ARITHMETIC DEGREES:

INITIAL SEGMENTS, ω-REA OPERATORS AND THE ω-JUMP

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The arithmetic degrees as a whole and the degrees of the ω-REA sets in particular are considered. Chapters 1 through 4 deal with ω-REA sets and operators. We prove that every non-arithmetic ω-REA set can be cupped up to $\emptyset^{\omega}$ via a set of minimal arithmetic degree. A simplified version of Harrington's construction of incomparable ω-REA sets is presented in Chapter 2. We extend the method in Chapters 3 and 4 to clarify the analogy between ω-REA sets for the arithmetic degrees and recursively enumerable sets for the Turing degrees. We show, for example, that all the ω-jump classes $\underset{\sim}{H}_{n+1} - \underset{\sim}{H}_{n}$, $\underset{\sim}{L}_{n+1} - \underset{\sim}{L}_{n}$, and $\underset{\sim}{I}$ contain an ω-REA set and that there exists a minimal pair of ω-REA sets (which join to $\underline{0}^{\omega}$) thereby carrying out some steps of a program proposed by Jockusch and Shore. The most well-known question that we answer is that of the range of the ω-jump on degrees below $\emptyset^{\omega}$. Chapter 3 contains a proof that a degree $\underline{d}$ is the ω-jump of a degree below $\emptyset^{\omega}$ if and only if $\underline{d}$ contains a set ω-REA in $\emptyset^{\omega}$. An analogous argument provides a new simple proof of the Sacks Jump Theorem. In Chapters 5 and 6 we prove the analog of the Lachlan-Lebeuf Theorem and the Abraham-Shore

Theorem. Thus any $\aleph_1$-size upper semi-lattice with least element and the countable predecessor property is isomorphic to an initial segment of the arithmetic degrees.

Biographical Sketch

The author was born in Maryland in December, 1957. He earned a B.S. degree from S.U.N.Y. at Binghamton in 1979. He has been a graduate student since 1979 at Cornell University, fulfilling the requirements for an M.S. degree in Computer Science and earning a Ph.D. degree in Mathematics with an Operations Research minor.

Dedicated to the dazzling

Yonn Mi Kouh

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Introduction

Degree structures induced by notions of computability have received much attention from recursion theorists. The canonical example of such a structure is of course the upper semi-lattice of Turing degrees. Turing reducibility to a subset of the natural numbers ω, or to a function $f: \omega \to \omega$, corresponds to the intuitive notion of algorithmic computability. The definition of the reducibility is quite natural; this perhaps explains in part the central role the T-degrees (i.e. Turing degrees) have played in the development of the field. It would not be too much of a simplification to say that the work on the T-degrees can be divided into two main parts: the study of the structure of the degrees as a whole, e.g. initial segments, homogeneity and automorphisms, and the study of the degrees of sets whose elements can be recursively enumerated, i.e. the degrees of r.e. sets. Indeed, Lerman's book [1983] covers topics of the first type almost exclusively, and Soare's forthcoming book [1986] deals extensively with topics of the second type (as well as other topics in the theory of r.e. sets).

A structure which is quite similar to the T-degrees is that of the arithmetic degrees. The definition of arithmetic reducibility is also quite natural ($f \leq_a g$ for $f,g: \omega \to \omega$ if f is definable in first-order arithmetic with g as parameter, and the a-degrees are of course the associated equivalence classes), but the a-degrees have received much less attention than the T-degrees. This is due in part to the lack of a suitable analog to the concept of a computation, which would induce a notion of "effective" enumeration, for example. This would then provide an analog for the concept of an r.e. set, i.e. an "effectively" enumerable set. The study of the appropriate entities for some other generalizations of Turing computability has flourished, e.g. the α-r.e. sets of α-recursion theory, recursion in higher types, and E or set recursion. Recent work of Jockusch and Shore [1984] has uncovered a possible counterpart in the arithmetic degrees to the notion of T-degree of an r.e. set however, namely the notion of ω-REA degree.

We study the arithmetic degrees from two perspectives. These approaches correspond to the classification given above for the work on the Turing degrees, and the general plan is to elucidate the relationships which exist between the two structures. As we will see, the study of these relationships can be illuminating. For example, one by-product of the work is a new simple proof of the traditional Sacks Jump Theorem.

In Chapters 5 and 6 we study the possible size $\aleph_1$ initial segments of the arithmetic degrees. Chapter 5 contains more detailed introductory remarks than given here. We prove the analog of the Abraham-Shore Theorem classifying the initial segments of the Turing degrees of size $\aleph_1$. That is, every upper semi-lattice with least element, the countable predecessor property and of size at most $\aleph_1$ is isomorphic to an initial segment of the a-degrees. This result in fact is a major justification for the above claim that the T- and a-degrees are quite similar in structure (note, however, that Shore [1984] has shown that the structures are not elementarily equivalent). The best previous result is that of Harding [1974] classifying the possible countable distributive initial segments. Our proof closely follows the treatment of Abraham and Shore [1985] and so is essentially in the style of Lerman [1983] except that we build trees for each element of the lattice rather than just one tree for the top element. The Abraham-Shore approach is specifically designed for the extension from the countable case to lattices of size $\aleph_1$, but we feel that it is the best presentation of the countable case even without this extension in mind.

A corollary to the embedding theorem answers a question of Odifreddi [1983]. Namely, the ordering of the T-degrees below $\emptyset^\omega$ and the ordering of the a-degrees below $\emptyset^\omega$ are not elementarily equivalent. ($\emptyset^\omega$, the ω-jump of $\emptyset$, is the truth set for arithmetic or equivalently the effective join of all the finite iterations of the Turing jump applied to $\emptyset$.)

Chapters 1-4 are devoted to the study of ω-REA sets and operators. Jockusch and Shore [1984] first defined these notions as instances of the definitions of α-REA sets and operators for any $\alpha < \omega_1^{CK}$ (REA stands for "recursively enumerable in and above"). An α-REA operator is called a <u>pseudo-jump</u> operator because of its relation to the α-jump. A 1-REA operator is one of the form $A \mapsto A \oplus W_e^A$ and has various properties analogous to the Turing jump, e.g. for every e and every set C with $\emptyset' \leq_T C$, there is an A such that $A \oplus W_e^A \equiv_T A \oplus \emptyset' \equiv_T C$ (Jockusch and Shore [1983]). This is the counterpart of the Friedberg

Jump Theorem. Analogously to the α-jump, α-REA operators are defined by composing sequences of 1-REA operators.

The REA hierarchy has yielded solutions to many problems in areas which at first seem to have few interconnections. For example, Jockusch and Shore [1984] obtained results on the difference hierarchy, they proved that $\underline{0}^{\omega}$ is the base of a cone of minimal covers in the Turing degrees, and they gave results on definability including the fact that the collection of degrees of arithmetic sets is definable in the first order theory of the Turing degrees. Jockusch-Shore [1983] contains a finite injury proof of the nontriviality of the standard jump classes (H_n, L_n, and I) of r.e. sets, using 1-REA operators.

Our interest in level ω of the REA hierarchy is of course due to the fact that the jump operator for the arithmetic degrees is the ω-jump. Indeed Jockusch and Shore [1984] gave some indications that the study of ω-REA operators would be relevant to the a-degrees and ω-jump. For example, they established that there is an incomplete high arithmetic degree (i.e. a degree $\underline{a}$ with $\underline{a} <_a \underline{0}^{\omega}$ and $\underline{a}^{\omega} \equiv_a \underline{0}^{\omega+\omega}$). Prior to this all arithmetic degrees $\underline{a}$ constructed satisfied $\underline{a}^{\omega} \equiv_a \underline{a} \vee \underline{0}^{\omega}$, and hence all known degrees $\underline{a}$ below $\underline{0}^{\omega}$ were low (i.e. $\underline{a}^{\omega} \equiv_a \underline{0}^{\omega}$). They also noted that $\emptyset^{\omega}$ is the complete ω-REA set, and that not all $A \leq_a \emptyset^{\omega}$ are ω-REA. These results suggest that the concept of ω-REA degree is a possible counterpart in the context of arithmetic degrees to the notion of r.e. degree in the context of Turing degrees. Therefore a wide array of problems immediately arises, e.g.: Is there a High_n-Low_n hierarchy for the arithmetic degrees (defined with the ω-jump analogously to the definition of the standard jump classes H_n, L_n)? Is there a minimal pair of ω-REA degrees? Is no ω-REA degree minimal?

The first question to arise after the suggestion of such an analogy would probably be that of the existence of an ω-REA set of intermediate degree, that is, the analog of Post's problem. Fortunately Harrington answered this difficult question affirmatively, before the notion of ω-REA set was even isolated, in his first proof of the existence of arithmetically incomparable arithmetic singletons [1975]. He produced arithmetically incomparable low sets which happened to be ω-REA by the nature of his construction (an ω-REA set is a Π^0_2-singleton), and thus established the analog of the Friedberg-Muchnik theorem.

Harrington's basic construction can be modified in various ways to answer other questions along these lines. For example in Chapters 3 and 4 we answer the first two questions above affirmatively (we conjecture that the answer to the third is also "yes"). His construction has never appeared in print; furthermore, his second proof of the existence of arithmetically incomparable arithmetic singletons [1976] (which produces singletons which are not ω-REA) is much easier to understand, so perhaps has eclipsed the original solution. We therefore present Harrington's construction in Chapter 2 (in a somewhat simplified form) before giving the extensions.

We also give in Chapter 3 a complete answer to the question (raised by Odifreddi [1982] and others) of the range of the ω-jump on degrees below $\emptyset^\omega$. The results of Jockusch and Shore [1984] prompted them to ask if every set ω-REA in $\emptyset^\omega$ has the same degree as the ω-jump of some set A below $\emptyset^{\omega}$. We prove (Theorem 3.8 and Corollary 3.11) that this is the case; in fact the analog of the Sacks Jump Theorem holds, i.e. A may be taken to be ω-REA (it is in this sense that we mean our answer to the above question is complete). A surprising free bonus is a new, very short proof of the standard Sacks Jump Theorem (see, e.g. Soare [1986]) using only an easy application of the finite injury priority method and the recursion theorem (Corollary 3.9).

Our construction in Chapter 4 of an ω-REA minimal pair has the added feature that the sets produced join to $\emptyset^{\omega}$ in the a-degrees and thus (by Lachlan's Non-Diamond Theorem, see e.g. Soare [1986]) we have a simple sentence satisfied by the ordering of the a-degrees of ω-REA sets which is not satisfied by the ordering of the T-degrees of r.e. sets. Thus the proposed analogy is not perfect.

The constructions we give of ω-REA sets rely heavily on the recursion theorem. The amazing uniformities inherent in most recursion theoretic constructions are exploited throughout. In fact our proof of the Sacks Jump Theorem depends in a sense on two layers of uniformity. Inside Sacks' "shiny little box" [1967] is yet another shiny little box, which we open in Corollary 3.9.

We now discuss our notation briefly, which is basically standard. Lerman [1983], Soare [1986] and Rogers [1967] are recommended as references. The map $(x,y) \to \langle x,y \rangle$ is some fixed recursive bijection from $\omega \times \omega$ to ω. For A a subset of ω, and $n \in \omega$, $A^{[n]}$ = column n of A =

$\{x|<n,x> \in A\}$, $A^{[\leq n]} = \{<m,x>|<m,x> \in A \ \& \ m \leq n\}$ and $A^{[<n]}$ is defined similarly. Note that $A^{[0]} \neq A^{[\leq 0]}$. For A and B subsets of ω, $A =^* B$ if they differ on just a finite set, i.e. $(A-B) \cup (B-A)$ is finite. Let $A \oplus B = (\{0\}\times A) \cup (\{1\}\times B)$, $\bigoplus_{i\leq n} B_i = \{<j,x>|j \leq n \ \& \ x \in B_j\}$, and similarly define $\bigoplus_{i<\omega} B_i$. For a string $\tau: n \to \omega$, we write $|\tau|$ for its length, i.e. n. The concatenation operator for strings is *. Sometimes we write $<x_0, x_1, \ldots, x_{n-1}>$ for the string $\tau(i) = x_i$ for $i < n = |\tau|$. When this conflicts with the above pairing function the intent will be clear from the context. For functions $f,g: \omega \to \omega$, $f \leq_a g$ if f is definable in first order arithmetic with parameter g. Equivalently, $f \leq_a g \iff \exists n(f \leq_T g^{(n)})$, where $g^{(n)}$ is the n^{th} iterate of the Turing jump applied to g, and $\leq_T$ denotes Turing reducibility. Given an arithmetic degree $\underline{d}$, $\mathcal{D}_a(<\underline{d})$ is the partial ordering of the a-degrees below $\underline{d}$, and $\mathcal{D}_T$ $(<\underline{b})$ is defined similarly. As usual, $\{e\}^A$ is the e^{th} function partial recursive in A in some standard enumeration and W_e^A is its domain. For improved readability, we use at times such notations as $W(e;A)$, $\{e\}(A;x)$ and $\Phi_e(A)$ $(= \{e\}^A)$.

We can now give some definitions.

Definition I.1: A set A is ω-REA in X if there exists some recursive function f such that $A^{[0]} = X$ and $\forall i\ (A^{[i+1]} = W(f(i);A^{[\le i]}))$. We use $J^{\omega}(f,A)$ or just $J(f,x)$ as notation for this A. A set B is n-REA in X if for some τ with $|\tau| = n$, $B^{[0]} = X$, $\forall i < n\ (B^{[i+1]} = W(\tau(i);B^{[\le i]})$ and $\forall m > n\ (B^{[m]} = \emptyset)$. We use $J^n(f,X)$ as short hand for $J(f,X)^{[\le n]}$ so $J^n(f,X)$ is n-REA in X. An ω-REA operator is of course one of the form $X \mapsto J(f,X)$ for some recursive f, and we define n-REA operators similarly. (Remark: our definitions of α-REA operators for $\alpha \le \omega$ differ slightly from those of Jockusch and Shore, but the induced sets are the same up to 1-1 degree.)

Note that the ω-REA sets (i.e. the sets ω-REA in ∅) are uniformly recursive in $\emptyset^{\omega}$, in that given an index e, $\emptyset^{\omega}$ can determine whether {e} is total, and if so can determine an index e_0 so that $J(\{e\},\emptyset) = \{e_0\}^{\emptyset^{\omega}}$. Indeed the ω-REA sets are uniformly 1-1 reducible to $\emptyset^{\omega}$ in the same sense, and so $\emptyset^{\omega}$ is the complete ω-REA set ($\emptyset^{\omega} \overset{def}{=} \{\langle n,x\rangle \mid x \in \emptyset^{(n)}\}$ is of course itself ω-REA).

We actually will not throw away the indices corresponding to partial functions as above since another approach allows notational matters to behave more nicely. We absorb the process of waiting for {e}(i) to converge into the enumeration of the associated column. Note that given e, there is (uniformly in e) a recursive function g such that for all i and Y

$$W^Y_{g(i)} = \begin{cases} W^Y_{\{e\}(i)} & \text{if } \{e\}(i)\downarrow \\ \emptyset & \text{if } \{e\}(i)\uparrow \end{cases} .$$

<u>Definition I.2</u>: Let a_e be the g above, and let $J_e(X) = J(a_e,X)$ (so $J_e(X) = J(\{e\},X)$ if $\{e\}$ is total). We define the n-REA operator J^n_e similarly and indeed given any ω-REA operator A: $X \mapsto A(X)$, we use A^n to denote the associated n-REA operator.

Remark: On occasion expressions such as $W^Y_{\{e\}(i)}$ and $\{\{e\}(i)\}$ will appear with $\{e\}(i)\uparrow$. Such expressions are to be interpreted in the obvious way, e.g. $W^Y_{\{e\}(i)} = \emptyset$ and $\{\{e\}(i)\} = \lambda x.\uparrow$. This convention would seem to make the above definition of a_e superfluous, but we use it since it still simplifies the exposition in places.

Chapter 1

Preliminary results

In this chapter we prove some propositions which mostly highlight the parallels between the Turing degrees of r.e. sets and the arithmetic degrees of ω-REA sets. The first result, however, gives an instance in which the Turing jump and ω-jump are dissimilar.

Clearly $A' \leq_1 B'$ implies both A and $\overline{A}$ are r.e. in B, so $A' \leq_1 B'$ implies $A \leq_T B$ (where $C \leq_1 D$ if there is a recursive 1-1 function f such that $x \in C \Longleftrightarrow f(x) \in D$). Hence $A \leq_T B$ if and only if $A' \leq_1 B'$. The analogous statement for the ω-jump is false.

<u>Proposition 1.1</u>: There is an A such that $A^\omega \leq_1 \emptyset^\omega$ and $A \not\leq_a \emptyset$.

The standard ω-generic set below $\emptyset^\omega$ is such an A. In fact, one reason for including this result is that we now have an excuse to give a brief discussion of ω-generic sets which we will need later. B is ω-generic if B is Cohen generic for arithmetic.

<u>Definition 1.2</u>: Let L be some standard language for first-order arithmetic with a constant symbol $\overline{n}$ for each $n \in \omega$, and a symbol $\underline{B}$ with $x \in \underline{B}$ and $\overline{n} \in \underline{B}$ as the possible atomic formulas involving $\underline{B}$ (this restriction allows for a simplified definition of forcing). If ϕ is a sentence of L and σ is a binary string, then $\sigma \Vdash \phi$,

<u>σ forces ϕ</u> is defined inductively.

i) If ϕ is an atomic formula that does not contain the symbol $\underline{B}$ then $\phi \Vdash \phi$ iff ϕ is true.

ii) $\sigma \Vdash \bar{n} \in \underline{B}$ iff $\sigma(n) = 1$.

iii) $\sigma \Vdash \phi_0 \vee \phi_1$ iff $\sigma \Vdash \phi_0$ or $\sigma \Vdash \phi_1$.

iv) $\sigma \Vdash \exists x\psi(x)$ iff there is an n such that $\sigma \Vdash \psi(\bar{n})$.

v) $\sigma \Vdash \sim\psi$ iff $\forall\tau \supseteq \sigma\ (\tau \nVdash \psi)$.

A set <u>B forces ϕ</u>, $B \Vdash \phi$, if $\sigma \Vdash \phi$ for some $\sigma \subseteq B$. B is <u>n-generic</u> if for every Σ^0_n-sentence ϕ, $B \Vdash \phi$ or $B \Vdash \sim\phi$. B is <u>ω-generic</u> (or just <u>generic</u>) if B is n-generic for every n. The basic facts are that for an n-generic set B and a Σ^0_n-sentence ϕ, $B \Vdash \phi$ iff $B \models \phi$, and that $\{\sigma | \sigma \Vdash \phi\}$ if ϕ is Σ^0_n (or Π^0_n) is (uniformly) recursive in $\emptyset^{(n)}$. Also, n-generic sets are not Σ^0_n, so in particular ω-generic sets are not arithmetic. For more details see e.g. Odifreddi [1982].

<u>Proof of Proposition 1.1</u>: Let $\langle\phi_s | s \in \omega\rangle$ be an effective listing of the sentences of L. Let $\alpha_0 = \emptyset$, and given α_s, let α_{s+1} = the least $\tau \supseteq \alpha_s$ such that $\tau \Vdash \phi_s$ if there is such a τ, and $\alpha_{s+1} = \alpha_s$ otherwise. Let $A = \bigcup_s \alpha_s$, so A is generic and hence A is not arithmetic. We show that $A^\omega \leq_1 \emptyset^\omega$ to complete the proof.

The point is that we can effectively in s write down a formula δ_s of arithmetic such that $\delta_s(\sigma) \Longleftrightarrow \sigma = \alpha_s$ (we assume that binary strings have been Gödel numbered in some effective way so that σ is really an integer when appropriate for the context). This follows from the inductive definition of α_s and the definability of forcing. Also the jump is definable in arithmetic so there is a recursive function f such that $A \models \phi_{f(<n,x>)}$ iff $x \in A^{(n)}$. By the fact that A is generic and by its definition, $x \in A^{(n)} \Longleftrightarrow \forall\sigma(\delta_{f(<n,x>)+1}(\sigma) \to \sigma \Vdash \phi_{f(<n,x>)})$. This means that we may effectively in <n,x> find a formula of arithmetic which is true iff $x \in A^{(n)}$. Let g(<n,x>) be the Godel number for this formula and let h be a function so that $h(e) \in \emptyset^{(\omega)}$ iff the sentence with Godel number e is true, with g and h recursive. Then $x \in A^{(n)} \Longleftrightarrow h(g(<n,x>)) \in \emptyset^{\omega}$, so $A^{\omega} \leq_1 \emptyset^{\omega}$. □

In Jockusch and Shore [1984] it was shown by a direct argument using perfect forcing that not every arithmetic degree $\underline{a} \leq_a \underline{0}^{\omega}$ is the degree of an ω-REA set. Proposition 1.3 is this same result phrased more generally. It is the analog of the fact that 1-generics do not bound non-recursive r.e. degrees.

Proposition 1.3: Suppose that A is ω-REA, B is ω-generic, and $A \leq_a B$. Then A is arithmetic.

Proof: The fastest way to see this is to note that A is a Π^0_2 singleton (this follows directly from the definition of ω-REA sets), and no non-arithmetic arithmetic singletons are bounded by ω-generics, for suppose A is an arithmetic singleton, B is ω-generic and $A \leq_a B$. Then let $\psi_0(Y) \Longleftrightarrow Y = A$ and $\psi_1(B;x) \Longleftrightarrow x \in A$ where ψ_0, ψ_1 are arithmetic formulas. Thus B satisfies an arithmetic formula ψ_2 which says that $\psi_1(\underline{B})$ satisfies ψ_0, so some $\sigma \subseteq B$ forces ψ_2. To calculate

$A(x)$, take $\gamma \supseteq \sigma$ such that either $\gamma \Vdash \psi_1(\underline{B};\bar{x})$ or $\gamma \Vdash \sim\psi_1(\underline{B};\bar{x})$. Then in the first case $A(x) = 1$ and otherwise $A(x) = 0$. (If not take a generic extension $\tilde{B}$ of γ. Then $\psi_1(\tilde{B})$ satisfies ψ_0, so $\psi_1(\tilde{B};x) \Longleftrightarrow x \in A$, which is a contradiction.) Since there is a fixed n (which depends on the rank of ψ_1) so that given x, such a γ can be found for x uniformly effectively in $\emptyset^{(n)}$, we have $A \leq_a \emptyset$ as desired. □

Epstein [1975] showed that given an r.e. Turing degree $\underline{a} > \underline{0}$, there is a minimal degree $\underline{m}$ such that $\underline{a} \vee \underline{m} = \underline{0}' = \underline{m}'$. We show that the arithmetic version of this holds, but first isolate an important property of ω-REA sets needed in the proof. This property is a generalization of the well-known fact that $\forall n\,(X \geq_T \emptyset^{(n)}) \rightarrow X'' \geq_T \emptyset^{\omega}$, and is implicit in the proof of Proposition 1.13 of Jockusch and Shore [1984].

<u>Proposition 1.4</u>: Suppose A is ω-REA in Y and $\forall n\,(X \geq_T A^{[n]})$. Then $X'' \geq_T A$.

<u>Proof</u>: Let f be a recursive function giving the indices of the columns of A. We will inductively define effectively in X'' indices i_n such that $\{i_n\}^X = A^{[n]}$. Let i_0 be any index such that $\{i_0\}^X = A^{[0]}$. Suppose we have defined $i_0,\ldots,i_n$. $A^{[n+1]} = W_{f(n)}^{A^{[\leq n]}} = W_e^X$ for some e which we can find uniformly from $i_0,\ldots,i_n$. Use X'' to find an index i_{n+1} such that $\{i_{n+1}\}^X$ is total and $\forall x(x \in W_e^X \Longleftrightarrow \{i_{n+1}\}^X(x) = 1)$. There is such an index and the search can be carried out effectively in X''. □

The next result implies that any ω-REA set of intermediate degree is complemented in $\mathcal{D}_a$ ($\leq \underline{0}^{\omega}$). It is unknown if a complement of ω-REA degree can always be found, but in Chapter 4 we show that some ω-REA degrees have ω-REA complements, in contrast to Lachlan's Non-Diamond Theorem.

<u>Proposition 1.5</u>: If $A >_a \emptyset$ is ω-REA then there is an arithmetically minimal set M such that $A \oplus M \equiv_a \emptyset^{\omega} \equiv_a M^{\omega}$.

