10.3 Counting Quantifiers

In this section, we prove that, for each $x \in \mathtt{In}$, $\mathcal{B}_x$ is in Γ_x. In fact, we will also analyze the complexity of the axioms needed to guarantee that a Boolean algebra is in $\mathcal{B}_x$. We will prove that $\mathcal{B}_x$ is Γ_x-hard in the following sections.

Definition 10.3.1. We define unary predicates $\mathcal{I}_n$, $Atom_n$, $Atomless_n$ and $Atomic_n$ and the associated formulas in the language of Boolean algebras by induction:

$$\begin{aligned}
\mathcal{I}_0(x) &\Leftrightarrow x = 0;\\
Atom_n(x) &\Leftrightarrow \neg\mathcal{I}_n(x) \ \&\ \forall y \leqslant x(\mathcal{I}_n(y) \vee \mathcal{I}_n(x-y));\\
Atomless_n(x) &\Leftrightarrow \neg\exists y \leqslant x(Atom_n(y));\\
Atomic_n(x) &\Leftrightarrow \neg\exists y \leqslant x(\neg\mathcal{I}_n(y) \ \&\ Atomless_n(y));\\
\mathcal{I}_{n+1}(x) &\Leftrightarrow \exists y, z(Atomless_n(y) \ \&\ Atomic_n(z) \ \&\ x = y \vee z).
\end{aligned}$$

Let B be a Boolean algebra. Note that $\mathcal{I}_n(B) = \{x \in B : B \models \mathcal{I}_n(x)\}$ is the ideal of B such that $B^{[n]} = B/\mathcal{I}_n(B)$. Let $[x]_n$ denote the equivalence class of x in $B/\mathcal{I}_n(B)$, that is $[x]_n = \{y \in B : x \triangle y \in \mathcal{I}_n(B)\}$. Then $Atom_n(x)$ holds iff $[x]_n$ is an atom of $B^{[n]}$, $Atomless_n(x)$ holds iff $[x]_n$ is atomless in $B^{[n]}$, and $Atomic_n(x)$ holds iff $[x]_n$ is atomic in $B^{[n]}$. Observe that the formulas $\mathcal{I}_n$, $Atom_n$, $Atomless_n$ and $Atomic_n$ are $\exists_{4n}$, $\forall_{4n+1}$, $\forall_{4n+2}$ and $\forall_{4n+3}$ respectively in the language of Boolean algebras. (Of course, that a computable Boolean algebra B satisfies a $\exists_n$ or $\forall_n$ formula is a Σ_n or Π_n relation, respectively.)

Definition 10.3.2. For $p, q < \omega$, we let $\mathcal{B}_{\langle p, \leqslant q, r\rangle} = \bigcup_{i \leqslant q} \mathcal{B}_{\langle p,i,r\rangle}$. Also let $l(\langle p, \leqslant q, r\rangle) = l(\langle p, q, r\rangle)$.

Lemma 10.3.3. *For $p, q < \omega$, x equal to either $\langle p, \leqslant q, 0\rangle$, $\langle p, \leqslant q, 1\rangle$ or $\langle p, \omega, 0\rangle$ and $n = l(x)$, $\mathcal{B}_x$ is in Π_n. Moreover, the corresponding classes of Boolean algebras are axiomatized by a $\forall_n$ sentence, a $\forall_n$ sentence, a $\forall_n$ sentence and a computable set of $\exists_{n-1}$ sentences (but not by any finite set of axioms), respectively. If $x = \langle p, \omega, 1\rangle$, then $\mathcal{B}_x$ is in Σ_n. Moreover, the corresponding class of Boolean algebras is axiomatized by a $\exists_n$ sentence and a computable set of $\exists_{n-2}$ sentences but is not finitely axiomatizable. (Of course, $\mathcal{B}_{\langle p,1,0\rangle} = \mathcal{B}_{\langle p,\leqslant 1,0\rangle}$ and $\mathcal{B}_{\langle p,0,1\rangle} = \mathcal{B}_{\langle p,\leqslant 0,1\rangle}$.)*

Proof: Consider $x = \langle p, \leqslant q, 0\rangle$, and let B be a computable Boolean algebra. B is in $\mathcal{B}_x$ if and only if B has first invariant at least p, but no more than q atoms in $B^{[p]}$, and no atomless members in $B^{[p]}$. Now, q atoms can generate at most 2^q non-equivalent members, so to say that there are at most q atoms it suffices to say

$$\neg\exists x_0, ..., x_{2^q}(\forall i, j \leqslant 2^q \neg\mathcal{I}_p(x_i \triangle x_j)),$$

which is a Π_{4p+1} predicate of B and indeed clearly equivalent to the truth of a $\forall_{4p+1}$ sentence. (Replace the bounded quantification by the corresponding conjunction.) This sentence also implies there are no atomless elements in $B^{[p]}$. For B to be in $\mathcal{B}_x$ we still need to say that B has first invariant at least p, i.e. $\neg\mathcal{I}_p(1)$ which is a $\forall_{4p}$ sentence.

Now consider $x = \langle p, \leqslant q, 1\rangle$. B is in $\mathcal{B}_x$ if and only if B has first invariant at least p, no more than q atoms in $B^{[p]}$, but more than 2^q elements in $B^{[p]}$. This is expressed by $\neg\mathcal{I}_p(1)$,

$$\neg\exists x_0, ..., x_q(\forall i \leqslant q(Atom_p(x_i)) \;\&\; \forall i < j \leqslant q(\neg\mathcal{I}_p(x_j \triangle x_i))),$$

and

$$\exists x_0, ..., x_{2^q}(\forall i, j \leqslant 2^q \neg\mathcal{I}_p(x_i \triangle x_j)).$$

Note that this is clearly equivalent to the truth in B a $\forall_{4p+2}$ sentence.

Consider now $x = \langle p, \omega, 0\rangle$. B is in $\mathcal{B}_x$ if and only if $[1]_p$ is atomic in $B^{[p]}$ and there are infinitely many atoms:

$$Atomic_p(1) \;\&$$
$$\forall m \exists x_1, ..., x_m(\forall i \leqslant m(Atom_p(x_i)) \;\&\; \forall i < j \leqslant m(\neg\mathcal{I}_p(x_j \triangle x_i))).$$

Observe that this is a Π_{4p+3} predicate on B which is equivalent to the truth of a $\forall_{4p+3}$ sentence and a computable set of Σ_{4p+2} sentences (one for each m). If this class were finitely axiomatizable then, by the completeness of the associated theory, some finite subset of this list of axioms would suffice to axiomatize the class. This, however, is obviously impossible since any finite subset has a an algebra with invariant $\langle p, q, 0\rangle$ for some q.

Finally let $x = \langle p, \omega, 1\rangle$. Then $B \in \mathcal{B}_x$ if and only if, in $B^{[p]}$, 1 is the sum of an atomless element and an atomic element, and there are infinitely many atoms. This is expressed by

$$\exists yz(1 = y \vee z \ \& \ Atomic_p(y) \ \& \ Atomless_p(z) \ \& \ \neg\mathcal{I}_p(z)) \ \& \\ \forall m \exists x_1, ..., x_m(\forall i \leqslant m(Atom_p(x_i)) \ \& \ \forall i < j \leqslant m(\neg\mathcal{I}_p(x_j \triangle x_i))))$$

which is a Σ_{4p+4} predicate on B which is equivalent to the truth of a $\exists_{4p+4}$ sentence and a computable set of Σ_{4p+2} sentences. The argument that this class is not finitely axiomatizable is the same as for $\langle p, \omega, 0\rangle$. □

It follows that, for all $n \in \omega$, $\mathcal{B}_n$ is in Σ^0_n if 4 divides n and it is in Π^0_n otherwise.

Lemma 10.3.4. *For $x = \langle p, q, r\rangle$ with $p < \omega$ and either $r = 0$ & $1 < q < \omega$, or $r = 1$ & $0 < q < \omega$, $\mathcal{B}_x$ is in $\Sigma_n \wedge \Pi_n$, where $n = l(x)$. Moreover, the corresponding classes of Boolean algebras are axiomatized by a sentence in $\exists_n$ and one in $\forall_n$.*

PROOF: For $r = 0$, observe that $\mathcal{B}_{\langle p,q,0\rangle}$ consists of the Boolean algebras B in $\mathcal{B}_{\langle p,\leqslant q,0\rangle}$ which are not in $\mathcal{B}_{\langle p,\leqslant q-1,0\rangle}$. By Lemma 10.3.3, B in $\mathcal{B}_{\langle p,\leqslant q,0\rangle}$ is guaranteed by a $\forall_{4p+1}$ sentence and B not in $\mathcal{B}_{\langle p,\leqslant q-1,0\rangle}$ is expressible by a $\exists_{4p+1}$ sentence. Similarly, for $r = 1$, observe that $\mathcal{B}_{\langle p,q,1\rangle}$ consists of the Boolean algebras B in $\mathcal{B}_{\langle p,\leqslant q,1\rangle}$ (guaranteed by a $\forall_{4p+2}$ sentence) which are not in $\mathcal{B}_{\langle p,\leqslant q-1,1\rangle}$ (expressible by a $\exists_{4p+2}$ sentence). □

Lemma 10.3.5. *$\mathcal{B}_{\langle\omega,0,0\rangle}$ is in $\Pi_{\omega+1}$. The corresponding class of Boolean algebras is axiomatized by a computable set of $\forall_n$ sentences with one for each n.*

PROOF: A computable Boolean algebra B is in $\mathcal{B}_{\langle\omega,0,0\rangle}$ if for all p, $B^{[p]}$ is non-empty. In other words if

$$\forall p < \omega(\neg\mathcal{I}_p(1)).$$

Since $0^{(\omega)}$ knows whether $\mathcal{I}_p(1)$ for each p uniformly in p, $\mathcal{B}_{\langle\omega,0,0\rangle}$ is co-c.e. in $0^{(\omega)}$, or equivalently $\Pi^0_{\omega+1}$. □

Note that these Lemmas establish the axiomatizabilty of the classes of Boolean algebras by sentences of the complexity required in Theorem 10.2.13. The second part of this theorem follows from Theorem 10.6.1(2).

Now that we have that, for each x, $\mathcal{B}_x$ is in Γ_x, we turn to proving that $\mathcal{B}_x$ is Γ_x-hard. We first need to introduce the concepts of dense Boolean algebras and back-and-forth relations.

10.4 Dense Boolean Algebras

We start by defining the Tarski invariants on elements of a Boolean Algebra.

Definition 10.4.1. Let B be a Boolean algebra and $a \in B$. We let $B \upharpoonright a$ be the Boolean algebra whose domain is $\{b \in B : b \leqslant a\}$, $1_{B \upharpoonright a} = a$, $0_{B \upharpoonright a} = 0$, $\vee_{B \upharpoonright a}$ and $\wedge_{B \upharpoonright a}$ are the restrictions of the corresponding operations in B, and the complement of b in $B \upharpoonright a$ is $a - b$. We let $\mathrm{inv}^B(a) = \mathrm{inv}(B \upharpoonright a)$. When no confusion should arise, we may might write $\mathrm{inv}(a)$ instead of $\mathrm{inv}^B(a)$.

Definition 10.4.2. A Boolean algebra B is *dense* if for every $b \in B$,

1. $\forall k < \mathrm{inv}_1(b)(\exists a \leqslant b(\mathrm{inv}(a) = \langle k, \omega, 0\rangle))$ and
2. if $\mathrm{inv}_1(b) = \omega$ or $\mathrm{inv}_2(b) = \omega$, then there is an $a \leqslant b$ such that $\mathrm{inv}_1(a) = \mathrm{inv}_1(b) = \mathrm{inv}_1(b-a)$ and $\mathrm{inv}_2(a) = \mathrm{inv}_2(b) = \mathrm{inv}_2(b-a)$.

Goncharov [Gon97, 2.3.2] proves that any two countable dense Boolean algebras with the same invariant are isomorphic. Moreover, he proves that every countable Boolean algebra B has an elementary extension B^* which is dense. This then shows that any two countable Boolean algebras with the same invariant are elementarily equivalent and so establishes Tarski's theorem.

We let D_x denote the dense Boolean algebra with invariant x. All of them are computably (even decidably) presentable by Morozov [Mor82].

Definition 10.4.3. We define an addition operation on the set `In` of invariants as follows:

$$\sum_{i \leqslant m} \langle p_i, q_i, r_i\rangle = \langle p_0, q_0, r_0\rangle + \dots + \langle p_m, q_m, r_m\rangle = \langle p, q, r\rangle,$$

where

$$\begin{aligned} p &= \max\{p_i : i \leqslant m\}, \\ q &= \sum\{q_i : i \leqslant m \;\&\; p_i = p\}, \\ r &= \max\{r_i : i \leqslant m \;\&\; p_i = p\}. \end{aligned}$$

We then say that $\langle p_0, q_0, r_0\rangle, \dots, \langle p_m, q_m, r_m\rangle$ is a *partition* of $\langle p, q, r\rangle$. (Here, we are using the convention that $\omega + q = q + \omega = \omega$.)

Definition 10.4.4. We say that $a_0, \dots, a_m \in B$ form a *partition* of $a \in B$ if $\bigvee_{i \leqslant m} a_i = a$ and for all $i \leqslant m$,

$$a_i \wedge \bigvee_{j \leqslant m, j \neq i} a_j = 0.$$

Observe that if $a_0, \dots, a_m$ form a partition of 1, then $B \cong B \upharpoonright a_0 \times \dots \times B \upharpoonright a_m$. We then say that $B \upharpoonright a_0, \dots, B \upharpoonright a_m$ form a *partition* of B.

Now consider an arbitrary tuple $\bar{b} = (b_0, \dots, b_n)$ of members of B. This generates a partition of B as follows. Let $A_0 = \{b_0, 1 - b_0\}$. Let $A_i = \{a - b_i, b_i \cap a : a \in A_i\}$ for $0 \leqslant i \leqslant n$. Then $\{B \upharpoonright a : a \in A_n, a \neq 0\}$ is *the partition of B generated by $\bar{b}$.*

Lemma 10.4.5. *If* $a_0, ..., a_{m-1}$ *form a partition of* a*, then* $\mathrm{inv}(a_0), ..., \mathrm{inv}(a_{m-1})$ *form a partition of* $\mathrm{inv}(a)$*.*

Proof: See [Gon97, Lemma 2.2.4] for a proof of the lemma when $m = 2$. The general case follows easily by induction. □

When we are dealing with dense Boolean algebras, the converse of the previous lemma also holds.

Lemma 10.4.6. *A Boolean algebra* B *is dense if and only if, for every* $b \in B$ *and every partition* $x_0, ..., x_m$ *of* $\mathrm{inv}(b)$*, there exists a partition* $a_0, ..., a_m$ *of* b *such that, for each* $i \leqslant m$*,* $\mathrm{inv}(a_i) = x_i$*.*

Proof: The denseness conditions are just special cases of the partition property.

To see that, if B is dense, then B has the partition property, make use of the denseness conditions along with Lemma 10.4.7 below. □

Lemma 10.4.7. *[Gon97, Lemma 2.2.6] Let* B *be a Boolean algebra,* $b \in B$*, and* $x = \langle p, q, r\rangle \in \mathtt{In}$*.*

1. *If* $p < \mathrm{inv}_1(b)$*,* $q < \omega$*, and* $r \leqslant 1$*, then there is an* $a \leqslant b$ *such that* $\mathrm{inv}(b-a) = \mathrm{inv}(b)$*.*
2. *If* $p = \mathrm{inv}_1(b)$*,* $q \leqslant \mathrm{inv}_2(b)$*, and* $r \leqslant \mathrm{inv}_3(b)$*, then there is an* $a \leqslant b$ *such that* $\mathrm{inv}(a) = x$*. Moreover, if* $q < \mathrm{inv}_2(b)$ *or* $r = 1$*, then we can also require that* $\mathrm{inv}_1(b - a) = \mathrm{inv}_1(b)$*,* $\mathrm{inv}_2(b - a) = q$*, and* $\mathrm{inv}_3(b - a) = \mathrm{inv}_3(b)$*, where we take* $\omega - \omega$ *to be* 0*.*

Corollary 10.4.8. *The product of dense Boolean algebras is dense.*

Proof: Consider $x, y \in \mathtt{In}$. We want to prove that $D_x \times D_y \cong D_{x+y}$. The element 1 of D_{x+y} has invariant $x + y$. So, by the lemma above, there exists a partition a, b of 1 such that $\mathrm{inv}(a) = x$ and $\mathrm{inv}(b) = y$. Since a, b is a partition of 1, $D_{x+y} \cong D_{x+y} \restriction a \times D_{x+y} \restriction b$. Since D_{x+y} is dense, so are $D_{x+y} \restriction a$ and $D_{x+y} \restriction b$. Therefore

$$D_x \times D_y \cong D_{x+y} \restriction a \times D_{x+y} \restriction b \cong D_{x+y}.$$

□

10.5 Back-and-Forth relations

In this section we define back-and-forth relations between structures and state the properties about them that we need. We refer the reader to [AK00] for more information on these relations.

Definition 10.5.1. Let K be a class of structures for a fixed language. For each $n < \omega$, we define the *standard back-and-forth relation* $\leqslant_n$ on pairs $(A, \bar{a})$, where $A \in K$ and $\bar{a}$ is a tuple in A. First suppose that $\bar{a}$ in A and $\bar{b}$ in B are tuples of the same length. Then,

1. $(A, \bar{a}) \leqslant_1 (B, \bar{b})$ if and only if all Σ_1 formulas true of $\bar{b}$ in B are true of $\bar{a}$ in A.

2. For $n > 1$, $(A, \bar{a}) \leqslant_n (B, \bar{b})$ if and only if for each $\bar{d}$ in B, and each $1 \leqslant k < n$, there exists a $\bar{c}$ in A with $|\bar{c}| = |\bar{d}|$ such that $(B, \bar{b}, \bar{d}) \leqslant_k (A, \bar{a}, \bar{c})$.

Now, we extend the definition of $\leqslant_n$ to tuples of different lengths. For $\bar{a}$ in A and $\bar{b}$ in B, let $(A, \bar{a}) \leqslant_n (B, \bar{b})$ if and only if $|\bar{a}| \leqslant |\bar{b}|$ and for the initial segment $\bar{b}'$ of $\bar{b}$ of length $|\bar{b}|$, we have $(A, \bar{a}) \leqslant_n (B, \bar{b}')$. We may write $A \leqslant_n B$ instead of $(A, \emptyset) \leqslant_n (B, \emptyset)$.

One observation that might give the reader some intuition about the back-and-forth relation is that $(A, \bar{a}) \leqslant_n (B, \bar{b})$ if and only if all the Π_n infinitary formulas true of $\bar{a}$ in A are true of $\bar{b}$ in B. (See [AK00, Proposition 15.1]; see [AK00, Chapter 6] for information on infinitary formulas.) Also observe that if $k < n$ and $(A, \bar{a}) \leqslant_n (B, \bar{b})$ then $(A, \bar{a}) \equiv_k (B, \bar{b})$, where $(A, \bar{a}) \equiv_k (B, \bar{b})$ if and only if $(A, \bar{a}) \leqslant_k (B, \bar{b})$ and $(A, \bar{a}) \geqslant_k (B, \bar{b})$.

The only structures we will be dealing with are Boolean algebras. The following lemma gives us a way of computing the back-and-forth relations on Boolean algebras without having to refer to the definition given above.

Lemma 10.5.2. *[AK00, 15.13] Suppose that A and B are Boolean algebras. Then $A \leqslant_1 B$ if and only if A is infinite or can be split into at least as many disjoint parts as B (i.e., if A is generated by p atoms, then B is generated by k atoms, for some $k \leqslant p$). For $n > 1$, $A \leqslant_n B$ if and only if, for any l with $1 \leqslant l < n$ and any finite partition of B into $B_1, ..., B_k$, there is a corresponding partition of A, $A_1, ..., A_k$, such that $B_i \leqslant_l A_i$.*

We will be interested in analyzing the back-and-forth relation among the dense Boolean algebras. Since each isomorphism type of a dense Boolean algebra is determined by its invariant, we translate the back-and-forth relation to one on the set of invariants:

Definition 10.5.3. Given $x, x' \in \mathtt{In}$ and $n < \omega$ we let $x \leqslant_n x'$ if $D_x \leqslant_n D_{x'}$.

The back-and-forth relations on the set of invariants can be computed using the following lemma.

Lemma 10.5.4. *Consider $x, x' \in \mathtt{In}$. Then $x \leqslant_1 x'$ if and only if either $l(x) > 1$, or $x = \langle 0, q, 0 \rangle$, $x' = \langle 0, q', 0 \rangle$ and $q \geqslant q'$. For $n > 1$, $x \leqslant_n x'$ if and only if, for any partition $y'_1, ..., y'_k$ of x', there is a corresponding partition $y_1, ..., y_k$ of x such that $y'_i \leqslant_{n-1} y_i$.*

Proof: Immediate from Lemma 10.5.2, noting that D_x is infinite if and only if $l(x) > 1$, and that if $l(x) = 0$ then $x = \langle 0, q, 0\rangle$ for some $1 \leqslant q < \omega$, so for D_x to be such that it can be split into at least as many disjoint parts as D'_x we must have $x' = \langle 0, q', 0\rangle$ for some $q' \leqslant q$. □

The above considerations reduce computing the back-and-forth relations on `In` to a combinatorial task, which we will do in Theorem 10.6.1. To complete the proofs of our hardness results we also make use of the concept of k-friendliness., which we now introduce. Again, we refer the reader to [AK00, Chapter 15] for more information.

Definition 10.5.5. A pair of structures $\{A_0, A_1\}$ is *k-friendly* if the structures A_i are computable, and for $n < k$, the standard back-and-forth relations $\leqslant_n$ on $(A_i, \overline{a})$, for $\overline{a} \in A_i$, are c.e., uniformly in n.

Theorem 10.5.6. *[AK00, 18.6] Let A_0 and A_1 be structures such that $A_1 \leqslant_k A_0$ and $\{A_0, A_1\}$ is k-friendly. Then for any Π^0_k set S, there is a uniformly computable sequence of structures $\{C_n\}_{n\in\omega}$ such that*

$$C_n \cong \begin{cases} A_0 & \text{if } n \in S \\ A_1 & \text{otherwise} \end{cases}$$

This theorem can be restated as follows.

Corollary 10.5.7. *Let A_0 and A_1 be n-friendly structures and $\mathcal{B}_{A_0}$ and $\mathcal{B}_{A_1}$ be subsets of ω such that every index of a computable copy of A_0 is in $\mathcal{B}_{A_0}$ and every index of a computable copy of A_1 is in $\mathcal{B}_{A_1}$. Then*

$$A_1 \leqslant_n A_0 \Rightarrow (\Sigma_n, \Pi_n) \leqslant_{\mathtt{m}} (\mathcal{B}_{A_1}, \mathcal{B}_{A_0}).$$

10.6 The Σ_n and the Π_n cases (Theorem 10.2.4)

We start by giving a complete analysis of the back-and-forth relations on the set of invariants, or equivalently, on the dense Boolean algebras. The proof of the following theorem is purely combinatorial and all it uses about the back-and-forth relations on `In` is Lemma 10.5.4.

Theorem 10.6.1. *Let $x = \langle p, q, r\rangle$ and $x' = \langle p', q', r'\rangle$ be invariants with $l(x) = l$ and $l(x') = l'$, and let $n \geqslant 1$. The following conditions determine whether $x \leqslant_n x'$.*

Case 1: If $l < n \vee l' < n$, then $x \leqslant_n x'$ iff $x = x'$.
Case 2: If $l > n \;\&\; l' > n$, then $x \leqslant_n x'$ always.
Case 3: If $l = n \;\&\; l' = n$, then $x \leqslant_n x'$ iff $q \geqslant q'$.
Case 4: If $l > n \;\&\; l' = n$, then $x \leqslant_n x'$ iff $n \neq 4p' + 4$.
Case 5: If $l = n \;\&\; l' > n$, then $x \leqslant_n x'$ iff $n = 4p + 4$.

PROOF: The proof is by induction on n. The case $n = 1$ follows trivially from Lemma 10.5.4 (recall l, $l' \geqslant 1$ by definition of level). Consider $n > 1$ and assume the theorem holds for all $m < n$.

Case 1: Suppose that either $l < n$ or $l' < n$. Clearly if $x = x'$ then $x \leqslant_n x'$. Now suppose $x \leqslant_n x'$. Then $x \equiv_{n-1} x'$. By induction hypothesis this can only happen either if $x = x'$ or if $l > n-1$ and $l' > n-1$. Therefore, since either $l < n$ or $l' < n$, we must have $x = x'$.

Case 2: Suppose $l > n$ and $l' > n$. We have to show that given any finite partition $y'_1, ..., y'_k$ of x', there is a corresponding partition $y_1, ..., y_k$ of x, such that $y'_i \leqslant_{n-1} y_i$. Assume $y'_1, ..., y'_k$ are ordered such that for some $j \leqslant k$, $y'_1, ..., y'_j$ have level $\geqslant n$ and $y'_{j+1}, ..., y'_k$ have level $< n$. It is not hard to observe that, whatever n is, since $l(x) > n$, it is always the case that there exists $y_1, ..., y_j$ of level $\geqslant n$ such that $\sum_{i \leqslant j} y_i = x$. Note that by induction hypothesis, since $l(y_i) > n-1$ and $l(y'_i) > n-1$, $y'_i \leqslant_{n-1} y_i$ for all $i \leqslant j$. For $i > j$ let $y_i = y'_i$. Another easy general observation is that for every $y, z \in \texttt{In}$ with $l(y) \leqslant l(z) - 2$, $z + y = z$. Then $\sum_{i \leqslant k} y_i = x + \sum_{i=j+1}^{k} y'_i = x$. So $y_1, ..., y_k$ is the desired partition of x.

Case 3: Assume $l = l' = n$. Note that $p = p'$ and $r = r'$. Also if $n = 4p+3$ or $n = 4p+4$, then $q = q' = \omega$ and therefore $x = x'$. So suppose n is either $4p+1$ or $4p+2$.

First suppose $q \geqslant q'$; we want to show that $x \leqslant_n x'$. Consider a partition $y'_1, ..., y'_k$ of x' with $y'_i = \langle p_i, q_i, r_i \rangle$. Note that, necessarily, for some $i \leqslant k$, $p_i = p$ and $r_i = r$; without loss of generality suppose that $p_1 = p$ and $r_1 = r$. Let $y_1 = \langle p, q_1 + (q - q'), r \rangle = \langle p, q - q', r \rangle + y'_1$, and for $i > 1$ let $y_i = y'_i$. Observe that

$$\sum_{i \leqslant k} y_i = \langle p, q - q', r \rangle + \sum_{i \leqslant k} y'_i = \langle p, q - q', r \rangle + x' = x.$$

Also, since $l(y_1) = l(y'_1) = n > n - 1$, by Case 2 of the inductive hypothesis $y'_1 \leqslant_{n-1} y_1$. So $y_1, ..., y_k$ is the desired partition.

Now suppose $q < q'$; we want to show that $x \not\leqslant_n x'$.

If $n = 4p+1$ or equivalently $r = 0$, consider the partition $y'_i = \langle p, 1, 0 \rangle$ for $i \leqslant q'$ of $x' = \langle p, q', 0 \rangle$. It is not hard to see that any partition, $y_1, ..., y_{q'}$, of x cannot have more than q elements at level n. So, for some $i \leqslant q'$, $l(y_i) < n = l(y'_i)$. Then, by either case 1 or case 4 of the induction hypothesis, $y'_i \not\leqslant_{n-1} y_i$.

If $n = 4p+2$, consider the partition $y'_i = \langle p, 1, 0 \rangle$ for $i \leqslant q'$ and $y_{q'+1} = \langle p, 0, 1 \rangle$ of $x' = \langle p, q', 1 \rangle$. Suppose toward a contradiction that there is a partition $y_1, ..., y_{q'+1}$ of x such that for all $i \leqslant q'+1$, $y'_i \leqslant_{n-1} y_i$. By induction hypothesis, $\langle p, 1, 0 \rangle \leqslant_{n-1} y_i$ implies that $y_i = \langle p, 1, 0 \rangle$. So, for all $i \leqslant q'$, $y_i = \langle p, 1, 0 \rangle$. But then, since $q < q'$, it cannot be the case that $\sum_{i < q'+1} y_i = \langle p, q, 1 \rangle = x$.

Case 4: Suppose now that $l > n$ and $l' = n$.

First suppose that $n \neq 4p' + 4$. Then, any partition $y'_1, ..., y'_k$ of x' must have some member at level n. Assume $l(y'_1) = n$. Note that there is a y_1 of level $l > n$ such that $y_1 + \sum_{1 < i \leqslant k} y'_i = x$. Also observe that since $l(y_1) > n - 1$ and

$l(y_1') > n-1$, $y_1' \leqslant_{n-1} y_i$. Then, if for $1 < i \leqslant k$ we let $y_i = y_i'$, we obtain the desired partition of x.

Now suppose that $n = 4p' + 4$ and hence $x' = \langle p', \omega, 1\rangle$; we want to show that $x \not\leqslant_n x'$. Consider the following partition of x': let $y_1' = \langle p', \omega, 0\rangle$ and $y_2' = \langle p', 0, 1\rangle$. Suppose toward a contradiction that y_1, y_2 is a partition of x such that $y_i' \leqslant_{n-1} y_i$ for $i \leqslant 2$. Then by induction hypothesis we must have $y_1 = \langle p', \omega, 0\rangle$ and $y_2 = \langle p', 0, 1\rangle$. But then $l(y_1 + y_2) = n < l(x)$, contradicting $y_1 + y_2 = x$.

Case 5: The last case is $l = n$ and $l' > n$.

Suppose first that $n = 4p + 4$, so $x = \langle p, \omega, 1\rangle$. Let $y_1', ..., y_k'$ be a partition of x'. Let $y_i = y_i'$ if $l(y_i') < n$ and $y_i = \langle p, \omega, 1\rangle$ otherwise. Note that $\sum_{i \leqslant k} y_k = x$, and that for i with $l(y_i') > n$, since $l(y_i) > n-1$, $y_i' \leqslant_{n-1} y_i$. So, $y_1, ..., y_k$ is the desired partition.

Note that if $l(x^*) \geqslant n+1$ and $l(x') \geqslant n+1$, then $x' \leqslant_n x^*$ by Case 2. So to show $x \not\leqslant_n x'$ it suffices to show $x \not\leqslant_n x^*$ for any x^* of level $n+1$.

Suppose $n = 4p+1$ or $n = 4p+2$, so $x = \langle p, q, r\rangle$ with $q < \omega$ and x' has level $n+1$; we want to show that $x \not\leqslant_n x'$. By case 3, $\langle p, q, r\rangle \not\leqslant_n \langle p, q+1, r\rangle$. By case 4, $x' \leqslant_n \langle p, q+1, r\rangle$. So we must have $\langle p, q, r\rangle \not\leqslant_n x'$.

Last, suppose $n = 4p+3$. Then $x = \langle p, \omega, 0\rangle$ and let $x' = \langle p, \omega, 1\rangle$; we want to show that $x \not\leqslant_n x'$. Consider the partition $y_1' = \langle p, \omega, 0\rangle$ and $y_2' = \langle p, 0, 1\rangle$ of x'. Suppose toward a contradiction that there is a partition y_1, y_2 of x such that $y_i' \leqslant_{n-1} y_i$. Now, $\langle p, 0, 1\rangle \leqslant_{n-1} y_2$ implies, by induction hypothesis, that $y_2 = \langle p, 0, 1\rangle$. But then y_2 cannot be part of a partition of $\langle p, \omega, 0\rangle$. Contradiction. □

Corollary 10.6.2. *Let A and B be computable presented Boolean algebras such that the functions inv^A and inv^B are computable. Then $\{A, B\}$ is n-friendly for every $n < \omega$.*

Proof: Let A_0 and A_1 be in $\{A, B\}$, $\bar{a}_0$ be a tuple in A_0, $\bar{a}_1$ be a tuple in A_1, and $n < \omega$. We will show how to decide whether $(A_0, \bar{a}_0) \leqslant_n (A_1, \bar{a}_1)$ computably. If $|a_0| > |a_1|$, then $(A_0, \bar{a}_0) \not\leqslant_n (A_1, \bar{a}_1)$. So suppose $|a_0| \leqslant |a_1|$. By truncating $\bar{a}_1$ if necessary, we can assume without loss of generality that they have the same length. Each tuple $\bar{a}_i$ generates a partition of A_i. We can then effectively compute the invariants of the partition, $y_{i,0}, \ldots, y_{i,k}$. By [AK00, Lemma 15.12], $(A_0, \bar{a}_0) \leqslant_n (A_1, \bar{a}_1)$ iff $y_{0,j} \leqslant_n y_{1,j}$ for $0 \leqslant j \leqslant k$. Then we can use Theorem 10.6.1 to decide this. □

In [Mor82], Morozov uniformly constructs dense Boolean algebras of each invariant which are decidable. While decidability does not quite give the computability of the inv functions on these algebras, it is not hard to see that they are in fact computable. (The argument is a tedious one by induction with several cases. Enough of it to give the ideas is carried out in Shore [Sho04, Proposition 6.5] when that proof is specialized to these algebras.) Therefore, by Corollary 10.6.2, these Boolean algebras are n-friendly for each n. Then, from Corollary 10.5.7 we obtain the following:

Corollary 10.6.3. *For every $p < \omega$,*

$$\begin{aligned}
(\Sigma_{4p+1}, \Pi_{4p+1}) &\leqslant_{\mathtt{m}} (\mathcal{DB}_{\langle p,0,1\rangle}, \mathcal{DB}_{\langle p,1,0\rangle})\\
(\Sigma_{4p+2}, \Pi_{4p+2}) &\leqslant_{\mathtt{m}} (\mathcal{DB}_{\langle p,\omega,0\rangle}, \mathcal{DB}_{\langle p,0,1\rangle})\\
(\Sigma_{4p+3}, \Pi_{4p+3}) &\leqslant_{\mathtt{m}} (\mathcal{DB}_{\langle p,\omega,1\rangle}, \mathcal{DB}_{\langle p,\omega,0\rangle})\\
(\Sigma_{4p+4}, \Pi_{4p+4}) &\leqslant_{\mathtt{m}} (\mathcal{DB}_{\langle p,\omega,1\rangle}, \mathcal{DB}_{\langle p+1,1,0\rangle})
\end{aligned}$$

Theorem 10.2.4 and the corresponding lines of Theorem 10.2.10 now follow from this corollary and Lemma 10.3.3.

We can also now derive the second part of Theorem 10.2.13 and Corollary 10.2.14. As remarked above, $D_x \equiv_n D_{x'}$ implies that the same $\forall_n$ formulas are true in D_x and $D_{x'}$ ([AK00, Proposition 15.1]). Case (2) of Theorem 10.6.1 then implies that, for every $m < \omega$, if $l(B), l(B') > m$ then $B \equiv_m B'$ as required for the second part of Theorem 10.2.13. As for Corollary 10.2.14, if D_x were axiomatized by sentences in $\exists_m$ and $\forall_m$ for $m < l(x)$ then, by the second part of Theorem 10.2.13, $D_{x'} \equiv_m D_x$ for any x' with $l(x') > n$ and so we would have $D_{x'} \equiv D_x$ for a contradiction.

10.7 The $\Sigma_n \wedge \Pi_n$ cases (Theorem 10.2.6)

Now we prove that, for $x = \langle p, q, r\rangle$ with $0 < q + r < \omega$, $\mathcal{DB}_x$ is $\Sigma^0_{l(x)} \wedge \Pi^0_{l(x)}$-hard. We first prove it for $x \neq \langle p, 2, 0\rangle$. Later, using a more complicated proof, we prove it for $x = \langle p, 2, 0\rangle$.

Lemma 10.7.1. *For $2 < q < \omega$, $\mathcal{DB}_{\langle p,q,0\rangle}$ is $(\Sigma_{4p+1} \wedge \Pi_{4p+1})$-hard. For $0 < q < \omega$, $\mathcal{DB}_{\langle p,q,1\rangle}$ is $(\Sigma_{4p+2} \wedge \Pi_{4p+2})$-hard. Moreover, the reductions proving hardness produce, in the case that n is not in the $\Sigma_{4p+1} \wedge \Pi_{4p+1}$ or $\Sigma_{4p+2} \wedge \Pi_{4p+2}$ set, an index in $\mathcal{DB}_{\langle p,\bar{q},0\rangle}$ or $\mathcal{DB}_{\langle p,\bar{q},1\rangle}$, respectively, as required in Theorem 10.2.10.*

Proof: Let $2 < q < \omega$. Consider two Σ_{4p+1} formulas $\phi(n)$ and $\psi(n)$. We want to construct a computable function f such that $\forall n(\phi(n)\ \&\ \neg\psi(n)) \Leftrightarrow f(n) \in \mathcal{DB}_{\langle p,q,0\rangle})$. Since $q > 2$, by Theorem 10.6.1, $\langle p, q, 0\rangle \leqslant_{4p+1} \langle p, 1, 0\rangle$ and $\langle p, q - 1, 0\rangle \leqslant_{4p+1} \langle p, 1, 0\rangle$. So by Corollary 10.5.7 there are computable g and h such that $\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,q-1,0\rangle}$, $\neg\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,1,0\rangle}$, $\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,q,0\rangle}$, and $\neg\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,1,0\rangle}$. Associating Boolean algebras with their indices, let $f(n) = g(n) \times h(n)$ and note that, by Corollary 10.4.8, $f(n)$ is an index for a dense Boolean algebra. Then if $\phi(n)\ \&\ \neg\psi(n)$, we have $\mathrm{inv}(f(n)) = \mathrm{inv}(g(n)) + \mathrm{inv}(h(n)) = \langle p, q - 1, 0\rangle + \langle p, 1, 0\rangle = \langle p, q, 0\rangle$. If $\phi(n)\ \&\ \psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, q-1, 0\rangle + \langle p, q, 0\rangle = \langle p, 2q-1, 0\rangle$, if $\neg\phi(n)\ \&\ \psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, 1, 0\rangle + \langle p, q, 0\rangle = \langle p, q+1, 0\rangle$, and if $\neg\phi(n)\ \&\ \neg\psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, 1, 0\rangle + \langle p, 1, 0\rangle = \langle p, 2, 0\rangle$. Thus f has the required properties.

Now suppose $0 < q < \omega$ and that $\phi(n)$ and $\psi(n)$ are Σ_{4p+2}. We now wish to construct a computable function f such that $\forall n(\phi(n)\ \&\ \neg\psi(n) \Leftrightarrow f(n) \in \mathcal{DB}_{\langle p,q,1\rangle})$. Again by Theorem 10.6.1 and Corollary 10.5.7 there are computable

g and h such that $\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,q,1\rangle}$, $\neg\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,0,1\rangle}$, $\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,q+1,1\rangle}$, and $\neg\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,0,1\rangle}$. Now let $f(n) = g(n) \times h(n)$ and note that f has the required properties. Indeed, if $\phi(n)$ & $\neg\psi(n)$, we have $\mathrm{inv}(f(n)) = \mathrm{inv}(g(n)) + \mathrm{inv}(h(n)) = \langle p,q,1\rangle + \langle p,0,1\rangle = \langle p,q,1\rangle$. If $\phi(n)$ & $\psi(n)$ then $\mathrm{inv}(f(n)) = \langle p,q,1\rangle + \langle p,q+1,1\rangle = \langle p,2q+1,1\rangle$, if $\neg\phi(n)$ & $\psi(n)$ then $\mathrm{inv}(f(n)) = \langle p,0,1\rangle + \langle p,q+1,1\rangle = \langle p,q+1,1\rangle$, and if $\neg\phi(n)$ & $\neg\psi(n)$ then $\mathrm{inv}(f(n)) = \langle p,0,1\rangle + \langle p,0,1\rangle = \langle p,0,1\rangle$. $\square$

To finish the proof of Theorem 10.2.6 and the corresponding parts of Theorem 10.2.10, we still need to prove that, for every p, $\mathcal{DB}_{\langle p,2,0\rangle}$ is $(\Sigma_{4p+1} \wedge \Pi_{4p+1})$-hard via reductions with an appropriate outcome in the case that n is not in the given $\Sigma_{4p+1} \wedge \Pi_{4p+1}$ set. We need the following definition.

Definition 10.7.2. Let $\{B_i\}_{i\in\omega}$ be a sequence of Boolean algebras. We define $\prod^{\omega}_{i\in\omega} B_i$, the *weak product* of $\{B_i\}_{i\in\omega}$, to be the Boolean algebra with domain the set of infinite strings $\bar{b} = (b_0, b_1, ...)$ such that $\forall i(b_i \in B_i)$ and for some i_0, either $\forall j \geqslant i_0(b_j = 0)$ or $\forall j \geqslant i_0(b_j = 1)$. The operations and constants of $\prod^{\omega}_{i\in\omega} B_i$ are defined coordinatewise in the obvious way, with $\overline{0} = (0_{B_0}, 0_{B_1}, \ldots)$, $\overline{1} = (1_{B_0}, 1_{B_1}, \ldots)$, and so forth.

Observation 10.7.3. We make two observations. One is that

$$\prod^{\omega}_{i\in\omega} B_i \;\cong B_0 \;\times \prod^{\omega}_{i\in\omega, i>0} B_i \;\cong\; B_0 \times B_1 \times ... \times B_n \times \prod^{\omega}_{i\in\omega, i>n} B_i$$

The second one is that

$$\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle} \cong D_{\langle p+1,1,0\rangle}.$$

PROOF: The first observation is clear. To see the second, we will show that $\mathrm{inv}(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}) = \langle p+1,1,0\rangle$, and that $\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}$ is dense. Note that if $\overline{1} = \overline{x} \vee \overline{y}$, then we may assume without loss of generality that there exists i_0 such that $\forall j \geqslant i_0(x_j = 1)$, as either $\overline{x}$ or $\overline{y}$ must have this form. Now since $[1]_p$ is neither atomic nor atomless in $D^{[p]}_{\langle p,\omega 1\rangle}$, $[1]_p$ is neither atomic nor atomless in $(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle})^{[p]}$. Hence $\mathrm{inv}_1(\overline{1}) > p$. Now if $\overline{b} \in \prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}$ is such that $\exists i_0 \forall j > i_0(b_j = 0)$, then $\overline{b} \in \mathcal{I}_{p+1}(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle})$. If $\overline{b}$ is such that $\exists i_0 \forall j > i_0(b_j = 1)$, then $\overline{1} \triangle \overline{b} \in \mathcal{I}_p(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle})$. Thus $[\overline{1}]_p$ is an atom in $(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle})^{[p]}$, and hence $\mathrm{inv}(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}) = \langle p+1,1,0\rangle$. For denseness, let $\overline{b} \in \prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}$. If $\overline{b} = (b_0, \ldots, b_{i_0}, 0, 0, \ldots)$ then

$$(\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}) \restriction \overline{b} \cong D_{\langle p,\omega,1\rangle} \restriction b_0 \times \cdots \times D_{\langle p,\omega,1\rangle} \restriction b_{i_0},$$

which is dense by Corollary 10.4.8. The denseness condition for b follows. If $\overline{b} = (b_0, \ldots, b_{i_0}, 1, 1, \ldots)$ then, by the first observation,

$$\mathrm{inv}(\overline{b}) = \mathrm{inv}^{D_{\langle p,\omega,1\rangle}}(b_0) + \cdots + \mathrm{inv}^{D_{\langle p,\omega,1\rangle}}(b_{i_0}) + \mathrm{inv}^{\prod^{\omega}_{i\in\omega} D_{\langle p,\omega,1\rangle}}(\overline{1}) = \langle p+1,1,0\rangle.$$

Note that $(0,\ldots,0,1,0,\ldots) < \bar{b}$ and has invariant $\langle p,\omega,1\rangle$. So the denseness condition for $\bar{b}$ follows from denseness below $(0,...,0,1,0,...)$ and the fact that below $(0,\ldots,0,1,0,\ldots)$ there is an element of invariant $\langle p,\omega,0\rangle$. $\square$

Lemma 10.7.4. *For every $p<\omega$, $\mathcal{DB}_{\langle p,2,0\rangle}$ is $(\Sigma_{4p+1}\wedge\Pi_{4p+1})$-hard. Moreover, the reductions proving hardness produce, in the case that n is not in the $\Sigma_{4p+1}\wedge\Pi_{4p+1}$ set, an index in $\mathcal{DB}_{\langle p,\bar{q},0\rangle}$ as required in Theorem 10.2.10.*

Proof: Consider two Σ_{4p+1} formulas $\phi(n)$ and $\psi(n)$. We want to construct a computable function f such that for every n

$$\phi(n)\ \&\ \neg\psi(n) \Leftrightarrow f(n)\in\mathcal{DB}_{\langle p,2,0\rangle}.$$

We start by finding a Π_{4p} formula $\hat{\phi}(n,x)$ such that $\phi(n)\Leftrightarrow\exists x\hat{\phi}(n,x)$, and such that if $\phi(n)$, then there is at most one x such that $\hat{\phi}(n,x)$. Since $\phi\in\Sigma_{4p+1}$, $\phi(n)=\exists x\forall w\bar{\phi}(n,x,w)$ for some $\bar{\phi}\in\Sigma_{4p-1}$. Let $\hat{\phi}(n,x)$ be the formula

$$x=\langle y,z\rangle\ \&\ \forall w\bar{\phi}(n,y,w)\ \&\ \forall y'<y\exists w\leqslant z(\neg\bar{\phi}(n,y',w))\ \& \\ \forall z'<z\exists y'\leqslant y\forall w\leqslant z'\bar{\phi}(n,y',w) \quad (10.7.1)$$

which has the desired properties. Indeed, it is clear that if $\exists x\hat{\phi}(n,x)$ then $\phi(n)$. Now suppose $\phi(n)$ holds. So $\exists x\forall w\bar{\phi}(n,x,w)$. Choose y least such that $\forall w\bar{\phi}(n,y,w)$. Then for each $y'<y$ there is a minimal w such that $\neg\bar{\phi}(n,y',w)$. Let z be the maximum of these w. Then $\hat{\phi}(n,\langle y,z\rangle)$ holds. Suppose also $\hat{\phi}(n,\langle\tilde{y},\tilde{z}\rangle)$. Then the third condition in (10.7.1) gives $y=\tilde{y}$ and the fourth condition gives $z=\tilde{z}$.

We also define $\hat{\psi}(n,x)$ to be a Π_{4p} formula such that for all n, $\psi(n)\Leftrightarrow\exists x\hat{\psi}(n,x)$, but if $\psi(n)$, then there are exactly two x such that $\hat{\psi}(n,x)$. We define $\hat{\psi}$ as we did with $\hat{\phi}$ but replace "$x=\langle y,z\rangle$" with "$x=\langle y,z\rangle\vee x=\langle y,z\rangle+1$".

Let g be a computable function such that $\forall n,x(\hat{\phi}(n,x)\Rightarrow g(n,x)\in\mathcal{DB}_{\langle p,1,0\rangle})$ and $\forall n,x(\neg\hat{\phi}(n,x)\Rightarrow g(n,x)\in\mathcal{DB}_{\langle p-1,\omega,1\rangle})$. Such a g exists by Corollary 10.6.3. Let h do the same with $\hat{\psi}$. Think of $g(n,x)$ and $h(n,x)$ as computable dense Boolean algebras, rather than as indices for such. For each n and x let $B_{n,x}$ be $g(n,\frac{x}{2})$ if x is even and $h(n,\frac{x-1}{2})$ if x is odd. Let $f(n)=\prod_{x\in\omega}^{\omega}B_{n,x}$. If $\phi(n)$ & $\neg\psi(n)$, then there is exactly one x such that $\hat{\phi}(n,x)$, so along the even components of $f(n)$ there is one copy of $D_{\langle p,1,0\rangle}$ with all others $D_{\langle p-1,\omega,1\rangle}$. As $\forall x(\neg\hat{\psi}(n,x))$, along the odd components there are copies of $D_{\langle p-1,\omega,1\rangle}$. Hence by the two observations about the product,

$$\begin{aligned}\mathrm{inv}(f(n)) &= \mathrm{inv}(D_{\langle p-1,\omega,1\rangle}\times\ldots\times D_{\langle p-1,\omega,1\rangle}\times D_{\langle p,1,0\rangle}\times\prod_{x\in\omega}^{\omega}D_{\langle p-1,\omega,1\rangle}) \\ &= \langle p-1,\omega,1\rangle+\cdots+\langle p-1,\omega,1\rangle+\langle p,1,0\rangle+\langle p,1,0\rangle \\ &= \langle p,2,0\rangle.\end{aligned}$$

Moreover, the resulting product is dense by Observation 10.7.3 and Corollary 10.4.8. Similarly, if $\neg\phi(n)$ & $\neg\psi(n)$, then we get only copies of $D_{\langle p-1,\omega,1\rangle}$, so $\mathrm{inv}(f(n)) = \langle p,1,0\rangle$. If $\phi(n)$ & $\psi(n)$, then we get three copies of $D_{\langle p,1,0\rangle}$, the rest $D_{\langle p-1,\omega,1\rangle}$, so $\mathrm{inv}(f(n)) = \langle p,4,0\rangle$, and if $\phi(n)$ & $\psi(n)$, then we get two copies of $D_{\langle p,1,0\rangle}$, the rest $D_{\langle p-1,\omega,1\rangle}$, so $\mathrm{inv}(f(n)) = \langle p,3,0\rangle$. Again, in every case the resulting algebra is dense by Observation 10.7.3 and Corollary 10.4.8. Thus f has the desired properties. □

10.8 The $\Pi_{\omega+1}$ case (Theorem 10.2.8)

We first prove that $\mathcal{B}_\omega = \mathcal{B}_{\langle\omega,0,0\rangle}$ is $\Pi^0_{\omega+1}$-hard. As in the previous section we will need to define some operations on Boolean Algebras.