<u>Proof</u>: The proof is an elaboration of that of Theorem 3.2 of Jockusch and Shore [1984]. By the previous proposition we have $\forall i\, \exists j$ $(A^{[j]} \not\leq_T \emptyset^{(i)})$. Let g(i) be the least such j. Note that $A^{[g(i)]} \leq_T \emptyset^{(i+1)}$. We want the construction to be effective in $\emptyset^{\omega}$, so note that $g \leq_T \emptyset^{\omega}$. We construct pairs of binary trees $\langle T_s, I_s \rangle \leq_T \emptyset^{(s)}$ (although not uniformly). [A <u>binary tree</u> is a function T from the set of binary strings to itself which satisfies $T(\sigma) \subseteq T(\tau) \Longleftrightarrow \sigma \subseteq \tau$. For a set A, we let $T[A] = \bigcup_{\sigma \subset A} T(\sigma)$. The set of <u>branches of T</u>, [T], is $\{T[A] \mid A \subseteq \omega\}$. We write $T' \subseteq T$ if range $(T') \subseteq$ range (T).] We guarantee that

(1) $\forall X \in [T_s]\ \forall \tau\ (X \supseteq T_s(\tau) \rightarrow X^{(s)} \supseteq I_s(\tau))$.

We show at the end of the proof that $B \leq_a M \rightarrow B = \{n\}^{M^{(n)}}$ for some n. This observation allows the minimality requirements for M to be handled in a simple fashion - we need to consider only one reduction from each iterate of the jump applied to M.

Let $\langle T_0, I_0\rangle = \langle id, id\rangle$. Suppose we are given $\langle T_s, I_s\rangle \leq_T \emptyset^s$ such that (1) holds. Define the intermediate pair of trees $\langle T'_{s+1}, I'_{s+1}\rangle$ with $T'_{s+1} \subseteq T_s$ by induction on levels. Let $T'_{s+1}(\emptyset) = T_s(i)$, where i is chosen so that we diagonalize against the s^{th} arithmetic function, and let $I_{s+1}(\emptyset) = \emptyset$. Say we have defined T'_{s+1} and I'_{s+1} through level n, $|\tau| = n$, and $T'_{s+1}(\tau) = T_s(\gamma)$. Given Y, let $Y^* = \{\sigma | \sigma \subseteq Y\}$ (recall that σ's Godel number is written as σ when no confusion should result). $(A^{[g(s)]})^*$ is not r.e. in $\emptyset^{(s)}$, for otherwise $A^{[g(s)]} \leq_T \emptyset^{(s)}$. Thus either

a) $\exists m[m \notin (A^{[g(s)]})^* \text{ and } \exists\beta \supseteq \gamma * 0^m *1(\{n\}^{I_s(\beta)}(n)\downarrow)]$

or b) $\exists m[m \in (A^{[g(s)]})^* \text{ and } \forall\beta \supseteq \gamma * 0^m *1(\{n\}^{I_s(\beta)}(n)\uparrow)]$.

Take the least m satisfying a) or b), and if a) holds, let β be the least string satisfying the condition. If b) holds, let $\beta = \gamma * 0^m *1$. Note that finding β is effective in $\emptyset^{(s+1)}$ since $(A^{[g(s)]})^* \leq_T \emptyset^{(s+1)}$ and $I_s \leq_T \emptyset^{(s)}$. Let $T'_{s+1}(\tau * j) = T_s(\beta * \emptyset^{(s+1)}(n) * j)$ for $j = 0,1$. $I'_{s+1}(\tau * j)$ is defined in the obvious manner, i.e. $I'_{s+1}(\tau * j) = I'_{s+1}(\tau) * i$, where $i = 1$ if we could force convergence (case a) and 0 otherwise (note that I_s is not strictly speaking a tree, i.e. $\tau_0 \neq \tau_1$ need not imply $I_s(\tau_0) \neq I_s(\tau_1)$).

Before defining $\langle T_{s+1}, I_{s+1}\rangle \subseteq \langle T'_{s+1}, I'_{s+1}\rangle$, note that (1) holds for T'_{s+1} and I'_{s+1} replacing T_s and I_s, and note that if $A \oplus M \geq_T \emptyset^{(s)}$ and $M \in [T'_{s+1}]$, then $A \oplus M \geq_T \emptyset^{(s+1)}$. This follows from the observation that $A^{[g(s)]}$, M and T_s are sufficient to recover the construction along the path of M, and $\emptyset^{(s+1)}$ is coded there. The next steps are simply to make M minimal. Either

a) There is a τ such that

i) $\exists x\ \forall\beta \supseteq \tau\ (\{s+1\}^{I'_{s+1}(\beta)}(x)\uparrow)$

or ii) condition i) fails and $\forall\beta_0,\beta_1 \supseteq \tau\ \forall x\ (\{s+1\}^{I'_{s+1}(\beta_0)}(x)\downarrow$ & $\{s+1\}^{I'_{s+1}(\beta_1)}(x)\downarrow$ implies they are equal)

or b) $\forall\tau\ \exists\beta_0,\beta_1 \supseteq \tau\ \exists x\ (\{s+1\}^{I'_{s+1}(\beta_0)}(x)\downarrow \neq \{s+1\}^{I'_{s+1}(\beta_1)}(x)\downarrow)$.

$\emptyset^{\omega}$ can tell which case we are in. If a) holds, let $T_{s+1} = \mathrm{Ext}(T'_{s+1},\tau)$ and $I_{s+1} = \mathrm{Ext}(I_{s+1},\tau)$, where $\mathrm{Ext}(T,\alpha)$ is the tree T' with $T'(\sigma) = T(\alpha * \sigma)$ for all σ. If b) holds, build a splitting tree inductively. Let $T_{s+1}(\emptyset) = T'_{s+1}(\emptyset)$ and $I_{s+1}(\emptyset) = I'_{s+1}(\emptyset)$. If $T_{s+1}(\tau) = T'_{s+1}(\gamma)$ and $I_{s+1}(\tau) = I'_{s+1}(\gamma)$ then $T_{s+1}(\tau * j) = T'_{s+1}(\gamma * \beta_j)$, where the β_j are least witnesses for a splitting above $I'_{s+1}(\gamma)$, and let $I_{s+1}(\tau * j) = I'_{s+1}(\gamma * \beta_j)$. In either case $\langle T_{s+1}, I_{s+1}\rangle \leq_T \emptyset^{(s+1)}$, and I_{s+1} still has the property that the (s+1)-jump of any branch of T_{s+1} conforms to what I_{s+1} says about it, that is, (1) holds with s replaced by s+1.

Now let M be $\bigcup_s T_s(\emptyset)$. Induction on s using the observation after the definition of T'_{s+1} and I'_{s+1} gives $\forall s\ (A \oplus M \geq_T \emptyset^{(s)})$. Thus $(A \oplus M)'' \geq_T \emptyset^{\omega}$, and since $A \oplus M \leq_T \emptyset^{\omega}$, we have $A \oplus M \equiv_a \emptyset^{\omega}$.

Next we argue that M is minimal. Suppose $B \leq_a M$. Then there is an n and an e such that $B = \{e\}^{M^{(n)}}$. Let g be recursive so that

$g(e,k)$ is arbitrary if $k < n$ and otherwise $B = \{g(e,k)\}^{M^{(k)}}$. By the recursion theorem, take $\bar{k} \geq n$ so that $\{g(e,\bar{k})\} = \{\bar{k}\}$. Then $B = \{\bar{k}\}^{M^{(\bar{k})}}$. Hence $B \leq_a M$ implies that $B = \{\bar{k}\}^{M^{(\bar{k})}}$ for some $\bar{k}$. (The point is merely that we actually handled all requirements.) If, at stage $\bar{k}$ in defining $\langle T_{\bar{k}}, I_{\bar{k}}\rangle$ from $\langle T'_{\bar{k}}, I'_{\bar{k}}\rangle$, we were in case a), then $B \leq_T \emptyset^{(\bar{k})}$. Otherwise, $B \oplus \emptyset^{(\bar{k})} \geq_T M$, because using B and $I_{\bar{k}}$ we can calculate the path of M in $T_{\bar{k}}$ (for that matter we can even directly calculate $M^{(\bar{k})}$). Hence $B \leq_a M$ implies $B \leq_a \emptyset$ or $B \equiv_a M$. Finally, we observe that $M \in [T_s] \to M^{(s)} \leq_T M \oplus \emptyset^{(s)}$, and hence $M^{\omega} \equiv_a M \oplus \emptyset^{\omega} \equiv_a \emptyset^{\omega}$. □

<u>Remark</u>: If we drop the part of the construction devoted to making M minimal (i.e. let $\langle T_{s+1}, I_{s+1}\rangle = \langle T'_{s+1}, I'_{s+1}\rangle$), we can get $A \oplus M \equiv_T \emptyset^{\omega}$, by coding $g(s)$ into the path of M through T'_{s+1}. That is, the β we choose in the definition of T'_{s+1} should extend $\gamma * 0^{\langle m, g(s)\rangle} * 1$. From this we'd get $A \oplus M \geq_T \emptyset^{(s+1)}$ <u>uniformly</u>. The refinement of $\langle T'_{s+1}, I'_{s+1}\rangle$ into $\langle T_{s+1}, I_{s+1}\rangle$ keeps $\langle T_{s+1}, I_{s+1}\rangle \leq_T \emptyset^{(s+1)}$, but loses uniformity.

In the next three chapters we present the main construction of ω-REA sets, and its extensions which produce minimal pairs of ω-REA sets and solve the range of the ω-jump problem, for example. An individual column of any one of the ω-REA sets built must be given by an r.e. enumeration, by definition. Thus the mysteries of priority arguments are prominent in these chapters. It is interesting how the enumerations of the columns can be made to fit together to achieve a common goal. One can, for instance, spread restraint for a certain goal throughout the constructions so that each construction is affected by only finitely many new negative requirements, and so not prevented from achieving its own goals. In Chapter 4 the columns use a signalling device to enable the next column in line to know whether it needs to act for a certain requirement.

It should be noted, however, that a construction of an ω-REA arithmetic degree could conceivably avoid r.e. constructions altogether. For example, full approximation constructions or constructions of sets Δ^0_2 in the input could be used. Our next lemma shows that if one uses constructions of bounded arithmetic complexity for the columns of a set, it has ω-REA a-degree. This observation does not help for the present constructions, however. We will see that the construction of one of our operators depends on some other operator being specified by r.e. enumerations (since we build columns to have a specific jump, and this of course must be r.e. in the jump of the input).

Definition 1.6: A set A is ω-Δ^0_n in X if there is a recursive function f such that $A^{[0]} = X$ and $\forall i(A^{[i+1]}$ = the $\Delta^0_n(A^{[\leq i]})$ set whose index is f(i)). One defines ω-Σ^0_n and ω-Π^0_n operators analogously. Thus an ω-Σ^0_1 operator is an ω-REA one.

Proposition 1.7: Suppose that A is ω-Δ^0_n in X. Then A has the same a-degree as a set ω-REA in X.

<u>Proof</u>: Define the ω-REA set C by $C^{[0]} = X$, $C^{[1]} = X'$, ..., $C^{[n-1]} = X^{(n-1)}$, $C^{[n]} = A^{[1]}$, $C^{[n+1]} = A^{[\leq 1]'}$, ... $C^{[2n-1]} = A^{[\leq 1](n-1)}$, $C^{[2n]} = A^{[2]}$,... . Then $C \geq_T A$ and $C^{[m]} \leq_T A^{(n-1)}$ for all m. Thus $C \equiv_a A$ (by Proposition 1.4 or just directly since $C^{[m]}$ is in fact recursive in $A^{(n-1)}$ uniformly in m). □

Chapter 2

Incomparable ω-REA sets

We present the simplest version of the main construction of ω-REA sets in this chapter. This is the foundation on which the extensions of future chapters rest. We prove the existence of incomparable ω-REA sets which join to $\emptyset^{\omega}$. Harrington's construction of incomparable ω-REA sets [1975] used explicit requirements to accomplish the diagonalization. Surprisingly, the technique presented here which makes the sets join to $\emptyset^{\omega}$ also allows for a simplified construction since diagonalization occurs automatically.

The same basic lemma is employed iteratively to define the ω-REA operators. Roughly speaking, this lemma tells us how to build the first two columns. The result is fed in as input and the lemma builds the next two columns, and so on. When actually used, the constructions of the lemma need to be changed by adding finitely many new requirements. Therefore we are interested in not only the result of the lemma, but also in the constructions.

There are some notational difficulties inherent in the material presented. For example, for the purposes of a construction of an individual column, set letters such as B and C are preferable. However, when we piece the constructions together to form an ω-REA operator, indexing of the columns is needed. Changes in notation which seem to improve readability in one place often have negative impacts in other places. After this word of caution, we now present "the Z,B,C-lemma" in order to give more detailed motivational remarks.

Lemma 2.1 (Harrington [1975]): Given any set Z and W r.e. in Z', there are B r.e. in Z and C r.e. in $Z \oplus B$ such that $(Z \oplus B \oplus C)' \equiv_T Z' \oplus B \oplus C \equiv_T Z' \oplus W$. Furthermore, this result is uniform; that is, indices for B and C are obtainable effectively from an index for W as a set r.e. in Z', as are indices for the Turing equivalences. Moreover, these indices are independent of Z.

Recall that this lemma will be used iteratively to build an ω-REA operator two columns at a time. Z will always be $J_i^{2n}(X)$ for some i and n. Given an ω-REA operator J_e, we use this lemma to produce an ω-REA operator J_i such that for all X and n

(1) $J_i^{2n}(X)' \equiv_T J_i^{2n}(X) \oplus X' \equiv_T J_e^n(X')$

uniformly in X and n.

Note that for n = 1, setting $Z = \{0\} \times X$ and taking W to be the first column of $J_e(X')$, the lemma gives (1) directly (where B and C have been renamed $J_i(X)^{[1]}$ and $J_i(X)^{[2]}$, respectively). To iterate, say we have (1) for n = m. Then $J_e^{m+1}(X')$ is r.e. in $J_i^{2m}(X)'$. Think of $J_i^{2m}(X)$ as Z. The lemma gives $J_i(X)^{[2m+1]}$, i.e. B, and $J_i(X)^{[2m+2]}$, i.e. C, with $J_i^{2m+2}(X)' \equiv_T J_i^{2m}(X)' \oplus J_i(X)^{[2m+1]} \oplus J_i(X)^{[2m+2]} \equiv_T J_i^{2m}(X)' \oplus J_e(X')^{[m+1]}$, uniformly. Rewriting, we deduce (1) for n = m+1. By the uniformities present i is a recursive function of e. So by the recursion theorem we can produce an ω-REA operator A such that for all X, $\bigoplus_n (A^{2n}(X)') \equiv_T A(X) \oplus X' \equiv_T A(X')$. To produce ω-REA sets with various properties we alter the constructions for each column slightly. For example, we can make $A(X)' \equiv_T \bigoplus_n (A^{2n}(X)')$. Thus for all i, $A(\emptyset)^{(i)} \equiv_T A(\emptyset) \oplus \emptyset^{(i)} \equiv_T A(\emptyset^{(i)})$, uniformly in i (just induct using the basic relation $A(X)' \equiv_T A(X) \oplus X' \equiv_T A(X'))$. Hence $A(\emptyset)$ is low. If we also guarantee that $A(\emptyset)$ is not arithmetic (which follows if $A(X) >_T X$ for all X: if $A(\emptyset) \leq_T \emptyset^{(n)}$, then $A(\emptyset^{(n)}) \equiv_T A(\emptyset) \oplus \emptyset^{(n)} \leq_T \emptyset^{(n)}$, which contradicts $A(\emptyset^{(n)}) >_T \emptyset^{(n)}$), then we have a solution to the analog of Post's problem.

Proof of the Z,B,C lemma: The proof we give is Harrington's, although it has been simplified slightly by the use of Lachlan's "hat trick." The construction of B is similar to the usual one for the Sacks Jump Theorem (e.g. see Soare [1986]) but needs to be jazzed up a bit.

Remark: We cannot simply choose B as in the usual Sacks Jump Theorem such that $(Z \oplus B)' \equiv_T Z' \oplus W$, and then choose C. For example, if $W = Z''$, then $Z' = B$ would satisfy the above Turing equivalence, but the only C's r.e. in $Z \oplus B$ satisfying $Z' \oplus B \oplus C \equiv_T Z' \oplus W$ do not satisfy $(Z \oplus B \oplus C)' \equiv_T Z' \oplus W$ since the first equivalence gives $Z' \oplus C \equiv_T Z''$ and thus $(Z \oplus B \oplus C)' = (Z \oplus Z' \oplus C)' \equiv_T Z'''$. The construction of B must aid in controlling the jump of $Z \oplus B \oplus C$.

There is an S r.e. in Z such that for all e, $S^{[e]} = \omega$ iff $e \notin W$ and $S^{[e]} = n = \{0,\dots,n-1\}$ for some n iff $e \in W$. B will satisfy as in the usual Sacks construction $B^{[e]} =^* S^{[e]}$ for all e.

<u>The construction of C</u>: We are given Z and B and must construct C r.e. in $Z \oplus B$. We have requirements P_0^C, N_0^C, P_1^C, ... listed in order of decreasing priority. P_e^C is a positive requirement (in that its action is to cause elements to be enumerated into C) which attempts to code in where $B^{[e]}$ stops switching. In the end this coding will enable us to show that $Z' \oplus B \oplus C \geq_T W$. The requirements of the form N_e^C are negative requirements (in that they attempt to prevent elements from entering C). These requirements help insure that $(Z \oplus B \oplus C)'$ does not get too high. The superscript "C" will be dropped when clear from the context.

N_e: N_e restrains y at stage s+1 if $y \notin C_s$ and $\{e\}_s(Z \oplus B \oplus C_s; e)\downarrow$ and $y <$ the C-use of $\{e\}_s(Z \oplus B \oplus C_s; e)$ (where the C-use is the greatest value of C_s consulted in the computation — N_e is merely trying to hold any convergences).

P_e: P_e wants to put y into C at stage s+1 if $y = \langle e,x \rangle$ and $\exists x'(\langle e,x' \rangle \leq s,\ x' > x$, and $x \in B^{[e]} \leftrightarrow x' \notin B^{[e]})$.

<u>Stage 0</u>: $C_0 = \emptyset$.

<u>Stage s+1</u>: We put y into C_{s+1} if some P_e for $e \leq s$ wants to and no N_i for $i < e$ restrains y.

Note that if $\forall e(B^{[e]} =^* S^{[e]})$, then C is given by a simple finite injury construction, since $B^{[e]}$ is either finite or cofinite.

<u>The construction of B</u>: We are given Z, S r.e. in Z as defined from (an index for) W above, and an index for the r.e. operation C. That is, given sets $\tilde{Z}$ and $\tilde{B}$, and given s, the above C-construction defines $C_s(\tilde{Z},\tilde{B})$.

<u>Notation</u>:

$$\text{Let } b_s = \begin{cases} \text{least } x \text{ such that } x \in B_s - B_{s \dot{-} 1} \\ \qquad \text{if } B_s - B_{s \dot{-} 1} \neq \emptyset \\ s \qquad \text{otherwise.} \end{cases}$$

Let $\gamma(t,\tilde{Z},\tilde{B}) = \mu x \geq t\ \forall x' \geq x$ (during the first t stages of the C construction applied to $\tilde{Z}$, $\tilde{B}$, the value of $\tilde{B}(x')$ is not consulted). If $\tilde{Z}$ is understood, then we write just $\gamma(t,\tilde{B})$. In the specific construction with which we are now dealing, $\forall\tilde{B}\forall t(\gamma(t,\tilde{B}) \leq t)$, and in fact in almost all recursion theoretic constructions of this type it is safe to assume that at least u stages are required in order to use the oracle up through u. Later, however, we will not be able to assume this. We will be adding a requirement to the C construction which will need more than $B \upharpoonright t$ at stage t. Introducing γ now will make the discussion of the modified construction easier when we need it in Chapter 4. For now, it is safe to think of "$\gamma(t,\tilde{B})$" as "t".

Even though it abuses notation slightly, let

$$\hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s); e) = \begin{cases} \{e_s\}(Z \oplus B_s \oplus C_t(Z,B_s); e) & \text{if this converges and the B-use of the computation is less than } b_s \text{ and } \gamma(t,B_s) < b_s. \\ \uparrow & \text{otherwise.} \end{cases}$$

Let T = "the set of true stages in the enumeration of B" = $\{s | B_s \upharpoonright b_s = B \upharpoonright b_s\}$. Note:

(2) If $s \in T$ and $\hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s); e)\downarrow$, then $\forall s' \geq s\ (\hat{\Phi}_{e,s'}(Z \oplus B_{s'} \oplus C_t(Z,B_{s'}); e)\downarrow =$ $\hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s); e) = \{e\}(Z \oplus B \oplus C_t(Z,B); e))$, by the same computation.

This is because, after stage s, B does not change in any way which could affect C_t or the use of the computation.

We have requirements P_0^B, N_0^B, P_1^B, ... listed in order of decreasing priority. Again, the P_e^B's are positive and the N_e^B's are negative, and the superscript "B" is dropped when clear from the context.

P_e: P_e wants to put $\langle e,x \rangle$ into B at stage s+1 if $x \in S_s^{[e]}$.

N_e: If $t \leq s$, N_e restrains y at stage s+1 because of t if $y \notin B_s$, $y < \gamma(t,B_s)$ and

(3) $\hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s); e)\downarrow$, $u \le t$ and

$\forall t'(u \le t' < t \to C_{t'}(Z,B_s)\upharpoonright u \ne C_t(Z,B_s)\upharpoonright u)$,

where u is the use of the computation.

Note that if N_e succeeds in keeping all such y out of B throughout the rest of the construction, then $\{e\}(Z \oplus B \oplus C_t(Z,B); e)\downarrow =$ $\hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s); e)$ since B does not change below the use and B does not change in the part that C can view in the first t stages of its construction. The reason for the last clause of (3) is so that the same computation does not give rise to infinitely much restraint — we will be able to show that there are only finitely many t such that N_e permanently restrains something because of t. B may permanently restrain something due to a false computation (C may not have settled down yet), but this will not happen too often.

The construction of B is as usual, that is, if a requirement P_e for $e \le s$ wants to put something into B at stage s+1, it does, unless restrained by some N_i for $i < e$.

Sublemma 1: If P_e^B wants to put y into B then P_e^B puts y into B unless some stronger requirement restrains y because of some fixed t for all sufficiently large stages. In such a situation, y is said to be permanently restrained with cause.

Proof: Say P_e wants to put y into B at stage s. Let s_0 be the first true stage after s. If no N_i for $i < e$ restrains y at stage s_0+1, then P_e puts y into B. If N_i restrains y at stage s_0+1 because of t then by (2), we have the same computation for all $s' \geq s_0+1$, i.e. (3) holds with s' replacing s. Since $B_{s'} \restriction \gamma(t, B_{s_0}) = B_{s_0} \restriction \gamma(t, B_{s_0})$ we have $\gamma(t, B_{s'}) = \gamma(t, B_{s_0}) > y$ so y is restrained because of t at stage s'.

<u>Sublemma 2</u>: For each e,

i) $B^{[e]}$ and $S^{[e]}$ differ on only a finite set.

ii) $C^{[e]}$ is finite.

iii) Only finitely many y are permanently restrained by N_e^C.

iv) Only finitely many y are permanently restrained with cause by N_e^B.

Let U be either $Z' \oplus W$ or $Z' \oplus B \oplus C$, and let $R_e^B = \{y | y$ is permanently restrained with cause by some N_i^B with $i < e\}$, and $R_e^C = \{y | y$ is permanently restrained by some N_i^C with $i < e\}$.

v) Canonical indices for $C^{[e]}$, $B^{[e]}$, R_{e+1}^B and R_{e+1}^C are computable uniformly from U.

vi) U can uniformly decide if $\{e\}(Z \oplus B \oplus C; e)\downarrow$.

<u>Proof</u>: By induction. Assume i) through v) for e' < e. By sublemma 1 (and the fact that $\forall t(\lim_s \gamma(t, B_s) = \gamma(t, B) < \infty$ so the set of y's

permanently restrained because of some fixed t is finite) we immediately have i) and hence we have ii), since $\lim_x B^{[e]}(x)$ exists. The standard argument now shows that iii) holds.

<u>Claim</u>: U can compute the canonical indices for $C^{[e]}$ and $B^{[e]}$. By induction hypothesis we have canonical indices for R_e^B and R_e^C.