In 10.8.2 we will define a binary operation, $*$, on presentations of Boolean algebras that corresponds, via the Interval Algebra operator, to the usual product on linear orderings. The only properties we will use of $*$ are the following.

Proposition 10.8.1. *Let B_0 and B_1 be Boolean algebras.*

1. *If $\mathrm{inv}(B_0) = \langle p,1,0\rangle$ and $\mathrm{inv}(B_1) = \langle p_1,q_1,r_1\rangle$, then $\mathrm{inv}(B_0 * B_1) = \langle p+p_1,q_1,r_1\rangle$.*
2. *If $\mathrm{inv}(B_0) = \langle p,\omega,0\rangle$, then $\mathrm{inv}(B_0 * B_1) = \langle p,\omega,0\rangle$.*

Moreover,

$$D_{\langle p,1,0\rangle} * D_{\langle p_1,q_1,r_1\rangle} \cong D_{\langle p+p_1,q_1,r_1\rangle} \quad \textit{and} \quad D_{\langle p,\omega,0\rangle} * B_1 \cong D_{\langle p,\omega,0\rangle}.$$

We will prove Proposition 10.8.1 in subsection 10.8.2, but use it now to prove Theorem 10.2.8. We will also make use of the following uniform version of Theorem 10.5.6.

Proposition 10.8.2. *([AK00, 18.9]) For each k, let A_k and B_k be structures such that $A_k \leqslant_k B_k$ and $\{A_k, B_k\}$ is k-friendly, and let S_k be a Σ^0_k set, all uniformly in k. If $f(n,k)$ is a computable function then there is a uniformly computable sequence $\{C_{n,k}\}_{n\in\omega,k\in\omega}$ such that*

$$C_{n,k} \cong \begin{cases} A_k & \textit{if } f(n,k) \in S_k \\ B_k & \textit{otherwise.} \end{cases}$$

Theorem 10.8.3. *$\mathcal{B}_\omega$ is $\Pi^0_{\omega+1}$-hard.*

PROOF: Suppose $S \in \Pi_{\omega+1}$, and f is a computable function such that

$$n \in S \Leftrightarrow \forall j (f(n,j) \notin 0^{(j)}).$$

We begin with a uniformly computable sequence $\langle A_{n,k} : n,k \in \omega\rangle$ of dense Boolean algebras such that

- $f(n,k) \in 0^{(k)} \Rightarrow A_{n,k} = D_{\langle k,\omega,0\rangle}$, and
- $f(n,k) \notin 0^{(k)} \Rightarrow A_{n,k} = D_{\langle k,1,0\rangle}$

Such a sequence exists by Proposition 10.8.2, Theorem 10.6.1 and the comment after Corollary 10.6.2.

Now define $K_{n,j}$ by recursion: $K_{n,1} = A_{n,1}$ and $K_{n,j+1} = K_{n,j} * A_{n,j+1}$. Let $K_n = \prod_{j\in\omega}^{\omega} K_{n,j}$. Let us next compute $\mathrm{inv}(K_n)$. First suppose that $n \in S$. Then, for every k, $\mathrm{inv}(A_{n,k}) = \langle k,1,0\rangle$, and then by Proposition 10.8.1

$$\mathrm{inv}(K_{n,j}) = \mathrm{inv}(A_{n,1}) + \cdots + \mathrm{inv}(A_{n,j}) = \langle 1+2+\cdots+j,1,0\rangle = \langle \frac{j(j+1)}{2},1,0\rangle.$$

Therefore $\mathrm{inv}_1(K_n) \geqslant \mathrm{inv}_1(K_{n,j}) = \frac{j(j+1)}{2}$ for every j. So, $\mathrm{inv}(K_n) = \langle \omega,0,0\rangle$. On the other hand, if $n \notin S$ there is a first j_0 such that $f(n,j_0) \in 0^{(j_0)}$. Then, again by Proposition 10.8.1,

$$\mathrm{inv}(K_{n,j_0}) = \mathrm{inv}(A_{n,1}) + \cdots + \mathrm{inv}(A_{n,j_0-1}) + \mathrm{inv}(A_{n,j_0}) = \langle \frac{j_0(j_0+1)}{2},\omega,0\rangle,$$

and for $j \geqslant j_0$, $\mathrm{inv}(K_{n,j})$ is constant and equal to $\langle \frac{j_0(j_0+1)}{2},\omega,0\rangle$. Therefore, for every j, $K_{n,j}^{[\frac{j_0(j_0+1)}{2}]}$ is atomic. It is not hard to see that then $K_n^{[\frac{j_0(j_0+1)}{2}]}$ is also atomic, and hence $\mathrm{inv}(K_n) = \langle \frac{j_0(j_0+1)}{2},\omega,0\rangle$. □

An interesting corollary is the following one about the complexity of deciding whether two Boolean algebras are elementarily equivalent. White [Whi00, 6.2.4] showed that for arbitrary structures this problem is as complicated as it can be. We prove the same when the structures are restricted to be Boolean algebras. Let $EE(BA)$ be the set of pairs $\langle i,j\rangle$ such that the computable Boolean algebras with indices i and j are elementarily equivalent. It clear that $EE(BA)$ is $\Pi^0_{\omega+1}$ because

$$\langle i,j\rangle \in EE(BA) \Leftrightarrow \forall\varphi \in \mathcal{L}^{BA}(B_i \models \varphi \Leftrightarrow B_j \models \varphi),$$

(where B_i and B_j are the computable Boolean algebras with indices i and j respectively and $\mathcal{L}^{BA}$ is the first order language of Boolean Algebras) and $0^{(\omega)}$ can tell whether $B_i \models \varphi$ uniformly in i and φ.

Corollary 10.8.4. *$EE(BA)$ is $\Pi^0_{\omega+1}$ complete.*

Proof: We already showed that $EE(BA)$ is in $\Pi^0_{\omega+1}$. We have to show that $EE(BA)$ is $\Pi^0_{\omega+1}$-hard. Consider $S \in \Pi_{\omega+1}$. Let K_n be as in the proof of the theorem above and let k_n be a computable index for K_n. Let d_ω be a computable index for $D_{\langle\omega,0,0\rangle}$. Then

$$n \in S \Leftrightarrow \mathrm{inv}(K_n) = \langle\omega,0,0\rangle \Leftrightarrow \langle d_\omega,k_n\rangle \in EE(BA).$$

□

10.8.1 $(\Sigma^0_{\omega+1}, \Pi^0_{\omega+1}) \leqslant_{\mathtt{m}} (\mathcal{DB}_{\langle\bar{\omega},\omega,0\rangle}, \mathcal{DB}_{\langle\omega,0,0\rangle})$

We now complete the proof of Theorem 10.2.10. We verify the last line of the table by improving the proof of Theorem 10.8.3 in which we showed that, given $S \in \Pi_{\omega+1}$, there are Boolean algebras K_n such that $n \in S \Leftrightarrow \mathrm{inv}_1(K_n) = \omega$. The $K_{n,j}$ as defined in the proof of Theorem 10.8.3 are dense because of Proposition 10.8.1. But when $n \in S$, K_n is not dense. We slightly modify the definition of K_n to make it dense.

Proposition 10.8.5. $(\Sigma^0_{\omega+1}, \Pi^0_{\omega+1}) \leqslant_{\mathtt{m}} (\mathcal{DB}_{\langle\bar{\omega},\omega,0\rangle}, \mathcal{DB}_{\langle\omega,0,0\rangle})$.

Proof: Let S and $K_{n,j}$ be as in the proof of Theorem 10.8.3. Now, instead of taking a product over ω, we define a componentwise product over $2^{<\omega}$. For $\sigma \in 2^{<\omega}$ let $K_{n,\sigma} = K_{n,|\sigma|}$. Let

$$\widetilde{K}_n = \prod_{\sigma \in 2^{<\omega}} K_{n,\sigma}$$

where $\prod_{\sigma\in 2^{<\omega}} B_\sigma$ is the set of $\langle b_\sigma : \sigma \in 2^{<\omega}\rangle \in \Pi_{\sigma\in 2^{<\omega}} B_\sigma$ such that for some n_0 we have that for every $\sigma \in 2^{n_0}$ either $\forall\tau \supseteq \sigma(b_\tau = 0)$ or $\forall\tau \supseteq \sigma(b_\tau = 1)$. The operations and constants for $\prod_{\sigma\in 2^{<\omega}} B_\sigma$ are defined componentwise.

As in the proof of Theorem 10.8.3, if $n \notin S$, then $\mathrm{inv}(\widetilde{K}_n) = \langle k, \omega, 0\rangle$ for some $k < \omega$, and if $n \in S$ then $\mathrm{inv}(\widetilde{K}_n) = \langle \omega, 0, 0\rangle$. If $n \notin S$, then denseness follows immediately from componentwise denseness as in Observation 10.7.3. Suppose $n \in S$, and $b \in \widetilde{K}_n$. Then, for each σ, $\mathrm{inv}(K_{n,\sigma}) = \langle \frac{|\sigma|(|\sigma|+1)}{2}, 1, 0\rangle$, and hence, as in the proof of Theorem 10.8.3, if $\mathrm{inv}_1(b) < \omega$ then for some n_0, for every $\sigma \in 2^{n_0}$, $\forall\tau \supseteq \sigma(b_\tau = 0)$, and if $\mathrm{inv}_1(b) = \omega$ then for some σ, $\forall\tau \supseteq \sigma(b_\tau = 1)$. If $\mathrm{inv}_1(b) < \omega$, then the denseness conditions for b are satisfied as in Observation 10.7.3. Suppose $\mathrm{inv}(b) = \langle \omega, 0, 0\rangle$. Then, there is some $\sigma \in 2^{<\omega}$ such that $\forall\tau \supseteq \sigma(b_\tau = 1)$. Now consider a defined by $\forall\tau \not\supseteq \sigma(a_\tau = b_\tau)$, $\forall\tau \supseteq \sigma^\frown 0(a_\tau = 0)$, and $\forall\tau \supseteq \sigma^\frown 1(a_\tau = 1)$. Observe that $a \leqslant b$ and $\mathrm{inv}(a) = \mathrm{inv}(b - a) = \langle \omega, 0, 0\rangle$ as desired to prove the denseness condition for b. $\square$

10.8.2 Interval Algebras and the $*$ operation

In this subsection we will show how to obtain a Boolean algebra from a linear ordering and vice versa. This will allow us to use operations on linear orderings on the corresponding Boolean algebras. We refer the reader to Monk [Mon89, I.6.15] and Goncharov [Gon97, 1.6 and 3.2] for general information on interval algebras. The goal of this section is to define a computable operator $*$ satisfying Proposition 10.8.1.

Definition 10.8.6. If L is a linear ordering with a first element, $\mathrm{IntAlg}(L)$ is the Boolean algebra of finite unions of half open intervals $[a, b)$ of L where b can be ∞. (The understanding here is that $[a, \infty) = \{x : x \geqslant a\}$.)

It is clear that if L is computable then so is $\mathrm{IntAlg}(L)$. The converse is also true:

Lemma 10.8.7. *[Gon97, 3.2.22] There is a computable operator* lin *that, given a countable Boolean algebra B, returns a linear ordering* $\mathrm{lin}(B)$ *such that* $\mathcal{B} \cong \mathrm{IntAlg}(\mathrm{lin}(B))$.

Definition 10.8.8. The product of linear orderings, $L_0 \cdot L_1$, is gotten by replacing each element of L_1 by a copy of L_0 (and so, it is the ordering on pairs $\langle x_1, y_1\rangle \in L_0 \times L_1$ given by $\langle x_1, y_1\rangle < \langle x_2, y_2\rangle \Leftrightarrow y_1 < y_2 \vee (y_1 = y_2 \;\&\; x_1 < x_2)$).

Given two Boolean algebras B_0 and B_1 we let

$$B_0 * B_1 = \mathrm{IntAlg}(\mathrm{lin}(B_0) \cdot \mathrm{lin}(\mathcal{B}_1)).$$

Note that $B_0 * B_1$ depends on the presentations of B_0 and B_1.

Now we show how to describe the analysis of the Tarski invariants of $\mathrm{IntAlg}(L)$ in terms of L.

Definition 10.8.9. A subset S of L is *convex* if $x, y \in S$ and $x < z < y$ implies that $z \in S$. An equivalence relation $\sim$ on L is *convex* if every one of its equivalence classes is convex.

Proposition 10.8.10. *[Gon97, 1.6,3.2][Mon89, I.6.15]There is a one-one correspondence between ideals I of* $\mathrm{IntAlg}(L)$ *and convex equivalence relations $\sim$ on L such that* $\mathrm{IntAlg}(L)/I \cong \mathrm{IntAlg}(L/\sim)$. *Here $L/\sim$ is the linear ordering of equivalence classes $[x]$, $[y]$ of $\sim$ given by $[x] < [y] \Leftrightarrow \forall w \sim x \forall z \sim y(w < z)$. The convention here is that if a final segment of L is collapsed to a single equivalence class, then it is removed from $L/\sim$ and its role is taken by ∞. For a given ideal I, the corresponding equivalence relation $\sim$ is given by $x \sim y \Leftrightarrow [x, y) \in I$ for $x \leqslant y \in L$.*

Definition 10.8.11. We denote $L/\sim_T$ by $L^{[1]}$ where $\sim_T$ is the equivalence relation corresponding to $\mathcal{I}$ and so

$$\mathrm{IntAlg}(L^{[1]}) \cong \mathrm{IntAlg}(L)/\mathcal{I}(\mathrm{IntAlg}(L)) = \mathrm{IntAlg}(L)^{[1]}$$

The following lemma is key for the proof of Proposition 10.8.1. The sum over M, $\sum_{i\in M} L_i$, of linear orderings L_i, $i \in M$, is gotten by replacing each element i of M by a copy of L_i. Observe that when for every i, $L_i \cong L$ we have that $\sum_{i\in M} L_i \cong L \cdot M$.

Lemma 10.8.12. *[Sho04, 5.8] If, for every $i \in \omega$,* $\mathrm{inv}_1(L_i) \geqslant 1$ *for every L_i and $L = \sum_{i\in M} L_i$ then $L^{[1]} = \Sigma_{i\in M} L_i^{[1]}$.*

Corollary 10.8.13. *If* $\mathrm{inv}_1(K) \geqslant 1$ *then* $(K \cdot M)^{[1]} = K^{[1]} \cdot M$.

Lemma 10.8.14. *Let B_0 and B_1 be Boolean algebras.*

1. *If* B_0 *is the trivial Boolean algebra, i.e.* $\mathrm{inv}(B_0) = \langle 0, 1, 0\rangle$, $B_0 * B_1 \cong B_1$.

2. *If* B_0 *is atomic and has infinitely many atoms, then* $B_0 * B_1$ *is atomic and* $\mathrm{inv}(B_0 * B_1) = \langle 0, \omega, 0\rangle$.

3. *If* $\mathrm{inv}(B_0) = \langle p, 1, 0\rangle$ *and* $\mathrm{inv}(B_1) = \langle p_1, q_1, r_1\rangle$, *then* $\mathrm{inv}(B_0 * B_1) = \langle p + p_1, q_1, r_1\rangle$.

4. *If* $\mathrm{inv}(B_0) = \langle p, \omega, 0\rangle$, *then* $\mathrm{inv}(B_0 * B_1) = \langle p, \omega, 0\rangle$.

Proof: For (1), if $\mathrm{inv}(B_0) = \langle 0, 1, 0\rangle$, then $\mathrm{lin}(B_0) \cong \mathbf{1}$. Hence $\mathrm{lin}(B_0) \cdot \mathrm{lin}(B_1) \cong \mathrm{lin}(B_1)$, and so $B_0 * B_1 \cong B_1$.

For (2), consider a non-zero $[x, y) \subseteq \mathrm{lin}(B_0) \cdot \mathrm{lin}(B_1)$. There is some non-zero $[x_0, y_0) \subseteq [x, y)$ with $[x_0, y_0)$ contained in a copy of $\mathrm{lin}(B_0)$. As B_0 is atomic, there is an atom below $[x_0, y_0)$, and hence below $[x, y)$. Thus $B_0 * B_1$ is atomic. Since B_0 has infinitely many atoms, so does $B_0 * B_1$, hence $\mathrm{inv}(B_0 * B_1) = \langle 0, \omega, 0\rangle$.

For parts (3) and (4) we first make a general observation. If $\mathrm{inv}_1(B_0) = p$, then

$$\begin{aligned}
(B_0 * B_1)^{[p]} &= \mathrm{IntAlg}(\mathrm{lin}(B_0) \cdot \mathrm{lin}(B_1))^{[p]} \\
&\cong \mathrm{IntAlg}((\mathrm{lin}(B_0) \cdot \mathrm{lin}(B_1))^{[p]}) \\
&= \mathrm{IntAlg}((\mathrm{lin}(B_0)^{[1]} \cdot \mathrm{lin}(B_1))^{[p-1]}) && \text{(by Corollary 10.8.13)} \\
&= \mathrm{IntAlg}((\mathrm{lin}(B_0)^{[2]} \cdot \mathrm{lin}(B_1))^{[p-2]}) && \text{(again by Corollary 10.8.13)} \\
&\;\;\vdots \\
&= \mathrm{IntAlg}(\mathrm{lin}(B_0)^{[p]} \cdot \mathrm{lin}(B_1))
\end{aligned}$$

For (3), we have that $\mathrm{lin}(B_0)^{[p]} = 1$, so $(B_0 * B_1)^{[p]} = \mathrm{IntAlg}(1 \cdot \mathrm{lin}(B_1)) \cong B_1$. Hence $\mathrm{inv}(B_0 * B_1) = \langle p + p_1, q_1, r_1\rangle$.

Finally, for (4), we have that $(B_0 * B_1)^{[p]} \cong \mathrm{IntAlg}(\mathrm{lin}(B_0)^{[p]} * \mathrm{lin}(B_1))$, and so, since $B_0^{[p]}$ is atomic and has infinitely many atoms, $\mathrm{lin}(B_0)^{[p]} * \mathrm{lin}(B_1)$ is also atomic and has infinitely many atoms as in part (2). The result follows. □

The first part of Proposition 10.8.1 follows from the lemma above. This first part was all we used in the proof of Theorem 10.8.3. We now prove the second part, used to prove Proposition 10.8.5.

Lemma 10.8.15. *1.* $D_{\langle p,1,0\rangle} * D_{\langle p_1,q_1,r_1\rangle} \cong D_{\langle p+p_1,q_1,r_1\rangle}$.

2. $D_{\langle p,\omega,0\rangle} * B \cong D_{\langle p,\omega,0\rangle}$.

Proof: We have seen, by Lemma 10.8.14, that the invariants are as claimed, so it remains to check denseness. Consider $B_0 * B_1$ where B_0 is dense, and an element of the interval algebra $b = [x, y)$ for which we want to verify the density conditions. If x and y belong to the same copy of $\mathrm{lin}(B_0)$ in the product $\mathrm{lin}(B_0) \cdot \mathrm{lin}(B_1)$, then we are done by the assumed density of B_0. If they are in adjacent copies of $\mathrm{lin}(B_0)$, then one of the two subintervals lying within single copies into which b can be decomposed is responsible for the hypothesis of the density condition holding and an application of density for that subinterval within its copy supplies the desired witness for density. Thus we may assume that there is a copy of $\mathrm{lin}(B_0)$ between x and y.

For (1), $B_0 = D_{\langle p,1,0\rangle}$, and $B_1 = D_{\langle p_1,q_1,r_1\rangle}$, so $\mathrm{lin}(B_0)^{[p]} = \mathbf{1}$ and so $(\mathrm{lin}(B_0)\cdot \mathrm{lin}(B_1))^{[p]} = \mathrm{lin}(B_1)$. We may assume that y is ∞ or the first element of some copy of $\mathrm{lin}(B_0)$. In either case, $\mathrm{inv}_1(b) \geqslant p$ and the image of b in $(\mathrm{lin}(B_0)\cdot \mathrm{lin}(B_1))^{[p]}$ is the interval of $\mathrm{lin}(B_1)$ corresponding to the copies of $\mathrm{lin}(B_0)$ starting with x and ending with y. We now take the witness for density in $\mathrm{lin}(B_1)$ and pull it back to $\mathrm{lin}(B_0)\cdot \mathrm{lin}(B_1)$.

For (2), $B_0 = D_{\langle p,\omega,0\rangle}$, and $B_1 = B$. So $\mathrm{lin}(B_0)^{[p]}$ is atomic and has infinitely many atoms. Thus $\mathrm{inv}(b) = \langle p, \omega, 0\rangle$ and the required witnesses for the first and second denseness conditions can be found within a copy of $\mathrm{lin}(B_0)$ contained in b. $\square$

Part IV

Miscellaneous

Chapter 11
Two results on effective randomness (*with* ***Barbara F. Csima and Bjørn Kjos-Hanssen***).

The first section of this chapter is joint work with Barbara F. Csima and will appear in the Proceedings of the American Mathematical Society. The second section is joint work with Bjørn Kjos-Hanssen.

The two sections in this chapter are completely independent. On the first one we show that there is a minimal pair of K-degrees, answering a question posed by Downey and Hirschfeldt. On the second section we show that there are non-continuously-random reals all the way up in the hyperarithmetic hierarchy. The question of what is the complexity of non-continuously-random reals was posed by Slaman at the annual ASL meeting in Stanford, in March 2005, where he presented a joint paper with Reimann [RS].

11.1 A minimal pair of K-degrees

11.1.1 Introduction and Notation

K-reducibility is defined with the intention of measuring the relative randomness of infinite binary strings, which we refer to as reals. This reducibility was defined using a function, K, that assigns to each finite binary string the length of its shortest description, in a sense we will specify. The idea being that if a string is random, there should not be any short way of describing it. The precise definition of K is given below, though the proofs presented in this paper use only the two properties of K listed at the end of this section.

The *prefix-free Kolomogorov complexity* of a string $\sigma \in 2^{<\omega}$ is defined to be the length of the shortest program $p \in 2^{<\omega}$ such that $U(p) = \sigma$, where U is a universal prefix-free Turing machine. That is, U is universal for machines V with the property that if $V(\tau)\downarrow$, then $V(\tau')\uparrow$ for all $\tau' \supset \tau$. We denote the Kolmogorov complexity of σ by $K(\sigma)$. This definition is independent of the choice of universal machine U, up to additive constant. The advantage of restricting to prefix-free machines is that otherwise the Kolmogorov complexity would contain extra information about the length of the string. For more background on Kolmogorov complexity, see Li and Vitányi [LV97], and Downey and Hirschfeldt [DH].

Prefix-free Kolmogorov complexity is used to define a notion of randomness for real numbers. A real $\gamma \in 2^\omega$ is K*-random* (or Levin-Gaćs-Chaitin random) if for all n, $K(\gamma \upharpoonright n) \geqslant n - \mathcal{O}(1)$. This notion has been extensively studied and coincides with other notions of randomness based on measure theory or unpredictability [DH], [DHNT]. We can also use K to define what it means for a real to be far from being random. We say a real is K*-trivial* if for all n, $K(\gamma \upharpoonright n) \leqslant K(n) + \mathcal{O}(1)$; that is, every initial segment is as simple as possible. But what of relative randomness of reals? K-reducibility was introduced to study notions of relative randomness.