<u>Case 1</u>: $U = Z' \oplus B \oplus C$. Let $\hat{x} = \mu x(x \notin R_e^C \ \& \ x \notin C^{[e]})$. U can compute $\hat{x}$. Since $C^{[e]} = \hat{x} - R_e^C$, U can compute the canonical index for $C^{[e]}$. Since $\hat{x}$ puts a bound on where $B^{[e]}$ changes, U can compute a canonical index for $B^{[e]}$.

<u>Case 2</u>: $U = Z' \oplus W$. $S^{[e]} = \omega \Longleftrightarrow e \notin W$. If $S^{[e]} = n$, Z' can compute n. So $Z' \oplus W$ can compute the canonical index for $S^{[e]}$. By Sublemma 1, $B^{[e]} = S^{[e]} - R_e^B$ so we have the canonical index for $B^{[e]}$. Let $\tilde{x} = \mu x(\forall x' > x(x' \in B^{[e]} \Longleftrightarrow x \in B^{[e]}))$. Since $C^{[e]} = \tilde{x} - R_e^C$ we have the canonical index for $C^{[e]}$.

We have uniformly from U an index for $B^{[\leq e]}$ as a recursive set. We have a canonical index for $C^{[\leq e]}$ uniformly from U.

Let $\Theta(e,s,t,u)$ assert:

(3) holds and $B_s^{[\leq e]} \restriction \gamma(t,B_s) = B^{[\leq e]} \restriction \gamma(t,B_s)$ and $C_t^{[\leq e]}(Z,B_s) \restriction u = C^{[\leq e]} \restriction u$.

Note that if $\{e\}(Z \oplus B \oplus C; e)\downarrow$ then $\Theta(e,s,t,u)$ for some s,t,u, for let u = use of the computation. Choose $t \geq u$ so that $C_t^{[\leq e]} \restriction u =$

$C^{[\le e]}\restriction u$ and $\forall t'(u \le t' < t \to C_{t'}\restriction u \ne C_t\restriction u)$. Next choose s so that $\forall s' \ge s(\gamma(t,B_{s'}) = \gamma(t,B))$ and so that $B_s^{[\le e]}\restriction\gamma(t,B) = B^{[\le e]}\restriction\gamma(t,B)$. Then $\Theta(e,s,t,u)$ holds. Also note that given indices for $B^{[\le e]}$ and $C^{[\le e]}$, Z', hence U, can effectively tell if $\exists s,t,u(\Theta(e,s,t,u))$.

Claim 1: If $\Theta(e,s,t,u)$ then:

a) $\forall s' \ge s(\gamma(t,B_{s'}) = \gamma(t,B))$.

b) $B\restriction\gamma(t,B) = B_s\restriction\gamma(t,B)$.

c) $C\restriction u = C_t(Z,B_s)\restriction u$.

d) $\{e\}(Z \oplus B \oplus C;\ e)\downarrow = \hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s);\ e)$, by the same computation.

Let $\sigma(x) = \mu x'(x' \ge x$ and $B_{x'}\restriction\gamma(x,B) = B\restriction\gamma(x,B))$.

e) $\forall s' \ge \sigma(t)(\Theta(e,s',t,u))$.

f) $\forall s',u',t'(\Theta(e,s',t',u') \to s' \ge \sigma(t)$ and $t' = t$ and $u' = u)$.

Proof: a) & b): Since no P_i for $i \le e$ puts anything into B below $\gamma(t,B_s)$ after stage s by assumption, and since $u \le t \le \gamma(t,B_s)$, induct on s' to see that $s' \ge s$ implies $\gamma(t,B_{s'}) = \gamma(t,B_s) = \gamma(t,B)$ and $B_s\restriction\gamma(t,B) = B_{s'}\restriction\gamma(t,B)$. This is because N_e protects against any action for positive requirements which could harm the computation.

c): By b) $C_t(Z,B) = C_t(Z,B_s)$. Thus $\{e\}(Z \oplus B \oplus C_t(Z,B))\downarrow = \hat{\Phi}_{e,s}(Z \oplus B_s \oplus C_t(Z,B_s))$, so N_e^C tries to preserve $C\restriction u$. Since P_i^C for $i \le e$ cannot change $C_t\restriction u$ any longer, N_e^C succeeds.

d), e), f): These follow from b) & c) and the minimality of t insisted on in (3).

Let $\hat{u},\hat{t}$ = the unique u,t such that $\Theta(e,\sigma(t),t,u)$, if they exist. Since we have a canonical index for the <u>finite</u> set $C^{[\leq e]}$ from U and since $B \leq_T Z' \leq_T U$, we can effectively in U find $\tilde{t}$ such that $C_{\tilde{t}}^{[\leq e]}(Z,B) = C^{[\leq e]}$.

<u>Claim 2</u>: a) N_e^C permanently restrains y $\iff$ $\hat{t}$ exists and $y \in \hat{u} - C_{\hat{t}}(Z,B)$.

b) N_e^B permanently restrains y with cause $\iff$ either

i) $\hat{t}$ exists and N_e^B restrains y at stage $\sigma(\hat{t})$ because of $\hat{t}$, or

ii) N_e^B restrains y at stage $\sigma(\tilde{t})$ because of t and $t < \tilde{t}$.

<u>Proof</u>: We have already established "$\Leftarrow$" for a) and b).

Assume N_e^C permanently restrains something. Then $\{e\}(Z \oplus B \oplus C;\, e)\downarrow$, so $\hat{t}$ exists. But then N_e^C restrains exactly $\hat{u} - C_{\hat{t}}(Z,B)$, so we have the "only if" for a).

Assume N_e^B permanently restrains y because of t. If $t < \tilde{t}$, we have ii). If $t \geq \tilde{t}$, then $C^{[\leq e]} = C_t^{[\leq e]}(Z,B) = C_t^{[\leq e]}(Z,B_{\sigma(t)})$. Since $B_{\sigma(t)} \restriction \gamma(t,B) = B \restriction \gamma(t,B)$, we have that N_e^B restrains y because of t at stage $\sigma(t)$, and for some $u \leq t$ $\Theta(e,\sigma(t),t,u)$. Thus i) holds.

Since U can uniformly effectively decide the truth of the assertions on the right of the equivalence in Claim 2, and since $\{e\}(Z \oplus B \oplus C;\, e)\downarrow \iff \exists s,t,u(\Theta(e,s,t,u))$ (so U can calculate the jump of $Z \oplus B \oplus C$), we have finished the proof of Sublemma 2.

Since for all e, $\lim_x B^{[e]}(x) = \overline{W}(e)$, we have $Z' \oplus W \leq_T (Z \oplus B)'$. Since $Z' \oplus W \geq_T (Z \oplus B \oplus C)'$ and $Z' \oplus B \oplus C \geq_T (Z \oplus B \oplus C)'$ we get $(Z \oplus B \oplus C)' \equiv_T Z' \oplus B \oplus C \equiv_T Z' \oplus W$ as desired. All of this was done with the proper uniformities, so Lemma 2.1 is proved. □

Corollary 2.2: Given Z, W r.e. in Z', and D r.e. in Z with $(Z \oplus D)' \leq_T Z'$, there are $\tilde{B}$ r.e. in Z and C r.e. in $Z \oplus \tilde{B}$ satisfying the same Turing equivalences as in Lemma 2.1 and with $\tilde{B}^{[0]} = D$.

Proof: The proof is almost identical to the lemma, especially if we think of $\tilde{B}^{[0]}$ as $B^{[-1]}$ and $\tilde{B}^{[i+1]}$ as $B^{[i]}$. The proof then goes through almost unchanged notationally. In the construction of B, put x into $B^{[-1]}$ as soon as it turns up in the enumeration of D from Z. We then have in the proof an index for $B^{[\leq e]}$ as a set recursive in D uniformly in U. Since $(Z \oplus D)' \leq_T Z'$, U can still tell if $\exists s,t,u(\Theta(e,s,t,u))$. □

Theorem 2.3: There are ω-REA sets A_0 and A_1 such that $A_0 \oplus A_1 \equiv_T \emptyset^\omega$ and $A_i^\omega \equiv_T \emptyset^\omega$ for i = 0,1.

Corollary 2.4 (Harrington [1975]): There are arithmetically incomparable arithmetic singletons.

The corollary follows immediately from Theorem 2.3 (recall that ω-REA sets are Π^0_2-singletons). The proof of the theorem is somewhat simpler than Harrington's original construction of incomparable ω-REA sets, however. Use of Corollary 2.2 allows us to build the low ω-REA sets so that they join to $\emptyset^\omega$, and hence there is no need for explicit diagonalization, so Harrington's construction can be streamlined.

Proof of Theorem 2.3: We construct ω-REA operators A_0 and A_1 and sometimes denote the set $A_i(\emptyset)$ by just A_i for simplicity. By the Sacks Splitting Theorem (see Soare [1986]) there are indices ℓ_0 and ℓ_1 that uniformly split X' into low r.e.-in-X sets. That is, for all X, $W^X_{\ell_0} \oplus W^X_{\ell_1} \equiv_T X'$ (uniformly), and for i = 0,1, $W^X_{\ell_i} >_T X$ and $(W^X_{\ell_i})' \equiv_T X'$. Let d_i be a given integer parameter. The construction of $A_i(X)$ is as follows. Column 2m+1 and column 2m+2 are the B and C respectively of the Z,B,C-lemma with $Z = A_i^{2m}(X)$ (recall that $A_i^{2m}(X)$ denotes $A_i(X)^{[\leq 2m]}$) and with $a_{d_i}(m)$ the index for W as a set r.e. in Z' (recall that

$$W^Y_{a_{d_i}(m)} = \begin{cases} \emptyset & \text{if } \{d_i\}(m)\uparrow \\ W^Y_{\{d_i\}(m)} & \text{if } \{d_i\}(m)\downarrow). \end{cases}$$

However, the B and C constructions are modified by the inclusion of finitely many negative requirements of strongest priority which try to make $\{e\}(A_i(X); e)\downarrow$ for e < m. This part of the construction acts to make $A_i(X)' \equiv_T \oplus_m (A_i^{2m}(X)')$. The construction of column 1 is such that $A_i(X)^{[1][0]} = W^X_{\ell_i}$. That is, we use Corollary 2.2 unaltered to construct $A_i(X)^{[1]}$ and $A_i(X)^{[2]}$. The recursion theorem is used to piece everything together, i.e. to insure that $A_i(X)' \equiv_T A_i(X) \oplus X' \equiv_T A_i(X')$.

Perhaps we should discuss more fully the role of the integer parameter d_i. We use the Z,B,C-lemma with $Z = A_i^{2m}(X)$ to define $A_i(X)^{[2m+1]}$ and $A_i(X)^{[2m+2]}$. Our goal is to have $A_i^{2n}(X)' \equiv_T A_i^{2n}(X) \oplus X'$ $\equiv_T A_i^n(X')$ for all n uniformly, for then, since we shall also insure that $A_i(X)' \equiv_T \oplus_{n<\omega} (A_i^{2n}(X)')$, we would have $A_i(X)' \equiv_T A_i(X) \oplus X' \equiv_T A_i(X')$

as desired. Assume by induction that we have $A^{2m}(X)' \equiv_T A_i^{2m}(X) \oplus X'$ $\equiv_T A_i^m(X')$. Then $A_i(X')^{[m+1]}$ is r.e. in $A_i^{2m}(X)'$, since $A_i^{2m}(X)' \equiv_T$ $A_i^m(X')$, so use of the Z,B,C-lemma with $Z = A^{2m}(X)$ to establish the basic equivalences with $n = m+1$ is justified. However, an <u>index</u> for $A_i(X')^{[m+1]}$ as a set r.e. in $A^{2m}(X)'$ (as required for the sake of uniformity) does not present itself immediately.

There are two approaches that could handle this issue. One would involve rather extensive book-keeping, e.g. naming the recursive functions which give the indices for the Turing equivalences of the Z,B,C-lemma and using these to calculate what index we should be using for the next W. Clearly this would be quite unpleasant. The approach we use is to simply observe that there is some recursive function which does this, and use the parameter d_i as its index. In the end, the recursion theorem (actually the Double Recursion Theorem (Smullyan [1961], or see Rogers [1967]) since we also use an index for the construction in the construction) applied to d_i gives the desired conclusions.

This seems to be one of the rare arguments in the literature which makes use of the Double Recursion Theorem. Actually in Chapters 3 and 4 we even use a "Multiple Recursion Theorem" which can be proved using a straightforward generalization of the proof of the Double Recursion Theorem.

An important feature of both the B_m and C_m constructions (i.e. the constructions of columns 2m+1 and 2m+2, respectively) is that if $Y_0 \upharpoonright s = Y_1 \upharpoonright s$, then using Y_0 or Y_1 as input into the enumeration will result in the same numbers being enumerated up through stage s. That is, stage s of the r.e. operator associated with any column depends only on the input below s.

For the purposes of the new requirements, we need to consider in the construction of column n what would happen at stage t in the construction of column n', for various t and n' > n, if the input looked a certain way. This is of course put on a sound footing via the recursion theorem. We can think of what follows as defining an index $\bar{e}_i$ with $\{\bar{e}_i\}^Y(n,s)$ = the canonical index of the set of numbers enumerated at stage s when applying the operator defined for column n to input Y. The definition of $\{\bar{e}_i\}^Y(n,s)$ depends on $\{\bar{e}_i\}^{\tilde{Y}}(n',t)$ for various n' > n, t < s and $\tilde{Y}$. If all computations with index $\bar{e}_i$ used in the definition of $\{\bar{e}_i\}^Y(n,s)$ converge, then $\{\bar{e}_i\}^Y(n,s)$ converges. But since we only look at t < s, we can do an easy induction on s to show that the index $\bar{e}_i$ has the property that $\forall Y \forall n \forall s (\{\bar{e}_i\}^Y(n,s)\downarrow)$. Thus use of the recursion theorem allows us to use stages of future columns as desired.

<u>Definition 2.5</u>: a) For i = 0 or 1, let $A_i(Y,n,s) = \{x \mid x$ is enumerated by stage s when we apply the operator associated with column n of A_i to $Y\} = \bigcup_{s' \leq s} \{x \mid x$ is in the set with canonical index $\{e_i\}^Y(n,s')\}$.

b) For i = 0 or 1 and $\tau \in \omega^{<\omega}$, define $A_i^{n,\tau}(Y)$ by induction on $|\tau|$. $A_i^{n,\emptyset}(Y) = Y^{[\leq n]}$ (note: $A_i^{n,\emptyset}(Y) \neq A_i^n(Y)$) and $A_i^{n,\tau * s}(Y) =$

$A_i^{n,\tau}(Y) \cup \{n + |\tau| + 1\} \times A_i(A_i^{n,\tau}(Y),\ n + |\tau| + 1,\ s)$.

In particular, $A_i^{n,\tau}(A_i^n(X))$ is the approximation to $A_i(X)$ obtained by starting with $A_i^n(X)$, feeding this into the enumeration of column n+1 and running for $\tau(0)$ stages, feeding the output of this and $A_i^n(X)$ into the enumeration of column n+2 and running for $\tau(1)$ stages, and so forth.

We now suppress writing "i" for the rest of the description of the construction, so we are given d.

<u>The C_m-construction</u>: We define the r.e. operator associated with column 2m+2 (for $m \geq 1$ — recall that C_0, or column 2, is constructed using Corollary 2.2 unaltered). We are given sets Z_m (= $A^{2m}(X)$ in practice) and B_m (= $A(X)^{[2m+1]}$ in practice) and the index $a_d(m)$ for a set r.e. in Z_m'. We have new negative requirements M_e for $e < m$, and a priority listing $M_0, M_1, \ldots, M_{m-1}, P_0, N_0, P_1, \ldots$.

<u>Definition 2.6</u>: For $\tau \in \omega^{<\omega}$, $\psi(e,\tau,n,Y) \iff \tau(0) > \tau(1) > \ldots > \tau(|\tau|-1)$, $\{e\}(A^{n,\tau}(Y);\, e)\downarrow$ in exactly $n + |\tau|$ steps, and $\tau(|\tau|-1) > n+|\tau|$. (By convention, if a computation converges in t steps, then it asks about neither any elements of the oracle in columns numbered greater than t, nor any elements greater than t in any column.)

The value of the set variable Y will depend on the context in which ψ is used. For a B_m-construction Y will be Z_m. For a C_m-construction Y will be $Z_m \cup (\{2m+1\} \times B_m)$. That is, Y will be the input into the r.e. operator being defined. Note that if $\psi(e,\tau,n,Y)$ holds, then there is an approximation of the ω-REA set being constructed which yields a convergence for the jump. The idea is that the M_e requirements act to hold onto such approximated convergences.

P_e: Exactly as in Lemma 2.1.

N_e: Exactly as in Lemma 2.1.

M_e: Let $Y = Z_m \cup (\{2m+1\} \times B_m)$. At stage s+1 M_e restrains y if $y \notin C_{m,s}$ and there is a τ such that $s \geq \tau(0)$, $\psi(e,\tau,2m+1,Y)$ and $y < \tau(0)$ where τ is chosen so that $\tau(0)$ is least possible.

The specific choice of notation here perhaps obscures the idea. In the construction of column 2m+2 (= C_m) applied to $A^{2m+1}(X)$, say M_e puts on restraint at stage s because $\psi(e,\tau,2m+1, A^{2m+1}(X))$ where τ is lexicographically least and $|\tau| > 1$. Then $C_{m,s}\upharpoonright\tau(0) = C_m\upharpoonright\tau(0) = C_{m,\tau(0)}\upharpoonright\tau(0)$, because M_e has higher priority than any positive requirement and first acted at stage $\tau(0) + 1$. Thus the approximation to the next column obtained by using $A^{2m+2}(X)$ as input and running for $\tau(1)$ stages is the same as the approximation obtained by using $A^{2m+1,<\tau(0)>}(A^{2m+1}(X))$ as input and running for $\tau(1)$ stages. This is because in $\tau(1)$ stages we do not use any part of column 2m+2 greater than $\tau(1)$, and $\tau(1) < \tau(0)$. So the construction of the next column will also act (at stage $\tau(1)$) to hold the computation, and so on down the line. It is for technical reasons (simplifying the proof of a future lemma) that we require the computation to take exactly $n+|\tau|$ steps. We could merely require that it take <u>no more than</u> $n+|\tau|$ steps.

<u>The B_m-construction</u>: If $m = 0$, use Corollary 2.2. Otherwise, we are given Z_m (= $A^{2m}(X)$ in practice), the index $a_d(m)$ for W as a set r.e. in Z_m', and the index for the r.e. operation C_m. We have new requirements M_e for $e < m$, and a priority listing M_0, M_1, ..., M_{m-1}, P_0, N_0, P_1, ...

P_e: Just as in Lemma 2.1.

N_e: Just as in Lemma 2.1.

M_e: At stage s+1, M_e restrains y if $y \notin B_{m,s}$, and there is a τ such that $\psi(e,\tau,2m,Z_m)$, $s \geq \tau(0)$, and $y < \tau(0)$ where τ is such that $\tau(0)$ is least possible.

A(X) is defined as previously indicated, i.e. $A(X)^{[0]} = X$ and for all m, $A(X)^{[2m+1]}$ = the result of applying the B_m-construction to input $Z_m = A^{2m}(X)$, and $A(X)^{[2m+2]}$ = the result of applying the C_m-construction to input $Z_m = A^{2m}(X)$ and $B_m = A(X)^{[2m+1]}$.

<u>Claim 1</u>: $A(X)' \equiv_T \bigoplus_{m<\omega} (A^{2m}(X)')$.

<u>Proof</u>: If $\{e\}(A(X); e)\downarrow$, then either $\{e\}(A(X); e)\downarrow$ in $\leq 2e+2$ steps, or $\exists\tau(\psi(e,\tau,2e+2, A^{2e+2}(X)))$ (the significance of 2e+2 is that it gives the first place where M_e is considered). We will show that the implication here can be reversed.

Assume that in the construction of column n+1 (for some n), M_e puts on restraint because $\psi(e,\gamma,n,A^n(X))$, where γ is lexicographically least, and that $|\gamma| > 1$. By the comments following the statement of $M_e^{C_m}$, we have that $\psi(e, <\gamma(1), \ldots, \gamma(|\gamma|-1)>, n+1, A^{n+1}(X))$ holds. Thus in the construction of column n+2, M_e acts at stage $\gamma(1)+1$. We therefore have that M_e acts in the construction of column 2e+3 because of τ implies $\forall m < |\tau|\ (A^{2e+2,\tau}(X)^{[2e+3+m]} \upharpoonright \tau(m) = A(X)^{[2e+3+m]} \upharpoonright \tau(m))$. In words, $A^{2e+2,\tau}(X)$ and $A(X)$ are identical as oracles as far as the computation $\{e\}(\cdot;e)$ is concerned. Thus $\exists\tau\psi(e,\tau,2e+2,A^{2e+2}(X)) \rightarrow \{e\}(A(X);\ e)\downarrow$. So we have $\{e\}(A(X);\ e)\downarrow \iff$ either

a) $\{e\}(A(X);\ e)\downarrow$ in $\leq 2e+2$ steps

or b) M_e acts in the construction of column 2e+3 (i.e. $\exists\tau\psi(e,\tau,2e+2,A^{2e+2}(X))$).

Thus we have a way to decide if $\{e\}(A(X);\ e)\downarrow$ uniformly from $\bigoplus_{m<\omega} (A^{2m}(X)')$. This establishes Claim 1.

Next we note that the argument for Lemma 2.1 still goes through with the addition of the new requirements M_e, because Z_m' can effectively determine what the actions of the appropriate M_e's are. This is clear for the new B_m construction, i.e. Z_m' can effectively determine the action of $M_e^{B_m}$ for $e < m$. At first glance it may appear to require $(Z_m \oplus B_m)'$ for the requirements $M_e^{C_m}$. However, note that $\psi(e,\tau,2m+1,A^{2m+1}(X)) \rightarrow \exists t > \tau(0)(\psi(e,t*\tau,2m,A^{2m}(X))$ and $[\psi(e,\gamma,2m,A^{2m}(X))$ and we act for γ in construction of $B_m] \rightarrow |\gamma| = 1$ or $\psi(e, <\gamma(1), \ldots, \gamma(|\gamma|-1)>, 2m+1, A^{2m+1}(X))$. Thus Z_m'

(and hence "U_m" in either of the two possible cases) can effectively determine the action of the M_e's in C_m's construction also. Hence for all m and X, $A^{2m+2}(X)' \equiv_T A^{2m}(X)' \oplus A(X)^{[2m+1]} \oplus A(X)^{[2m+2]} \equiv_T A^{2m}(X)' \oplus W(a_d(m); A^{2m}(X)')$ uniformly, and thus

(4) $A^{2m+2}(X)' \equiv_T A^{2m+2}(X) \oplus X' \equiv_T A^{2m}(X)' \oplus W(a_d(m); A^{2m}(X)')$
uniformly.

<u>Claim 2</u>: We can choose d so that the ω-REA operator defined satisfies $A(X)' \equiv_T A(X) \oplus X' \equiv_T A(X')$ uniformly in X.

<u>Proof</u>: Let ê be an index such that $J_{\hat{e}} = A$ (so ê is a recursive function of d). The proof is really just a computation which shows that the recursion theorem is applicable. We first define a recursive function f. Let f(0) be such that $W(f(0); \{0\} \times X') \equiv_T W(a_d(0); (\{0\} \times X)')$. Thus, from (4), $J^2_{\hat{e}}(X)' \equiv_T J^2_{\hat{e}}(X) \oplus X' \equiv_T J^1(f,X')$. Assume that $f\upharpoonright m$ has been defined so that for all X

(5) $J^{2m}_{\hat{e}}(X)' \equiv_T J^{2m}_{\hat{e}}(X) \oplus X' \equiv_T J^m(f,X')$, uniformly.

Choose f(m) so that

(6) $W(f(m); J^m(f,X')) = W(a_d(m); J^{2m}_{\hat{e}}(X)')$.

Hence $J^{2m+2}_{\hat{e}}(X)' \equiv_T J^{2m+2}_{\hat{e}}(X) \oplus X' \equiv_T J^{m+1}(f,X')$ uniformly. So we have by induction (5) and (6) for all m. Thus from (5) there is a recursive function g such that for all m

(7) $W(g(m);\ J_{\hat{e}}^{2m}(X)') = W(a_{\hat{e}}(m);\ J^{m}(f,X'))$.