For two reals α and β in 2^ω we let

$$\alpha \leqslant_K \beta \Leftrightarrow (\forall n)\ K(\alpha \upharpoonright n) \leqslant K(\beta \upharpoonright n) + \mathcal{O}(1),$$

i.e., if there exists a constant C such that $(\forall n)\ K(\alpha \upharpoonright n) \leqslant K(\beta \upharpoonright n) + C$. The K-degrees are defined as equivalence classes under this quasiordering.

As is usual when considering a reducibility, we want to understand the structure of the K-degrees. We know that the K-degrees have a bottom element that corresponds to the K-degree of the K-trivial reals. Yu, Ding, and Downey showed that there are uncountably many K-degrees, indeed $2^{\aleph_0}$ many among the K-random reals ([YDD04], see [DHNT]). When restricting attention to c.e. reals (reals with nice approximations), Downey, Hirschfeldt, and LaForte have shown density and existence of join [DHL04]. A result of Solovay is that K-reducibility does not imply Turing reducibility (see [DH]).

A natural question to ask when studying a reducibility is if there exists a minimal pair. Rod Downey and Denis Hirschfeldt asked this question for the K-degrees. That is, they asked whether there exist non-K-trivial reals α and β in 2^ω such that whenever $\gamma \in 2^\omega$ is such that $\gamma \leqslant_K \alpha$ and $\gamma \leqslant_K \beta$ then γ is K-trivial. Here we answer this question affirmatively with a simple and elegant construction of a minimal pair. We do it by first constructing a unbounded nondecreasing function f which forces K-triviality in the sense that a real γ is K-trivial if and only if $(\forall n)\ K(\gamma \upharpoonright n) \leqslant K(n) + f(n) + \mathcal{O}(1)$. This function will likely be useful in showing other results about K-reducibility.

If a real is K-trivial, then there is some constant which witnesses its K-triviality. We say a real γ is *K-trivial(C)* if for all n, $K(\gamma \upharpoonright n) \leqslant K(n) + C$, where $K(n) = K(0^n)$. Then, we have that γ is K-trivial if and only if it is K-trivial(C) for some C. We say that γ *appears to be K-trivial(C) at n* if for all $m \leqslant n$, $K(\gamma \upharpoonright m) \leqslant K(m) + C$. We say that γ *stops appearing K-trivial(C) at n* if it appears K-trivial(C) at $n-1$ but not at n. Throughout the paper, γ will always denote a real, i.e. $\gamma \in 2^\omega$.

The properties of K that we will use are.

Property 11.1.1 (Zambella, see [DHNS03]). *For every C, there are only finitely many reals that are K-trivial(C).*

Property 11.1.2. *For any $\sigma \in 2^{<\omega}$, $\sigma^\frown 0^\omega$ is K-trivial, and hence K-trivial(C) for some C.*

11.1.2 Construction of a minimal pair

Theorem 11.1.3. *There exists a minimal pair of K-degrees.*

To prove our theorem, we will use the following lemma, which is interesting in itself, and may have other applications.

Lemma 11.1.4. *There exists a unbounded nondecreasing function f such that for all reals $\gamma \in 2^\omega$, the following are equivalent.*

1. *γ is K-trivial.*

2. *For almost every n, $K(\gamma \upharpoonright n) \leqslant K(n) + f(n)$.*

3. *$(\forall n)$ $K(\gamma \upharpoonright n) \leqslant K(n) + f(n) + \mathcal{O}(1)$.*

Before proving Lemma 11.1.4, we show how Theorem 11.1.3 follows from it.

Proof of Theorem 11.1.3: Let f be as in Lemma 11.1.4. We will construct two non-K-trivial reals α and β such that $\min\{K(\alpha \upharpoonright n), K(\beta \upharpoonright n)\} \leqslant K(n)+f(n)$. This will give us a minimal pair because if $\gamma \leqslant_K \alpha$ and $\gamma \leqslant_K \beta$, then $K(\gamma \upharpoonright n) \leqslant K(n) + f(n) + \mathcal{O}(1)$, and hence γ is K-trivial.

We construct α and β as the limits of two sequences of finite strings, $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$, which satisfy that, for every s, $\alpha_s \subset \alpha_{s+1}$, $\beta_s \subset \beta_{s+1}$ and $|\alpha_s| = |\beta_s|$. We denote $|\alpha_s|$ by n_s. To get $\min\{K(\alpha \upharpoonright n), K(\beta \upharpoonright n)\} \leqslant K(n) + f(n)$, we ensure that if $n_s \leqslant n < n_{s+1}$, then $K(\alpha \upharpoonright n) \leqslant K(n) + f(n)$ if s is odd, and $K(\beta \upharpoonright n) \leqslant K(n)+f(n)$ if s is even. To make α and β non-K-trivial, we ensure that for every s there is some n, $n_s \leqslant n < n_{s+1}$, such that either $K(\alpha \upharpoonright n) > K(n) + s$, or $K(\beta \upharpoonright n) > K(n) + s$ depending or whether s is even or odd.

Construction of α and β: Stage 0: Let $\alpha_0 = \beta_0 = \emptyset$. Stage $s+1$: Suppose first that s is even. Let $\alpha'_{s+1} \supset \alpha_s$ be such that $K(\alpha'_{s+1}) \geqslant K(|\alpha'_{s+1}|) + s$. Such an α'_{s+1} must exist because not every extension of α_s is K-trivial$(s-1)$. Let C_{s+1} be such that $\alpha'_{s+1}{}^\frown 0^\omega$ is K-trivial(C_{s+1}). Choose $n_{s+1} > |\alpha'_{s+1}|$ such that $f(n_{s+1}) \geqslant C_{s+1}$. Finally, let $\alpha_{s+1} = \alpha'_{s+1}{}^\frown 0^\omega \upharpoonright n_{s+1}$ and $\beta_{s+1} = \beta_s{}^\frown 0^\omega \upharpoonright n_{s+1}$. If s is odd do the same as above but with roles of α and β reversed. $\Diamond$

It is clear from the construction that for s even there is some n, $n_s \leqslant n < n_{s+1}$, such that $K(\alpha \upharpoonright n) > K(n)+s$, namely $|\alpha'_{s+1}|$. Also, for every n, $n_{s+1} \leqslant n < n_{s+2}$,

$$\begin{aligned} K(\alpha \upharpoonright n) = K(\alpha_{s+2} \upharpoonright n) = K(\alpha_{s+1}{}^\frown 0^\omega \upharpoonright n) &= K(\alpha'_{s+1}{}^\frown 0^\omega \upharpoonright n) \\ &\leqslant K(n) + C_{s+1} \leqslant K(n) + f(n_{s+1}) \leqslant K(n) + f(n). \end{aligned}$$

Analogously for s odd. $\square$

Proof of Lemma 11.1.4: Clearly (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) for any unbounded nondecreasing function. We now show that (3) $\Rightarrow$ (1). That is, we construct an unbounded nondecreasing function f such that, for any real γ, if $(\forall n)$ $K(\gamma \upharpoonright n) \leqslant K(n) + f(n) + \mathcal{O}(1)$, then γ is K-trivial.

We first define an unbounded nondecreasing function f_0 such that $(\forall n)$ $K(\gamma \upharpoonright n) \leqslant K(n) + f_0(n)$ implies that γ is K-trivial(0). We do it by defining a sequence $n_0 < n_1 < n_2 < \cdots$, and letting $f_0(n) = k$ for every n such that $n_{k-1} < n \leqslant n_k$ (where $n_{-1} = -1$).

As there are only finitely many reals that are K-trivial(2), we can choose n_0 such that any γ that is K-trivial(2), but not K-trivial(0), has stopped appearing

K-trivial(0) by n_0. Suppose now that we have already defined n_k. Let n_{k+1} be such that any γ that is K-trivial($k+3$), but not K-trivial(0), has stopped appearing K-trivial(0) by n_{k+1}. We can do this because there are only finitely many reals that are K-trivial($k+3$). Except when $k=0$, we also require n_{k+1} to be such that any γ which stopped appearing K-trivial(0) at some m, $n_{k-1} < m \leqslant n_k$, does not appear to be K-trivial($k+1$) by n_{k+1}. Note that such n_{k+1} has to exist. Indeed, by definition of n_{k-1}, $\gamma \restriction m$ can have no K-trivial($k+1$) real extending it. So by König's Lemma, the tree of apparently K-trivial($k+1$) extensions of $\gamma \restriction m$ must be finite.

We claim that f_0 is as wanted. Suppose that γ is a real such that $(\forall n)\ K(\gamma \restriction n) \leqslant K(n) + f_0(n)$; we want to show that actually $(\forall n)\ K(\gamma \restriction n) \leqslant K(n)$. Clearly γ appears to be K-trivial(0) up to length n_0. Assume for a contradiction that γ is not K-trivial(0). Let $k > 0$ be least such that γ stops appearing K-trivial(0) at some m, $n_{k-1} < m \leqslant n_k$. Then by definition of n_{k+1}, γ stops appearing K-trivial($k+1$) by n_{k+1}. That means that there is some $m \leqslant n_{k+1}$ such that $K(\gamma \restriction m) \geqslant K(m) + k + 2 > K(m) + f_0(m)$, a contradiction.

There is nothing special about 0 in this proof. In the same way we can construct, for each i, a function f_i such that $f_i(0) = i$ and $(\forall n)\ K(\gamma \restriction n) \leqslant K(n) + f_i(n)$ implies that γ is K-trivial(i). Just choose n_0 such that any γ that is K-trivial($i+2$), but not K-trivial(i), has stopped appearing K-trivial(i) by n_0. Then given n_k, let n_{k+1} be such that any γ that is K-trivial($i+k+3$), but not K-trivial(i), has stopped appearing K-trivial(i) by n_{k+1}. For $k \neq 0$, also require n_{k+1} to be such that any γ which stopped appearing K-trivial(i) at some m, $n_{k-1} < m \leqslant n_k$, does not appear to be K-trivial($i+k+1$) by n_{k+1}. Let $f_i(n) = i + k$ for every n such that $n_{k-1} < n \leqslant n_k$.

For each $n \in \omega$, let $f(n) = \min\{f_{2i}(n) - i : i \in \omega\}$, which exists because $(\forall i, n) f_{2i}(n) - i \geqslant i$. Note that f is a nondecreasing function. It is also unbounded because for each j, if we let n be such that $(\forall i < j) f_{2i}(n) > 2j$, then $j \leqslant f(n)$. Now, suppose that γ is a real such that $(\forall n)\ K(\gamma \restriction n) \leqslant K(n) + f(n) + i$ for some i. Then $(\forall n)\ K(\gamma \restriction n) \leqslant K(n) + f_{2i}(n)$, and hence γ is K-trivial($2i$). So every γ such that $K(\gamma \restriction n) \leqslant K(n) + f(n) + \mathcal{O}(1)$ is K-trivial. □

11.2 Non-Continuously Random reals

Reimann and Slaman [RS] started to consider random reals with respect to measures other than the Lebesgue measure. They introduced the notion of continuously random reals, and proved that every real which is not continuously random is hyperarithmetic. We show here that there are reals which are not continuously random all the way up in the hyperarithmetic hierarchy .

Definition 11.2.1. (Reimann and Slaman [RS]) Let μ be a probability measure defined in the Borel sets of 2^ω.

1. A *Martin-Löf test for* μ is a sequence $\langle U_n \rangle_{n\in\mathbb{N}}$ of subsets of $2^{<\omega}$ which is

uniformly computable in μ and such that for each n,

$$\sum_{\sigma \in U_n} \mu([\sigma]) \leqslant 2^{-n}.$$

We use $[\sigma]$ to denote the basic open set $\{X \in 2^\omega : \sigma \subset X\}$.

2. A real $X \in 2^\omega$ is *random with respect to* μ, or simply μ*-random*, if for every Martin-Löf test, $\langle U_n \rangle_{n \in \mathbb{N}}$, for μ, we have that

$$X \notin \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in U_n} [\sigma].$$

3. We say that $X \in 2^\omega$ is *not continuously random* if for every continuous measure μ, X is not μ-random. (Recall that μ is continuous if for every $Y \in 2^\omega$, $\mu(\{Y\}) = 0$.)

4. Let NCR be the set of reals which are not continuously random.

Remark 11.2.2. Observe that we can code μ by a real number. The reason being that by Carathéodory's theorem, μ is determined by $\langle \mu([\sigma]) : \sigma \in 2^{<\omega} \rangle$. Other ways of representing μ are possible too (see [RS], where a different representation is used). So, when we say that $\langle U_n \rangle_{n \in \mathbb{N}}$ is computable in μ, we mean computable in the representation of μ.

Lemma 11.2.3. *Let μ be a continuous probability measure on 2^ω and let let $[T] \subset 2^\omega$ be a countable Π^0_1-class, where $T \subset 2^{<\omega}$ is a computable tree. There exists a Martin-Löf test, $\langle U_n \rangle_{n \in \mathbb{N}}$, for μ such that*

$$[T] \subseteq \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in U_n} [\sigma].$$

PROOF: For each $n \in \omega$, let $U_n = \{\sigma : \sigma \in 2^n \cap T\}$ and let $V_n = \bigcup_{\sigma \in U_n} [\sigma] \subseteq 2^\omega$. Observe that $\bigcap_{n \in \omega} V_n = [T]$. Since μ is continuous and $[T]$ countable, $\mu([T]) = 0$. So, $lim_n \mu(V_n) = 0$. For each $m \in \omega$, let n_m be the least n such that $\mu(V_n) \leqslant 2^{-m}$. Note that n_m can be uniformly computed from μ. Now, $\{U_{n_m} : m \in \mathbb{N}\}$ is a Martin-Löf test for μ as desired. □

In [CCS+86], Cenzer, Clote, Smith, Soare and Wainer studied countable Π^0_1-classes and their members. They assigned a rank below ω_1^{CK} to each member of a countable Π^0_1-class and they made the following definition.

Definition 11.2.4. [CCS+86] A real $X \in 2^\omega$ is ranked if it belongs so some countable Π^0_1-class.

If follows from the lemma above that every ranked real is in NCR. They also proved that for every $\alpha < \omega_1^{CK}$ there exists a ranked real Turing equivalent to $0^{(\alpha)}$. So, we get the following corollary.

Corollary 11.2.5. *For every hypearithmetic real Z, there exists an $X \in NCR$ such that $Z \leqslant_T X$. Moreover, for every computable ordinal α, there is a real $X \in NCR$ which is Turing equivalent to $0^{(\alpha)}$.*

Chapter 12
Invariants for scattered linear orderings up to equimorphisms

12.1 Introduction

We say that a linear ordering is *scattered* if the order type of the rationals does not embed in it. People have been interested in this class of linear ordering for a long time. One of the earliest results is the following, first proved by Hausdorff [Hau08], and rediscovered by Erdös and Hajnal [EH63].

Theorem 12.1.1 (Hausdorff). *Let $\mathbb{S}$ be the smallest class of linear orderings such that*

- $\mathbf{1} \in \mathbb{S}$*;*
- *if $\mathcal{A}, \mathcal{B} \in \mathbb{S}$, then $\mathcal{A} + \mathcal{B} \in \mathbb{S}$; and*
- *if κ is a regular cardinal and $\{\mathcal{A}_\gamma : \gamma \in \kappa\} \subseteq \mathbb{S}$, then both $\sum_{\gamma\in\kappa} \mathcal{A}_i$ and $\sum_{\gamma\in\kappa^*} \mathcal{A}_i$ belong to $\mathbb{S}$.*

Then $\mathbb{S}$ is the class of scattered linear orderings. (The notation used is explained in the background section below.)

Another important contribution of Hausdorff to the study of scattered linear orderings is the definition of the Hausdorff rank (see [Ros82, Chapter 5]). He first defined an operation on linear orderings which is similar to the Cantor-Bendixon derivative on topological spaces: Given a linear ordering $\mathcal{L}$, let $\mathcal{L}'$ be the linear ordering obtained by collapsing the elements which have only finitely many elements between them). Informally, the Hausdorff rank of $\mathcal{L}$ is the least ordinal α such that the αth iterate of this operation on $\mathcal{L}$ is finite. Here is the definition we will use.

Definition 12.1.2. Given a linear ordering $\mathcal{L}$ and an ordinal α, we define an equivalence relation $\approx_\alpha$ on $\mathcal{L}$ by transfinite induction as follows. Let $\approx_0$ be the identity relation. For $x, y \in \mathcal{L}$, let $x \approx_\alpha y$ if and only if for some $\beta < \alpha$, there are only finitely many $\approx_\beta$-equivalence classes between x and y. Let $\mathcal{L}^{(\alpha)}$ be the linear ordering which consists of the $\approx_\alpha$-equivalence classes ordered in the obvious way. We let the *Hausdorff rank* of $\mathcal{L}$, $\mathrm{rk}(\mathcal{L})$, be the least ordinal α such that $\mathcal{L}^{(\alpha)}$ is finite. If no such an α exists, we let $\mathrm{rk}(\mathcal{L}) = \infty$. We will usually omit the word Hausdorff and just refer to the rank of a linear ordering.

Hausdorff proved that a linear ordering is scattered if and only if $\mathrm{rk}(\mathcal{L}) \neq \infty$.

The definition above is slightly different from some other definitions of Hausdorff rank found in the literature, but is essentially the same. We prefer it to other definitions because it satisfies the following three properties. Let $\mathcal{A}$ and $\mathcal{B}$ be linear orderings. Then

1. if $\mathcal{A}$ embeds in $\mathcal{B}$, then $\mathrm{rk}(\mathcal{A}) \leqslant \mathrm{rk}(\mathcal{B})$;
2. $\mathrm{rk}(\mathcal{A}+\mathcal{B}) = \max(\mathrm{rk}(\mathcal{A}), \mathrm{rk}(\mathcal{B}))$;
3. $\mathrm{rk}(\mathcal{A}\cdot\mathcal{B}) = \mathrm{rk}(\mathcal{A}) + \mathrm{rk}(\mathcal{B})$.

After Hausdorff's results, the following important structural result about the class of scattered linear orderings was conjectured by Fraïsé in [Fra48]. It was proved by Richard Laver twenty three years later.

Theorem 12.1.3. *[Lav71] The scattered linear orderings are well-quasiordered by the relation of embeddablity.*

(A *well-quasiordering* is a quasiordering which has no infinite descending sequences and no infinite antichains.)

Moreover, Laver proved that the class of scattered linear orderings is a better-quasiordering. Better-quasiorderings are a particular case of well-quasiorderings introduced by Nash-Williams in [NW68]. Then, for example, using Nash-Williams' theorem on transfinite sequences [NW68], we get that the class of ideals of scattered linear orderings (i.e., downwards closed sets of linear orderings), ordered by the inclusion relation, is well-quasiordered too.

To prove Laver's result, indecomposable linear orderings play a very important role. A linear ordering $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} \preccurlyeq \mathcal{A}+\mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq \mathcal{B}$. Along with the theorem above, Laver proved some structural results about the class of σ-scattered linear orderings (see Definition 12.5.2). When we restrict these results to the class of scattered linear orderings we obtain the following theorem.

Theorem 12.1.4. *[Lav71]*

1. *Every scattered linear ordering can be written as a finite sum of indecomposable linear orderings.*
2. *Every indecomposable linear orderings is either a κ-sum or a κ^*-sum of indecomposable linear orderings of smaller rank, where κ is some regular cardinal.*

When a linear ordering $\mathcal{A}$ can be embedded into another linear ordering $\mathcal{B}$, we write $\mathcal{A} \preccurlyeq \mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic* if $\mathcal{A} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{A}$. If this is the case, we write $\mathcal{A} \sim \mathcal{B}$. Everything mentioned so far about scattered linear ordering is not really about isomorphism types of linear orderings, but actually about equimorphism types. The properties of being scattered, being indecomposable, and having a certain rank are preserved under equimorphisms. Also, the operation of taking finite sums, products and κ-sums are well-defined on equimorphism types.

In this paper we are interested in the structure of equimorphism types of scattered linear orderings. We use Laver's work and assign to each scattered linear ordering $\mathcal{L}$ a finite sequence $\mathtt{Inv}(\mathcal{L})$ of finite trees labeled by ordinals and signs in $\{-,+\}$. This assignment is an equimorphism invariant, that is, given scattered linear orderings $\mathcal{A}$ and $\mathcal{B}$, we have that

$$\mathcal{A} \sim \mathcal{B} \quad \Leftrightarrow \quad \mathtt{Inv}(\mathcal{A}) = \mathtt{Inv}(\mathcal{B}).$$

Let $\mathbb{S}$ denote the class of equimorphism types of scattered linear orderings and $\mathbb{H}$ the class of equimorphism types of scattered indecomposable linear orderings. From now on, indecomposable means scattered and indecomposable linear ordering, unless otherwise stated. Let $\mathbb{H}_\alpha = \{\mathcal{L} \in \mathbb{H} : \mathrm{rk}(\mathcal{L}) < \alpha\}$.

Jullien [Jul69, Theorem IV.6.2] proved the following. Let $\mathcal{L}$ be a scattered linear ordering and let $\langle \mathcal{A}_0, ..., \mathcal{A}_{n-1}\rangle$ be a sequence of indecomposables such that $\mathcal{L} = \mathcal{A}_0 + ... + \mathcal{A}_{n-1}$ and n is minimum possible. Then $\langle \mathcal{A}_0, ..., \mathcal{A}_{n-1}\rangle$ is unique up to equimorphism (see also Subsection 6.3.2). So, to define $\mathtt{Inv}(\mathcal{L})$, it is enough to define invariants for the class of indecomposable linear orderings. We will assign a finite tree $\mathtt{T}(\mathcal{A}_i)$ to each indecomposable linear ordering and then take

$$\mathtt{Inv}(\mathcal{L}) = \langle \mathtt{T}(\mathcal{A}_0), ..., \mathtt{T}(\mathcal{A}_{n-1})\rangle.$$

A linear ordering is *indecomposable to the left (right)* if, whenever $\mathcal{A}$ and $\mathcal{B}$ are linear orderings such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have that $\mathcal{L}$ is equimorphic to $\mathcal{A}$ (to $\mathcal{B}$). Another result of Jullien [Jul69, Theorem IV.3.3] is that every indecomposable linear ordering is either indecomposable to the right or to the left. (See also [Fra00, 6.3.4(3)] and [Ros82, Lemma 10.3], and see Chapter 7 for a reverse mathematics analysis of this statement.) Let $\epsilon_{\mathcal{L}}$ be $+$ if $\mathcal{L}$ is indecomposable to the right, and let $\epsilon_{\mathcal{L}}$ be $-$ if it is indecomposable to the left. Given $\mathcal{L} \in \mathbb{H}$, let $\mathbb{I}_{\mathcal{L}} = \{\mathcal{A} \in \mathbb{H} : 1 + \mathcal{A} + 1 \prec \mathcal{L}\}$. Note that $\mathbb{I}_{\mathcal{L}} \subseteq \mathbb{H}_{\mathrm{rk}(\mathcal{L})}$ and that $\mathbb{I}_{\mathcal{L}}$ is closed downwards. Subsets of $\mathbb{H}$ which are closed downwards are going to be called *ideals of* $\mathbb{H}$. We will prove in Corollary 12.2.6 that $\mathcal{L}$ is determined by $\epsilon_{\mathcal{L}}$ and $\mathbb{I}_{\mathcal{L}}$. We use this fact to define $\mathtt{T}(\mathcal{L})$, the invariant of $\mathcal{L}$.

Definition 12.1.5. We assign a finite tree, $\mathtt{T}(\mathcal{L})$, with labels in $\mathcal{O}n \times \{+,-\}$, to each $\mathcal{L} \in \mathbb{H}$ (where $\mathcal{O}n$ is the class of ordinals). Let $\{\mathcal{L}_1, ..., \mathcal{L}_k\}$ be the set of minimal elements of $\mathbb{H}_{\mathrm{rk}(\mathcal{L})} \smallsetminus \mathbb{I}_{\mathcal{L}}$. Define

$$\mathtt{T}(\mathcal{L}) = [\langle \mathrm{rk}(\mathcal{L}), \epsilon_{\mathcal{L}}\rangle; \mathtt{T}(\mathcal{L}_1), ..., \mathtt{T}(\mathcal{L}_k)].$$

That is, $\mathcal{T}(\mathcal{L})$ is a tree with a root labeled $\langle \mathrm{rk}(\mathcal{L}), \epsilon_{\mathcal{L}}\rangle$ and with k branches $\mathtt{T}(\mathcal{L}_1)$,..., $\mathtt{T}(\mathcal{L}_k)$.