An index for g as a recursive function is obtainable uniformly from d, that is $g = \{h(d)\}$ for some recursive h. Choose $\hat{d}$ so that $\{\hat{d}\} = \{h(\hat{d})\}$. (To be more precise, we are really using the Double Recursion Theorem (Smullyan [1961]) so that the use of an index for the construction in the construction is also justified.) Then by (6) and (7), $W(f(m);\ J^{m}(f,X')) = W(a_{\hat{d}}(m);\ J_{\hat{e}}^{2m}(X)') = W(g(m);\ J_{\hat{e}}^{2m}(X)') = W(a_{\hat{e}}(m);\ J^{m}(f,X'))$ for all m. Thus $J(f,X') = J_{\hat{e}}(X')$, and so by (5) and Claim 1, we get $J_{\hat{e}}(X)' \equiv_T \bigoplus_{m<\omega} (J_{\hat{e}}^{2m}(X)') \equiv_T J_{\hat{e}}(X) \oplus X' \equiv_T J_{\hat{e}}(X')$ uniformly in X. In other words we have

(8) $A(X)' \equiv_T A(X) \oplus X' \equiv_T A(X')$ uniformly in X.

This finishes the proof of Claim 2.

By induction on n, using (8), we easily get for all n and X

(9) $A(X)^{(n)} \equiv_T A(X) \oplus X^{(n)} \equiv_T A(X^{(n)})$ uniformly in X and n.

If we now "put i back in", we have defined for all X, $A_0(X)$ and $A_1(X)$.

Claim 3: $A_0(\emptyset) \oplus A_1(\emptyset) \equiv_T \emptyset^\omega$, and $A_i(\emptyset)$ is low for $i = 0,1$.

Proof: First we show that $\emptyset^{(n)} \leq_T A_0(\emptyset) \oplus A_1(\emptyset)$ uniformly in n for every n. $A_0(\emptyset) \oplus A_1(\emptyset) \geq_T \emptyset'$ since we have coded a Sacks split of $\emptyset'$ into the first columns of the $A_i(\emptyset)$. Thus $A_0(\emptyset) \oplus A_1(\emptyset) \geq_T A_0(\emptyset') \oplus A_1(\emptyset')$ by (8). Continue inductively. Assume that $A_0(\emptyset) \oplus A_1(\emptyset) \geq_T A_0(\emptyset^{(n)}) \oplus A_1(\emptyset^{(n)})$. We can calculate (in a uniform way) $\emptyset^{(n+1)}$ from $A_0(\emptyset^{(n)}) \oplus A_1(\emptyset^{(n)})$, so by (8) again we get $A_0(\emptyset) \oplus A_1(\emptyset) \geq_T \emptyset^{(n+1)} \oplus A_0(\emptyset^{(n)}) \oplus A_1(\emptyset^{(n)}) \geq_T A_0(\emptyset^{(n+1)}) \oplus A_1(\emptyset^{(n+1)})$ uniformly, and the induction continues.

The lowness of the $A_i(\emptyset)$ follows from (9). That is, $\emptyset^\omega \geq_T A_i(\emptyset) \oplus \emptyset^{(n)} \geq_T A_i(\emptyset)^{(n)}$ uniformly in n, so $\emptyset^\omega \geq_T A_i(\emptyset)^\omega$. □

Remark: Rather than using Corollary 2.2 as above one might argue that we should proceed as follows. Simply define $A_i(X)^{[1]}$ to be $W^X_{\ell_i}$. Next use the Z,B,C-lemma with $Z = A^1_i(X)$ to produce $A_i(X)^{[2]}$ and $A_i(X)^{[3]}$. Since $A^1_i(X)' \equiv_T X'$, we can guarantee that $A^3_i(X)' \equiv_T A^3_i(X) \oplus X' \equiv_T X' \oplus W(a_{d_i}(0),X')$. The method of the theorem can now be used to define A_i so that $\forall m\ (A^{2m+1}_i(X)' \equiv_T A^{2m+1}_i(X) \oplus X' \equiv_T A^m_i(X'))$ and $A_i(X)' \equiv_T \bigoplus_m (A^{2m+1}_i(X)')$. Thus, again $A_i(X)' \equiv_T A_i(X) \oplus X' \equiv_T A_i(X')$, and $A_0(\emptyset)$ and $A_1(\emptyset)$ are low ω-REA sets which join to $\emptyset^\omega$. This approach requires less work since we need not check that the proof of the Z,B,C-lemma generalizes as claimed in Corollary 2.2. In Chapter 4, however, the approach would introduce some minor difficulties. The presentation chosen seems to allow for a more "uniform" exposition.

Chapter 3

The range of the ω-jump on degrees below $\underline{0}^{\omega}$

Jockusch and Shore [1983] showed that given i, there is an e such that $\forall X(W_e^X \oplus W_i^{W_e^X} \equiv_T X'$ and $W_e^X >_T X)$, and in fact that e can be obtained uniformly from i. In our notation this says given j there is a k so that $\forall X(J_j^1(J_k^1(X)) \equiv_T X'$ and $J_k^1(X) >_T X)$. In unrelativized form, this says intuitively that for any r.e. operation, we can find a cone (a cone of degrees is of the form $\{\underline{d} \mid \underline{d} \geq \underline{b}\}$ for some fixed $\underline{b}$) with non-recursive r.e. base such that $\emptyset'$ looks (up to degree) like that operation. Jockusch and Shore used this result to give a finite injury proof that all the standard jump classes $H_{n+1} - H_n$, $L_{n+1} - L_n$ and I contain an r.e. set (an r.e. set A satisfies $A \in H_n$ if $A^{(n)} \equiv_T \emptyset^{(n+1)}$, $A \in L_n$ if $A^{(n)} \equiv_T \emptyset^{(n)}$, and $A \in I$ if $A \notin \bigcup_n (H_n \cup L_n)$). In [1984], they point out that the analogous theorem for ω-REA operators would in the same way establish the non-triviality of the ω-jump hierarchy for ω-REA sets. In fact it does more than that — we will deduce the analog of the Sacks Jump Theorem and give a new <u>finite</u> injury proof of the standard Sacks Jump Theorem.

We first prove the analog of the Jockusch-Shore result: given i, there is a cone with non-arithmetic ω-REA base such that $\emptyset^{\omega}$ looks like the operation J_i in that cone. This is a generalization of the fact that there is a non-arithmetic low ω-REA set, where i is simply the index for the ω-jump.

Theorem 3.1: Given i, there is an e such that $J_i(J_e(X)) \equiv_T X^{\omega}$ and $J_e(X) >_a X$. Further, this is uniform, in the sense that e is given by f(i) for some recursive f, and works for all X.

Before focusing on the lemma required to prove this theorem, we try to give a rough indication of the approach. In Theorem 2.3, we produced an ω-REA operator J_{e_0} such that $J_{e_0}(X)' \equiv_T J_{e_0}(X) \oplus X' \equiv_T J_{e_0}(X')$. We will show that the construction can be generalized so that given k, there is an ω-REA operator J_{e_1} so that $J^1_k(J_{e_1}(X)) \equiv_T J_{e_1}(X) \oplus X' \equiv_T J_{e_1}(X')$ (from which the above of course follows: Let k be an index for the Turing jump). It would be nice if, analogously to the induction which establishes $J_{e_0}(X)^{(n)} \equiv_T J_{e_0}(X) \oplus X^{(n)} \equiv_T J_{e_0}(X^{(n)})$, we could show that $J^n_k(J_{e_1}(X)) \equiv_T J_{e_1}(X) \oplus X^{(n)} \equiv_T J_{e_1}(X^{(n)})$, for this would establish the result. However, the induction breaks down for two reasons. The first is because $\{k\}(n)$ is not necessarily constant and so we should be using different indices on different "levels" of the construction; for example $J_{e_1}(X')$ should behave appropriately with respect to $\{k\}(1)$, not $\{k\}(0)$. The solution to this is to build an array of ω-REA operators $\{J_{e_1,j} | j \in \omega\}$, with $J_{e_1,n}$ used for handling $\{k\}(n)$. The second reason the induction fails is that the 1-REA operators involved do not in general satisfy $J^1_k(Y_0) \equiv_T J^1_k(Y_1)$ if $Y_0 \equiv_T Y_1$ (indeed the question of whether there are <u>any</u> nontrivial such k is open). For example, suppose we have that $J^1_k(J_{e_1,0}(X)) \equiv_T J_{e_1,0}(X) \oplus X' \equiv_T J_{e_1,1}(X')$. We would like to next argue that $J^2_k(J_{e_1,0}(X)) \equiv_T J_{e_1,0}(X) \oplus X'' \equiv_T J_{e_1,2}(X'')$. The problem is that we do not know in advance what index $J_{e_1,1}$ should have been constructed to handle, even though there is an index $\bar{e}$ such that $J_i^2(J_{e_1,0}(X)) \equiv_T J_{\bar{e}}^1(J_{e_1,1}(X'))$. The index $\bar{e}$ depends (uniformly!) on the various parameters of the construction. By now the way to deal with this should be clear — the recursion theorem.

Definition 3.2: Let $\tilde{J}_e^1(X) = X \oplus W_e^X$. This is so that we can avoid writing "$J_{e_0}^1(X)$ where $\{e_0\}(0) = e$".

Lemma 3.3: Given b and d, there is an ω-REA operator $\hat{A}$ such that for all $X, \hat{A}(X) >_T X$ and $\tilde{J}_b^1(\hat{A}(X)) \equiv_T \bigoplus_{m<\omega} (\hat{A}^{2m}(X)') \equiv_T \hat{A}(X) \oplus X' \equiv_T \bigoplus_{m<\omega} \tilde{J}_{a_d(m)}^1(\hat{A}^{2m}(X)')$, uniformly.

Proof: The proof is a modification of what we have already done. We use the M requirements to keep $\tilde{J}_b^1(\hat{A}(X))$ down, but must do some coding of X' to get $\tilde{J}_b^1(\hat{A}(X)) \geq_T X'$. The idea here is like the proof of the theorem of Jockusch and Shore discussed above. We reserve column 0 in every C_m for this coding.

Fix a uniform 1-1 recursive-in-X enumeration $\{k_s^X | s < \omega\}$ of $K^X = X'$ (i.e. for any X use the same index). Again we use the recursion theorem (as explained before Definition 2.5) to justify the use of $\hat{A}^{n,\tau}(Y)$ in the construction, where $\hat{A}^{n,\tau}(Y)$ is defined analogously to $A_i^{n,\tau}(Y)$ in Theorem 2.3.

<u>The $\hat{C}_m$-construction</u>: We are given input $\hat{Z}_m$ and $\hat{B}_m$, and an index $a_d(m)$ for a set r.e. in $\hat{Z}'_m$. We have requirements M_i for $i < m$, $\hat{P}$, and P_e, N_e for $e < \omega$, with priority listing $M_0, \ldots, M_{m-1}, \hat{P}, P_0, N_0, P_1, \ldots$.

P_e: Just as in the C_m-construction of Theorem 2.3, except that P_e puts elements into $\hat{C}_m^{[e+1]}$ (we are reserving $\hat{C}_m^{[0]}$ for coding). That is, P_e wants to put $y = \langle e+1, x\rangle$ into $\hat{C}_m$ at stage s+1 if $\exists x'(\langle e,x'\rangle \leq s \; \& \; x' > x \; \& \; x \in \hat{B}_m^{[e]} \Longleftrightarrow x' \notin \hat{B}_m^{[e]})$.

N_e: Just as in the C_m-construction.

<u>Definition 3.4</u>: $\hat{\psi}(e,\tau,n,Y) \Longleftrightarrow \tau(0) > \tau(1) > \ldots > \tau(|\tau|-1)$ and $\{b\}(\hat{A}^{n,\tau}(Y); e)\downarrow$ in exactly $n+|\tau|$ steps and $\tau(|\tau|-1) > n+|\tau|$. We are interested in $\hat{\psi}$ rather than ψ because we are trying to control $\tilde{J}_b^1(\hat{A}(X))$ rather than $\hat{A}(X)'$.

M_e: Just as in the C_m-construction with ψ, Z_m and B_m replaced respectively by $\hat{\psi}$, $\hat{Z}_m$, and $\hat{B}_m$.

$\hat{P}$: Let $r(s+1)$ be the least element of $\{\langle 0,x\rangle | x \in \omega\}$ which is greater than the restraint imposed by all the M_e's at stage s+1. At stage s+1, put $r(s+1)$ in $\hat{C}_m$ if $k_s^X = m$ (where $X = Z_m^{[0]}$).

The $\hat{B}_m$-construction: We are given $\hat{Z}_m$, an index for the $\hat{C}_m$-construction, and an index $a_d(m)$ for a set r.e. in $\hat{Z}'_m$. If $m = 0$, to guarantee that $\hat{A}(X) >_T X$, we use Corollary 2.2 unaltered to put a low-in-X, non-recursive-in-X set into column 0 of $\hat{B}_0$ (here $\{0\}\times X$ is $\hat{Z}_0$, of course). Otherwise, we have requirements $M_0, M_1, \ldots, M_{m-1}$, $P_0, N_0, P_1, \ldots$, given in order of decreasing priority.

P_e, N_e, and M_e: Just as in the B_m-construction, but with the appropriate entities "hatted".

$\hat{A}(X)$ is of course now defined by applying the $\hat{B}_0$-construction to $\hat{Z}_0 = \{0\}\times X$ to get $\hat{A}(X)^{[1]}$, and in general applying the $\hat{B}_m$-construction to $\hat{A}^{2m}(X)$ to get $\hat{A}(X)^{[2m+1]}$, and the $\hat{C}_m$-construction to $\hat{A}^{2m}(X)$ and $\hat{A}(X)^{[2m+1]}$ to get $\hat{A}(X)^{[2m+2]}$.

The argument for claim 1 of Theorem 2.3 gives us $W(b;\hat{A}(X)) \leq_T \bigoplus_{m<\omega}(\hat{A}^{2m}(X)')$, so

(1) $\tilde{J}^1_b(\hat{A}(X)) \leq_T \bigoplus_{m<\omega}(\hat{A}^{2m}(X)')$.

Claim: $\tilde{J}^1_b(A(X)) \geq_T \hat{A}(X) \oplus X'$.

Proof: Suppose we wish to answer the question "is $m \in X'$?" Let $R = \{e < m | e \in W(b; A(X))$ and $\{b\}(A(X); e)\downarrow$ in greater than 2m+1 steps$\}$. In the $\hat{C}_m$-construction, M_e imposes restraint if and only if $e \in R$. Once an M_e imposes restraint, it never acts again. Look at the total restraint imposed by all the M_e for $e \in R$. For a stage s_0

after all these M_e have acted, $m \in X' \iff \exists \langle 0,y\rangle \leq r(s_0)$ ($\langle 0,y\rangle \in \hat{C}_m = \hat{A}(X)^{[2m+2]}$). This gives a uniform way to decide if $m \in X'$ effectively from $\tilde{J}^1_b(A(X))$.

Since Z'_m can decide the action of the new requirements in the C_m- and B_m-constructions, the argument for the Z,B,C lemma goes through with almost no change. So inductively we get for all m, $\hat{A}^{2m+2}(X)' \equiv_T \hat{A}^{2m+2}(X) \oplus X' \equiv_T \tilde{J}^1_{a_d(m)}(\hat{A}^{2m}(X)')$ uniformly, from the basic Turing equivalences of the Z,B,C-lemma. Hence, we have

$$\bigoplus_{m<\omega}(\hat{A}^{2m}(X)') \equiv_T \hat{A}(X) \oplus X' \equiv_T \bigoplus_{m<\omega}\tilde{J}^1_{a_d(m)}(\hat{A}^{2m}(X)'),$$

from which the lemma follows, when combined with (1) and the Claim. □

<u>Proof of Theorem 3.1</u>: As in Harrington's construction of incomparable ω-REA sets, we build an array $\{A_k : k<\omega\}$ of ω-REA operators. Lemma 2.3 is used to build A_k for each k. Given $\tilde{b}$ and $\tilde{d}$ as input, we use the lemma to construct A_k such that for all k

$$\text{(2)} \quad \tilde{J}^1_{a_{\tilde{b}}(k)}(A_k(X)) \equiv_T \bigoplus_{m<\omega}(A_k^{2m}(X)') \equiv_T$$

$$A_k(X) \oplus X' \equiv_T \bigoplus_{m<\omega}\tilde{J}^1_{a_{\tilde{d}}(\langle k,m\rangle)}(A_k^{2m}(X)'), \text{ uniformly.}$$

We can use the recursion theorem to obtain an index $\hat{d}$ such that if we use $\tilde{d} = \hat{d}$, then for all k,

$$\text{(3)} \quad \bigoplus_{m<\omega}(A_k^{2m}(X')) \equiv_T A_k(X) \oplus X' \equiv_T A_{k+1}(X'), \text{ uniformly in k and X.}$$

The argument for this is almost identical to the analogous argument of Theorem 2.3 (i.e. the proof of Claim 2 of that theorem), so we omit it.

Next we show that the recursion theorem gives us a $\hat{b}$ so that if we take $\tilde{b} = \hat{b}$, then for all k, $J_i^k(A_0(X)) \equiv_T A_{k+1}(X^{(k+1)})$ uniformly and thus $J_i(A_0(X)) \equiv_T X^{\omega}$. As for most of these types of calculations, this is not particularly illuminating (nor is it presented in the way it was discovered), but is crucial to the proof.

We define a recursive function h. Let h(0) be such that $W(h(0); A_0(X)) = W(a_{\tilde{b}}(0); A_0(X))$ and thus $J^1(h,A_0(X)) \equiv_T \tilde{J}^1_{a_{\tilde{b}}(0)}(A_0(X)) \equiv_T A_1(X')$, using (2) and (3). Suppose that $h\restriction k$ has been defined so that

$$(4)\quad J^k(h,A_0(X)) \equiv_T \tilde{J}^1_{a_{\tilde{b}}(k-1)}(A_{k-1}(X^{(k-1)})) \equiv_T A_k(X^{(k)}) \text{ uniformly}$$

Let h(k) be such that $W(h(k), J^k(h,A_0(X)) = W(a_{\tilde{b}}(k); A_k(X^{(k)}))$, and so (4) holds with "k+1" replacing "k" throughout. Now define a recursive function g so that for all k, $W(g(k); A_k(X^{(k)})) = W(a_i(k); J^k(h,A_0(X)))$. An index for g as a recursive function is obtainable effectively from $\tilde{b}$ (formally, from $\tilde{b}$, $\tilde{d}$, and an index for the construction — we really are using a version of the recursion theorem which says that given recursive functions $f_0, f_1, \ldots, f_n$, there are integers $\bar{e}_0, \ldots, \bar{e}_n$ such that for all $\ell \in \{0,\ldots,n\}$, $\{\bar{e}_\ell\} = \{f_\ell(\bar{e}_0,\ldots,\bar{e}_n)\}$). So take $\hat{b}$ such that if we used $\tilde{b} = \hat{b}$, then $\{\hat{b}\} = g$. Then, for all k, $W(h(k); J^k(h,A_0(X)) = W(a_{\hat{b}}(k); A_k(X^{(k)})) = W(g(k); A_k(X^{(k)})) = W(a_i(k); J^k(h,A_0(X)))$. Thus $J(h,A_0(X)) = J_i(A_0(X))$ and $J_i^k(A_0(X)) \equiv_T A_k(X^{(k)})$ uniformly, as desired.

Now, by an easy induction using the fact that for all Y and k, $A_k(Y) \oplus Y' \equiv_T A_{k+1}(Y')$ uniformly, we get that for all k and X, $A_0(X) \oplus X^{(k)} \equiv_T A_k(X^{(k)})$ uniformly. Thus $J_i(A_0(X)) \equiv_T X^{\omega}$, and $A_0(X) >_a X$. The second part follows since for all Y and k, $A_k(Y) >_T Y$, so if $A_0(X) \leq_T X^{(n)}$, then $X^{(n)} \geq_T A_0(X) \oplus X^{(n)} \geq_T A_n(X^{(n)})$ and we have a contradiction. If we take e so that $J_e = A_0$, then we have Theorem 3.1 (e is a recursive function of i by the recursion theorem with parameters). □

<u>Definition 3.5</u>: For $A \leq_a \emptyset^{\omega}$, $n \geq 0$, let $A \in \underset{\sim}{H}_n$ if $A^{\omega \cdot n} \equiv_a \emptyset^{\omega \cdot (n+1)}$, $A \in \underset{\sim}{L}_n$ if $A^{\omega \cdot n} \equiv_a \emptyset^{\omega \cdot n}$ and $A \in \underset{\sim}{I}$ if $A \notin \bigcup_n (\underset{\sim}{H}_n \cup \underset{\sim}{L}_n)$. Define the relativized jump classes $\underset{\sim}{H}_n^X$, $\underset{\sim}{L}_n^X$ and $\underset{\sim}{I}^X$ similarly.

<u>Corollary 3.6</u>: All the jump classes $\underset{\sim}{H}_{n+1} - \underset{\sim}{H}_n$, $\underset{\sim}{L}_{n+1} - \underset{\sim}{L}_n$ and $\underset{\sim}{I}$ are non-empty. In fact, there are incomparable ω-REA sets in each one.

<u>Proof</u>: The argument that gives an ω-REA set in each $\underset{\sim}{H}_{n+1} - \underset{\sim}{H}_n$, $\underset{\sim}{L}_{n+1} - \underset{\sim}{L}_n$, and in $\underset{\sim}{I}$ is just as in Jockusch and Shore [1983] for the corresponding result for r.e. sets, so we state it only briefly. If f is as in the statement of Theorem 3.1, and j_0 is an index for the ω-jump, then by induction on n, $J_{f^{2n+1}(j_0)}(X) \in \underset{\sim}{L}_{n+1}^X - \underset{\sim}{L}_n^X$ and $J_{f^{2n+2}(j_0)}(X) \in \underset{\sim}{H}_{n+1}^X - \underset{\sim}{H}_n^X$ for all X (here of course $f^m(j_0) \overset{\text{def}}{=} f(f(\ldots(j_0))\ldots)$, where f is applied m times). To produce something in $\underset{\sim}{I}^X$, let e_0 be such that $\{e_0\}^X = \{f(e_0)\}^X$ for all X. Then

$J_{e_0}(X) \in \underset{\sim}{I}^X$, since for all n,

$$\underbrace{J_{e_0}(J_{e_0}(\ldots J_{e_0}}_{\text{2n applications}}(X))\ldots) \equiv_T X^{\omega\cdot n},$$

so for all n

$$X^{\omega\cdot n} <_a \underbrace{J_{e_0}(J_{e_0}(\ldots J_{e_0}}_{\text{2n+1 applications}}(X))\ldots) <_a X^{\omega\cdot(n+1)}.$$

To get incomparable ω-REA sets in each jump class (other than $\underset{\sim}{H}_0$ or $\underset{\sim}{L}_0$) one defines functions f_0 and f_1 just like f, but also we guarantee that column 0 of column 1 of $A_{j,k}(X)$ is $W^X_{\ell_j}$, where ℓ_0 and ℓ_1 are chosen so that $W^X_{\ell_0} \oplus W^X_{\ell_1} \equiv_T X'$ uniformly. (In other words, we can use the trick of Theorem 2.3 in this context.) Then for every i with $J_i(Y) >_a Y$ for all Y, we have $J_{f_0(i)}(X) \overset{\text{def}}{=} A_{0,0}(X) |_a A_{1,0}(X) \overset{\text{def}}{=} J_{f_1(i)}(X)$. To see this, we use the fact that $\forall k \forall Y (A_{j,k}(Y) \oplus Y' \equiv_T A_{j,k+1}(Y'))$ uniformly, as before, to show that $A_{0,0}(X) \oplus A_{1,0}(X) \equiv_T X^\omega$. Then if $J_{f_0(i)}(X)$ and $J_{f_1(i)}(X)$ were arithmetically comparable, we would have $X^\omega \equiv_T J_i(J_{f_j(i)}(X)) >_a J_{f_j(i)}(X) \equiv_a X^\omega$ for either j = 0 or 1. Actually, we even have $J_{f_0(i_0)}(X) |_a J_{f_1(i_1)}(X)$ for any i_0, i_1 such that for j = 0,1, $J_{i_j}(Y) >_a Y$ for every Y, by the same argument. Therefore, the first part of the proof applied to f_0 and to f_1 produces incomparable ω-REA members of every jump class; indeed, all the sets with indices given by an iterate of f_j are incomparable with any set with index given by an iterate of f_{1-j}. □

Corollary 3.7: There are recursive functions f and g such that for all i and X, $J_i(J_{f(i)}(X)) \equiv_T X^{\omega}$, and this Turing equivalence has index g(i) (that is, g(i) codes a pair of indices for Turing reductions establishing the equivalence).

Proof: We have already argued for the existence of f. We need only examine the proof of Theorem 3.1 to see that we actually have such a g as well. This follows from all of the uniformities and the fact that when applying the recursion theorem, the fixed points may be expressed as recursive functions of the parameter i. □

Theorem 3.8: A Turing degree $\underline{b}$ is the degree of the ω-jump of an ω-REA set if and only if $\underline{b}$ is the degree of a set ω-REA in $\emptyset^{\omega}$.

Proof: If $A \leq_T \emptyset^{\omega}$ then the set C given by $C^{[0]} = \emptyset^{\omega}$, $C^{[1]} = A$ and $\forall n(C^{[n+1]} = A^{(n)})$ is ω-REA in $\emptyset^{\omega}$ and has the same Turing degree (1-1 degree even) as A^{ω}.

Suppose now that $\deg(J_i(\emptyset^{\omega})) = \underline{b}$. We will (uniformly in i) produce an ω-REA set whose ω-jump has degree $\underline{b}$. See Figure 3.1 as an aid in visualizing the proof.

There is a recursive h such that if $\emptyset^{\omega} \equiv_T D$ via k (that is, k codes a pair of indices establishing the equivalence) then $J_i(\emptyset^{\omega}) \equiv_T J_{h(i,k)}(D)$. Take f and g as in Corollary 3.7 so that for all Y and j, $J_j(J_{f(j)}(Y)) \equiv_T Y^{\omega}$ via g(j). Thus $J_j(J_{f(j)}(J_{f(f(j))}(\emptyset)))$ $\equiv_T J_{f(f(j))}(\emptyset)^{\omega}$. By the recursion theorem, take j_0 so that $\{j_0\} =$

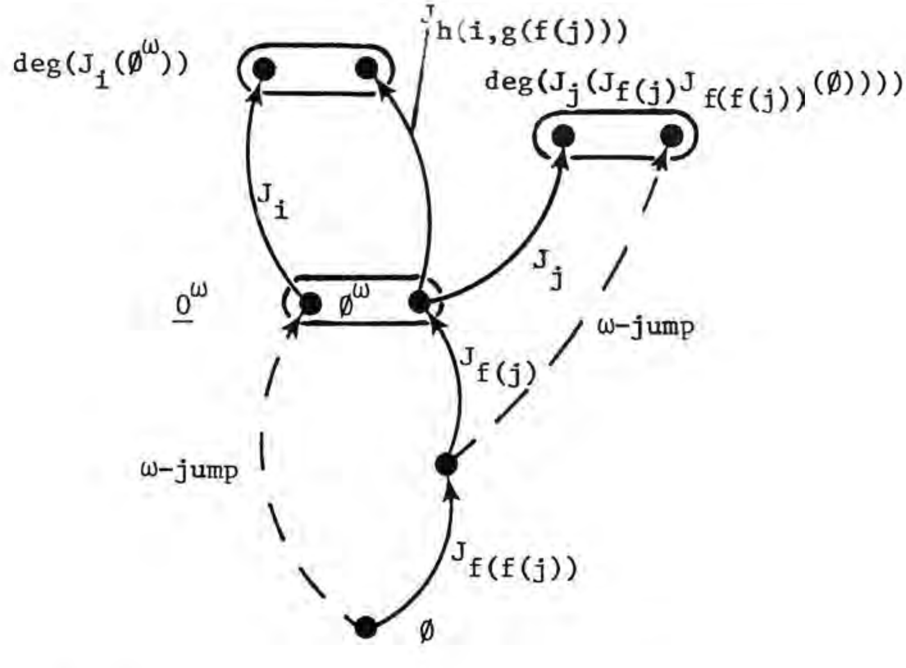

Figure 3.1. a) The dots represent sets, the ovals represent degrees, and the arrows are the appropriate pseudo-jump operators.

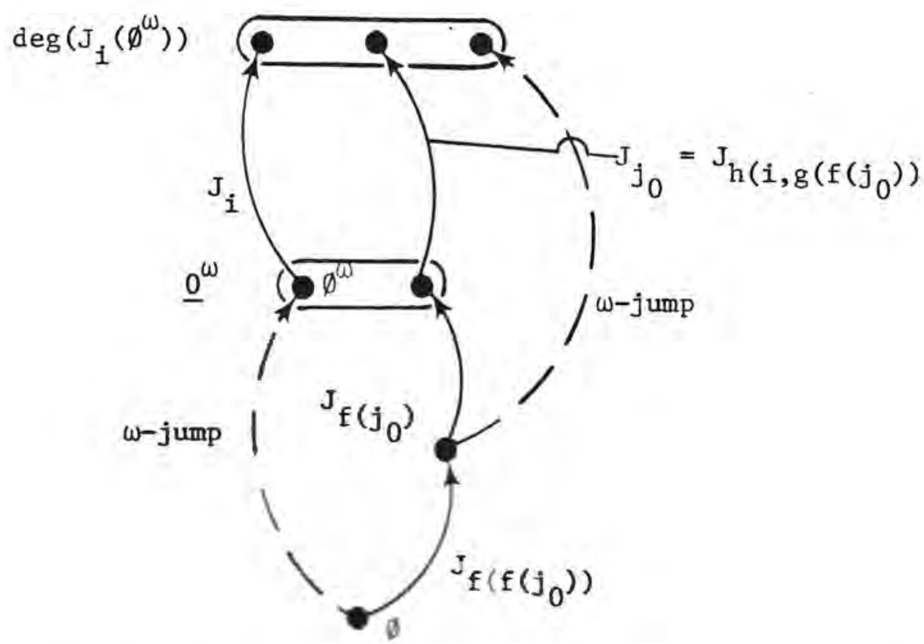

Figure 3.1. b) The situation after the invocation of the recursion theorem.

$\{h(i,g(f(j_0))\}$. Note that $J_{f(j_0)}(J_{f(f(j_0))}(\emptyset)) \equiv_T \emptyset^\omega$ via $g(f(j_0))$. Thus

$$J_{f(f(j_0))}(\emptyset)^\omega \equiv_T J_{j_0}(J_{f(j_0)}(J_{f(f(j_0))}(\emptyset))) =$$

$$J_{h(i,g(f(j_0)))}(J_{f(j_0)}(J_{f(f(j_0))}(\emptyset)) \equiv_T J_i(\emptyset^\omega) .$$

So $J_{f(f(j_0))}(\emptyset)$ is the ω-REA set whose ω-jump has degree $\underline{b}$. By the recursion theorem with parameters, we have the index uniformly from i. □

Corollary 3.9: (Sacks Jump Theorem) Given i, there is (uniformly in i) an e such that $\tilde{J}^1_e(\emptyset)' \equiv_T \tilde{J}^1_i(\emptyset')$.

Proof: The argument given for Theorem 3.8, combined with the Jockusch-Shore analog of Theorem 3.1 and the observation that their proof has the extra uniformity analogous to that of Corollary 3.7, gives the result (with the "~" notation, we need the uniform relativized version of the recursion theorem). □

It is interesting that the set $\tilde{J}^1_{f(f(j_0))}(\emptyset)$ is really constructed via the finite injury method (in fact, by a particularly simple construction). Of course, the argument that we did the "right" finite injury construction appeals to the properties of the finite injury construction $\tilde{J}^1_{f(j_0)}$. However, we need to know only one (parametrized) finite injury construction in order to prove the Sacks Jump Theorem.

We may think of a pseudo-jump operator as a "hop". Some hops go nowhere and some hops are jumps, but all hops go (non-strictly) up in degree. Jockusch and Shore taught us that for every choice of second hop, there is a first hop such that (up to Turing degree) two hops are a jump. That is, "all hops have mates". Theorem 3.1 says the same thing in the context of arithmetic degrees. The proof given here of the Sacks Jump Theorem and its arithmetic analog, i.e. that a jump followed by a hop is the same (up to degree) as some hop followed by a jump, succeeds by showing that they both are the same as a sequence of three appropriately chosen hops.

It is possible that an argument along these lines would give an easier proof of the Z,B,C-lemma. However, it is not immediately clear that this would serve any useful purpose, since the result of the Z,B,C-lemma is not all that the overall constructions use. We also need to mix in new requirements, and it is not clear how this could be accomplished with a proof in the above style.

The following corollary was pointed out by Jockusch [1984]. He thought perhaps that if e is such that $X <_T \tilde{J}_e^1(X)$ for all X then there would exist an r.e. minimal pair A,B with $\tilde{J}_e^1(A) \equiv_T \tilde{J}_e^1(B) \equiv_T \emptyset'$. This would give, for example, a pseudo-jump-style proof of the existence of a minimal pair in every non-trivial jump class. The proof of Corollary 3.9 can be used to show that at least the conjecture does not hold uniformly.

<u>Corollary 3.10</u>: There is no recursive function f such that for all e and $X, J_e(J_{f(e)}(X)) \equiv_T X'$ uniformly and $[\forall Y(Y <_T J_e(Y)) \rightarrow \deg(J_{f(e)}(\emptyset))$ is cappable]. (A degree is cappable if it is half of a minimal pair.)

Proof: The argument given in Corollary 3.9 for the Sacks Jump Theorem applied to such an f would show that every degree 1-REA in $\emptyset'$ is the jump of a cappable degree. However, this is not true by Shore [1987] or Cooper [1987]. □

Corollary 3.11: An a-degree (T-degree) $\underline{b}$ is in the range of the ω-jump on a-degrees (T-degrees) below $\underline{0}^{\omega}$ iff $\underline{b}$ is the a-degree (T-degree) of a set ω-REA in $\emptyset^{\omega}$.

Proof: The only part of the statement which has not been proved above is that $A \leq_a \emptyset^{\omega}$ implies A^{ω} has arithmetic degree that of some set ω-REA in $\emptyset^{\omega}$. Suppose $A \leq_T \emptyset^{\omega+n}$. Define C by $C^{[0]} = \emptyset^{\omega}$, $C^{[1]} = \emptyset^{\omega+1}, \ldots, C^{[n]} = \emptyset^{\omega+n}$, $C^{[n+i+1]} = A^{(i)}$ for $i \geq 0$. Then C is ω-REA in $\emptyset^{\omega}$ and $C \equiv_a A^{\omega}$. □

Note: It is not the case that $A \leq_a \emptyset^{\omega}$ implies A^{ω} has T-degree that of some set ω-REA in $\emptyset^{\omega}$. Take $\underline{m}$ such that $\underline{m} \leq_T \underline{0}^{\omega+3}$, $\underline{m} \nleq_T \underline{0}^{\omega+2}$, and $\underline{m}$ is a minimal cover of $\underline{0}^{\omega}$. Then by a standard forcing-and-coding argument, there is an ω-generic A with $A^{\omega} \equiv_T A \oplus \emptyset^{\omega} \equiv_T M$, where M is such that deg M = $\underline{m}$. Thus $A \leq_a \emptyset^{\omega}$, but $A^{\omega} \equiv_T M$ is not of ω-REA in $\emptyset^{\omega}$ T-degree. If so, M is above a set B, say, which is r.e. in $\emptyset^{\omega}$ and strictly above $\emptyset^{\omega}$, and thus M is not a minimal cover of $\emptyset^{\omega}$. (Recall that if all columns of a set ω-REA in Y are recursive in X, then the set is recursive in X". If $A^{\omega} \equiv_T J_e(\emptyset^{\omega})$, let $B = J_e^{\bar{n}}(\emptyset^{\omega})$, where $\bar{n}$ is the number of the first column not recursive in $\emptyset^{\omega}$.)

We have shown that not only does the arithmetic analog of the Shoenfield Jump Theorem hold, but the analog of the Sacks Jump Theorem holds as well. The fact that the same basic argument for the range of the ω-jump also gives a new proof of the Sacks Jump Theorem is surprising. This gives a very striking analogy between ω-REA operators and ω-jump, and r.e. operators and Turing jump. The weakness of the argument seems to be that it does not really give a direct construction of the ω-REA set with the desired ω-jump. Such a construction would probably be quite illuminating. It most likely would yield new insights into the operation of ω-jump and would also employ techniques which could perhaps be used to deduce new facts about the ω-REA sets. A more direct construction for the analog of the Shoenfield Jump Theorem would also be illuminating, even though the result is weaker. Both of these problems seem difficult, especially since the range of the ω-jump on degrees below $\underline{0}^{\omega}$ had been an open question for some time, and it has proved quite resilient to direct attack.

Chapter 4

Minimal Pairs and Diamond

In this chapter we extend the basic construction further to show that there is an arithmetically minimal pair of ω-REA sets which join to $\underline{0}^{\omega}$.

Theorem 4.1: There are ω-REA sets A_0 and A_1 such that A_0, A_1 form a minimal pair, i.e. $\forall B((B \leq_a A_0 \ \& \ B \leq_a A_1) \rightarrow B \leq_a \emptyset)$ and $A_0 |_a A_1$, moreover $A_0 \oplus A_1 \equiv_a \emptyset^{\omega}$.

Proof: We construct two arrays of ω-REA operators $\{A_{i,k} | k \in \omega\}$ for $i = 0,1$, and let $A_i = A_{i,0}(\emptyset)$. We will guarantee that $A_{i,k}(X)' \equiv_T A_{i,k}(X) \oplus X' \equiv_T A_{i,k+1}(X')$ uniformly, and hence, by induction, we get $A_{i,k}(X)^{(n)} \equiv_T A_{i,k}(X) \oplus X^{(n)} \equiv_T A_{i,k+n}(X^{(n)})$ uniformly in X, n, k and i.

We also fix (an index for) some (partial) recursive function α, and guarantee that if $\Phi_{\alpha(0,k)}(A_{0,k}(X)) = \Phi_{\alpha(1,k)}(A_{1,k}(X)) = D$, then $D \leq_T X'$.

Suppose then that $D \leq_a A_0$ and $D \leq_a A_1$. Then there is an e and an n such that $\Phi_e(A_0^{(n)}) = D = \Phi_e(A_1^{(n)})$ (note: the trick of Chapter 1, namely that we can assume e = n, serves no purpose here). Let $k = \langle e,n \rangle$ (by convention $k \geq n$). Then, for i = 0 or 1, $D \leq_T A_i^{(n)} = A_{i,0}(\emptyset)^{(n)} \leq_T A_{i,0}(\emptyset)^{(k)} \equiv_T A_{i,k}(\emptyset^{(k)})$. By the uniformities we have D recursive in $A_{i,k}(\emptyset^{(k)})$ uniformly in e, n, i and k, hence uniformly in i and k. Thus $D = \Phi_{\overline{\alpha}(i,k)}(A_{i,k}(\emptyset^{(k)}))$ for some recursive $\overline{\alpha}$ (whose index is computable effectively from the various parameters of the construction). But by the recursion theorem, we can take (the index for) α so that $\alpha = \overline{\alpha}$. Therefore, $D \leq_T \emptyset^{(k+1)}$, and hence D is arithmetic. This style of argument is similar to Harrington's diagonalization technique. In his construction, diagonalization for a given index was taken care of at another level of the construction, and the recursion theorem tied things together in much the same way.

For now we suppress the mention of the parameter k, and attempt to motivate the constructions. We have the P_e's and N_e's of the C_m and B_m constructions, as well as the M_e's for controlling the jump of the ω-REA set under construction. The new idea is that we try to force $\{\alpha(0)\}(A_0(X);x)\downarrow \neq \{\alpha(1)\}(A_1(X);x)$ for some x if it is possible, and if it is not (and $\Phi_{\alpha(i)}(A_i(X))$ is total for i = 0,1), then what is computed is recursive in X'. This is accomplished by using a single requirement Q of strongest priority. Q is both a positive and a negative requirement. The negative aspect of Q is to hold a diagonalization if found. The positive aspect is to "leave word" so that the next column can do its job in holding the diagonalization.

As in the M_e requirements, Q uses approximations of the form $A_i^{0,\tau}(X)$ for τ a decreasing string (i.e. $\tau(0) > \tau(1) > \ldots > \tau(|\tau|-1)$) in the construction. We will often abbreviate $A_i^{0,\tau}(X)$ as $A_i^{\tau}(X)$. Suppose in the construction of column 1 of $A_0(X)$, we see at stage s+1 decreasing strings τ_0 and τ_1 and an $x \leq s$ such that $s = \tau_0(0) = \tau_1(0)$ and $\{\alpha(0)\}(A_0^{\tau_0}(X);x)\downarrow \neq \{\alpha(1)\}(A_1^{\tau_1}(X);x)\downarrow$ where the computation with parameter i for i = 0 or 1 converges in fewer than $|\tau_i|$ and fewer than $\tau_i(|\tau_i|-1)$ steps. We would like to make these computations permanent to guarantee that $\Phi_{\alpha(0)}(A_0(X)) \neq \Phi_{\alpha(1)}(A_1(X))$. Perhaps at first it seems that there is no problem in doing this just as in the M_e requirements. Unfortunately, without doing something else, columns numbered greater than 1 will not know to cooperate. For example, if $|\tau_0| > 1$, the construction of column 2 of $A_0(X)$ must impose restraint at stage $\tau_0(1)+1$. Column 2 could even look back at column 1 and calculate a stage t by which the enumeration of column 1 has settled down on $\tau_0(1)$, but this information would not help. It may be the case that the stage s in the enumerations of the first columns at which we noticed the chance to diagonalize is much greater than t due to the first column of $A_1(X)$ not settling down. Since in the enumeration of $A_0(X)^{[2]}$ we do not have access to $A_1(X)^{[1]}$ we could wait forever at stage $\tau_0(1)+1$ trying to decide if it is a critical stage in some diagonalization. Of course, holding $A_0^{1,\beta}(A_0^1(X))$ whenever $\{\alpha(0)\}(A_0^{1,\beta}(A_0^1(X));x)\downarrow$ so that we need not decide this at all would entail infinitely much restraint. The reason the M_e's did not have this problem is that any one column only had to worry about finitely many convergences. Here we must worry about <u>any</u> x being the argument for a potential diagonalization.

The solution to the problem outlined above is straightforward, but it introduces new difficulties into the construction. If we act to diagonalize in each column 1, we code this fact in simply by enumerating into column 1 an appropriate member of the current complement of column 1. Let $B_{i,0,s}$ = the set enumerated in the construction of column 1 of $A_i(X)$ after stage s. Let $\overline{B}_{i,0,s} = \{\Gamma_s^0 < \Gamma_s^1 < \Gamma_s^2 < \ldots\}$, thus we are now thinking of markers sitting on the complement of $B_{i,0,s}$. If we act to diagonalize at stage s+1 because of τ_0, τ_1, then if $|\tau_i| > 1$, enumerate $\Gamma_s^{\tau_i(1)}$ into $B_{i,0,s+1}$, that is "kick" the markers. Note that $\Gamma_s^{\tau_i(1)} \geq \tau_i(1)$, so we are not changing the part of $B_{i,0,s}$ that we'd like to protect. Now at stage r+1 in the enumeration of column 2 of $A_0(X)$, we need only wait for a stage t by which the r+1st element of the complement of $B_{i,0,t}$ has settled down. If Q has not acted by this stage in column 1, then r is not $\tau(1)$ in any string τ giving an approximated computation that we are trying to hold, so it is safe to continue.

The new problem the above coding strategy introduces is that we have lost the seemingly crucial property that stage t of the enumeration of a column depends only on the oracle up through t. This is why the proof of the Z,B,C - lemma was written in the way that it was, so we have foreseen some of the troubles at least. We will see that the property still holds in a restricted sense.

The constructions

We will still code a low-in-X set into $A_i(X)^{[1][0]}$. This again makes explicit diagonalization unnecessary, as well as establishing "diamond", i.e. that the diamond lattice $\Diamond$ is embeddable into the ω-REA a-degrees preserving top and bottom. In the present context, however, Q will perhaps keep some finite part of the low-in-X set out. We shall think of the coding as being controlled by a requirement L. Explicit diagonalization as in Harrington [1975] would mesh into this construction easily (the reader may want to try this as an exercise, using the last sentence of the third paragraph of this proof as a hint), but we use the present technique since it establishes more.

Let ℓ_0 and ℓ_1 be as in the proof of Theorem 2.3, i.e. $W^X_{\ell_0} \oplus W^X_{\ell_1} \equiv_T X'$ uniformly and for $i = 0,1$, $W^X_{\ell_i} >_T X$ and $(W^X_{\ell_i})' \equiv_T X'$. Let integers d_0, d_1 and indices for α_0, α_1 be given. We can again by the recursion theorem use an index for the construction in the construction. More on this point later.

<u>The $B_{i,0}$-construction</u>: We have requirements Q, L, P_0, N_0, P_1, ... listed in order of decreasing priority. We are given Z_0 (= $\{0\}\times X$) and an index $a_{d_i}(0)$ for a set W r.e. in Z_0', and the index for the r.e. operation $C_{i,0}$ to be defined later.

P_e and N_e: These requirements are just as in Theorem 2.3 (so just as in the Z,B,C - lemma).

L: At stage s+1, L wants to put $\langle 0,x\rangle$ into $B_{i,0}$ if $x \in W^X_{\ell_i,s}$.

Q: At stage s+1, if Q has not yet acted, Q <u>wants to act</u> if $\exists x$, τ_0, τ_1 such that $x \leq s$, $\tau_0(0) = s = \tau_1(0)$, both τ are decreasing strings, $\{\alpha(0)\}(A_0^{\tau_0}(x);x)\downarrow \neq \{\alpha(1)\}(A_1^{\tau_1}(X);x)\downarrow$ and the computation with parameter j for j = 0,1 converges in fewer than $|\tau_j|$ and fewer than $\tau_j(|\tau_j|-1)$ steps.

<u>stage 0</u>: $B_{i,0,0} = \emptyset$.

<u>stage s+1</u>: First, determine if Q wants to act at this stage. If so, Q <u>acts for τ_i</u> (where x, τ_0, τ_1 is the least triple which satisfies the definition of Q wanting to act) as follows.

<u>case 1</u>: If $|\tau_i| = 1$, then Q permanently restrains future elements from entering $B_{i,0}\restriction\tau_i(0)$, so $B_{i,0,s}\restriction\tau_i(0) = B_{i,0}\restriction\tau_i(0)$.

<u>case 2</u>: If $|\tau_i| > 1$, then Q permanently restrains future elements from entering $B_{i,0}\restriction\tau_i(1)$. Also, if $\overline{B}_{i,0,s} = \{\Gamma_s^0 < \Gamma_s^1 < \Gamma_s^2 < \ldots\}$, then Q enumerates $\Gamma_s^{\tau_i(1)}$ into $B_{i,0,s+1}$. (The slight awkwardness here is forced by the need for coding. It would be nice to be able to restrain up to $\tau_i(0)$ in both cases, but the positive action may interfere. Of course, one could take the positive action and then restrain but this is also awkward.)

Next, determine what elements the other negative requirements restrain, and what elements the positive requirements want to put into $B_{i,0,s+1}$. Put in those elements not restrained by a requirement of higher priority.

<u>The $B_{i,m}$-construction</u> (for m > 0): We have requirements Q, M_0, M_1, ..., M_{m-1}, P_0, N_0, P_1, ... listed in order of priority. We are given $Z_{i,m}$($=A_i^{2m}(X)$ in practice), an index $a_{d_i}(m)$ for a set r.e. in $Z'_{i,m}$, and the index for the r.e. operation $C_{i,m}$ to be defined later.

M_e, P_e, and N_e: Just as in Theorem 2.3.

Q: $C_{i,m-1}$ is the r.e. operator associated with column 2m of the construction. To avoid confusion, the Q's for different columns may be labelled with superscripts. At stage s+1, if $Q^{B_{i,m}}$ has not yet acted, calculate the stage $\delta(s)$ by which the $s+1^{st}$ element of the complement of $C_{i,m-1}(Z_{i,m}^{[<2m]})$ has settled down, that is the stage $\delta(s)$ by which Γ^s has reached its limit in the construction of $C_{i,m-1}$ with input the columns of Z_m up to 2m-1. This calculation is intended to proceed as follows. Calculate $C_{i,m-1,t}(Z_{i,m}^{[<2m]})$ for $t = 0,1,2,\ldots$ until t is found so that the $s+1^{st}$ element of the complement of $C_{i,m-1,t}(Z_{i,m}^{[<2m]})$ is the same as the $s+1^{st}$ element of the complement of $Z_{i,m}^{[2m]}$ (with pathological $Z_{i,m}$'s $\delta(s)$ will not become defined, i.e. this stage will <u>stall</u>, but if $Z_{i,m}$ really is $A_i^{2m}(X)$, $\delta(s)$ will be defined). If $Q^{C_{i,m-1}}$ does not act in the $C_{i,m-1}$-construction with input $Z_{i,m}^{[<2m]}$ by stage $\delta(s)$, do nothing. If $Q^{C_{i,m-1}}$ does act before $\delta(s)$ in the $C_{i,m-1}$-construction, let τ be such that $Q^{C_{i,m-1}}$ acted for τ.