The set of minimal elements of $\mathbb{H}_{\mathrm{rk}(\mathcal{L})} \smallsetminus \mathbb{I}_{\mathcal{L}}$ is finite because there are no infinite antichains in $\mathbb{H}$, and determines $\mathbb{I}_{\mathcal{L}}$ because $\mathbb{H}$ is well-founded, and hence for $\mathcal{A} \in \mathbb{H}_{\mathrm{rk}(\mathcal{L})}$, $\mathcal{A} \in \mathbb{I}_{\mathcal{L}}$ if and only if for no $i \leqslant k$, $\mathcal{L}_i \preccurlyeq \mathcal{A}$.

The rest of the chapter is dedicated to prove that these invariants are actually equimorphism invariants and to show that they are somewhat constructive. We do the latter by showing that the definition of the embeddablity relation on the invariants is relatively simple, and that we can easily characterize the finite trees that correspond to invariants. We also compute the invariants of every linear ordering which is a product of linear orderings of the form ω^α or $(\omega^\alpha)^*$.

Let $\mathcal{T}r$ be the class $\{\mathtt{T}(\mathcal{L}) : \mathcal{L} \in \mathbb{H}\}$ and let $\mathcal{I}n = \{\mathtt{Inv}(\mathcal{L}) : \mathcal{L} \in \mathbb{S}\}$. In Section 12.2 we define a relation $\preccurlyeq$ on $\mathcal{I}n$ that such that $\mathtt{Inv}\colon \langle \mathbb{S}, \preccurlyeq\rangle \to \langle \mathcal{I}n, \preccurlyeq\rangle$ is an isomorphism. We define $\preccurlyeq$ in a way such that, given $S, T \in \mathcal{I}n$, we can tell whether $S \preccurlyeq T$ via a finite manipulation of symbols, assuming we can compare the ordinals that appear in the labels of S and T and their cofinalities.

In Proposition 12.2.16 we characterize the finite sequences of finite trees with labels in $\mathcal{O}n \times \{-,+\}$ which belong to $\mathcal{I}n$. This characterization is based on Proposition 12.2.14, where we characterize the finite trees with labels in $\mathcal{O}n \times \{-,+\}$ which belong to $\mathcal{T}r$. All the conditions in these characterizations can be checked using a finite algorithm, but one: 12.2.14.4, which requires the computation of the cofinality of an ideal. This condition always holds when we are dealing with countable linear orderings. So, we do have a characterization of the elements of $\mathcal{I}n_{\omega_1} = \{\mathtt{Inv}(\mathcal{L}) : \mathcal{L} \in \mathbb{S} \,\&\, \mathrm{rk}(\mathcal{L}) < \omega_1\}$ via a finite algorithm.

To find invariants of linear orderings, it is necessary to find the minimal linear orderings of the complements of ideals. The first result in this direction is the following.

Theorem 12.1.6 (Hausdorff, see [Ros82]). *Let κ be a regular cardinal and $\mathcal{L}$ a scattered linear ordering. Then $\kappa \leqslant |\mathcal{L}|$ if and only if either $\kappa \preccurlyeq \mathcal{L}$ or $\kappa^* \preccurlyeq \mathcal{L}$.*

Since a scattered linear ordering has rank $\geqslant \kappa$ if and only if it has size $\geqslant \kappa$, it follows that $\{\kappa, \kappa^*\} \subset \mathbb{S}$, is the set of minimal equimorphism types of rank κ. For each ordinal α, since $\mathbb{S}$ is well-quasiordered, there exists a finite set $\mathbb{F}_\alpha$ of minimal equimorphism types of rank α. In Section 12.3, we explicitly define the elements of $\mathbb{F}_\alpha$ for each α. In Section 12.4 we find the invariants of these minimal equimorphism types.

As we mentioned before, the class of ideals of $\mathbb{H}$, ordered by inclusion, is also a well-quasiordering. In Section 12.3, for each α, we also explicitly define a finite set of ideals of $\mathbb{H}$ of rank α which contains all the minimal ideals of rank α. Then, in Section 12.4, we find the invariants of these ideals. We will use these invariants to describe an algorithm that checks if an ideal of $\mathbb{H}$ has a certain rank, as needed to verify condition 12.2.14.3.

Many ideas in this chapter originated in Chapter 9, where we proved that every hyperarithmetic linear ordering is equimorphic to a computable one, extending an old result of Spector about hyperarithmetic ordinals. The definitions of the invariants and the definition of minimal ideals of a certain rank is essentially done there for the case of countable linear ordering, but it is not stated as so. Peter Cholak, after a talk we gave in Notre Dame, suggested that there might be some relation between the work in Chapter 9 and equimorphism invariants. An important tool used in Chapter 9 is the concept of signed trees. Some results that were already proved in Chapter 9 only for countable linear orderings, we prove here for arbitrary cardinality and without using signed trees. Even though the proofs with signed trees are cleaner, we do not know how to generalize the concept of signed trees to arbitrary cardinality preserving its nice properties. At the end of Section 12.2 we mention how the results in Chapter 9 could be deduced as an application of the results on equimorphism invariants proved here.

In the last section we mention extensions of our results to the class of σ-scattered linear orderings and some questions that are left open.

12.1.1 Background on linear orderings

The reader can find background information about linear orderings in the introductory chapter of [Ros82]. Basic knowledge about ordinals is assumed. The reader can learn about ordinals in any basic textbook in set theory, as for example [Kun80]. Now, we define the notation we will use and we prove some basic lemmas about indecomposable linear orderings. Let $\mathcal{A} = \langle A, \leqslant_{\mathcal{A}} \rangle$ be a linear ordering. The *reverse* linear ordering of $\mathcal{A}$ is $\mathcal{A}^* = \langle A, \geqslant_{\mathcal{A}} \rangle$. We let $\mathcal{A}^+ = \mathcal{A}$ and $\mathcal{A}^- = \mathcal{A}^*$. Let $\mathcal{B} = \langle B, \leqslant_{\mathcal{B}} \rangle$ be another linear ordering. The *product*, $\mathcal{A} \cdot \mathcal{B}$, of $\mathcal{A}$ and $\mathcal{B}$ is obtained by substituting a copy of $\mathcal{A}$ for each element of $\mathcal{B}$. That is: $\mathcal{A} \cdot \mathcal{B} = \langle A \times B, \leqslant_{\mathcal{A}\cdot\mathcal{B}} \rangle$ where $\langle x, y \rangle \leqslant_{\mathcal{A}\cdot\mathcal{B}} \langle x', y' \rangle$ iff $y <_{\mathcal{B}} y'$ or $y = y'$ and $x \leqslant_{\mathcal{A}} x'$. The *sum*, $\sum_{i \in \mathcal{A}} \mathcal{B}_i$, of a set of linear orderings $\{\mathcal{B}_i\}_{i \in A}$ indexed by another linear ordering $\mathcal{A}$, is constructed by substituting a copy of $\mathcal{B}_i$ for each element $i \in A$. So, for example, $\mathcal{A} \cdot \mathcal{B} = \sum_{i \in \mathcal{B}} \mathcal{A}$. When $\mathcal{A} = \{0 < 1 < \cdots < n-1\}$, we sometimes write $\mathcal{B}_0 + \ldots + \mathcal{B}_{m-1}$ or $\sum_{i=0}^{m-1} \mathcal{B}_i$ instead of $\sum_{i \in \mathcal{A}} \mathcal{B}_i$.

The powers of ω are defined as follows. $\omega^0 = 1$ and for an ordinal α, $\omega^\alpha = \sup\{\omega^\beta \cdot \omega : \beta < \alpha\}$. We write $\omega^{\alpha *}$ for $(\omega^\alpha)^*$. Cantor proved that for every ordinal α there exists a finite sequence of ordinals $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_k$ such that

$$\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + \ldots + \alpha^{\alpha_k}.$$

This decomposition is called the *Cantor Normal Form* of α. It can be proved by induction that $\mathrm{rk}(\omega^\alpha) = \alpha$. Moreover, ω^α is the least ordinal of rank α. It is also known, and not hard to prove, that an ordinal is indecomposable if and only if it is of the form ω^α.

Given a set X, we denote the set of finite sequences from X by $X^{<\omega}$.

Given a partial ordering $\mathcal{P} = \langle P, \leqslant_{P} \rangle$, we say that a set $A \subset P$ is *cofinal* in $\mathcal{P}$ if $\forall x \in P \exists y \in A (x \leqslant_{P} y)$. The *cofinality* of $\mathcal{P}$, $\mathrm{cf}(\mathcal{P})$, is the least cardinal κ such that there is a cofinal set of $\mathcal{P}$ of size κ. Note that if $\{x_\gamma : \gamma \in \kappa\}$ is an increasing sequence cofinal in $\mathcal{P}$, then $\mathcal{P}$ has cofinality κ. A *regular* cardinal is one whose cofinality is itself.

Definition 12.1.7. Given an indecomposable linear ordering let κ be the cofinality of $\mathcal{L}^{\epsilon_{\mathcal{L}}}$, and let $\tau(\mathcal{L}) = \kappa^{\epsilon_{\mathcal{L}}}$.

Note that $\mathcal{L}$ can be written as a sum $\mathcal{L} = \sum_{i \in \tau(\mathcal{L})} \mathcal{L}_i$ where the $\mathcal{L}_i$ have smaller rank than $\mathcal{L}$. Using theorem 12.1.4.1, we can assume that $\mathcal{L}_i \in \mathbb{H}$ for each i.

Lemma 12.1.8. *Let $\mathcal{L} \in \mathbb{H}$ and suppose that $\mathcal{L} \preccurlyeq \sum_{i\in\mathcal{A}} \mathcal{B}_i$.*

1. *If $\tau(\mathcal{L}) \not\preccurlyeq \mathcal{A}$, then for some $i \in \mathcal{A}$, $\mathcal{L} \preccurlyeq \mathcal{B}_i$.*
2. *If $\epsilon_{\mathcal{L}} = +$ and $\tau(\mathcal{L}) < \mathrm{cf}(\mathcal{A})$, then, there is an initial segment $\mathcal{A}_0$ of $\mathcal{A}$, of cofinality either $\tau(\mathcal{L})$ or 1, such that $\mathcal{L} \preccurlyeq \sum_{i\in\mathcal{A}_0} \mathcal{B}_i$.*
3. *If $\mathcal{A} \preccurlyeq \tau(\mathcal{L})$ and $1 + \mathcal{L} + 1 \preccurlyeq \sum_{i\in\mathcal{A}} \mathcal{B}_i$, then, for some $i \in \mathcal{A}$, $\mathcal{L} \preccurlyeq \mathcal{B}_i$.*

Proof: Suppose that $\epsilon_{\mathcal{L}} = +$. Let g be an embedding $\mathcal{L} \to \sum_{i\in\mathcal{A}} \mathcal{B}_i$, and let $\{x_\gamma : \gamma \in \tau(\mathcal{L})\}$ be an increasing cofinal sequence in $\mathcal{L}$. For each $\gamma < \tau(\mathcal{L})$, let $a_\gamma \in \mathcal{A}$ be such that $g(x_\gamma) \in B_{\alpha_\gamma}$.

In the first case, we have that there has to be a $\delta < \tau(\mathcal{L})$ such that $\forall\gamma \geqslant \delta(a_\gamma = a_\delta)$. Then, since $\mathcal{L}$ is indecomposable to the right, we have that $\mathcal{L} \preccurlyeq \mathcal{B}_{a_\delta}$.

In the second case, let $A_0 = \{y \in A : \exists\gamma < \tau(\mathcal{L})\ (y \leqslant_{A} a_\gamma)\}$. Note that $\mathcal{L} \preccurlyeq \sum_{i\in\mathcal{A}_0} \mathcal{B}_i$. If there is an a_δ as above, then it is the maximal element of A_0 and $\mathrm{cf}(A_0) = 1$. Otherwise, $\mathrm{cf}(A_0) = \tau(L)$.

The last part follows from the first one. □

Lemma 12.1.9. *Let $\mathcal{L} \in \mathbb{H}$. Then $\tau(\mathcal{L}) = (\mathrm{cf}(\mathrm{rk}(\mathcal{L})) \vee \omega)^{\epsilon_{\mathcal{L}}}$.*

(Here, $\alpha \vee \beta$ denotes the maximum of α and β.)

Proof: Without loss of generality suppose that $\epsilon_{\mathcal{L}} = +$ and assume that $\mathcal{L}$ has a first element a. If $\mathrm{rk}(\mathcal{L})$ is a successor ordinal, then $\mathcal{L}$ is an ω-sum of smaller indecomposables, so $\tau(\mathcal{L}) = \omega$. So, suppose that $\kappa = \mathrm{cf}(\mathrm{rk}(\mathcal{L})) \geqslant \omega$, and let $\{\alpha_\gamma : \gamma < \kappa\}$ be increasing and cofinal in $\mathrm{rk}(\mathcal{L})$. For each $\gamma < \kappa$, let $x_\gamma \in \mathcal{L}$ be such that $a \not\approx_{\alpha_g} x_\gamma$ and $\forall\delta < \gamma(x_\delta \leqslant_{L} x_\gamma)$. We have constructed an increasing, cofinal sequence in $\mathcal{L}$ of size κ. Therefore $\tau(\mathcal{L}) = \kappa$. □

We let $s\mathcal{R}eg$ be the class of linear orderings τ such that either τ or τ^* is a regular cardinal.

Lemma 12.1.10. *Suppose that $\mathcal{L} \in \mathbb{H}$, $\epsilon_{\mathcal{L}} = +$, and $\{\mathcal{L}_\gamma : \gamma \in \alpha\}$ is such that for each γ, $\mathcal{L}_\gamma + 1 \preccurlyeq \mathcal{L}$ and $\alpha \prec \tau(\mathcal{L})$. Then $(\sum_{\gamma\in\alpha} \mathcal{L}_\gamma) + 1 \prec \mathcal{L}$. If $\alpha = \tau(\mathcal{L})$, then $(\sum_{\gamma\in\alpha} \mathcal{L}_\gamma) \preccurlyeq \mathcal{L}$.*

Sketch of the Proof: Use transfinite induction on α. When α is limit use that $\alpha < \mathrm{cf}(\mathcal{L})$. □

Lemma 12.1.11. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be linear orderings such that $\mathcal{A} \cdot \mathcal{B} \preccurlyeq \mathcal{C} \cdot \mathcal{D}$ and $\mathcal{A}$ has a either a first or a last element . Then either $\mathcal{A} \preccurlyeq \mathcal{C}$ or $\mathcal{B} \preccurlyeq \mathcal{D}$.*

Proof: Let a be either the first or the last element of $\mathcal{A}$, and let $g\colon \mathcal{A} \cdot \mathcal{B} \to \mathcal{C} \cdot \mathcal{D}$ be an embedding. Suppose $\mathcal{B} \not\preccurlyeq \mathcal{D}$. Then, there has to exist $b_0 \leqslant_{\mathcal{B}} b_1 \in \mathcal{B}$ such that $g(\langle a, b_0\rangle)$ and $g(\langle a, b_1\rangle)$ belong to the same copy of $\mathcal{C}$. It follows that $\mathcal{A} \preccurlyeq \mathcal{C}$. □

12.2 The Invariants

We show in this section that the invariants $\mathtt{Inv}(\cdot)$ defined in the introduction behave well. First we show that an indecomposable linear ordering $\mathcal{L}$ is determined by $\epsilon_{\mathcal{L}}$ and $\mathbb{I}_{\mathcal{L}}$. This result is essential for showing that different equimorphism types get different invariants. Then, we show how to define an ordering on $\mathcal{T}r$ which coincides with the embeddablity relation on indecomposables. We use this relation to define an ordering on $\mathcal{I}n$ which coincides with the embeddablity relation on scattered linear orderings.

In the last subsection we characterize the finite trees with labels in $\mathcal{O}n\times\{-,+\}$ which belong to $\mathcal{T}r$, and the sequences in $\mathcal{T}r^{<\omega}$ which belong to $\mathcal{I}n$.

12.2.1 Equimorphism invariants

We start by defining the Hausdorff rank of an ideal and proving some basic properties about it.

Definition 12.2.1. Given an ideal $\mathbb{I}\subset\mathbb{H}$, we define its *rank* to be $\mathrm{rk}(\mathbb{I})=\sup\{\mathrm{rk}(\mathcal{L})+1:\mathcal{L}\in\mathbb{I}\}$.

Lemma 12.2.2. *Let $\mathcal{L}\in\mathbb{H}$. Then*

1. $\mathrm{rk}(\mathbb{I}_{\mathcal{L}})=\mathrm{rk}(\mathcal{L})$*;*
2. *if* $\mathrm{cf}(\mathbb{I}_{\mathcal{L}})>\omega$*, then* $\mathrm{cf}(\mathbb{I}_{\mathcal{L}})=\mathrm{cf}(\mathrm{rk}(\mathbb{I}_{\mathcal{L}}))$*;*
3. *if* $\mathrm{cf}(\mathbb{I}_{\mathcal{L}})\leqslant\omega$*, then* $\mathrm{cf}(\mathrm{rk}(\mathbb{I}_{\mathcal{L}}))\leqslant\omega$*.*

Proof: Write $\mathcal{L}$ as $\sum_{\gamma\in\tau(\mathcal{L})}\mathcal{L}_\gamma$ where each $\mathcal{L}_\gamma\in\mathbb{H}$. Without loss of generality assume that $\epsilon_{\mathcal{L}}=+$ and let $\kappa=\tau(\mathcal{L})=\mathrm{cf}(\mathrm{rk}(\mathcal{L}))\vee\omega$.

Suppose first that $\kappa>\omega$. For each $\gamma\in\kappa$, we define $\alpha_\gamma<\kappa$ and $\mathcal{L}^\gamma\in\mathbb{H}$ by transfinite recursion as follows. Let $\alpha_0=1$ and $\mathcal{L}^0=L_0$. Let α_γ be such that for every $\delta<\gamma$, $\alpha_\delta<\alpha_\gamma$ and $\mathcal{L}^\delta\prec\sum_{\beta<\alpha_\gamma}\mathcal{L}_\beta$. Let $\mathcal{L}^\gamma=(\sum_{\delta<\alpha_\gamma}\mathcal{L}_\delta)\cdot\omega^{\epsilon_{\mathcal{L}}}$. Note that, since $\mathcal{L}$ is indecomposable to the right, $\mathcal{L}^\gamma\preccurlyeq\mathcal{L}$. The existence of α_γ follows from Lemma 12.1.8. It also follows that $\mathcal{L}^\gamma\in\mathbb{I}_{\mathcal{L}}$. Also, if $\mathcal{A}\in\mathbb{I}_{\mathcal{L}}$, then $\mathcal{A}\preccurlyeq\mathcal{L}^\gamma$ for some $\gamma\in\kappa$. So $\{\mathcal{L}^\gamma:\gamma\in\kappa\}$ is an increasing cofinal sequence in $\mathbb{I}_{\mathcal{L}}$ of size κ, and hence $\mathrm{cf}(\mathbb{I}_{\mathcal{L}})=\kappa$ and $\mathrm{rk}(\mathbb{I}_{\mathcal{L}})=\sup\{\mathrm{rk}(\mathcal{L}^\gamma)+1:\gamma\in\kappa\}\leqslant\mathrm{rk}(\mathcal{L})$. For any two $x,y\in\mathcal{L}$, the interval $[x,y]_{\mathcal{L}}$ embeds into $\mathcal{L}^\gamma$ for some $\gamma\in\kappa$, and hence $x\approx_{\mathrm{rk}(\mathcal{L}^\gamma)+1}y$. It follows that $\mathrm{rk}(\mathcal{L})\leqslant\sup\{\mathrm{rk}(\mathcal{L}^\gamma)+1:\gamma\in\kappa\}$, and hence $\mathrm{rk}(\mathbb{I}_{\mathcal{L}})=\mathrm{rk}(\mathcal{L})$.

Suppose now that $\kappa=\omega$. Then, it follows from lemma 12.1.8.3 that $\{\mathcal{L}_\gamma:\gamma\in\omega\}$ is cofinal in $\mathbb{I}_{\mathcal{L}}$ and hence $\mathrm{cf}(\mathbb{I}_{\mathcal{L}})\leqslant\omega$ and $\mathrm{rk}(\mathbb{I}_{\mathcal{L}})=\sup\{\mathrm{rk}(\mathcal{L}_\gamma)+1:\gamma\in\omega\}\leqslant\mathrm{rk}(\mathcal{L})$. For any two $x,y\in\mathcal{L}$, the interval $[x,y]_{\mathcal{L}}$ embeds into a finite sum of $\mathcal{L}^\gamma$'s, and hence $x\approx_{\sup\{\mathrm{rk}(\mathcal{L}_\gamma)+1:\gamma\in\omega\}}y$. It follows that $\mathrm{rk}(\mathcal{L})\leqslant\sup\{\mathrm{rk}(\mathcal{L}^\gamma)+1:\gamma\in\kappa\}$, and hence $\mathrm{rk}(\mathbb{I}_{\mathcal{L}})=\mathrm{rk}(\mathcal{L})$. $\square$

Definition 12.2.3. Given an ideal $\mathbb{I} \subset \mathbb{H}$ and a $\tau \in s\mathcal{R}eg$, we say that a linear ordering $\mathcal{L}$ is a *τ-unbounded sum of* $\mathbb{I}$, and we write $\mathcal{L} = \mathbb{I} \cdot \tau$, if $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_{i \in \tau} \mathcal{B}_i$$

where $\{\mathcal{B}_i : i \in \tau\} \subseteq \mathbb{I}$, and any other linear ordering of that form embeds in $\mathcal{L}$. Note that, up to equimorphism, there exists at most one τ-unbounded sum of $\mathbb{I}$.

The idea for the definition above derived from Laver's regular unbounded sums and shuffle sums [Lav71].

The τ-unbounded sum of an ideal $\mathbb{I}$ does not always have to exists. But in some cases we do know it exists:

Lemma 12.2.4. *Let $\mathbb{I} \subset \mathbb{H}$ be an ideal and $\tau \in s\mathcal{R}eg$ be such that* $\mathrm{cf}(\mathbb{I}) \leqslant |\tau|$. *Then, the unbounded sum $\mathbb{I} \cdot \tau$ exists.*

PROOF: Let $\kappa = |\tau|$ and let $\{\mathcal{L}_\xi : \xi < \kappa\}$ be cofinal in $\mathbb{I}$. Let $\pi \colon \tau \to \kappa$ be a function such that for every $\delta \in \kappa$, the set $\pi^{-1}[\delta]$ is cofinal in κ. Let $\mathcal{L}$ be the equimorphism type of $\sum_{\gamma \in \tau} \mathcal{L}_{\pi(\gamma)}$. Using Lemma 12.1.10, it is not hard to prove that if $\mathcal{B} = \sum_{i \in \tau} \mathcal{B}_i$, and $\{\mathcal{B}_i : i \in \tau\} \subseteq \mathbb{I}$, then $\mathcal{B} \preccurlyeq \mathcal{L}$. So, $\mathcal{L}$ is the τ-unbounded sum of $\mathbb{I}$. $\square$

Corollary 12.2.5. *Let $\mathcal{L} \in \mathbb{H}$. Then $\mathbb{I}_{\mathcal{L}} \cdot \tau(\mathcal{L})$ exists and equals $\mathcal{L}$.*

PROOF: Clearly $\mathcal{L} \preccurlyeq \mathbb{I}_{\mathcal{L}} \cdot \tau(\mathcal{L})$. That $\mathbb{I}_{\mathcal{L}} \cdot \tau(\mathcal{L}) \preccurlyeq \mathcal{L}$ follows from Lemma 12.1.10. $\square$

Corollary 12.2.6. *If $\mathcal{A}, \mathcal{B} \in \mathbb{H}$ are such that $\epsilon_{\mathcal{A}} = \epsilon_{\mathcal{B}}$ and $\mathbb{I}_{\mathcal{A}} = \mathbb{I}_{\mathcal{B}}$, then $\mathcal{A} = \mathcal{B}$.*

It follows that the invariants $\mathtt{T}(\cdot)$ defined in the introduction are one to one on $\mathbb{H}$. Thus, the invariants $\mathtt{Inv}(\cdot)$ are one to one on $\mathbb{S}$.