<u>case 1</u>: $|\tau| = 2m$. Then we need not do anything, since the column 2m+1 which we are constructing is not involved with the diagonalization Q tries to hold.

<u>case 2</u>: $|\tau| > 2m$. If $\tau(2m) \neq s$ then we need not do anything at stage s+1. If $\tau(2m) = s$, then $Q^{B_{i,m}}$ <u>wants to act for τ at stage s+1</u>.

<u>stage 0</u>: $B_{i,m,0} = \emptyset$.

<u>stage s+1</u>: First see if for some τ $Q^{B_{i,m}}$ wants to act for τ at stage s+1. If so, we have two cases. If $|\tau| = 2m+1$, then Q <u>acts for τ</u> by permanently restraining future elements from entering $B_{i,m}\restriction\tau(2m)$. If $|\tau| > 2m+1$ (so the next column needs signalling) then Q <u>acts for τ</u> by permanently restraining future elements from entering $B_{i,m}\restriction\tau(2m+1)$ and enumerating $\Gamma_s^{\tau(2m+1)}$ into $B_{i,m,s+1}$ where $\overline{B}_{i,m,s} = \{\Gamma_s^0 < \Gamma_s^1 < \ldots\}$.

Next, determine the restraint of the negative requirements and desired actions of the positive requirements and act accordingly, as usual.

<u>The $C_{i,m}$-construction</u>: We have requirements Q, M_0, ..., M_{m-1}, P_0, N_0, P_1, ... (if m = 0, there are no M_i's). We are given $Z_{i,m}$ and $B_{i,m}$ ($=A_i^{2m}(X)$ and $A_i(X)^{[2m+1]}$ in practice) and the index $a_{d_i}(m)$ for a set r.e. in $Z'_{i,m}$. The M_e, P_e and N_e requirements are just as in Theorem 2.3. Q acts just as in the description given above for $Q^{B_{i,m}}$. That is, at stage s+1, $Q^{C_{i,m}}$ looks back at the enumeration of the previous column to calculate a stage $\delta(s)$ by which Q in that column would need to act if $Q^{C_{i,m}}$ was to act at stage s+1. The action of $Q^{C_{i,m}}$ is as described there.

As in Theorem 2.3, we think of the construction as defining an index $\bar{e}_i$ such that $\{\bar{e}_i\}^Y(n,s)$ = canonical index for the set of numbers enumerated at stage s, when applying the operator defined for column n to input Y. So in the construction when we used, for example, $C_{i,m-1,s}(Z^{[<2m]})$ we really meant $\bigcup_{t\leq s}$ $\{x | x \in$ the set with canonical index $\{\bar{e}_i\}^{Z^{[<2m]}}(2m,t)\}$. As usual an application of the recursion theorem allows this. We also want however that $\forall n \geq 1\ \forall s$ $(\{e_i\}^{A_i(X)^{[<n]}}(n,s)\downarrow)$. In fact, we need more to show that the approximations that the M_e and Q requirements use are such that these requirements achieve their goals. In Theorem 2.3 the required convergences followed easily by induction on s since a convergence on s

relied only on various convergences with lower stage numbers. Here in the requirement Q we look back at the previous column at higher stage numbers, however. The next lemma deals with these issues.

Definition 4.2: If Y is the input to the enumeration operator associated with column n+1 of construction i, call Y a good input to column n+1 if $Y = A_i^{m,\tau}(A_i^m(X))^{[\leq n]}$ for some $m \leq n$ and some string τ. That is, Y is good if it is an approximation to $A_i^n(X)$ of the type we use in the construction.

We say a stage s in the application of the operator for column n to input Y stalls if $\{e_i\}^Y(n,s)\uparrow$.

Lemma 4.3: For all n, if Y is a good input to column n+1 and $Y^{[n]}$ is coinfinite, then $\forall s(\{e_i\}^Y(n+1,s)\downarrow)$.

Proof: The only worry is that the "looking back" aspect of Q may make a stage stall. If Y is a good input, this will not happen, however. Proceed by induction on s. For all n and T, $\{e_i\}^T(n+1,0)\downarrow =$ the canonical index for $\emptyset$. Suppose that we have the statement of the lemma for all $s' \leq s$. The definition of stage s+1 of the enumeration operator associated with column n+1 applied to Y requires calculation of $\{e_i\}^{A_i^{n,\tau}(Y)^{[\leq n']}}(n'+1,t)$ for various τ, $n' > n$ and $t \leq s$ (in the requirements M_e (and Q if n = 0)). If Y is good, so is $A_i^{n,\tau}(Y)^{[\leq n']}$ (for column n') and the last column is coinfinite. So all of these computations terminate by the induction hypothesis. The only other way that stage s+1 in column n+1 with good input could stall is in the calculation of $\delta(s)$. This entails examining column n of the input and enumerating from columns less than n until things settle down appropriately. The key point is that Y good implies the last column is <u>defined by</u> the same enumeration we are comparing it to. If some stage in the enumeration of column n stalls, we do not get this far because things settle down at the stage just before the first stalling stage. (Suppose s_0 is least so that $\{e_i\}^{Y^{[<n]}}(n,s_0+1)\uparrow$. Then, by definition, $Y^{[n]} = \bigcup_{s'\leq s_0} \{x|x \in$ set with canonical index $\{e_i\}^{Y^{[<n]}}(n,s')\}$. When we check $Y^{[n]}$ against the enumeration of $Y^{[n]}$ from $Y^{[<n]}$, we get these equal at stage s_0, so the s+1st element of the compliment has settled down, so the calculation of $\delta(s)$ terminates.) Otherwise the assumption that $Y^{[n]}$ is coinfinite implies that the s+1st element of the compliment exists, so we still terminate. □

<u>Definition 4.4</u>: If Q acts in the first columns in the constructions of $A_i(X)$ for i = 0,1, suppose Q acts for τ_0, τ_1. For $1 \leq n \leq |\tau_i|$, let q(n,i) = the <u>critical stage</u> for column n of construction i = $\tau_i(n-1)+1$. For $n > |\tau_i|$ let $q(n,i) = \omega$. If Q does not act, let $q(n,i) = \omega$ for all n and for i = 0,1.

<u>Lemma 4.5</u>: Let $t_0 < q(i,n)$ be given, and suppose that $T_0^{[0]}=T_1^{[0]}=X$, and that $T_0\restriction t_0 = T_1\restriction t_0$. Suppose further that no stages t' for $t' \leq t_0$ stall in the application of the operator for column n of construction

i to T_0, and similarly for T_1. Then the set enumerated after t_0 stages with input T_0 is the same as the set enumerated after t_0 stages with input T_1.

Proof: By induction on n, if $Y^{[0]} = X$, then in the enumeration of column n with input Y, Q can act for τ only if Q acts for τ in column 1 of $A_i(X)$. The definition of Q acting guarantees that Q can act for τ only at the critical stage. Thus the requirement Q does not act in the first t_0 stages since t_0 is before the critical stage. The only other requirements which might possibly involve the input above the stage number are the M_e's (in the approximations). But, Q does not act in column n until after t_0 implies Q does not act in any column n' > n in any approximation of the form $A_i^{n-1,t'*\gamma}(A_i(X)^{[<n]})$ where $t' \leq t_0$ (the δ(s) calculations will all involve stages before Q acts in the previous approximated column). Since also no stages stall whether the input is T_0 or T_1 the effect of the construction is the same as if Q were not present in the instructions. Thus, since $T_0 \restriction t_0 = T_1 \restriction t_0$, T_0 and T_1 are interchangeable as input to the first t_0 stages of the construction of column n. □

The next lemma shows that the approximations Q uses are indeed useful.

Lemma 4.6: Assume that $\forall n \geq 1$ ($A_i(X)^{[n]}$ is coinfinite). If Q acts for τ_i in column 1 of $A_i(X)$, then $\forall n$ $(1 \leq n \leq |\tau_i| - 1 \rightarrow A_i^{\tau_i}(X)^{[n]} \upharpoonright \tau_i(n) = A_i(X)^{[n]} \upharpoonright \tau_i(n)$ and $n = |\tau_i| \rightarrow A_i^{\tau_i}(X)^{[n]} \upharpoonright \tau_i(n-1) = A_i(X)^{[n]} \upharpoonright \tau_i(n-1))$. Thus Q acts $\Rightarrow \Phi_{\alpha(0)}(A_0(X)) \neq \Phi_{\alpha(1)}(A_1(X))$.

Proof: We proceed by induction on n. If n = 1, then in the construction at stage $\tau_i(0)+1$ Q acts and permanently restrains elements less than either $\tau_i(1)$ or $\tau_i(0)$ depending on whether $|\tau_i| > 1$ or $|\tau_i| = 1$. Since this occurs at stage $\tau_i(0)+1$, we have the desired conclusion. Suppose we have the appropriate equalities for all $n \in \{1,\ldots,n_0\}$, and that Q acts for τ_i in the construction of column n_0 with input $A_i(X)^{[<n_0]}$, at stage $\tau_i(n_0-1)$. Then by Lemmas 4.3 and 4.5, and the fact that τ_i is decreasing, we have $A_i^{\tau_i}(X)^{[n_0+1]} = A_i^{n_0,<\tau_i(n_0)>}(A_i^{n_0}(X))^{[n_0+1]}$. In words, we have that the set enumerated after $\tau_i(n_0)$ stages is the same whether we apply the operator for column n_0+1 to input $A_i^{\tau_i}(X)^{[\leq n_0]}$ or $A_i(X)^{[\leq n_0]}$. Since Q acted at stage $\tau_i(n_0-1)+1$ in the construction of column n_0 with input $A_i(X)^{[<n_0]}$, Q at stage $\tau_i(n_0)+1$ in the construction of column n_0+1 with input $A_i(X)^{[\leq n_0]}$ calculates $\delta(\tau_i(n_0)) > \tau_i(n_0-1)+1$, so Q knows to act. Thus Q acts at stage $\tau_i(n_0)+1$ and preserves the appropriate (depending on the length of τ_i) amount of the set being enumerated to establish the equality for $n = n_0+1$.

Since the above establishes that if Q acts then the sets constructed equal the approximations we act for, on the part which the computations can see, we have that Q acts implies $\Phi_{\alpha(0)}(A_0(X)) \neq \Phi_{\alpha(1)}(A_1(X))$.

Lemma 4.7: Suppose that Q does not act in the construction of the first columns of the $A_i(X)$ for i = 0,1. Suppose also that $\Phi_{\alpha(0)}(A_0(X)) = \Phi_{\alpha(1)}(A_1(X)) = D$. Then $D \leq_T X'$.

Proof: We show how to calculate D(x) effectively in X'. Let τ_0 be any decreasing string with $|\tau_0| > 1$, such that $\{\alpha(0)\}(A_0^{\tau_0}(X);x)\downarrow$, and this computation takes fewer than $|\tau_0|$ and fewer than $\tau_0(|\tau_0|-1)$ steps to converge, and such that $B_{0,0,\tau_0(0)}\upharpoonright\tau_0(1) = B_{0,0}\upharpoonright\tau_0(1)$ (i.e. the first column of $A_0(X)$ has settled down on $\tau_0(1)$). There is such a τ_0 since $\Phi_{\alpha(0)}(A_0(X))$ is total. If the computation converges to y, then D(x) = y, for suppose not. Then let τ_1 be defined as above, but with the parameter 0 changed to 1, and with the stipulation that the computation be the correct one. Again $\Phi_{\alpha(1)}(A_1(X))$ total implies the existence of such a τ_1. Let u = $\max\{\tau_0(0),\tau_1(0)\}$. Note that at stage u of both constructions, since $B_{i,0,u}\upharpoonright\tau_i(1) = B_{i,0}\upharpoonright\tau_i(1) = B_{i,0,\tau_i(0)}\upharpoonright\tau_i(1)$ for i = 0,1, we would have wanted to act for Q because of $x,\langle u,\tau_0(1), \ldots, \tau_0(|\tau_0|-1)\rangle,\langle u,\tau_1(1), \ldots, \tau_1(|\tau_1|-1)\rangle$ (this last assertion uses Lemma 4.5). This is a contradiction since Q wants to act implies Q acts. Thus to calculate D(x) we find the least τ_0 satisfying the above restrictions, and the value calculated is correct. Finding such a τ_0 is effective in $A_0(X)^{[\leq 1]}$, hence in X'. □

In the way we have phrased the construction, the M_e's still act at most once, so if Q acts in column n, it may (by signalling column n+1) interfere with the M_e's of column n. However, Q's interference is limited to some finite number of columns (corresponding to $|\tau_i|$). Furthermore, X' can determine uniformly if Q acts and if so for what τ_i. The mode of argument of Lemma 4.6 establishes that if M_e acts in a column $n > |\tau_i|$, then $\{e\}(A_i(X);e)\downarrow$. Conversely we get that if $\{e\}(A_i(X);e)\downarrow$ in more than 2e+2 steps then M_e acts in column 2e+2. Hence, as in Theorem 2.3, $A_i(X)' \equiv_T \bigoplus_{m<\omega} (A_i^{2m}(X)')$ uniformly.

Next we note that the argument for the Z,B,C-lemma still goes through with the addition of the Q requirements, since X', hence $(Z_{i,m})'$ can determine the action of Q in the $C_{i,m}$ or $B_{i,m}$ constructions. (Note: Since each column of $C_{i,m}$ is finite, $C_{i,m}$ is coinfinite. If $W(a_{d_i}(m), Z'_{i,m}) = W$ is infinite, then $B_{i,m}$ is coinfinite since $\lim_x (B_{i,m}^{[e]}(x) = \overline{W}(x))$. So it is an easy matter to get the coinfiniteness of the columns required to make sure that the calculations of δ stop, and keep uniformity; just use $W \oplus \omega$ in place of W.) Thus, if we put the parameter k back in, we have for all m and X, $A_{i,k}^{2m+2}(X)' \equiv_T$ $A_{i,k}^{2m}(X)' \oplus A_{i,k}(X)^{[2m+1]} \oplus A_{i,k}(X)^{[2m+2]} \equiv_T A_{i,k}^{2m}(X)' \oplus W(a_{d_i}(<k,m>);$ $A_{i,k}^{2m}(X)')$, uniformly. A recursion theorem argument, applied to the parameters d_i in the style of Claim 2 of Theorem 2.3 establishes $A_{i,k}(X)' \equiv_T A_{i,k}(X) \oplus X' \equiv_T A_{i,k+1}(X')$ for i = 0,1 and all k, uniformly in i and k. Thus we have, after an application of the recursion theorem to an index for α, using Lemmas 4.6 and 4.7 and the argument given in the motivation for the proof that $A_{0,0}(\emptyset) \geq_a D$ and $A_{1,0}(\emptyset) \geq_a D$ imply that $\emptyset \geq_a D$.

Finally we argue that $A_{0,0}(\emptyset) |_a A_{1,0}(\emptyset)$ and $A_{0,0}(\emptyset) \oplus A_{1,0}(\emptyset) \equiv_a \emptyset^{\omega}$. The fact that $A_{i,0}(\emptyset)^{(n)} \equiv_T A_{i,0}(\emptyset) \oplus \emptyset^{(n)}$ uniformly implies that $A_{i,0}(\emptyset)$ is low. In the construction of column 1 of $A_{i,k}(X)$, all but finitely much of $W^X_{\ell_i}$ is coded in (since Q acts at most once). Thus $\forall X \forall k(A_{0,k}(X) \oplus A_{1,k}(X) \geq_T X')$, but not necessarily uniformly. Still, induction on n shows that $\forall n(A_{0,0}(\emptyset) \oplus A_{1,0}(\emptyset) \geq_T A_{0,0}(\emptyset) \oplus A_{0,1}(\emptyset) \oplus \emptyset^{(n)} \geq_T A_{0,n}(\emptyset^{(n)}) \oplus A_{1,n}(\emptyset^{(n)}) \geq_T \emptyset^{(n+1)})$. Therefore $A_{0,0}(\emptyset) \oplus A_{1,0}(\emptyset) \geq_a \emptyset^{\omega}$. Thus we have a low minimal pair joining to $\underline{0}^{\omega}$ in the a-degrees. □

<u>Corollary 4.8</u>: The theory of the r.e. T-degrees and the theory of the ω-REA a-degrees are not elementarily equivalent.

<u>Proof</u>: Immediate from Theorem 4.1 and Lachlan's Non-Diamond Theorem (see e.g. Soare [1986]). □

<u>Corollary 4.8</u>: Given B with $\emptyset <_a B <_a \emptyset^{\omega}$, there is an ω-REA A with $A |_a B$.

<u>Proof</u>: The result is again in immediate consequence of Theorem 4.1. A would be either A_0 or A_1 of the theorem, since if both are above B then $B \equiv_a \emptyset$, if both are below B then $B \geq_a \emptyset^{\omega}$, and if B is between them, A_0 and A_1 are comparable. □

It is unknown whether the result holds uniformly for ω-REA sets (as it does for r.e. sets and Turing reducibility), i.e. whether there is a recursive f such that $\forall e\ (\emptyset <_a J_e(\emptyset) <_a \emptyset^{\omega} \rightarrow J_e(\emptyset) |_a J_{f(e)}(\emptyset))$.

Chapter 5

Introduction to initial segments

The study of initial segments of various degree structures has been of major interest to recursion theorists. The Turing degrees naturally have received the most attention, and some of the major milestones on the road to our present-day list of possible initial segments of the T-degrees are as follows: a minimal degree, Spector [1956]; all countable distributive lattices, Lachlan [1968]; all finite lattices, Lerman [1971]; all countable upper semi-lattices with 0, Lachlan and Lebeuf [1976]; all $\aleph_1$-size upper semi-lattices with 0 and the countable predecessor property, Abraham and Shore [1985]. These results are not only interesting philosophically (the initial segments of the T-degrees are as rich as possible) but have also been put to powerful use such as in analysing the global structure of the T-degrees (S. Simpson [1977] or Nerode and Shore [1979]). Perhaps the ultimate application has been Shore's refutation of the homogeneity conjecture [1979].

The tools developed to embed various lattices as initial segments of the Turing degrees have in general been applicable to other degree structures as well. For instance, the Abraham-Shore result holds of the tt- and wtt-degrees also, as can be shown with almost no extra effort. In the other direction (more general reducibilities) the changes needed are not quite as straightforward.

Sacks [1971] introduced the technique of perfect forcing and used it to establish, for example, the existence of minimal arithmetic, hyperarithmetic, and constructibility degrees (the latter of course requires some extra set theoretic hypotheses or can be seen as a relative consistency result). It can be argued that these proofs are really just Spector's minimal degree construction translated into the appropriate setting, although the translation (especially for hyperarithmetic and constructibility degrees) is somewhat delicate. Indeed for hyperarithmetic or constructibility degrees, it is unknown which countable upper semi-lattices may be embedded as initial segments. Note, however, that by recent work of Lubarsky and Shore, it is a theorem of ZFC that some countable upper semi-lattices are not embeddable as initial segments of the c-degrees.

We continue with the translation of initial segments results into the arithmetic degree case by showing that all upper semi-lattices (u.s.l.'s) of size $\aleph_1$, with least element and the property that each element has at most countably many precedessors, are isomorphic to initial segments of the arithmetic degrees. The previous best result was that of Harding [1974] that all countable distributive lattices with 0 are embeddable as initial segments of the a-degrees. His proof uses the framework of Yates [1976]. The approach we follow is in the style of Lerman's book [1983], or more accurately we follow Abraham and Shore [1985], which is basically in the style of Lerman.

The proof uses local forcing on arithmetic trees (or Cohen forcing for arithmetic relativized to arithmetic trees) as in Sacks [1971]. The main contribution here seems to be the explication of the similarity between a Turing computation $\{e\}^{\tau}(x)$ converging for τ on a tree and the e^{th} arithmetic sentence (with a function parameter) being forced by a string on a tree of generic paths. To control the arithmetic degree of a function g one of course must control $g^{(n)}$ for every n. Using the idea of local forcing and controlling $\phi_e(g)$, however, allows for a proof which is remarkably similar notationally to the Turing degree case.

As a corollary to the embedding of countable non-distributive lattices as initial segments of the arithmetic degrees we solve, using techniques of Shore [1981] and [1982], Problem 2 of Odifreddi [1983]: $\mathcal{D}_T (\leq \underline{0}^{\omega}) \not\equiv \mathcal{D}_a (\leq \underline{0}^{\omega})$, that is, the orderings of the a- and T-degrees below $\emptyset^{\omega}$ are not elementarily equivalent. This relies on the fact that our embedding may be carried out in the a-degrees below the degree of a presentation of the given lattice.

We start by giving a brief discussion of local forcing on arithmetic trees, which is essentially standard. For more details see, e.g., Odifreddi [1982]. We do, however, use a function symbol in our language rather than a unary predicate symbol since we do not restrict ourselves to binary trees. Next we prove Sacks' theorem that there is a minimal a-degree since we feel that it shows the exploitation of the analogy between arithmetic and Turing computations mentioned above most clearly. We will follow the notation of Abraham and Shore [1985] as closely as possible.

<u>Definition 5.1</u>: a) $\mathcal{S}$ is the set of all strings.

b) If $f: \omega \to [\omega]^{<\omega}$ (where $[\omega]^{<\omega}$ denotes the set of all finite subsets of ω), then $\mathcal{S}_f$ is the set of f-strings, that is all σ such that $\forall x < |\sigma| \, (\sigma(x) \in f(x))$.

c) An f-tree (for $f: \omega \to [\omega]^{<\omega}$) is a map $T: \mathcal{S}_f \to \mathcal{S}_f$ such that $\forall \sigma, \tau \in \mathcal{S}_f \, (\sigma \subseteq \tau \Longleftrightarrow T(\sigma) \subseteq T(\tau))$.

d) $\tau \in T$, i.e. τ is on T, iff $\exists \sigma (\tau = T(\sigma))$.

e) $[T]$ = the set of branches of $T = \{h \mid \exists g(h = \bigcup_{\sigma \subseteq g} T(\sigma) \stackrel{\text{def}}{=} T[g])\}$.

f) T is arithmetic if it is arithmetic as a function. (All trees we mention will be arithmetic.)

g) T' is a subtree of T, $T' \subseteq T$, if range $T' \subseteq$ range T.

Definition 5.2: Let L^* be a language for first-order arithmetic augmented by a function symbol $\underline{G}$. We assume that L^* includes a constant symbol $\bar{n}$ for each non-negative integer n. The only terms allowed as arguments for $\underline{G}$ are variables or constant symbols (this allows for a simplified definition of forcing). For T an arithmetic tree with $T: S_f \to S_f$ we define the forcing relation $\sigma \Vdash^T \psi$ for ψ a sentence of L^* and $\sigma \in T$ as follows.

i) If ψ is an atomic formula that does not contain $\underline{G}$, then $\sigma \Vdash^T \psi$ iff ψ is true in arithmetic.

ii) $\sigma \Vdash^T \underline{G}(\bar{n}) = \bar{m}$ iff $\sigma(n) = m$.

iii) $\sigma \Vdash^T \psi_0 \vee \psi_1$ iff $\sigma \Vdash^T \psi_0$ or $\sigma \Vdash^T \psi_1$.

iv) $\sigma \Vdash^T \exists x \psi(x)$ iff $\exists n(\sigma \Vdash^T \psi(\bar{n}))$.

v) $\sigma \Vdash^T \sim\psi$ iff $\forall \rho \supseteq \sigma(\rho \in T \to (\rho \nVdash^T \psi))$.

Given $g \in [T]$, $g \Vdash \psi$ iff $\sigma \Vdash \psi$ for some $\sigma \subseteq g$. g is T-generic if $\forall\psi$ ($g \Vdash^T \psi$ or $g \Vdash^T \sim\psi$), and g is n-T-generic if $\forall\psi$ (rank$(\psi) \leq n$ $\to g \Vdash^T \psi$ or $g \Vdash^T \sim\psi$). If $\sigma \Vdash^T \psi$ or $\sigma \Vdash^T \sim\psi$ we say that σ T-decides ψ or just σ decides ψ and similarly for g.

Clause v) is where the change as been made from standard Cohen forcing in arithmetic (Feferman [1965]). The definition of local forcing is due to Sacks [1971].

Proposition 5.3 (definability of local forcing): Let $<\psi_e : e \in \omega>$ be an effective Gödel numbering of the sentences of L. If T is an arithmetic tree, then for fixed n, $\{<\sigma,e> | \text{rank}(\psi_e) \leq n \text{ and } \sigma \Vdash^T \psi_e\}$ is arithmetic (uniformly in n, and hence $\{<\sigma,e> | \sigma \Vdash^T \psi_e\} \leq_T \emptyset^{\omega}$).

Proposition 5.4: If g is n-T-generic and rank(ψ) $\leq$ n, then $g \models \psi$ iff $g \Vdash^T \psi$.

Proof: A straightforward induction establishes this result. □

Definition 5.5: Given a formula $\psi(x,y)$ of L^* with free variables x and y, $\psi^g = \{(m,n) | g \models \psi(\bar{m},\bar{n})\}$.

Note: ψ^g is not necessarily a function, although when building initial segments we are mostly concerned with those cases in which ψ^g is a function (e.g. for the sake of diagonalization). We also define ψ^g to be the set $\{m | g \models \psi(\bar{m})\}$ when ψ has just one free variable. This will allow for a reduction in technicalities in some cases.

Proposition 5.6: Given an arithmetic tree T and a fixed integer n, there is an arithmetic tree $T' \subseteq T$ such that $\forall g \in [T']$ (g is n-T-generic). Also if ψ has rank at most n, then for some fixed m, $\forall g \in [T']$ $(\psi^g \leq_T g \oplus \emptyset^{(m)})$.

Proof: To build T', use Proposition 5.3. That is, define T' inductively so that for all σ with $|\sigma| = \ell$, $T'(\sigma) \Vdash^T \phi_\ell \vee \sim\phi_\ell$ where ϕ_ℓ is the ℓ^{th} formula of rank at most n. Hence all branches of T' are n-T-generic, and the rest of the statement follows from Propositions 5.3 and 5.4. □

We now give a short proof of Sacks' theorem that there is a minimal a-degree. Our proof shows clearly how the standard argument for a minimal T-degree can be transformed into a proof of the analogous result for a-degrees. This same basic theme underlies all of the initial segment results to come.

Theorem 5.6 (Sacks [1971]): There is a minimal arithmetic degree.

Proof: We build a sequence $Id = T_0 \supseteq T_1 \supseteq \ldots$ of arithmetic binary trees, and take $g \in \bigcap_{n\in\omega} [T_n]$. Let $\{\phi_e : e \in \omega\}$ be an effective listing of all formulas of L^* with free variable x, with $\text{rank}(\phi_e) \leq e$ for all e. For $\sigma, \tau \in T$, we say σ and τ T-e-split if $\exists x(\sigma \Vdash^T \phi_e(\bar{x})\ \&\ \tau \Vdash^T \sim\phi_e(\bar{x})$, or vice versa).

Given T_e, we construct $T_{e+1} \subseteq T_e$ so that $\forall g \in [T_{e+1}]\ (\phi_e^g \leq_a \emptyset$ or $g \leq_a \phi_e^g)$. Let T_e' be the standard e-T_e-generic subtree of T_e given by Proposition 5.6. Now either

i) $\exists\sigma\forall\tau_0,\tau_1 \supseteq \sigma(T_e'(\tau_0)$ and $T_e'(\tau_1)$ do not T_e-e-split)

or ii) $\forall\sigma\ \exists\tau_0,\tau_1 \supseteq \sigma(T_e'(\tau_0)$ and $T_e'(\tau_1)$ T_e-e-split).

If i) holds, let $T_{e+1} = \text{Ext}(T_e',\sigma)$ (i.e. $T_{e+1}(\tau) = T_e'(\sigma*\tau)$). Note that then $\forall g \in [T_{e+1}]\ (\phi_e^g \leq_a \emptyset)$. In fact, ϕ_e^g is the same for all such g.

If ii) holds, define T_{e+1} inductively by levels. Let $T_{e+1}(\emptyset) = T_e'(0)$. Given $T_{e+1}(\gamma) = T_e'(\sigma)$, let τ_0 and τ_1 be the first pair we find extending σ which yield a T_e-e-splitting. Let $T_{e+1}(\gamma * i) = T'_e(\tau_i)$ for i = 0,1. Note that T_{e+1} is arithmetic, by the definability of local forcing, since T_e is arithmetic. Also, given ϕ_e^g for some $g \in [T_{e+1}]$, we have $g \leq_a \phi_e^g$. To calculate the path of g arithmetically in ϕ_e^g, proceed by induction on levels. Say we know $T_{e+1}(\gamma) \subseteq g$. Now there is an x such that $T_{e+1}(\gamma * 0) \Vdash^{T_e} \phi_e(\bar{x})$ and $T_{e+1}(\gamma * 1) \Vdash^{T_e} \sim\phi_e(\bar{x})$ (or vice versa). Since g is e-T-generic (and rank $(\phi_e) \leq e$) ϕ_e^g agrees with what is forced along its path, so we know which of $\gamma * 0$ or $\gamma * 1$ follows the path of g. Since the forcing relation for formulas of bounded rank is arithmetic, we have $g \leq_T \phi_e^g \oplus \emptyset^{(m)}$ for some m.

Now if $g = \bigcup_e T_e(\emptyset)$, then $g \in \bigcap_e [T_e]$ and thus $\forall e$ (ϕ_e^g is arithmetic or $g \leq_a \phi_e^g$). Hence g has minimal arithmetic degree (g is not arithmetic since $\forall n \exists T$ (g is n-T-generic)). □

Note the similarity of this proof to the standard one for the existence of a minimal Turing degree. Indeed given a computation lemma saying that if T is an e-splitting tree (defined appropriately) then $\forall g \in [T]$ $(g \leq_a \phi_e^g)$, then we could arrange the construction notationally to be identical to that of a minimal T-degree. One benefit of working in the a-degrees is that we may arrange for the minimal degree to be below $\underline{0}^{\omega}$ essentially for free since $\emptyset^{\omega}$ can answer all questions needed in the construction. The analogous result for the T-degrees, that there is a minimal $\underline{m} \leq_T \underline{0}'$ actually takes some work, and lattice embeddings below $\underline{0}'$ take a considerable amount of work (Lerman [1983]). We will see that the lattice embeddings in the a-degrees can be done below $\underline{0}^{\omega}$, again for free, as long as the lattice has a presentation arithmetic in $\emptyset^{\omega}$.

Note also that the proof guarantees that $\forall e\, \exists m\, (\phi_e^g \leq_T g \oplus \emptyset^{(m)})$. Hence $\forall n\, (g^{(n)} \leq_T g \oplus \emptyset^{\omega})$. Thus $g^{\omega} \equiv_a g \oplus \emptyset^{\omega}$, i.e. $g \in \underset{\sim}{GL}_1$ (the arithmetic generalized high-low hierarchy is defined in analogy with the Turing case, that is, $g \in \underset{\sim}{GL}_n$ iff $g^{\omega \cdot n} \equiv_a (g \oplus \emptyset^{\omega})^{\omega \cdot (n-1)}$ and $g \in \underset{\sim}{GH}_n$ iff $g^{\omega \cdot n} \equiv_a (g \oplus \emptyset^{\omega})^{\omega \cdot n}$). All known arithmetically minimal functions are in $\underset{\sim}{GL}_1$.

Chapter 6

Initial segments

We classify in this chapter the possible initial segments of the a-degrees of size $\aleph_1$ by proving the analog of the Abraham-Shore Theorem [1985]. Thus any $\aleph_1$-size upper semi-lattice with least element and the property that each element has at most countably many predecessors is isomorphic to an initial segment of the a-degrees.

We start by classifying the countable initial segments. Even in this case we follow the approach of Abraham and Shore quite closely. Rather than working on one tree as in Lerman [1983], we will have trees for each x in the given u.s.l. We follow this approach for two reasons. One is that the set-up is tailor made for the extension to $\aleph_1$-size u.s.l.'s. The other reason is our belief that even without larger u.s.l.'s in mind, the approach is superior. The definition of the reductions $G_x \leq G_y$, for $x \preceq y$ in the u.s.l. and G_x, G_y candidates for where x,y are sent by the embedding map, is more natural, and technicalities are reduced.

We need several definitions. We follow notation of Abraham and Shore [1985] as closely as possible. In fact, definitions 6.1 - 6.5 are taken directly from that paper except for minor changes, some necessitated by the consideration of arithmetic computability (e.g. we use arithmetic rather than recursive trees).

Definition 6.1: Let L be a finite u.s.l. with 0 (and thus a lattice with 1).

a) $\theta \subseteq \omega^L$ is a u.s.l. table for L iff

i) $\forall\alpha,\beta \in \theta\ (\alpha(0) = \beta(0) = 0)$

ii) $\forall\alpha,\beta \in \theta\ \forall x,y \in L\ [x \preceq y\ \&\ \alpha(y) = \beta(y) \rightarrow \alpha(x) = \beta(x)]$

iii) $\forall\alpha,\beta \in \theta\ \forall x,y,z \in L\ [x \vee y = z\ \&\ \alpha(x) = \beta(x)\ \&$

$\alpha(y) = \beta(y) \rightarrow \alpha(z) = \beta(z))]$

iv) $\forall x,y \in L\ [x \npreceq y \rightarrow \exists\,\alpha,\beta \in \theta\ (\alpha(y) = \beta(y)\ \&\ \alpha(x) \neq \beta(x))]$.

b) If $L' \subseteq L$ and θ is a table for L then $\theta\restriction L'$ is the table $\{\alpha\restriction L' \mid \alpha \in \theta\}$. If $x \in L$, then $\theta\restriction x$ is $\{\alpha(x) \mid \alpha \in \theta\}$.

c) If $\alpha,\beta \in \theta$ and $x \in L$ then α is congruent to β modulo x, $\alpha \equiv_x \beta$, iff $\alpha(x) = \beta(x)$.

Every finite u.s.l. has a finite table (Lerman [1983], Appendix B.2.2). We will use trees with branchings given by a table θ for L, i.e. the tree T_x associated with $x \in L$ will have (essentially) branchings given by $\theta\restriction x$ (actually the tables will be more complicated). Then if $G_x \in [T_x]$ and $y \in L$ satisfies $y \preceq x$, the associated G_y on T_y can be defined from the path of G_x in T_x. That is, the path of G_y at the n^{th} level is given by $\alpha(y)$, where α is any row of the table such that $\alpha(x)$ matches the path of G_x at the n^{th} level. The listed conditions then guarantee that $G_0 \equiv_a \emptyset$, $y \preceq x \rightarrow G_y \leq_a G_x$, and $x \vee y = z \rightarrow G_x \oplus G_y \equiv_a G_z$. Also, condition iv) allows for diagonalization, i.e. if $x \npreceq y$ we can insure that $G_x \nleq_a G_y$. In order to handle infimum requirements and to allow the finite lattice L to be extended (so that we can embed a given countable u.s.l. L) we require that the tables satisfy additional conditions.

Definition 6.2:

a) If θ and Ψ are tables for L then Ψ extends θ if $\theta \subseteq \Psi$. Ψ is an admissible extension of θ, $\theta \subseteq_a \Psi$, if $\forall\alpha \in \Psi\ \exists\beta \in \theta\ \forall\gamma \in \theta\ \forall x \in L\ (\alpha \equiv_x \gamma \rightarrow \alpha \equiv_x \beta)$.

b) $\theta = \langle \theta_i | i \in \omega \rangle$ is a sequential (weakly homogeneous) table for L iff

i) Each θ_i is a finite table for L.

ii) $\forall i \in \omega\ (\theta_i \subseteq_a \theta_{i+1})$

iii) $\forall i \in \omega\ \forall\alpha,\beta \in \theta_i\ \forall x,y,z \in L\ [x \wedge y = z \ \&\ \alpha \equiv_z \beta \rightarrow \exists\gamma_0,\gamma_1,\gamma_2 \in \theta_{i+1}\ (\alpha \equiv_x \gamma_0 \equiv_y \gamma_1 \equiv_x \gamma_2 \equiv_y \beta)]$.

iv) $\forall i \in \omega\ \forall\, \alpha_0,\alpha_1,\beta_0,\beta_3 \in \theta_i\ [\forall x \in L\ (\alpha_0 \equiv_x \alpha_1 \rightarrow \beta_0 \equiv_x \beta_3) \rightarrow \exists\beta_1,\beta_2 \in \theta_{i+1}\ \exists f_0,f_1,f_2\colon\ \theta_i \rightarrow \theta_{i+1}\ [f_0(\alpha_0) = \beta_0 \ \&\ f_0(\alpha_1) = \beta_1 \ \&\ f_1(\alpha_0) = \beta_1 \ \&\ f_1(\alpha_1) = \beta_2 \ \&\ f_2(\alpha_0) = \beta_2 \ \&\ f_2(\alpha_1) = \beta_3 \ \&\ \forall y \in L\ \forall\alpha,\beta \in \theta_i\ (\alpha \equiv_x \beta \rightarrow f_0(\alpha) \equiv_x f_0(\beta) \ \&\ f_1(\alpha) \equiv_x f_1(\beta) \ \&\ f_2(\alpha) \equiv_x f_2(\beta))]]$.

As Abraham and Shore point out, condition iv) is taken from Lerman [1971] rather than Lerman [1983] or Lachlan-Lebeuf [1976] since three functions rather than two are actually required. However, the role this condition plays in the proof is the same, and will not concern us.

Definition 6.3:

a) A sequential table $\theta = \langle\theta_i | i \in \omega\rangle$ is recursive if there is a recursive function giving canonical indices for the θ_i.

b) If θ is a sequential table for L we write $\theta\restriction L' = \langle\theta_i\restriction L' | i \in \omega\rangle$ and $\theta\restriction x = \langle\theta_i\restriction x | i \in \omega\rangle$ for $L' \subseteq L$ and $x \in L$. Thus $\theta\restriction x$ is a map from ω to $[\omega]^{<\omega}$.

c) If θ is a sequential table for L and Ψ is one for $L' \supseteq L$ then Ψ refines θ if there is a recursive h such that $\Psi_i\restriction L \subseteq_a \theta_{h(i)}$.

We can now give precise definitions of the trees and forcing conditions. We use abstract forcing machinery since it is useful in the size-$\aleph_1$ case, and besides is quite standard (see, e.g. Lerman [1983]).

Definition 6.4: Let L be a countable u.s.l. with least element 0. We define the notion of forcing P appropriate to L as follows.

a) A condition P consists of a finite sub u.s.l. L_P of L containing 0, a recursive (extendible) sequential table $\theta_P = \langle\theta_{P,i} | i \in \omega\rangle$ for L_P (the definition of an extendible table, given in Abraham-Shore, will not concern us here), for each $x \in L_P$ a (uniform) arithmetic $\theta_P\restriction x$ tree (we define uniform trees in Definition 6.5), and a commutative system of maps $F_{P,x,y}: [T_x] \overset{\text{onto}}{\to} [T_y]$ for each $y \preceq x$ in L_P which are induced by the table θ_P. That is, if $G_x = T_x[g]$ then $F_{P,x,y}[G_x] = G_y$ is $T_y[h]$ where $h(n) = \alpha(y)$ for any $\alpha \in \theta_n$ with $\alpha(x) = g(n)$. Property ii) of Definition 6.1 insures that this is well defined.

Intuitively, to calculate G_y from G_x, just use T_x to see which way G_x turns at a given level, and then the table to look up an appropriate row giving the direction that the path of G_y turns. The computation is arithmetic since the trees are.

b) A condition Q <u>refines</u> P, $Q \leq P$, if $L_Q \supseteq L_P$, $T_{Q,x} \subseteq T_{P,x}$ for $x \in L_P$, $F_{Q,x,y} = F_{P,x,y} \upharpoonright [T_{Q,x}]$ for $y \preceq x$ in L_P.

c) A set of conditions is <u>dense</u> if all conditions P have an extension Q in the set.

d) If $\mathcal{C}$ is a class of dense sets then $\mathcal{G} \subseteq \mathcal{P}$ is a <u>$\mathcal{C}$-generic filter</u> if

i) $\forall P \in \mathcal{G}\ \forall Q \geq P\ (Q \in \mathcal{G})$

ii) $\forall P,Q \in \mathcal{G}\ \exists R \in \mathcal{G}\ (R \leq P\ \&\ R \leq Q)$

iii) $\forall D \in \mathcal{C}\ (\mathcal{G} \cap D \neq \emptyset)$.

As usual the goal is to define an appropriate countable class $\mathcal{C}$ of dense sets such that from any $\mathcal{C}$-generic filter one may define the embedding map. The map will be $x \mapsto \bigcup \{T_{P,x}(\emptyset) \mid P \in \mathcal{G}$ and $x \in L_P\}$, so in particular the image of x is a branch of every tree $T_{P,x}$ with $P \in \mathcal{G}$ and $x \in L_P$.

We will often refine a condition P to one Q with $L_Q = L_P$ and $\theta_Q = \theta_P$ by defining a suitable subtree of $T_{P,1}$ (where 1 is 1_P) and taking projections. We now define the required terminology.

Definition 6.5:

a) An f-tree T' is uniform if $\forall n\ \forall j \in f(n)\ \exists \tau_j\ (\forall \sigma \in S_f$ with $|\sigma| = n)\ (T'(\sigma * j) = T'(\sigma) * \tau_j)$. Moreover, the τ_j's associated with a given level must have constant length as j varies. Thus the possible extensions of a node on a given level in a uniform tree all have the same length and are the same across the level.

Let θ be a sequential table for L.

b) If $x \preceq y$ in L and $\sigma \in S_{\theta\restriction y}$, then the y-projection of σ on x, $f_{y,x}(\sigma)$, is the $\tau \in S_{\theta\restriction x}$ with $|\tau| = |\sigma|$ and $\tau(n) = \alpha(x)$ for any $\alpha \in \theta$ with $\alpha(y) = \sigma(n)$. This is well defined by property ii) of the tables.

c) If $x \preceq y$ in L and $\sigma, \tau \in S_{\theta\restriction y}$ then σ is congruent to τ modulo x,y, $\sigma \equiv_{x,y} \tau$, if $f_{y,x}(\sigma) = f_{y,x}(\tau)$. If y is clear from the context we often write $\sigma \equiv_x \tau$ and $f_x(\sigma)$.

d) A uniform $\theta\restriction 1$-tree, S, is distinguished if $\forall x \in L\ \forall \sigma, \tau \in S_{\theta\restriction 1}$ $(\sigma \equiv_x \tau \leftrightarrow S(\sigma) \equiv_x S(\tau))$.

e) If $\theta = \theta_P$, $L = L_P$, $T = T_{P,1}$, and S is a distinguished $\theta\restriction 1$-tree we can define a condition $Q = S(P) \leq P$ by setting $L_Q = L$, $\theta_Q = \theta$, $F_{P,x,y} = F_{P,x,y}\restriction [T_{Q,x}]$ and $T_{Q,x} = T_{P,x} \circ S_x$ where S_x is defined by $S_x(\sigma) = f_x(S(\tau))$ for any $\tau \in S_{\theta\restriction 1}$ with $f_x(\tau) = \sigma$. Note that S_x is well defined since S is distinguished, and that $T_{Q,x}$ is uniform for all $x \in L$. Not also that the projection maps work out precisely as required.

We now start to define and prove dense the sets that will guarantee that any generic filter defines an embedding onto an initial segment of the a-degrees.

<u>Definition 6.6</u>: Let $D_{0,n} = \{P: |T_{P,x}(\emptyset)| \geq n \text{ for each } x \in L_P\}$, and let C_0 consist of all such sets. These sets guarantee that the functions defined by a C_0-generic filter are total.

<u>Lemma 6.7</u>: Each $D_{0,n}$ is dense.

<u>Proof</u>: Let $P \in \mathcal{P}$ and define $Q \leq P$ by taking $T_{Q,1} = \text{Ext}(T_{P,1},\sigma)$ for some $\sigma \in S_{\theta\restriction 1}$ with $|f_x(T_{P,1}(\sigma))| \geq n$ for every $x \in L_P$ and project to define the trees $T_{Q,x}$ (recall that $\text{Ext}(T,\sigma)$ is the tree T' defined by $T'(\tau) = T(\sigma * \tau)$). □

<u>Lemma 6.8</u>: If θ is a recursive extendible sequential table for L and L' is a finite extension of L then there is a recursive extendible sequential table Ψ for L' which refines θ.

<u>Proof</u>: This result is Lemma 2.9 of Abraham-Shore. They prove a more general version of it as Theorem 4.1. □

<u>Definition 6.9</u>: Let C_1 contain C_0 and the sets $D_{1,x} = \{P | x \in L_P\}$ for $x \in \mathcal{L}$.

<u>Lemma 6.10</u>: Each $D_{1,x}$ is dense.

<u>Proof</u>: Let $P \in \mathcal{P}$ and $x \in \mathcal{L}-L_P$ be given. Extend L_P to L by adding $x \vee y$ (if needed) for all $y \in L_P$. Then L is a finite sub u.s.l. of $\mathcal{L}$ containing

x. By Lemma 6.8 we can take Ψ to be a recursive sequential table for L refining θ_P by means of the recursive function h. To define $Q \leq P$ with $Q \in D_{1,x}$, let $L_Q = L$ and $\theta_Q = \Psi$. For $y \in L_P$ let $T_{Q,y} \subseteq T_{P,y}$ be defined as follows. $T_{Q,y}(\emptyset) = T_{P,y}(0^{h(0)})$ and if $T_{Q,y}(\sigma) = T_{P,y}(\tau)$ with $|\sigma| = n$ and $|\tau| = h(n)$, then for $i \in \Psi_n \restriction y \subseteq \theta_{h(n)} \restriction y$, $T_{Q,y}(\sigma * i) = T_{P,y}(\tau * i^{h(n+1)-h(n)})$. Note that the maps $F_{Q,z,y}$ for $y \preceq z$ in L_P induced by Ψ are the restrictions of $F_{P,z,y}$ to $[T_{Q,z}]$ as required. This follows from the definition of the phrase "ψ refines θ." The tree $T_{Q,y}$ for $y \in L-L_P$ is simply the $\psi \restriction y$-identity tree.

We now define what it means for a condition P to force a sentence of arithmetic (with function parameters). This use of the term "force" is of course not to be confused with the local forcing on the arithmetic trees which are part of the condition P. As usual local forcing is used to make the condition P force something.

<u>Definition 6.11</u>: Let $P \in \mathcal{P}$ and $\phi(\underline{G}_{x_1},\ldots,\underline{G}_{x_n})$, a sentence of arithmetic with function parameters $\underline{G}_{x_i}$ for $x_i \in L_P$, be given. Then <u>P forces ϕ</u>, written $P \Vdash \phi$, if for any $G \in [T_{P,1}]$, $\phi(G_{x_1},\ldots,G_{x_n})$ holds where $G_{x_i} = F_{P,1,x_i}[G]$.

<u>Definition 6.12</u>: Let $\{f_e | e < \omega\}$ be a recursive list of all the formulas of arithmetic with one function parameter and two free variables u and v. Let

$$f_e^G = \{(m,n) | G \models f_e(\underline{G},\overline{m},\overline{n})\}.$$

In order to make the embedding an isomorphism we wish to guarantee that for any $e \in \omega$ and $x,y \in L$ with $y \nleq x$, $f_e^{G_x} \neq G_y$. It may of course be the case that $f_e^{G_x}$ is not even a function. The next lemma shows that we can diagonalize, but first we define the appropriate dense sets, as called for by our abstract forcing machinery.

<u>Definition 6.13</u>: For $e \in \omega$ and $x,y \in L$, let $D_{2,e,x,y} = \{P \mid y \nleq x \rightarrow P \Vdash f_e^{G_x} \neq G_y\}$. Let C_2 contain C_1 and all $D_{2,e,x,y}$.

<u>Lemma 6.14</u>: The $D_{2,e,x,y}$ are dense.

We first give a construction which we need repeatedly. We simply show how to refine a given condition so that a desired tree in it is comprised of branches which are sufficiently generic.

<u>Definition 6.15</u>: A tree T is an <u>n-decision tree via T'</u> (or simply an <u>n-decision tree</u> or <u>decision tree</u>) if $T' \supseteq T$ and $\forall f \in [T]$ (f is n-T'-generic).