12.2.2 Ordering of invariants.

In this subsection we define a relation $\preccurlyeq$ on $\mathcal{T}r$ that such that $\mathtt{T}$ is an isomorphism between $\langle \mathcal{T}r, \preccurlyeq \rangle$ and $\langle \mathbb{H}, \preccurlyeq \rangle$. We then define a relation $\preccurlyeq$ on $\mathcal{I}n$ such that $\mathtt{Inv} \colon \langle \mathbb{S}, \preccurlyeq \rangle \to \langle \mathcal{I}n, \preccurlyeq \rangle$ is an isomorphism.

Notation 12.2.7. Given $T = [\langle \alpha, \epsilon_T \rangle; T_0, ..., T_{k-1}] \in \mathcal{T}r$, let $\mathrm{rk}(T) = \alpha$ and $\tau(T) = \mathrm{cf}(\alpha)^{\epsilon_T}$.

Definition 12.2.8. Given $S = [\langle \alpha, \epsilon_S \rangle; S_0, ..., S_{l-1}]$ and $T = [\langle \beta, \epsilon_T \rangle; T_0, ..., T_{k-1}] \in \mathcal{T}r$ we let $S \preccurlyeq T$ if,

- either $\alpha \leqslant \beta$, $\tau(S) \preccurlyeq \tau(T)$ and $\forall i < k\ (\mathrm{rk}(T_i) \geqslant \alpha \vee \exists j < l (S_j \preccurlyeq T_i))$,
- or $\alpha < \beta$, $\tau(S) \not\preccurlyeq \tau(T)$ and $\forall i < k\ (T_i \not\preccurlyeq S)$.

Proposition 12.2.9. *For $\mathcal{A}, \mathcal{B} \in \mathbb{H}$, $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $\mathtt{T}(\mathcal{A}) \preccurlyeq \mathtt{T}(\mathcal{B})$.*

PROOF: Using Lemmas 12.1.8 and 12.1.10, one can show that $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if

- either $\tau(\mathcal{A}) \preccurlyeq \tau(\mathcal{B})$ and $\mathbb{I}_{\mathcal{A}} \subseteq \mathbb{I}_{\mathcal{B}}$,
- or $\tau(\mathcal{A}) \not\preccurlyeq \tau(\mathcal{B})$ and $\mathcal{A} \in \mathbb{I}_{\mathcal{B}}$.

The proposition then follows from the following observation. Let $\alpha = \mathrm{rk}(\mathcal{A})$ and $\beta = \mathrm{rk}(\mathcal{B})$, let $\{\mathcal{A}_0, ..., \mathcal{A}_{l-1}\}$ be the set of minimal elements of $\mathbb{H}_\alpha \setminus \mathbb{I}_{\mathcal{A}}$, and let $\{\mathcal{B}_0, ..., \mathcal{B}_{k-1}\}$ be the set of minimal elements of $\mathbb{H}_\beta \setminus \mathbb{I}_{\mathcal{B}}$. Then, $\mathbb{I}_{\mathcal{A}} \subseteq \mathbb{I}_{\mathcal{B}}$ if and only if $\alpha \preccurlyeq \beta$ and for each $i < k$, either $\mathcal{B}_i \notin \mathbb{H}_\beta$ or there exists $j < l$ such that $\mathcal{A}_j \preccurlyeq \mathcal{B}_i$. Also, $\mathcal{A} \in \mathbb{I}_{\mathcal{B}}$ if and only if $\alpha < \beta$ and for each $i < k$, $\mathcal{B}_i \not\preccurlyeq \mathcal{A}$. □

Definition 12.2.10. Given $S = \langle S_0, ..., S_l \rangle \in \mathcal{I}n$ and $T = [\langle \alpha, \epsilon_T \rangle; T_0, ..., T_{k-1}] \in \mathcal{T}r$ we let $S \preccurlyeq T$ if

- either $\epsilon_T = +$, $S_l \preccurlyeq T$ and $\forall j \in \{0, ..., l-1\}$ $(\mathrm{rk}(S_j) < \alpha \ \& \ \forall i < k (T_i \not\preccurlyeq S_j))$,
- or $\epsilon_T = -$, $S_0 \preccurlyeq T$ and $\forall j \in \{1, ..., l\}$ $(\mathrm{rk}(S_j) < \alpha \ \& \ \forall i < k (T_i \not\preccurlyeq S_j))$.

Lemma 12.2.11. *Let $\mathcal{A} \in \mathbb{S}$ and $\mathcal{L} \in \mathbb{H}$. Then, $\mathtt{Inv}(\mathcal{A}) \preccurlyeq \mathtt{T}(\mathcal{L})$ if and only if $\mathcal{A} \preccurlyeq \mathcal{L}$.*

PROOF: Write $\mathcal{A}$ as a sum of indecomposables, $\mathcal{A}_0 + ... + \mathcal{A}_l$, and assume without loss of generality that $\epsilon_{\mathcal{L}} = +$. Since $\mathcal{L}$ is indecomposable to the right, it is not hard to see that $\mathcal{A} \preccurlyeq \mathcal{L}$ if and only if $\mathcal{A}_l \preccurlyeq \mathcal{L}$ and for every $j = 0, ..., l-1$, $\mathcal{A}_j + 1 \preccurlyeq \mathcal{L}$. □

Definition 12.2.12. Now, given $S = \langle S_0, ..., S_l \rangle \in \mathcal{I}n$ and $T = \langle T_0, ..., T_k \rangle \in \mathcal{I}n$ we let $S \preccurlyeq T$ if

$$\bigvee_{0 = i_0 \leqslant ... \leqslant i_k \leqslant i_{k+1} = l+1} \left(\bigwedge_{n \leqslant k} \langle S_{i_n}, S_{i_n + 1}, ..., S_{i_{n+1} - 1} \rangle \preccurlyeq T_n \right).$$

Proposition 12.2.13. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{S}$. Then, $\mathtt{Inv}(\mathcal{A}) \preccurlyeq \mathtt{T}(\mathcal{B})$ if and only if $\mathcal{A} \preccurlyeq \mathcal{B}$.*

PROOF: The proof is straightforward using basic properties of indecomposables. See Corollary 9.4.3. □

12.2.3 The class of invariants

Now we are interested in characterize $\mathcal{T}r$ and $\mathcal{I}n$. Given $T_0, ..., T_{k-1} \in \mathcal{T}r$ and an ordinal α, let $\mathcal{I}^{\alpha}_{T_0,...,T_{k-1}} = \{S \in \mathcal{T}r : \mathrm{rk}(S) < \alpha \ \& \ \forall i < k(T_i \not\preccurlyeq S)\}$. Given and ideal $\mathcal{I} \subset \mathcal{T}r$, let $\mathrm{rk}(\mathcal{I}) = \sup\{\mathrm{rk}(T) + 1 : T \in \mathcal{I}\}$.

Proposition 12.2.14. *A tree $T = [\langle\alpha, \epsilon\rangle; T_0, ..., T_{k-1}]$ with labels in $s\mathcal{R}eg$ belongs to $\mathcal{T}r$ if and only if*

1. *for each i, $T_i \in \mathcal{T}r$ and $\mathrm{rk}(T_i) < \alpha$;*
2. *$T_0, .., T_{k-1}$ are mutually $\preccurlyeq$-incomparable;*
3. *$\mathrm{rk}(\mathcal{I}^{\alpha}_{T_0,...,T_{k-1}}) = \alpha$;*
4. *$\mathrm{cf}(\mathcal{I}^{\alpha}_{T_0,...,T_{k-1}}) \vee \omega = \mathrm{cf}(\alpha) \vee \omega$;*
5. *for no i, $\tau(T_i) \prec \tau(T)$.*

Proof: First we prove the implication from left to right. Suppose $T = \mathtt{T}(\mathcal{L})$ with $\mathcal{L} \in \mathbb{H}$. The first two parts are obvious from the definition of $\mathtt{T}$. The third and fourth follow from Lemma 12.2.2. For the last part suppose, toward a contradiction, that $\tau(T) = \kappa$ and $\tau(T_i) = \lambda < \kappa$. Let $\mathcal{L}_i$ be the minimal element of $\mathbb{H}_\alpha \smallsetminus \mathbb{I}_{\mathcal{L}}$ such that $T_i = \mathtt{T}(\mathcal{L}_i)$. Then, $\mathcal{L}_i = \sum_{j\in\lambda} \mathcal{L}_{i,j}$ for some $\mathcal{L}_{i,j} \in \mathbb{I}_{\mathcal{L}_i}$. For every j, $\mathcal{L}_{i,j} \prec \mathcal{L}_i$ and hence belong to $\mathbb{I}_{\mathcal{L}}$. By Lemma 12.1.10, $(\sum_\lambda \mathcal{L}_{i,j}) + 1 \preccurlyeq \mathcal{L}$. Therefore $\mathcal{L}_i \in \mathbb{I}_{\mathcal{L}}$, contradicting its choice.

Let us now prove the other direction. Suppose that T satisfies the five conditions above. Let $\mathbb{I} = \{\mathcal{A} \in \mathbb{H}_\alpha : \mathtt{T}(\mathcal{A}) \in \mathcal{I}^{\alpha}_{T_0,...,T_{k-1}}\}$. This ideal has rank α and cofinality $\mathrm{cf}(\alpha)$. Let $\tau = \mathrm{cf}(\alpha)^\epsilon = \tau(T)$. Let $\mathcal{L} = \mathbb{I} \cdot \tau$. We claim that $T = \mathtt{T}(\mathcal{L})$. Clearly $\epsilon_{\mathcal{L}} = \epsilon$, $\mathrm{rk}(\mathcal{L}) = \alpha$ and $\mathbb{I} \subseteq \mathbb{I}_{\mathcal{L}}$. Suppose toward a contradiction that $\mathbb{I}_{\mathcal{L}} \not\subseteq \mathbb{I}$. Let $\mathcal{L}_i$ be a minimal element of $\mathbb{I}_{\mathcal{L}} \smallsetminus \mathbb{I}$. In particular, $\mathcal{L}_i$ is a minimal element of $\mathbb{H}_\alpha \smallsetminus \mathbb{I}$, so $\mathtt{T}(\mathcal{L}_i) = T_i$ for some $i < k$. Since $\tau(\mathcal{L}_i) = \tau(T_i) \not\prec \tau$ and $1 + \mathcal{L}_i + 1 \prec \mathcal{L}$, we have that $\mathcal{L}_i$ embeds into one of the summands of $\mathbb{I} \cdot \tau\mathcal{L}$, and hence belongs to $\mathbb{I}$. This contradicts the choice of $\mathcal{L}_i$. □

Notation 12.2.15. If $T = [\langle\alpha, \epsilon\rangle; T_0, ..., T_{k-1}] \in \mathcal{T}r$, we let $\mathcal{I}_T = \mathcal{I}^{\alpha}_{T_0,...,T_{k-1}}$.

Proposition 12.2.16. *Let $T = \langle T_0, ..., T_k\rangle \in \mathcal{T}r^{<\omega}$. Then, $T \in \mathcal{I}n$ if and only if for no $i < k$ we have that*

1. *either $\epsilon_i = -$ and $T_{i+1} \in \mathcal{I}_{T_i}$,*
2. *or $\epsilon_{i+1} = +$ and $T_i \in \mathcal{I}_{T_{i+1}}$.*

Sketch of the Proof: Let $\mathcal{A}_0, ..., \mathcal{A}_k \in \mathbb{H}$ be such that $\mathtt{T}(\mathcal{A}_i) = T_i$. Then, $\langle T_0, ..., T_k\rangle \in \mathtt{Inv}$ if and only if $\langle \mathcal{A}_0, ..., \mathcal{A}_k\rangle$ is a minimal decomposition of $\mathcal{A}_0 + ... + \mathcal{A}_k$. It is not hard to see that if $\langle \mathcal{A}_0, ..., \mathcal{A}_k\rangle$ is not minimal decomposition, then for some i, either $\mathcal{A}_i + \mathcal{A}_{i+1} \preccurlyeq \mathcal{A}_i$ or $\mathcal{A}_i + \mathcal{A}_{i+1} \preccurlyeq \mathcal{A}_{i+1}$. Now, we have that

$\mathcal{A}_i + \mathcal{A}_{i+1} \preccurlyeq \mathcal{A}_{i+1}$ if and only if $\epsilon_{\mathcal{A}_{i+1}} = +$ and $\mathcal{A}_i + 1 \preccurlyeq \mathcal{A}_{i+1}$, and we have that $\mathcal{A}_i + \mathcal{A}_{i+1} \preccurlyeq \mathcal{A}_i$ if and only if $\epsilon_{\mathcal{A}_{i+1}} = -$ and $1 + \mathcal{A}_{i+1} \preccurlyeq \mathcal{A}_i$. The proposition follows. □

We note that all the conditions in the two propositions above can be easily checked via a finite manipulation of symbols, using maybe some basic operations on ordinals, except for 12.2.14.3 and 12.2.14.4. We will prove in the next sections that condition 12.2.14.3 can also be checked using such an algorithm (see the end of Section 12.4). But we do not know anything about condition 12.2.14.4. For the countable case, condition 12.2.14.4 always holds, and hence we can tell whether a sequence belongs to $\mathcal{I}n_{\omega_1}$ or not, using a finite algorithm. As a corollary we get that if α is a computable ordinal, then $\mathbb{S}_\alpha$ is computably presentable, which we already proved as Corollary 9.4.3. We could then use this result to prove that every equimorphism type in $\mathbb{S}_\alpha$ has a computably presentable member, getting a different proof of Theorem 9.2.3.

12.3 Minimal linear orderings

In this section we explicitly define the elements of $\mathbb{F}_\alpha$ for each α, where $\mathbb{F}_\alpha$ is the set of minimal equimorphism types of rank α. Also, for each α, we explicitly define a finite set of ideals of $\mathbb{H}$ which contains the set of minimal ideals of rank α.

Definition 12.3.1. Given an indecomposable ordinal $\alpha > 1$, and two signs ϵ_0 and ϵ_1, we define an equimorphism type $\mathrm{lin}(\langle \alpha, \epsilon_0, \epsilon_1 \rangle)$ as follows. Let $\{\alpha_\gamma : \gamma < \mathrm{cf}(\alpha)\}$ be an increasing sequence cofinal in α. Define

$$\mathrm{lin}(\langle \alpha, \epsilon_0, \epsilon_1 \rangle) \; = \sum_{\gamma \in \mathrm{cf}(\alpha)^{\epsilon_1}} (\omega^{\alpha_\gamma})^{\epsilon_0}.$$

Observe that, up to equimorphism, this definition is independent of the cofinal sequence chosen. We also let $\mathrm{lin}(\langle 1, + \rangle) = \omega$ and $\mathrm{lin}(\langle 1, - \rangle) = \omega^*$. We call these equimorphism types, *basic linear orderings*. We use `b.l.o.` to denote the set of codes for basic linear orderings:

$$\texttt{b.l.o.} = \{\langle \alpha, \epsilon_0, \epsilon_1 \rangle : \alpha = \omega^\delta, \delta \in \mathcal{O}n, \delta > 0 \;\&\; \epsilon_0, \epsilon_1 \in \{+, -\}\} \cup \{\langle 1, + \rangle, \langle 1, - \rangle\}.$$

Finite products of basic linear orderings will be called *finitely alternating linear orderings*. Let $\texttt{F.l.o.} = \texttt{b.l.o.}^{<\omega}$. Given $\vec{a} = \langle a_0, ..., a_n \rangle \in \texttt{F.l.o.}$, let

$$\mathrm{lin}(\vec{a}) = \mathrm{lin}(a_0) \cdot ... \cdot \mathrm{lin}(a_n).$$

Example 12.3.2. $\mathrm{lin}(\langle \alpha, +, + \rangle) = \omega^\alpha$ and $\mathrm{lin}(\langle \alpha, -, - \rangle) = \omega^{\alpha *}$.

Notation 12.3.3. For $a = \langle \alpha, \epsilon_0, \epsilon_1 \rangle \in \texttt{b.l.o.}$, let $\mathrm{rk}(a) = \alpha$ and $\epsilon_a = \epsilon_1$, and for $a = \langle 1, \epsilon \rangle$, let $\mathrm{rk}(a) = 1$ and $\epsilon_a = \epsilon$. Let $\mathrm{cf}(a) = \mathrm{cf}(\mathrm{rk}(a))$ and $\tau(a) = \mathrm{cf}(a)^{\epsilon_a}$. Given $\vec{a} = \langle a_0, ..., a_n \rangle \in \texttt{F.l.o.}$, let $\mathrm{rk}(\vec{a}) = \sum_{i \leqslant n} \mathrm{rk}(a_i)$, $\mathrm{cf}(\vec{a}) = \mathrm{cf}(a_n)$, $\epsilon_{\vec{a}} = \epsilon_{a_n}$, and $\tau(\vec{a}) = \tau(a_n)$.

Definition 12.3.4. Let δ be an ordinal with Cantor normal form $\delta = \omega^{\alpha_0} + ... + \omega^{\alpha_{k-1}}$ where $\alpha_0 \geqslant \alpha_1 \geqslant ... \geqslant \alpha_{k-1}$. Then, let

$$\texttt{F.l.o.}_\delta = \{\langle a_0, ..., a_{k-1}\rangle \in \texttt{F.l.o.} : \forall i < k(\mathrm{rk}(a_i) = \omega^{\alpha_i})\}.$$

Proposition 12.3.5. *Let $\mathcal{L} \in \mathbb{H}$ and δ be an ordinal. Then,*

$$\delta \leqslant \mathrm{rk}(\mathcal{L}) \quad \Leftrightarrow \quad (\exists \vec{a} \in \texttt{F.l.o.}_\delta)\ \mathrm{lin}(\vec{a}) \preccurlyeq \mathcal{L}.$$

Before proving this proposition we need to prove a few lemmas.

Lemma 12.3.6. *Let $\mathcal{L}$ be a scattered linear ordering of rank α. If we color $\mathcal{L}$ with finitely many colors, there is a monochromatic subset of $\mathcal{L}$ of rank α. In other words, if C is a finite set, then for every $f\colon \mathcal{L} \to C$, there exists $c \in C$ such that* $\mathrm{rk}(f^{-1}[c]) = \alpha$.

Proof: We use transfinite induction in α.

Suppose first that α is a limit ordinal. For every $\beta < \alpha$ there exists a color c_β such that $\mathrm{rk}(f^{-1}[c_\beta]) \geqslant \beta$. There is some color c such that $\{\beta < \alpha : c_\beta = c\}$ is unbounded in α. Then $f^{-1}[c]$ has rank α.

Suppose now that $\alpha = \beta + 1$. Then $L^{(\beta)}$ is infinite. Note that the set of $\approx_\beta$-equivalence classes which have rank β is infinite too. For each $x \in \mathcal{L}^{(\beta)}$ that has rank β, there exists a color c_x such that $f^{-1}[c_x] \cap x$ has rank β. (We are thinking of x as a segment of $\mathcal{L}$.) For some color c, $\{x \in \mathcal{L}^{(\beta)} : \mathrm{rk}(x) = \beta \ \&\ c_x = c\}$ is infinite. For that color c we have that $f^{-1}[c]$ has rank α. □

Lemma 12.3.7. *If $\mathcal{L}$ is a linear ordering of rank α and $\alpha = \beta + \gamma$, then, there exist indecomposable linear orderings $\mathcal{B}$ and $\mathcal{C}$ of ranks β and γ respectively such that $\mathcal{B} \cdot \mathcal{C} \preccurlyeq \mathcal{L}$*

Proof: Consider $\mathcal{L}^{(\beta)} = \mathcal{L}/\approx_\beta$. Note that $\mathcal{L} = \sum_{x \in \mathcal{L}^{(\beta)}} x$, viewing each equivalence class as a segment of $\mathcal{L}$. Let

$$\hat{\mathcal{L}} = \sum_{x \in \mathcal{L}^{(\beta)}, \mathrm{rk}(x) = \beta} x \ \subseteq \mathcal{L}.$$

Note that $\hat{\mathcal{L}}^{(\beta)} = \{x \in \mathcal{L}^{(\beta)} : \mathrm{rk}(x) = \beta\}$.

We claim that $\mathrm{rk}(\hat{\mathcal{L}}) = \alpha$. For each $\delta \geqslant \beta$, let $f_\delta \colon \hat{\mathcal{L}}^{(\delta)} \to \mathcal{L}^{(\delta)}$ be the obvious embedding: $f_\delta(y)$ is the unique $z \in \mathcal{L}^{(\delta)}$ such that $y \subseteq z$. We will prove that for each $\delta \geqslant \beta$, $f_\delta[\hat{\mathcal{L}}^{(\delta)}] = \{z \in \mathcal{L}^{(\delta)} : \mathrm{rk}(z) \geqslant \beta\}$. Clearly $f_\delta[\hat{\mathcal{L}}^{(\delta)}] \subseteq \{z \in \mathcal{L}^{(\delta)} : \mathrm{rk}(z) \geqslant \beta\}$. Now, suppose that $z \in \mathcal{L}^{(\delta)}$ and $\mathrm{rk}(z) \geqslant \beta$. z can be written as a sum of $\approx_\beta$-equivalence classes in $\mathcal{L}^{(\beta)}$. If it is only one $\approx_\beta$-equivalence class, then it is an equivalence class that belongs to $\hat{\mathcal{L}}^{(\beta)}$, and $z = f_\delta(z) \in f_\delta[\hat{\mathcal{L}}^{(\delta)}]$. If it contains more than one $\approx_\beta$-equivalence classes, since $\mathcal{L}^{(\beta)}$ is scattered, there are $x, w \in \mathcal{L}^{(\beta)}$, $x, w \subseteq z$ which are adjacent. Then, at least one of x and w belongs to $\hat{\mathcal{L}}^{(\beta)}$. This is because, if $\mathrm{rk}(x) < \beta$ and $\mathrm{rk}(w) < \beta$, the elements of x and the ones of w would be

$\approx_\beta$-equivalent. Suppose $x \in \hat{\mathcal{L}}^{(\beta)}$. Then $z = f_\delta(y)$, where y is the $\approx_\delta$-equivalence class in $\hat{\mathcal{L}}^{(\delta)}$ which contains x. It follows that for all δ, $\beta \leqslant \delta < \alpha$, $\hat{\mathcal{L}}^{(\delta)}$ is infinite, and hence $\mathrm{rk}(\hat{\mathcal{L}}) = \alpha$, proving our claim. Therefore, $\mathrm{rk}(\hat{\mathcal{L}}^{(\beta)}) = \gamma$.

Let $\mathcal{B}_0$,..., $\mathcal{B}_k$ be the minimal linear orderings of rank β. So, for each $x \in \hat{\mathcal{L}}^{(\beta)}$, there is an $i_x \leqslant k$ such that $\mathcal{B}_{i_x} \preccurlyeq x$. By the previous lemma, there is an $i \leqslant k$ such that $\hat{\mathcal{C}} = \{x \in \hat{\mathcal{L}}^{(\beta)} : i_x = i\}$ has rank γ. Let $\mathcal{C} \preccurlyeq \hat{\mathcal{C}}$ be an indecomposable of rank γ. Note that $\mathcal{B} = \mathcal{B}_i$ and $\mathcal{C}$ are as desired. □

Lemma 12.3.8. *Let $\alpha > 1$ be an indecomposable ordinal, and let $\mathcal{L} \in \mathbb{H}$. Then, $\alpha \leqslant \mathrm{rk}(\mathcal{L})$ if and only if for some $\epsilon_0, \epsilon_1 \in \{+,-\}$, $\mathrm{lin}(\langle \alpha, \epsilon_0, \epsilon_1\rangle) \preccurlyeq \mathcal{L}$.*

PROOF: The direction from right to left follows from the fact that $\mathrm{lin}(\langle \alpha, \epsilon_0, \epsilon_1\rangle)$ has rank α.

Assume, without loss of generality, that $\mathcal{L}$ is indecomposable and has rank α. So $|\tau(\mathcal{L})| = \mathrm{cf}(\alpha)$. Write $\mathcal{L}$ as $\sum_{\gamma \in \tau(\mathcal{L})} \mathcal{L}_\gamma$ and let $\epsilon_1 = \epsilon_{\mathcal{L}}$.

To prove the lemma it will be enough to show that for every $\beta < \alpha$ there exists γ such that either ω^β or $\omega^{\beta *}$ embeds in $\mathcal{L}_\gamma$. Because then, for some $\epsilon_0 \in \{-,+\}$, we have that for every $\beta < \alpha$ there is a $\gamma \in \tau(\mathcal{L})$ such that $\omega^{\beta \epsilon_0} \preccurlyeq \mathcal{L}_\gamma$, and hence, by Lemma 12.1.10, $\mathrm{lin}(\langle \alpha, \epsilon_0, \epsilon_1\rangle) \preccurlyeq \mathcal{L}$. Assume first that $\alpha = \omega^\delta$ and δ is a limit ordinal. Then, $\forall \beta < \alpha\ \exists \delta_0 < \delta\ (\beta < \omega^{\delta_0})$, and then, by inductive hypothesis, for some ϵ^0 and ϵ^1, $\mathrm{lin}(\omega^{\delta_0}, \epsilon^0, \epsilon^1) \preccurlyeq \mathcal{L}$, and hence $\beta^{\epsilon^0} \preccurlyeq \mathcal{L}$. Assume now that the lemma is true for ω^δ and let us prove it for $\alpha = \omega^{\delta+1}$. Consider $\beta < \alpha$. For some $n < \omega$, $\beta < \omega^\delta \cdot n$. We need to prove that either $\omega^{\omega^\delta \cdot n}$ or $\omega^{\omega^\delta \cdot n *}$ embeds in $\mathcal{L}$. By the previous lemma there exist $\mathcal{A}_1, ..., \mathcal{A}_{2n}$ of rank $\omega^\delta \cdot 2$ such that $\mathcal{A}_1 \cdot ... \cdot \mathcal{A}_{2n} \preccurlyeq \mathcal{L}$. We claim now that for each $i \leqslant 2n$, either ω^{ω^δ} or $\omega^{\omega^\delta *}$ embeds in $\mathcal{A}_i$. Note that by an application of the pigeon-hole principle, this claim implies that either $\omega^{\omega^\delta \cdot n} \preccurlyeq \mathcal{L}$ or $\omega^{\omega^\delta \cdot n *} \preccurlyeq \mathcal{L}$. By the previous lemma, there exists $\mathcal{B}_0$ and $\mathcal{B}_1$, both of rank ω^δ, such that $\mathcal{B}_0 \cdot \mathcal{B}_1 \preccurlyeq \mathcal{A}_i$. So, by the inductive hypothesis, there exists $\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3 \in \{+,-\}$ such that $\mathrm{lin}(\langle \omega^\delta, \epsilon^0, \epsilon^1\rangle) \preccurlyeq \mathcal{B}_0$ and $\mathrm{lin}(\langle \omega^\delta, \epsilon^2, \epsilon^3\rangle) \preccurlyeq \mathcal{B}_1$. If $\mathrm{cf}(\omega^\delta) = \omega^\delta$, then $\omega^{\omega^\delta \epsilon^1} \preccurlyeq \mathrm{lin}(\langle \omega^\delta, \epsilon^0, \epsilon^1\rangle) \preccurlyeq \mathcal{B}_0 \preccurlyeq \mathcal{A}_i$. So, assume that $\mathrm{cf}(\omega^\delta) < \omega^\delta$. If $\epsilon^2 = \epsilon^3$, then $\omega^{\omega^\delta \epsilon^2} \preccurlyeq \mathcal{B}_1 \preccurlyeq \mathcal{A}_i$ as wanted. Otherwise, either $\epsilon^0 = \epsilon^2$ or $\epsilon^0 = \epsilon^3$. In any case, since $\mathrm{cf}(\omega^\delta) \preccurlyeq \mathrm{lin}(\omega^\delta, \epsilon^2, \epsilon^3\rangle)$,

$$\omega^{\omega^\delta \epsilon^0} = \mathrm{lin}(\langle \omega^\delta, \epsilon^0, \epsilon^0\rangle) \preccurlyeq \mathrm{lin}(\langle \omega^\delta, \epsilon^0, \epsilon^1\rangle) \cdot \mathrm{lin}(\langle \omega^\delta, \epsilon^2, \epsilon^3\rangle) \preccurlyeq \mathcal{B}_0 \cdot \mathcal{B}_1 \preccurlyeq \mathcal{A}_i$$

as wanted. □

The proof of Proposition 12.3.5 follows easily from the previous two lemmas. For the case $\delta = 1$ use the fact that if $\mathrm{rk}(\mathcal{L}) \geqslant 1$, then either $\omega \preccurlyeq \mathcal{L}$ or $\omega^* \preccurlyeq \mathcal{L}$.

The set $\{\mathrm{lin}(\vec{a}) : \vec{a} \in \texttt{F.l.o.}_\delta\}$ is not exactly the set of minimal linear orderings of rank δ, but a superset of it. The problem is that there might be $\vec{a}, \vec{b} \in \texttt{F.l.o.}_\delta$ such that $\mathrm{lin}(\vec{a}) \prec \mathrm{lin}(\vec{b})$. For example, if $\kappa > \omega$ is a regular cardinal, then

$$\mathrm{lin}(\langle \kappa, +, +\rangle) = \kappa \ \prec\ \mathrm{lin}(\langle \kappa, -, +\rangle).$$

Also, if $\beta \geqslant \mathrm{cf}(\alpha)\cdot\omega$, then, since $\mathrm{cf}(\alpha)\cdot\omega^\beta = \omega^{\mathrm{cf}(\alpha)}\cdot\omega^\beta = \omega^\beta$,

$$\mathrm{lin}(\langle\langle\alpha,+,+\rangle,\langle\beta,+,+\rangle\rangle) \prec \mathrm{lin}(\langle\langle\alpha,+,-\rangle,\langle\beta,+,+\rangle\rangle).$$

Definition 12.3.9. Let δ be an ordinal with Cantor normal form $\delta = \omega^{\alpha_0} + \ldots + \omega^{\alpha_k}$ where $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_k$. Let $\mathbb{F}_\delta$ be the set consisting of $\mathrm{lin}(\vec{a})$ for $\vec{a} = \langle a_0, \ldots, a_{,k}\rangle \in \mathtt{F.l.o.}$ such that for each i,

1. if ω^{α_i} is regular and $a_i = \langle\omega^{\alpha_i}, \epsilon_{i,0}, \epsilon_{i,1}\rangle$, then $\epsilon_{i,0} = \epsilon_{i,1}$,

2. if $\omega^{\alpha_{i+1}} > \mathrm{cf}(\omega^{\alpha_i})$ and $a_{i+1} = \langle\omega^{\alpha_{i+1}}, \epsilon_{i+1,0}, \epsilon_{i+1,1}\rangle$,, then $\epsilon_{a_i} = \epsilon_{i+1,0}$.

Corollary 12.3.10. *For each ordinal δ, $\mathbb{F}_\delta$ is the set of minimal linear orderings of rank δ.*

Proof: From the example above it follows that for every $\vec{a} \in \mathtt{F.id.}_\delta$, either $\mathrm{lin}(\vec{a}) \in \mathbb{F}_\delta$, or some element of $\mathbb{F}_\delta$ embeds in $\mathrm{lin}(\vec{a})$. So, all we need to prove is that $\mathbb{F}_\delta \subset \mathbb{H}$ is an antichain. Let δ have Cantor normal form as in the definition above. By induction on k we show that if $\vec{a}, \vec{b} \in \mathtt{F.id.}_\delta$ are such that

$$\mathrm{lin}(\vec{a}) \in \mathrm{ide}(\vec{b}^\frown\langle\omega^{\alpha_k}, \epsilon_{\vec{b}}\rangle),$$

then $\vec{a} = \vec{b}$. Suppose that $\vec{a} = \vec{c}^\frown\langle\alpha_k, \epsilon^0_{\vec{a}}, \epsilon_{\vec{a}}\rangle$ and $\vec{b} = \vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{b}}, \epsilon_{\vec{b}}\rangle$, and that $\mathrm{lin}(\vec{a}) \preccurlyeq \mathrm{lin}(\vec{b})\cdot\omega^{\beta\epsilon_{\vec{b}}}$, for some $\beta < \omega^{\alpha_k}$.

If $\epsilon_{\vec{a}} \neq \epsilon_{\vec{b}}$, then $\mathrm{lin}(\vec{a}) \in \mathrm{ide}(\vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{b}}\rangle)$, which is impossible because $\mathrm{rk}(\vec{a}) = \mathrm{rk}(\vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{b}}\rangle)$. So $\epsilon_{\vec{a}} = \epsilon_{\vec{b}}$.

If ω^{α_k} is regular, then $\epsilon^0_{\vec{a}} = \epsilon_{\vec{a}} = \epsilon^0_{\vec{b}}$. If not, assume that $\beta > \mathrm{cf}(\omega^{\alpha_k})$ is indecomposable, and hence $\mathrm{cf}(\omega^{\alpha_k}) + \beta = \beta < \beta + \mathrm{cf}(\omega^{\alpha_k})$. Then, since

$$\omega^{\beta\epsilon^0_{\vec{a}}}\cdot\mathrm{cf}(\omega^{\alpha_k})^{\epsilon_{\vec{a}}} \preccurlyeq \mathrm{lin}(\omega^{\alpha_k}, \epsilon^0_{\vec{a}}, \epsilon_{\vec{a}}) \preccurlyeq \mathrm{lin}(\omega^{\alpha_k}, \epsilon^0_{\vec{b}}, \epsilon_{\vec{a}})\cdot\omega^{\beta\epsilon_{\vec{a}}},$$

we have that $\epsilon^0_{\vec{a}} = \epsilon^0_{\vec{b}}$, because otherwise we would have that $\epsilon^0_{\vec{a}} = \epsilon_{\vec{a}}$ and that $\omega^{\beta+\mathrm{cf}(\omega^{\alpha_K})} \preccurlyeq \omega^{\mathrm{cf}(\omega^{\alpha_k})+\beta}$, which is impossible.

What is left is to prove that $\vec{c} = \vec{d}$. Since $\mathrm{lin}(\langle\alpha_k, \epsilon^0_{\vec{a}}, \epsilon_{\vec{a}}\rangle) \not\preccurlyeq \omega^{\beta\epsilon_{\vec{b}}}$, $1+\mathrm{lin}(\vec{c})+1 \prec \mathrm{lin}(\vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{a}}, \epsilon_{\vec{a}}\rangle)$. If $\epsilon^0_{\vec{a}} = \epsilon_{\vec{a}}$, then we have that $\mathrm{lin}(\vec{c}) \in \mathrm{ide}(\vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{a}}\rangle)$. If not, then necessarily $\mathrm{ide}(\vec{c}^\frown\langle\alpha_k, \epsilon^0_{\vec{a}}\rangle) \subseteq \mathrm{ide}(\vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{a}}\rangle)$, so, we also have that

$$\mathrm{lin}(\vec{c}) \in \mathrm{ide}(\vec{d}^\frown\langle\alpha_k, \epsilon^0_{\vec{a}}\rangle) \subseteq \mathrm{ide}(\vec{d}^\frown\langle\alpha_{k-1}, \epsilon^0_{\vec{a}}\rangle)$$

If $\mathrm{cf}(\vec{c}) = \mathrm{cf}(\omega^{\alpha_{k-1}}) \not\preccurlyeq \omega^{\alpha_k}$, then actually $\mathrm{lin}(\vec{c}) \preccurlyeq \mathrm{lin}(\vec{d})$, and hence, by inductive hypothesis, $\vec{c} = \vec{d}$. Otherwise, $\epsilon_{\vec{c}} = \epsilon^0_{\vec{a}} = \epsilon^0_{\vec{b}} = \epsilon_{\vec{d}}$. So, again by inductive hypothesis, $\vec{c} = \vec{d}$. $\square$

12.3.1 Minimal ideals

As important as minimal linear orderings of a certain rank are minimal ideals. We will use minimal ideals to identify the ideals of $\mathcal{T}r$ which have a certain rank. As we did in the previous subsection for linear orderings, we will define a class of ideals called the finitely alternating ideals. We will define a set $\mathtt{F.id.}$ of codes for finitely alternating ideals and an operation $\mathrm{ide}(\cdot)$ that assigns an ideal to each member of $\mathtt{F.id.}$:

Definition 12.3.11. Given an indecomposable ordinal $\alpha > 1$, let $\mathrm{ide}(\langle\alpha,\epsilon\rangle) = \mathbb{I}(\omega^{\alpha\epsilon}) = \{\omega^{\beta\epsilon} : \beta < \alpha\}$. We call these ideals, *basic ideals.* Let $\mathtt{b.id.} = \{\langle\alpha,\epsilon\rangle : \alpha \in \mathcal{O}n,\ \alpha$ indecomposable & $\alpha > 1\}$. Given a finitely alternating linear ordering $\mathcal{L}$ and a basic ideal $\mathbb{I}$ we let $\mathcal{L}\cdot\mathbb{I} = \{\mathcal{A} \in \mathbb{H} : \exists\mathcal{B} \in \mathbb{I}\ \ (\mathcal{A} \preccurlyeq \mathcal{L}\cdot\mathcal{B})\}$. Ideals of this form are called *finitely alternating ideals.* Let $\mathtt{F.id.} = \{\vec{a}^\frown b : \vec{a} \in \mathtt{F.l.o.}, b \in \mathtt{b.id.}\}$. Given $\vec{b} = \vec{a}^\frown b \in \mathtt{F.id.}$, let $\mathrm{ide}(\vec{b}) = \mathrm{lin}(\vec{a})\cdot\mathrm{ide}(b)$.

Note that $\mathrm{lin}(\langle\alpha,\epsilon_0,\epsilon_1\rangle) = \mathrm{ide}(\langle\alpha,\epsilon_0\rangle)\cdot\mathrm{cf}(\alpha)^{\epsilon_1}$. Now, for each limit ordinal δ, we want to define a finite set of finitely alternating ideals which contains all the minimal ideals of rank δ.

Definition 12.3.12. Let δ have Cantor normal form $\delta = \omega^{\alpha_0} + \ldots + \omega^{\alpha_k}$ with $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_k > 1$. Then, let

$$\mathtt{F.id.}_\delta = \{\langle a_0, \ldots, a_{k-1}, a_k\rangle \in \mathtt{F.id.} : \forall i \leqslant k(\mathrm{rk}(a_i) = \omega^{\alpha_i})\}.$$

Proposition 12.3.13. *Let α be a limit ordinal and $\mathbb{I} \subseteq \mathbb{H}$ be an ideal. Then,*

$$\alpha \leqslant \mathrm{rk}(\mathbb{I}) \quad\Leftrightarrow\quad (\exists\vec{b} \in \mathtt{F.id.}_\alpha)\ \mathrm{ide}(\vec{b}) \subseteq \mathbb{I}.$$

PROOF: The implication from right to left follows from the fact that for each $\vec{b} \in \mathtt{F.id.}_\alpha$, $\mathrm{rk}(\mathrm{ide}(\vec{b})) = \alpha$.

Let us now prove the other direction, so assume that $\mathrm{rk}(\mathbb{I}) \geqslant \alpha$. Write α in Cantor's normal form and let ω^β be the last term. So $\alpha = \alpha_0 + \omega^\beta$, where ω^β divides α_0. Let $\{\mathcal{L}_\gamma : \gamma < \kappa\}$ be a cofinal sequence in $\mathbb{I}$. Assume that for all $\gamma < \kappa$, $\mathrm{rk}(\mathcal{L}_\gamma) > \alpha_0$ and let β_γ be such that $\mathrm{rk}(\mathcal{L}_\gamma) = \alpha_0 + \beta_\gamma$. Note that $\sup\{\beta_\gamma : \gamma < \kappa\} = \omega^\beta$. For each γ there exits $\mathcal{A}_\gamma$ and $\mathcal{B}_\gamma$ of ranks α_0 and β_γ, such that $\mathcal{A}_\gamma\cdot\mathcal{B}_\gamma \preccurlyeq \mathcal{L}_\gamma$. We can assume that each $\mathcal{A}_\gamma$ is a minimal linear ordering of rank α_0 and hence a finitely alternating one. There exists $\vec{a} \in \mathtt{F.l.o.}_{\alpha_0}$, such that $\{\beta_\gamma : \gamma < \kappa, \mathcal{A}_\gamma = \mathrm{lin}(\vec{a})\}$ is unbounded in ω^β. Let $A = \{\gamma < \kappa : \mathcal{A}_\gamma = \mathrm{lin}(\vec{a})\}$. By transfinite induction and using the same idea used in Lemma 12.3.8, we can prove that for each $\delta < \omega^\beta$ there exists $\gamma \in A$ such that either ω^δ of $\omega^{\delta*}$ embeds in $\mathcal{B}_\gamma$. So, for ϵ either $+$ or $-$, we have that $\mathrm{ide}(\vec{a}^\frown\langle\beta,\epsilon\rangle) = \mathrm{lin}(\vec{a})\cdot\mathrm{ide}(\omega^\beta,\epsilon) \subseteq \mathbb{I}$. $\square$

The set $\mathbb{FI}_\delta$ of minimal ideals of $\mathbb{H}$ of rank δ can be defined from $\mathtt{F.id.}_\delta$ using the ideas of Corollary 12.3.10.

12.4 Examples of Invariants

For each $\vec{a} \in \mathtt{F.l.o.} \cup \mathtt{F.id.}$ we define a finite set $\mathrm{com}(\vec{a}) \subset \mathtt{F.l.o.}$ which is the complement of $\vec{a}$ in the sense of Proposition 12.4.2 below. We will use these sets to compute the invariants of some finitely alternating linear orderings. Then, we will use these invariants to define an algorithm that checks whether condition 3 of Proposition 12.2.14 holds.

Definition 12.4.1. First, let $\mathrm{com}(\langle\rangle) = \{\langle 1,+\rangle, \langle 1,-\rangle\}$. Second, consider $\vec{a} \in \mathtt{F.l.o.}$. If $\vec{a} = \vec{d}^\frown\langle\beta,\epsilon_0,\epsilon_1\rangle$ with $\beta > 1$, let $\tau = \tau(\langle\beta,\epsilon_0,\epsilon_1\rangle)$ and let

$$\begin{array}{rcl} \mathrm{com}(\vec{a}) & = & \{\vec{c} \in \mathrm{com}(\vec{d}^\frown\langle\beta,\epsilon_0\rangle) : \tau(\vec{c}) \not\preccurlyeq \tau\}. \\ & \cup & \{\vec{c}^\frown\langle 1,+\rangle, \vec{c}^\frown\langle 1,-\rangle : \vec{c} \in \mathrm{com}(\vec{d}^\frown\langle\beta,\epsilon_0\rangle) \;\&\; \tau(\vec{c}) \preccurlyeq \tau\} \end{array}$$

Otherwise, $\vec{a} = \vec{d}^\frown\langle 1,\epsilon\rangle$. Let $\tau = \tau(\langle 1,\epsilon\rangle)$ and let

$$\begin{array}{rcl} \mathrm{com}(\vec{a}) & = & \{\vec{c} \in \mathrm{com}(\vec{d}) : \tau(\vec{c}) \not\preccurlyeq \tau\} \\ & \cup & \{\vec{c}^\frown\langle 1,+\rangle, \vec{c}^\frown\langle 1,-\rangle : \vec{c} \in \mathrm{com}(\vec{d}) \;\&\; \tau(\vec{c}) \preccurlyeq \tau\}. \end{array}$$

Third, consider $\vec{b} \in \mathtt{F.id.}$ and write $\vec{b}$ as $\vec{a}^\frown\langle\alpha,\epsilon\rangle$, where $\vec{a} \in \mathtt{F.l.o.}$ might be empty. Let $\mathbb{I} = \mathrm{ide}(\langle\alpha,\epsilon\rangle)$, let $\bar{\epsilon}$ be the opposite of ϵ, (i.e., $\bar{+} = -$ and $\bar{-} = +$) and let

$$\begin{array}{rcl} \mathrm{com}(\vec{b}) & = & \{\vec{c} \in \mathrm{com}(\vec{a}) : \tau(\vec{c}) \notin \mathbb{I}\} \\ & \cup & \{\vec{c}^\frown\langle\alpha,\epsilon,\epsilon\rangle, \vec{c}^\frown\langle 1,\bar{\epsilon}\rangle : \vec{c} \in \mathrm{com}(\vec{a}) \;\&\; \tau(\vec{c}) \in \mathbb{I}\} \end{array}$$

The definition above might look complicated at first. The motivation for the definition of $\mathrm{com}(\cdot)$ is just to make the following proposition work. Note that if $\vec{a} \in \mathtt{F.l.o.} \cup \mathtt{F.id.}$, and $\vec{c} \in \mathrm{com}(\vec{a})$, then $\mathrm{lin}(\vec{c})$ is a product of linear orderings of the form ω^γ or $\omega^{\gamma*}$.

Proposition 12.4.2. *Let $\vec{a} \in \mathtt{F.l.o.}$, $\vec{b} \in \mathtt{F.id.}$ and $\mathcal{L} \in \mathbb{H}$. Then*

1. *$\mathcal{L} \not\preccurlyeq \mathrm{lin}(\vec{a})$ if and only if $(\exists\vec{c} \in \mathrm{com}(\vec{a}))\ \mathrm{lin}(\vec{c}) \preccurlyeq \mathcal{L}$, and*
2. *$\mathcal{L} \notin \mathrm{ide}(\vec{b})$ if and only if $(\exists\vec{c} \in \mathrm{com}(\vec{b}))\ \mathrm{lin}(\vec{c}) \preccurlyeq \mathcal{L}$.*

We start by proving the implications from right to left.

Lemma 12.4.3. *Let $\vec{a} \in \mathtt{F.l.o.}$ and $\vec{b} \in \mathtt{F.id.}$. Then*

1. *for every $\vec{c} \in \mathrm{com}(\vec{a})$, $\mathrm{lin}(\vec{c}) \not\preccurlyeq \mathrm{lin}(\vec{a})$;*
2. *for every $\vec{c} \in \mathrm{com}(\vec{b})$, $\mathrm{lin}(\vec{c}) \notin \mathrm{ide}(\vec{b})$.*

PROOF: We use induction on the sizes of $\vec{a}$ and $\vec{b}$. We show only the second part, since the idea to prove the first one is very similar. Suppose $\vec{b} = \vec{a}^\frown\langle\alpha,+\rangle$ and consider $\vec{d} \in \mathrm{com}(\vec{b})$. Of course, the case $\vec{b} = \vec{a}^\frown\langle\alpha,-\rangle$ is analogous. Assume, toward a contradiction, that $\mathrm{lin}(\vec{d}) \in \mathrm{ide}(\vec{b})$. So $\mathrm{lin}(\vec{d}) \preccurlyeq \mathrm{lin}(\vec{a}) \cdot \mathcal{C}$ for some $\mathcal{C} \in \mathrm{ide}(\langle \alpha, + \rangle)$. There are three possibilities for $\vec{d}$: The first one is that $\tau(\vec{d}) \notin \mathrm{ide}(\langle \alpha, + \rangle)$ and $\vec{d} \in \mathrm{com}(\vec{a})$. In this case, $\tau(\vec{d}) \not\preccurlyeq \tau(\mathcal{C})$, and hence $\mathrm{lin}(\vec{d}) \preccurlyeq \mathrm{lin}(\vec{a})$, contradicting that $\vec{d} \in \mathrm{com}(\vec{a})$. The second case is that $\vec{d} = \vec{c}^\frown\langle 1, - \rangle$, where $\vec{c} \in \mathrm{com}(\vec{a})$, and $\tau(\vec{c}) \in \mathrm{ide}(\langle \alpha, + \rangle)$. In this case, since $\omega^* \not\preccurlyeq \mathcal{C}$, necessarily $\mathrm{lin}(\vec{c}) \preccurlyeq \mathrm{lin}(\vec{a})$, again contradicting its choice. The last case is that $\vec{d} = \vec{c}^\frown\langle \alpha, +, + \rangle$, where $\vec{c} \in \mathrm{com}(\vec{a})$, and $\tau(\vec{c}) \in \mathrm{ide}(\langle \alpha, + \rangle)$. Let $\mathcal{C}_1$ such that $\mathcal{C} \preccurlyeq \omega^{|\tau(\vec{c})|} \cdot \mathcal{C}_1 \prec \omega^\alpha$. Then $\omega^\alpha = \omega^{\alpha - |\tau(\vec{c})|} \not\preccurlyeq \mathcal{C}_1$. So, since $\mathrm{lin}(\vec{c}) \cdot (\omega^\alpha) = \mathrm{lin}(\vec{d}) \preccurlyeq \mathrm{lin}(\vec{a}) \cdot \mathcal{C} \preccurlyeq \mathrm{lin}(\vec{a}) \cdot \omega^{|\tau(\vec{c})|} \cdot \mathcal{C}_1$, we have that $\mathrm{lin}(\vec{c}) + 1 \preccurlyeq \mathrm{lin}(\vec{a}) \cdot \omega^{|\tau(\vec{c})|}$. Then, by Lemma 12.1.8.3, $\mathrm{lin}(\vec{c}) \preccurlyeq \mathrm{lin}(\vec{a})$, again contradicting that $\vec{c} \in \mathrm{com}(\vec{a})$. $\square$

Before proving the proposition we need a few definitions and observations about presentations of equimorphism types of indecomposable linear orderings.

Definition 12.4.4. An indecomposable linear ordering $\mathcal{L}$ is in *normal form* if

- either $\mathcal{L} = \sum_{i \in \omega^\epsilon} \mathcal{L}_i$, where each $\mathcal{L}_i$ is in normal form and appears infinitely often in the sum,
- or $\mathcal{L} = \sum_{\gamma \in \kappa^\epsilon} \mathcal{L}_\gamma$, where $\kappa > \omega$ is a regular cardinal, and for each $\alpha, \beta < \kappa$,

  $$\sum_{\gamma \in [\omega^\alpha \cdot \beta, \omega^\alpha \cdot (\beta+1))^\epsilon} \mathcal{L}_\gamma$$

  is an indecomposable linear ordering in normal form.

Note that being in normal form is a property of the isomorphism type of $\mathcal{L}$ and not its equimorphism type.

Given an indecomposable linear ordering $\mathcal{L}$ in normal form, we let $\mathtt{bSeg}(\mathcal{L})$ be the set of *building segments* of $\mathcal{L}$. More precisely, $\mathtt{bSeg}(1) = \{1\}$; if $\mathcal{L} = \sum_{\omega^\epsilon} \mathcal{L}_i$, then $\mathtt{bSeg}(\mathcal{L}) = \{\mathcal{L}\} \cup \bigcup_{i \in \omega} \mathtt{bSeg}(\mathcal{L}_i)$; and if $\mathcal{L} = \sum_{\gamma \in \kappa^\epsilon} \mathcal{L}_\gamma$, then

$$\mathtt{bSeg}(\mathcal{L}) = \{\mathcal{L}\} \cup \bigcup_{\alpha, \beta < \kappa} \mathtt{bSeg}\left(\sum_{\gamma \in [\omega^\alpha \cdot \beta, \omega^\alpha \cdot (\beta+1))^\epsilon} \mathcal{L}_\gamma \right).$$

Note that $\mathtt{bSeg}(\mathcal{L})$ is not a set of isomorphism types, but a set of subsets of $\mathcal{L}$.

Lemma 12.4.5. *Every indecomposable linear orderings is equimorphic to one in normal form.*

Proof: The proof is by transfinite induction. Let $\mathcal{L} \in \mathbb{H}$ be such that $\mathcal{L} = \sum_{\gamma \in \tau(\mathcal{L})} \mathcal{L}_\gamma$ and each $\mathcal{L}_\gamma$ is in normal form. If $\tau(\mathcal{L}) = \omega$, then note that since $\mathcal{L}$ is indecomposable, $\sum_{\gamma \in \omega} \mathcal{L}_\gamma \sim \sum_{n \in \omega} (\sum_{\gamma < n} \mathcal{L}_\gamma)$, which is in normal form. Suppose now that $\tau(\mathcal{L}) = \kappa > \omega$. For each $\gamma < \kappa$, we define $\mathcal{L}^\gamma \in \mathbb{H}$ by transfinite recursion, such that if $\delta < \gamma$, then $\mathcal{L}^\delta$ is an initial segment of $\mathcal{L}^\gamma$. Let $\mathcal{L}^0 = L_0$. If $\gamma = \delta + 1$, let $\mathcal{L}^\gamma = (\mathcal{L}^\delta + \mathcal{L}_\gamma) \cdot \omega$. If γ is a limit ordinal and $\mathrm{cf}(\gamma) = \omega$, let $\gamma_0 < \gamma_1 < \ldots$ be a cofinal sequence in γ and let $\mathcal{L}^\gamma$ be an ω-sum of $\{\mathcal{L}_\gamma, \mathcal{L}_{\gamma_0}, \mathcal{L}_{\gamma_1}, \ldots\}$ in which each term appears infinitely often. If $\mathrm{cf}(\gamma) > \omega$, let $\mathcal{L}^\gamma$ be the union of the $\mathcal{L}^\delta$, $\delta < \gamma$. Note that $\mathcal{L}^\kappa$ is in normal form and equimorphic to $\mathcal{L}$. □

Observation 12.4.6. Let $\mathcal{L}$ be an indecomposable linear ordering in normal form.

1. We note that $\langle \texttt{bSeg}(\mathcal{L}), \subseteq \rangle$ is well-founded. This is because $\langle \mathbb{H}, \preccurlyeq \rangle$ is well-founded, and if $\mathcal{A}, \mathcal{B} \in \texttt{bSeg}(\mathcal{L})$ and $\mathcal{A} \subsetneq \mathcal{B}$, then $\mathcal{A} \prec \mathcal{B}$.

2. Second we note that every two elements of $\texttt{bSeg}(\mathcal{L})$ are either disjoint or one contains the other.

3. Every $\mathcal{A} \in \texttt{bSeg}(\mathcal{L}) \smallsetminus \{\mathcal{L}\}$ has a *successor*, $\mathit{succ}(\mathcal{A})$, in $\texttt{bSeg}(\mathcal{L})$, that is, an element of $\texttt{bSeg}(\mathcal{L})$ which is the least one that strictly includes $\mathcal{A}$. This is because $\{\mathcal{B} \in \texttt{bSeg}(\mathcal{L}) : \mathcal{A} \subsetneq \mathcal{B}\}$ is well-ordered. Moreover, the successor of $\mathcal{A}$ is either an ω-sum or an ω^*-sum of members of $\texttt{bSeg}(\mathcal{L})$, infinitely many of which are isomorphic to $\mathcal{A}$.

4. If $\mathcal{A}, \mathcal{B} \in \texttt{bSeg}(\mathcal{L})$, there is a least $\mathcal{C} \in \texttt{bSeg}(\mathcal{L})$ which contains both. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are incomparable, this $\mathcal{C}$ is successor element. This is because if $\mathcal{C}$ is a τ-sum and $|\tau| > \omega$, then there is a smaller building segment that contains both $\mathcal{A}$ and $\mathcal{B}$.

5. Let $\mathbb{A}$ is an antichain of $\texttt{bSeg}(\mathcal{L})$, and let $\tilde{\mathbb{A}}$ be the upwards closure of $\mathbb{A}$. Define
$$\mathbb{B} = \mathbb{A} \cup \{\mathcal{B} \in \texttt{bSeg}(\mathcal{L}) : \mathcal{B} \notin \tilde{\mathbb{A}} \ \&\ \mathit{succ}(\mathcal{B}) \in \tilde{\mathbb{A}}\}.$$
We claim that $\mathbb{B}$ is a maximal antichain in $\texttt{bSeg}(\mathcal{L})$. Suppose not, and let $\mathcal{C} \in \texttt{bSeg}(\mathcal{L})$ be incomparable with all the elements of $\mathbb{B}$. Let $\mathcal{C}_1 \in \tilde{\mathbb{A}}$ be the least one in that contains $\mathcal{C}$. Then, $\mathcal{C}_1$ is the least upper bound $\mathcal{C}$ and some $\mathcal{B} \in \mathbb{A}$, so it is the successor of some $\mathcal{C}_2 \supseteq \mathcal{C}$. But then, $\mathcal{C}_2 \in \mathbb{B}$ contradicting the definition of $\mathcal{C}$.

Proof of Proposition 12.4.2: The direction from right to left follows from Lemma 12.4.3

Consider $\vec{a} \in \texttt{F.l.o.}$ and assume that $\mathcal{L} \not\preccurlyeq \mathrm{lin}(\vec{a})$. We want to show that for some $\vec{c} \in \mathrm{com}(\vec{a})$, $\mathrm{lin}(\vec{c}) \preccurlyeq \mathcal{L}$. If $\vec{a} = \langle\rangle$, then $\mathcal{L}$ is infinite, and then either $\omega \preccurlyeq \mathcal{L}$ or $\omega^* \preccurlyeq \mathcal{L}$. So, suppose that $\vec{a} = \vec{d}^\frown\langle \alpha, \epsilon_0, \epsilon_1 \rangle$ where $\alpha > 1$. The case $\vec{a} = \vec{d}^\frown\langle 1, \epsilon \rangle$ is similar, but simpler. Consider the set of building segments of $\mathcal{L}$ which do not belong to $\mathrm{ide}(\vec{d}^\frown\langle \alpha, \epsilon_0 \rangle)$. Suppose first that there is some $\hat{\mathcal{L}} \in \texttt{bSeg}(\mathcal{L}) \smallsetminus \{\mathcal{L}\}$ and $\hat{\mathcal{L}} \notin \mathrm{ide}(\vec{d}^\frown\langle \alpha, \epsilon_0 \rangle)$. Then there exists $\vec{c} \in \mathrm{com}(\vec{d}^\frown\langle \alpha, \epsilon_0 \rangle)$ such that $\mathrm{lin}(\vec{c}) \preccurlyeq \hat{\mathcal{L}}$. If $\tau(\vec{c}) \not\preccurlyeq \tau(\langle \alpha, \epsilon_0, \epsilon_1 \rangle)$, then $\vec{c} \in \mathrm{com}(\vec{a})$, and we are done. Otherwise, both $\vec{c}^\frown\langle 1, + \rangle$ and $\vec{c}^\frown\langle 1, - \rangle$ belong to $\mathrm{com}(\vec{a})$. Since either $\hat{\mathcal{L}} \cdot \omega$ or $\hat{\mathcal{L}} \cdot \omega^*$ embeds in $\mathcal{L}$, either $\mathrm{lin}(\vec{c}^\frown\langle 1, + \rangle) \preccurlyeq \mathcal{L}$ or $\mathrm{lin}(\vec{c}^\frown\langle 1, - \rangle) \preccurlyeq \mathcal{L}$. Suppose now that $\mathcal{L}$ is the only member of $\texttt{bSeg}(\mathcal{L})$ which is not in $\mathrm{ide}(\vec{d}^\frown\langle \alpha, \epsilon_0 \rangle)$. Let $\vec{c} \in \mathrm{com}(\vec{d}^\frown\langle \alpha, \epsilon_0 \rangle)$ be such that

$\mathrm{lin}(\vec{c}) \preccurlyeq \mathcal{L}$. Note that $\tau(\vec{c}) = \tau(\mathcal{L})$, because otherwise $\mathrm{lin}(\vec{c})$ would embed into a smaller building segment of $\mathcal{L}$. If $\tau(\vec{c}) \not\preccurlyeq \tau(\langle \alpha, \epsilon_0, \epsilon_1 \rangle)$, then $\vec{c} \in \mathrm{com}(\vec{a})$, and we are done. Otherwise, $\tau(\mathcal{L}) = \tau(\vec{c}) \preccurlyeq \tau(\langle \alpha, \epsilon_0, \epsilon_1 \rangle)$. But then, we have that $\mathcal{L} \preccurlyeq \mathrm{ide}(\vec{d}^\frown \langle \alpha, \epsilon_0 \rangle) \cdot \tau(\langle \alpha, \epsilon_0, \epsilon_1 \rangle) = \mathrm{lin}(\vec{a})$, contradicting our initial assumptions.

Consider now $\vec{b} = \vec{d}^\frown \langle \alpha, + \rangle \in \mathtt{F.id.}$. Let $\{\mathcal{L}_i : i \in \hat{L}\} \subseteq \mathtt{bSeg}(\mathcal{L})$ be the set of minimal building segments of $\mathcal{L}$ which do not embed in $\mathrm{lin}(\vec{d})$. For each $i \in \hat{L}$ there exists $\vec{c}_i \in \mathrm{com}(\vec{d})$ such that $\mathrm{lin}(\vec{c}_i) \preccurlyeq \mathcal{L}_i$. Note that for each $i \in \hat{L}$, $\tau(\vec{c}_i) = \tau(\mathcal{L}_i)$, because otherwise $\mathrm{lin}(\vec{c})$ would embed into a smaller building segment of $\mathcal{L}_i$. If for some $i \in \hat{L}$, $\tau(\vec{c}_i) \notin \mathrm{ide}(\langle \alpha, + \rangle)$, then $\vec{c}_i \in \mathrm{com}(\vec{a})$, so we are done. Suppose now that for every $i \in \hat{L}$, $\tau(\mathcal{L}_i) = \tau(\vec{c}_i) \in \mathrm{ide}(\langle \alpha, + \rangle)$.

Extend $\{\mathcal{L}_i : i \in \hat{L}\}$ to a maximal antichain $\{\mathcal{L}_i : i \in \bar{L}\} \subseteq \mathtt{bSeg}(\mathcal{L})$ as in Observation 12.4.6.5. For $i \in \hat{L}$, let $\kappa_i = \tau(\mathcal{L}_i)$, and for $i \in \bar{L} \smallsetminus \hat{L}$, let $\kappa_i = 1$. Let $\bar{\mathcal{L}} = \langle \bar{L}, \leqslant_{\bar{L}} \rangle$, where $\leqslant_{\bar{L}}$ is the ordering on $\bar{L}$ induced by the ordering on $\mathcal{L}$ in the obvious way. First, we observe that if some $\mathcal{A} \in \mathtt{bSeg}(\mathcal{L})$ extending some element of $\bar{\mathcal{L}}$ is indecomposable to the left, then we have that for some i, $\mathrm{lin}(\vec{c}_i) \cdot \omega^* \preccurlyeq \mathcal{L}$ and so we are done. So, we can assume that $\bar{\mathcal{L}}$ is an ordinal. If $\sum_{i \in \bar{\mathcal{L}}} \kappa_i \prec \omega^\alpha$ then we would have that $\mathcal{L} \in \mathrm{ide}(\vec{b})$, contradicting our assumptions. So, $\omega^\alpha \preccurlyeq \sum_{i \in \bar{\mathcal{L}}} \kappa_i$. We note that $\sum_{i \in \bar{\mathcal{L}}} \kappa_i = \sum_{i \in \hat{\mathcal{L}}} \kappa_i$. The reason is the following. Let $\mathcal{A} \in \mathtt{bSeg}(\mathcal{L})$ be a successor of some $\mathcal{L}_i$, $i \in \bar{L}$; it is also a successor of some $\mathcal{L}_j$, $j \in \hat{L}$. So, $\mathcal{A} = \sum_{k \in \bar{\mathcal{B}}} \mathcal{L}_k$, where $\bar{\mathcal{B}} \subseteq \bar{\mathcal{L}}$ has order type either ω or ω^*, and each $\mathcal{L}_k$ appears infinitely often in the sum. It is then not hard to show that $\sum_{k \in \bar{\mathcal{B}}} \kappa_k$ is equimorphic to $\sum_{k \in \bar{\mathcal{B}} \cap \hat{\mathcal{L}}} \kappa_k$. It follows that $\sum_{i \in \bar{\mathcal{L}}} \kappa_i = \sum_{i \in \hat{\mathcal{L}}} \kappa_i$. So, we have that $\omega^\alpha \preccurlyeq \sum_{i \in \hat{\mathcal{L}}} \kappa_i$.

It is known that ω^α is *strongly indecomposable* or *indivisible* (see [Fra00, 6.8.1]). That is, for every coloring of ω^α into finitely many colors, there is a monochromatic subset of ω^α equimorphic to ω^α. Therefore, there is some $\vec{c} \in \mathrm{com}(\vec{d})$ such that $\omega^\alpha \preccurlyeq \sum_{i \in \hat{\mathcal{L}}, \vec{c}_i = \vec{c}} \kappa_i = \tau(\vec{c}) \cdot \{i \in \hat{\mathcal{L}} : \vec{c}_i = \vec{c}\}$. Since α is indecomposable and $\tau(\vec{c}) < \alpha$, we have that $\tau(\vec{c}) + \alpha = \alpha$. Therefore $\omega^\alpha \preccurlyeq \{i \in \hat{\mathcal{L}} : \vec{c}_i = \vec{c}\}$, and hence $\mathrm{lin}(\vec{c}^\frown \langle \alpha, +, + \rangle) = \mathrm{lin}(\vec{c}) \cdot \omega^\alpha \preccurlyeq \mathcal{L}$. □

Now we use the results above to compute the invariants of linear orderings which are products of indecomposable ordinals or reverse indecomposable ordinals. Note that if ω^δ is an indecomposable ordinal, and $\delta = \omega^{\alpha_0} + ... + \omega^{\alpha_k}$, then

$$\omega^\delta = \mathrm{lin}(\langle \omega^{\alpha_0}, +, + \rangle, ..., \langle \omega^{\alpha_k}, +, + \rangle).$$

Notation 12.4.7. Consider $\vec{a} = \vec{c}^\frown \langle \alpha, \epsilon, \epsilon \rangle \in \mathtt{F.l.o.}$. Note that $\mathbb{I}_{\mathrm{lin}(\vec{a})} = \mathrm{ide}(\vec{c}^\frown \langle \alpha, \epsilon \rangle)$. Let us define $\mathcal{I}_{\vec{a}} = \vec{c}^\frown \langle \alpha, \epsilon \rangle$.

Corollary 12.4.8. *Let* $\vec{a} \in \mathtt{F.l.o.}$ *be such that* $\mathrm{lin}(\vec{a})$ *is a product of indecomposable ordinals and reverse ordinals. Then*

$$\mathtt{T}(\mathrm{lin}(\vec{a})) = [\langle \mathrm{rk}(\vec{a}), \epsilon \rangle; \quad \{\mathtt{T}(\mathrm{lin}(\vec{b})) : \vec{b} \in \mathrm{com}(\mathcal{I}_{\vec{a}}), \mathrm{rk}(\vec{b}) < \mathrm{rk}(\vec{a})\}].$$

Proof: The proof is straightforward from the proposition above and the definition of $\mathtt{T}(\cdot)$. □

Observation 12.4.9. Suppose that $T_0, ..., T_k \in \mathcal{T}r$. Let $\mathbb{I} = \{\mathrm{lin}(T) : T \in \mathcal{I}^{\alpha}_{T_0,...,T_k}\}$. We note that

$$\mathrm{rk}(\mathcal{I}^{\alpha}_{T_0,...,T_k}) = \alpha \quad \Leftrightarrow \quad \exists \vec{b} \in \mathtt{F.id.}_{\alpha}\ (\mathrm{ide}(\vec{b}) \subseteq \mathbb{I}).$$

We can check whether $\mathrm{ide}(\vec{b}) \subseteq \mathbb{I}$ because

$$\mathrm{ide}(\vec{b}) \subseteq \mathbb{I} \quad \Leftrightarrow \quad \forall i \leqslant k\ \exists \vec{c} \in \mathrm{com}(\vec{b})\ (\mathtt{T}(\mathrm{lin}(\vec{c})) \preccurlyeq T_i).$$

12.5 Open questions

In this section we mention how to extend our results to the class of σ-scattered linear orderings and some possible directions for future work.

Operations on $\mathcal{T}r$

The main question we leave open is whether, given a tree with labels in $\mathcal{O}n \times \{+, -\}$, we can tell if it belongs to $\mathcal{T}r$ via a finite manipulation of the symbols in the tree, using some basic operations on ordinals. Using our results, what is left to do is to find a procedure to check that an ideal in $\mathcal{T}r$ has a certain cofinality, the ideal being given by the minimal elements of its complement (see Proposition 12.2.14).

Then comes the question of which operations on $\mathcal{T}r$ can be done via a finite manipulation of symbols. An interesting operation is the product of linear orderings.

Another possible definition

A variant of the definition of the invariant $T(\mathcal{L})$ could be the following. Instead of using $\mathbb{I}_{\mathcal{L}}$, we can take $\mathbb{I}^{min}_{\mathcal{L}}$, the minimal ideal such that $\mathcal{L} = \mathbb{I}^{min}_{\mathcal{L}} \cdot \tau(\mathcal{L})$. Here is a proof that such a minimal ideal exists.

Lemma 12.5.1. *Let $\mathcal{L} \in \mathbb{H}$. There exists an ideal $\mathbb{J} \subset \mathbb{H}_{\mathrm{rk}(\mathcal{L})}$ which is the least one such that $\mathbb{J} \cdot \tau(\mathcal{L}) = \mathcal{L}$.*

Proof: As we mentioned in the introduction, since $\mathbb{H}$ is a better-quasiordering, the class of ideals of $\mathbb{H}$ is well-quasiordered by inclusion. Let $\mathbb{I}_0, ..., \mathbb{I}_{k-1}$ the set of minimal ideals $\mathbb{I}$ such that $\mathbb{I} \cdot \tau(\mathcal{L}) = \mathcal{L}$. We claim that $k = 1$. Suppose not, and let $\mathbb{I} = \mathbb{I}_0 \cap \mathbb{I}_1$. Let $\mathcal{A} \in \mathbb{I}_0$. If, $\tau(\mathcal{A}) \not\prec \tau(\mathcal{L})$, then $\mathcal{A}$ embeds into one of the summands of $\mathbb{I}_1 \cdot \tau(\mathcal{L}) = \mathcal{L}$, and hence $\mathcal{A} \in \mathbb{I}_0 \cap \mathbb{I}_1 = \mathbb{I}$. Therefore, every $\mathcal{A} \in \mathbb{I}_0 \setminus \mathbb{I}$ has $\tau(\mathcal{A}) \prec \tau(\mathcal{L})$. We claim that for each $\mathcal{A} \in \mathbb{I}_0 \setminus \mathbb{I}$, there exists an $\alpha_{\mathcal{A}} \prec \tau(\mathcal{L})$ and a set $\{\mathcal{B}_i : i < \alpha_{\mathcal{A}}\} \subseteq \mathbb{I}$, such that $\mathcal{A} \preccurlyeq \sum_{i \in \alpha_{\mathcal{A}}} \mathcal{B}_i$. The proof is by induction on the rank of $\mathcal{A}$. Since $\tau(\mathcal{A}) \prec \tau(\mathcal{L})$, $\mathcal{A} = \sum_{j \in \tau(\mathcal{A})} \mathcal{A}_j$ where each

$\mathcal{A}_j \in \mathbb{I}_0$ has smaller rank than $\mathcal{A}$. So, for each $\mathcal{A}_j$ there exists $\alpha_{\mathcal{A}_j} < \tau(\mathcal{L})$ and a set $\{\mathcal{B}_{j,k} : k < \alpha_{\mathcal{A}_j}\} \subseteq \mathbb{I}$, such that $\mathcal{A}_j \preccurlyeq \sum_{k\in\alpha_{\mathcal{A}_j}} \mathcal{B}_{j,k}$. Therefore

$$\mathcal{A} \preccurlyeq \sum_{\langle k,j\rangle \in \sum_{j\in\tau(\mathcal{A})} \alpha_j} \mathcal{B}_{j,k}.$$

Let $\alpha_{\mathcal{A}} = \sum_{j\in\tau(\mathcal{A})} \alpha_j$. Since $|\tau(\mathcal{L})|$ is regular, $\alpha_{\mathcal{A}} \in \tau(\mathcal{L})$.

Now, using Lemma 12.1.10, we get that $\mathbb{I}_0 \cdot \tau(\mathcal{L}) \preccurlyeq \mathbb{I} \cdot \tau(\mathcal{L})$ and hence that $\mathcal{L} = \mathbb{I} \cdot \tau(\mathcal{L})$. This contradicts the minimality of $\mathbb{I}_0$ and $\mathbb{I}_1$. $\square$

Invariants for Galvin's Class

The same idea we used to define invariants for $\mathbb{S}$ can be used to define equimorphism invariants for the class of σ-scattered linear orderings.

Definition 12.5.2. We say that $\mathcal{L}$ is *σ-scattered* if it is a countable union of scattered linear orderings.

This class was first studied by Galvin. The reason why one can define invariants for this class as we did for the class of scattered linear orderings is that versions of Theorems 12.1.1, 12.1.3 and 12.1.4 can be proved for this class. (Each of these theorems is due either to Galvin or to Laver; see [Lav71].) In this case, the labels of the trees should also include information about how the linear ordering is constructed from smaller ones. In other words, the label at the root of $\mathtt{T}(\mathcal{L})$ should now include $\tau(\mathcal{L})$, which now is an element of $s\mathcal{R}eg \cup \{\eta_{\alpha,\beta} : \langle\alpha,\beta\rangle \text{ is admissible}\}$.

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