<u>Lemma 6.16</u>: Fix n. Given a condition P and $x \in L_P$, there is a $Q \leq P$ such that $L_Q = L_P$, $\theta_Q = \theta_P$ and $T_{Q,x}$ is an n-decision tree.

<u>Proof</u>: The idea is that if a tree is uniform, then it has a uniform decision subtree. We define a distinguished uniform $\theta_P \restriction 1$-tree S and take Q to be the condition S(P) so that $T_{Q,x}$ is such a subtree of $T_{P,x}$.

Let $<\psi_e | e \in \omega>$ be a recursive listing of all sentences of arithmetic with rank at most n and one function parameter. Let $T' = T_{P,x}$ and $T'' = T_{P,1}$.

Let ρ be the least element of $S_{\theta\restriction x}$ such that $T'(\rho) \overset{T'}{\Vdash} \psi_0 \vee \sim\psi_0$ (such a ρ exists by the definition of forcing the negation of a sentence). Let $S(\emptyset) = \tau$ where $f_x(\tau) = \rho$. Suppose that $S(\sigma)$ has been defined for $\sigma \in S_{\theta\restriction 1}$ of length n. We first define a string $\overline{\tau}$ such that for all $\sigma \in S_{\theta\restriction 1}$ with $|\sigma| = n$, $T'(f_x(S(\sigma) * \overline{\tau})) \overset{T'}{\Vdash} \psi_{n+1} \vee \sim\psi_{n+1}$. This is straightforward; we define $\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \overline{\tau}$ so that $T'(f_x(S(\sigma_i) * \tau_i))$ decides ψ_{n+1} by letting σ_i run through the possible strings σ of length n (or if desired take only enough σ's to insure that their images under $f_x \circ S$ run through the possibilities in T'). Then set $S(\sigma * j) = S(\sigma) * \overline{\tau} * j$ for $j \in \theta_n \restriction 1$. The definition of <u>one</u> $\overline{\tau}$ is for uniformity. The consistency of local forcing (i.e. $(\tau' \supseteq \rho' \;\&\; \rho' \overset{T'}{\Vdash} \psi) \rightarrow \tau' \overset{T'}{\Vdash} \psi)$ implies that $f_x(S(\sigma * j))$ decides ψ_n for all σ of length n. One can prove by induction on the length of the strings that $\forall x \in L_P\ \forall\sigma,\tau \in S_{\theta\restriction 1}\ (\sigma \equiv_x \tau \leftrightarrow S(\sigma) \equiv_x S(\tau))$, so S is distinguished.

If we let $T = T_{Q,x} = T_{P,x} \circ S_x$, then T is an n-decision tree via $T' = T_{P,x}$. The definability of local forcing and the fact that all trees in P are arithmetic imply that all trees in Q are arithmetic. □

<u>Proof of Lemma 6.14</u>: Let $e \in \omega$, P and $x,y \in L$ with $y \not\leq x$ be given. We may assume by Lemma 6.10 that $y,x \in L_P$, and by Lemma 6.16 that $T_{P,x}$ is an n-decision tree via T' for some T' where $n = \text{rank}(f_e)$. Since $y \not\leq x$, we have $\alpha,\beta \in \theta_{P,0}$ such that $\alpha(x) = \beta(x)$ but $\alpha(y) \neq \beta(y)$. The

idea is the standard one, i.e. we can change G_y while keeping G_x the same and thereby force a difference between $f_e^{G_x}$ and G_y. Now $T_{P,y}(\alpha(y)) \neq T_{P,y}(\beta(y))$ while $T_{P,x}(\alpha(x)) = T_{P,x}(\beta(x))$. Let m, n_0, n_1 be such that $n_0 = T_{P,y}(\alpha(y))(m) \neq T_{P,y}(\beta(y))(m) = n_1$. Take an extension $T_{P,x}(\sigma)$ of $T_{P,x}(\alpha(x))$ on $T_{P,x}$ which decides both $f_e(\underline{G},\overline{m},\overline{n}_0)$ and $f_e(\underline{G},\overline{m},\overline{n}_1)$, and let $\tau_0 \supseteq <\alpha(1)>$ and $\tau_1 \supseteq_{T'} <\beta(1)>$ be elements of $S_{\theta_P}\restriction 1$ with $f_x(\tau_0) = f_x(\tau_1) = \sigma$. If $T_{P,x}(\sigma) \Vdash \sim f_e(\underline{G},\overline{m},\overline{n}_i)$ for either $i = 0$ or 1 (or both) let $S(\gamma) = \tau_i * \gamma$ for all γ and $Q = S(P)$. Then $Q \Vdash f_e^{G_x} \neq G_y$ since $f_e^{G_x}(m) \neq G_y(m)$ for any $G_x \in [T_{Q,x}]$ and $G_y \in [T_{Q,y}]$. Otherwise $T_{P,x}(\sigma) \Vdash^{T'} f_e(\underline{G},\overline{m},\overline{n}_i)$ for both $i = 0$ and $i = 1$. Then taking $S(\gamma) = \tau_i * \gamma$ for either i finishes the proof since then $Q \Vdash (f_e^{G_x}$ is not a function). $\square$

<u>Definition 6.17</u>: For $e \in \omega$ and $x \in L$, let $D_{3,e,x} = \{P \mid \exists y \preceq x\ (P \Vdash \phi_e^{G_x} \equiv_a G_y)\}$, where $<\phi_e(\underline{G},v) \mid e \in \omega>$ is an effective listing of all formulas of arithmetic with one function parameter and free variable v, and $\phi_e^{G_x} \overset{def}{=} \{m \mid G_x \models \phi_e(\underline{G},\overline{m})\}$. C_3 contains C_2 and the sets $D_{3,e,x}$.

<u>Lemma 6.18</u>: The $D_{3,e,x}$ are dense. In fact, if $x \in L_P$ then there is a $Q \leq P$ in $D_{3,e,x}$ with $L_Q = L_P$ and $\theta_Q = \theta_P$.

<u>Proof</u>: We closely follow Section 3 of Chapter VII of Lerman [1983], but need some notational changes since the conditions consist of trees for each element rather than just the top element of the lattice and since we are using arithmetic computations.

<u>Definition 6.19</u>: If T_x is a $\theta\restriction x$-tree from some condition L and T_x is a rank(ϕ_e)-decision tree via T' then T_x is an <u>e-splitting tree for $y \preceq x$</u> via T', where $y \in L$ if

i) $\forall\sigma \in S_{\theta\restriction x}\ \forall q,r \in \theta_{|\sigma|}\restriction x$

$[q \not\equiv_y r \to \langle T_x(\sigma * q),\ T_x(\sigma * r)\rangle$ forms an <u>e-splitting via T'</u>, that is, $\exists m(T_x(\sigma * q) \Vdash^{T'} \phi_e(\underline{G},\overline{m})$ and $T_x(\sigma * r) \Vdash^{T'} {\sim}\phi_e(\underline{G},\overline{m})$ or vice versa)]

and ii) $\forall\sigma,\tau \in S_{\theta\restriction x}\ [\langle T_x(\sigma),\ T_x(\tau)\rangle$ is an e-splitting via $T' \to \sigma \not\equiv_y \tau]$.

<u>Sublemma 6.20</u> (Computation Lemma): Let T_x be an e-splitting tree for $y \preceq x$, where T_x is the tree associated with x in some condition. Then $\forall G_x \in [T_x]\ (\phi_e^{G_x} \equiv_a F_{x,y}(G_x))$, where $F_{x,y}$ is the appropriate map from the condition.

<u>Proof</u>: The result follows from the definition of e-splitting tree and the definability of local forcing. First we show that $\phi_e^{G_x} \leq_T F_{x,y}(G_x) \oplus \emptyset^{(m)}$ for some fixed m for all $G_x \in [T_x]$. We know that forcing equals truth on T_x, so to decide if $n \in \phi_e^{G_x}$, we only need to decide if $G_x \Vdash^{T'} \phi_e(\underline{G},\overline{n})$. For any level ℓ we may calculate (arithmetically) from $F_{x,y}(G_x)$ (or recursively from $F_{x,y}(G_x) \oplus \emptyset^{(m)}$ for large enough m) a σ_ℓ with $|\sigma_\ell| = |\rho_\ell| = \ell$ and $\sigma_\ell \equiv_y \rho_\ell$ where ρ_ℓ is such that $G_x \supseteq T_x(\rho_\ell)$. Now just take ℓ large enough so that

$T_x(\sigma_\ell)$ decides $\phi_e(\underline{G},\overline{n})$. Property ii) of Definition 6.19 and the genericity of G_x guarantee that $T_x(\sigma_\ell) \Vdash^{T'} \phi_e(\underline{G},\overline{n})$ if and only if $n \in \phi_e^{G_x}$. So $\phi_e^{G_x} \leq_T F_{x,y}(G_x) \oplus \emptyset^{(m)}$, where m is chosen large enough. Next we show that $F_{x,y}(G_x) \leq_T \phi_e^{G_x} \oplus \emptyset^{(m)}$. We can inductively calculate the path which G_x induces in the tree for y because property i) in the definition of e-splitting tree allows us to eliminate one-by-one the equivalence classes mod y as possible extensions of the current path in T_x, until just one remains. That is, if we know that $\sigma_\ell \equiv_y \rho_\ell$, we can find a $q \in \theta_\ell \restriction x$ such that $\sigma_\ell * q \equiv_y \rho_{\ell+1}$ by a process of elimination. Hence we may calculate $F_{x,y}(G_x)$ from $\phi_e^{G_x} \oplus \emptyset^{(m)}$ for every $G_x \in [T_x]$ (for some appropriate fixed m). □

Thus to prove Lemma 6.18 it suffices to extend P to a condition Q with $T_{Q,x}$ an e-splitting tree from some $y \in L_P = L_Q$ with $y \preceq x$. We may assume that $T_{P,x} = T_x$ is a rank(ϕ_e)-decision tree via some T', by Lemma 6.16. We will now change our notation so as to conform with the Turing degree case and save much repetition of proofs from Lerman.

<u>Definition 6.21</u>: We say that $\phi_e^{T_x(\sigma)}(m)$ <u>converges</u>, $\phi_e^{T_x(\sigma)}(m)\downarrow$, if $T_x(\sigma) \Vdash^{T'} \phi_e(\underline{G},\overline{m}) \vee \sim\phi_e(\underline{G},\overline{m})$. If $\phi_e^{T_x(\sigma)}(m)\downarrow$ then $\phi_e^{T_x(\sigma)}(m) = 1$ if $T_x(\sigma) \Vdash^{T'} \phi_e(\underline{G},\overline{m})$ and $\phi_e^{T_x(\sigma)}(m) = 0$ otherwise. Also if $\langle T_x(\sigma), T_x(\tau)\rangle$ forms an e-splitting and $\sigma \equiv_y \tau$ then this is an <u>e-splitting mod y</u>.

With this notation there is one change from the Turing degree case. There can be no σ,m such that $\forall\tau \supseteq \sigma(\phi_e^{T_x(\sigma)}(m)\uparrow)$. Other than this we could follow Lerman's treatment given in VII 3.2 - VII 3.10 almost word for word. In this sequence of lemmas he shows (with some notational changes) how to construct a distinguished tree S such that if $Q = S(P)$ then $T_{Q,x}$ is an e-splitting tree for some $y \precsim x$, and thus $Q \in D_{3,e,x}$ by the computation Lemma. We will dispense with the repetition of this construction. We point out, however, that in order to carry out the construction we only need to be able to tell when $\phi_e^{T_x(f_x(\gamma))}(m)\downarrow$, when for $\sigma,\tau \in S_{\theta\restriction 1}$, $<T_x(f_x(\sigma)), T_x(f_x(\tau))>$ forms an e-splitting, and whether various congruence relations hold between certain strings. The table is recursive and so by the definability of local forcing the entire construction may be carried out arithmetically in the trees which comprise the given condition P. Hence the trees of Q are arithmetic as desired. □

<u>Theorem 6.22</u>: If L is a countable upper semi-lattice with least element then L is isomorphic to an initial segment of the arithmetic degrees.

<u>Proof</u>: Let G be a C_3-generic filter. The map $x \mapsto \deg(G_x)$ where $G_x = \bigcup \{T_{P,x}(\emptyset) \mid P \in G \ \& \ x \in L_P\}$ gives an isomorphism by C_2-genericity. By C_3-genericity the range of the map is an initial segment. □

We now will show that the orderings $\mathcal{D}_T(\leq \underline{0}^\omega)$ and $\mathcal{D}_a(\leq \underline{0}^\omega)$ are not elementarily equivalent. The result will follow from the fact that any lattice that has complexity arithmetic in $\emptyset^\omega$ is embeddable as an initial segment in the a-degrees below $\underline{0}^\omega$, while any lattice initial segment in the T-degrees below $\underline{0}^\omega$ has a more restricted complexity.

Definition 6.23: Given a countable lattice L, a presentation of L is an isomorphic lattice $L' = \langle \omega, \leq_{L'}, \wedge_{L'}, \vee_{L'} \rangle$ with domain ω. The degree of the presentation is $\leq_{L'} \oplus \wedge_{L'} \oplus \vee_{L'}$.

Corollary 6.24: Suppose L is a lattice with a presentation arithmetic in $\emptyset^\omega$. Then L is embeddable as an initial segment of the a-degrees below $\emptyset^\omega$.

Proof: Lerman ([1983], B.3.29) points out that we may divide a lattice L' into finite approximations $\{L_i' \mid i \in \omega\}$ to it with $\langle \theta_{i,j} \mid j \in \omega \rangle$ a sequential table for each L_i', such that there is a function h with h(i,j) a canonical index for $\theta_{i,j}$, with h recursive in a presentation of the lattice. Note that then the embedding construction may be carried out recursively in $\emptyset^\omega \oplus h$ (since, e.g., $\emptyset^\omega$ can (uniformly) decide what y to make T_x an e-splitting tree for, and similarly can answer all other questions needed in the construction), and hence L may be embedded arithmetically below $\emptyset^\omega$ as claimed. □

Corollary 6.25. The orderings of the a-degrees below $\emptyset^{\omega}$ and the T-degrees below $\emptyset^{\omega}$ are not elementarily equivalent. This solves Problem 2 of Odifreddi [1983].

Proof: We use the machinery developed in Nerode-Shore [1980] and Shore [1981], [1982]. Shore [1981], Definition 1.10 defines what it means for a degree $\underline{e}$ to effectively code a model of arithmetic (the details need not concern us here). Lemma 1.12 of that paper provides a proof that if $\underline{e}$ effectively codes a model and $\underline{e} \leq_T \emptyset^{\omega}$ then there is an $f \leq_T \emptyset^{\omega+3}$ such that $\deg\{f(n)\}^{\emptyset^{\omega}}$ codes the integer n. The subsets of ω coded by a model are the ones picked out by exact pairs, so if $\underline{e}$ effectively codes a standard model then the subsets of ω coded by that model in $\mathcal{D}_T$ $(\leq \underline{0}^{\omega})$ are of the form $W = \{n \mid \{f(n)\}^{\emptyset^{\omega}} \leq_T X,Y\}$ where $X \leq_T \emptyset^{\omega}$ and $Y \leq_T \emptyset^{\omega}$. Now $\leq_T$ on indices below $\emptyset^{\omega}$ is $\Sigma_3^{\emptyset^{\omega}}$, so all W coded in this way satisfy $W \in \Sigma_4^{\emptyset^{\omega}}$. In $\mathcal{D}_a(\leq \underline{0}^{\omega})$ however, there is an $\underline{e}$ which effectively codes a standard model and an exact pair $\underline{u},\underline{v} \leq_a \underline{0}^{\omega}$ such that $\underline{u},\underline{v}$ codes a set of degree $\underline{0}^{\omega+5}$. [Suppose we are given a presentation of a lattice that effectively codes a standard model of arithmetic, with $\vee$ and $\wedge$ given by $\cup$ and $\cap$ respectively and the atom $\{a_n\}$ representing the integer n, as in the Remark before Theorem 4.1 of Shore [1982]. We need only add the set $\{a_n \mid n \in \emptyset^{\omega+5}\}$ as a new generator to obtain an arithmetic-in-$\emptyset^{\omega}$ presentation of a lattice that effectively codes a standard model and codes $\emptyset^{\omega+5}$ (there is a single element, so an exact pair, above the appropriate set of atoms). Since this lattice is embeddable below $\emptyset^{\omega}$ in the a-degrees we have the claim. The definition of a lattice effectively coding a model of arithmetic guarantees that it is not distributive, so we need the embedding result of Corollary 6.24].

Given a sentence ϕ of second order arithmetic, one can effectively translate it to a sentence of partial orderings $\phi'(\underline{e})$ (ϕ' has other parameters in addition to $\underline{e}$) such that if $\underline{e}$ effectively codes a standard model then $\mathcal{D}_T(\leq \underline{0}^{\omega}) \models \phi'(\underline{e})$ if and only if $M \models \phi$, where M is the 2nd order model $<\omega, S, \epsilon, +, \times, \leq>$. Here S (the class of sets over which the set quantifiers range) is the class of subsets of ω that are coded by some exact pair $\underline{u}, \underline{v} \leq_T \underline{0}^{\omega}$. Similarly, if an a-degree $\underline{e} \leq_a \underline{0}^{\omega}$ effectively codes a standard model then $\mathcal{D}_a(\leq \underline{0}^{\omega}) \models \phi'(\underline{e})$ if and only if $M' \models \phi$ where $M' = <\omega, S', \epsilon, +, \times, \leq>$ and S' is the class of sets coded by exact pairs in $\mathcal{D}_a(\leq \underline{0}^{\omega})$. For details of the translation procedure see Nerode-Shore [1980].

Let $\psi(\underline{e})$ be the sentence of orderings which says that $\underline{e}$ effectively codes a standard model (so in addition to saying that $\underline{e}$ effectively codes a model, it says that every proper initial segment of the ω-like ordering given by the model that has an exact pair has a top element). Let ϕ be the following sentence of second order arithmetic: $\exists X[X^{[0]} = \emptyset \ \& \ \forall n(X^{[n+1]} = X^{[n]}{}') \ \& \ \forall Y(Y \in \Sigma_4^X)]$. Let $\gamma \equiv \forall \underline{e}(\psi(\underline{e}) \rightarrow \phi'(\underline{e}))$. Now $\mathcal{D}_T (\leq \emptyset^{\omega}) \models \gamma$ while $\mathcal{D}_a (\leq \underline{0}^{\omega}) \not\models \gamma$: That $\mathcal{D}_a (\leq \emptyset^{\omega}) \not\models \gamma$ is clear since there is an $\underline{e}$ that effectively codes a standard model below $\underline{0}^{\omega}$ and such that the set $\emptyset^{\omega+5}$ is coded in this model. Suppose that $\mathcal{D}_T (\leq \underline{0}^{\omega}) \models \psi(\underline{e})$ for some $\underline{e}$. Then $\underline{e}$ actually does code a standard model in $\mathcal{D}_T (\leq \underline{0}^{\omega})$ because the ideal generated by the codes for the standard integers has an exact pair below $\emptyset^{\omega}$. This is proved in Shore [1981], i.e.: the set of indices for the (representatives of the) degrees in the ideal is Σ_3^E, where $E \in \underline{e}$, and so there is an exact pair for the ideal recursive in $E \oplus \emptyset' \leq_T \emptyset^{\omega}$ by Lemma 2.1 of that paper. Similarly $\emptyset^{\omega}$ is coded by some exact pair below $\underline{0}^{\omega}$ for every effective standard model. Thus $\mathcal{D}_T$ $(\leq \underline{0}^{\omega})$ $\models \gamma$. □

Section 3 of Abraham-Shore [1985] shows how to embed any conceivable $\aleph_1$-size upper semi-lattice as an initial segment of the T-degrees (to be deemed conceivable, a u.s.l. need only satisfy the obvious requirements: it has a least element and satisfies the countable predecessor property). The main points are:

i) Given L of size $\aleph_1$ there is an end extension L' of L that may be divided up in a sufficiently nice way as $\cup\{L_\alpha | \alpha < \aleph_1\}$, with each L_α a countable u.s.l.

ii) Define the sequence of forcing notions $\langle P_\alpha | \alpha < \aleph_1 \rangle$ and C_5-generic filters G_α by simultaneous induction. P_0 is the notion of forcing appropriate for L_0 (as defined in the countable case) and G_0 is any C_5-generic filter (where C_5 is defined later). If P_α is defined and G_α is a C_5-generic filter on P_α then $P_{\alpha+1}$ is the collection of conditions P in the notion for forcing appropraite for $L_{\alpha+1}$ such that there is a $P' \in G_\alpha$ that represents P in the proper way.

iii) C_5 can be defined so that C_5-genericity of G_α implies the existence of a C_5-generic $G_{\alpha+1} \subseteq P_{\alpha+1}$. Here C_5 contains the Turing degree analog of our C_3.

Thus one gets an initial segment of the Turing degrees isomorphic to the u.s.l. L' (and so also one isomorphic to L) by sending $x \in L'$ to $G_x = \cup\{T_{P,x}(\emptyset) \mid \exists\alpha(P \in G_\alpha)$ and $x \in L_P\}$.

With very minor notational changes the same proof works in our setting. Indeed i) and ii) do not change at all. The only real changes come in proving the density of the $D_{2,e,x,y}$ and $D_{3,e,x}$ in that we need to talk about arithmetic computations, but the density of these for G_α imply the density of $G_{\alpha+1}$ in exactly the same manner as in Abraham-Shore. We thus get the following result.

<u>Theorem 6.25</u>: If L is a u.s.l. with least element and the countable predecessor property, and $|L| \leq \aleph_1$, then L is isomorphic to an initial segment of the arithmetic degrees.

References

Abraham, U. and Shore, R. A.

[1985] Initial segments of the degrees of size $\aleph_1$, to appear in Isr. Jour. Math.

Cooper, S. B.

[1987] A jump class of non-cappable degrees, to appear.

Epstein, R. L.

[1975] Minimal degrees of unsolvability and the full approximation construction, Memoirs Am. Math. Soc. 162.

Feferman, S.

[1965] Some applications of the notions of forcing and generic sets, Fund. Math. 56, 325-345.

Harding, C. J.

[1974] Forcing in Recursion Theory, Ph.D. thesis, University College of Swansea.

Harrington, L.

[1975] Arithmetically incomparable arithmetic singletons, mimeographed handwritten notes.

[1976] McLaughlin's conjecture, mimeographed handwritten notes.

Jockusch, C. G. Jr.

[1984] Private communication.

Jockusch, C. G. Jr. and Shore, R. A.

[1983] Pseudo-jump operators I: the r.e. case, Trans. Am. Math. Soc. 275, 599-609.

[1984] Pseudo-jump operators II: transfinite iterations, hierarchies and minimal covers, Jour. Symb. Logic 49, 1205-1236.

Lachlan, A. H.

[1968] Distributive initial segments of the degrees of unsolvability, Z. Math. Logik und Grund. der Math. 14, 457-472.

Lachlan, A. H. and Lebeuf, R.

[1976] Countable initial segments of the degrees of unsolvability, Jour. Symb. Logic 41, 289-300.

Lerman, M.

[1971] Initial segments of the degrees of unsolvability, Ann. Math. 93, 365-389.

[1983] Degrees of Unsolvability, Springer-Verlag, Berlin.

Nerode, A. and Shore, R. A.

[1979] Second order logic and first order theories of reducibility orderings, in Kleene Symposium, Barwise et al. eds., North Holland, Amsterdam, 181-200.

[1980] Reducibility orderings: theories, definability and automorphisms, Ann. Math. Logic 18, 61-89.

Odifreddi, P. G.

[1982] Forcing and reducibilities, Jour. Symb. Logic 48, 288-310.

[1983] On the first order theory of the arithmetical degrees, Proc. Am. Math. Soc. 87, 505-507.

Rogers, H. Jr.

[1967] Theory of Recursive Functions and Effective Computability, McGraw-Hill, New York.

Sacks, G. E.

[1967] On a theorem of Lachlan and Martin, Proc. Am. Math. Soc. 18, 140-141.

[1971] Forcing with perfect closed sets, in Axiomatic Set Theory I (Proc. of the 13th Symp. in Pure Math. of the Am. Math. Soc.), Scott, D. ed., Am. Math. Soc., Providence, 331-355.

Shore, R. A.

[1979] The homogeneity conjecture, Proc. Nat. Acad. Sci. USA 76, 4218-4219.

[1981] The theory of the degrees below 0', Jour. London Math. Soc. 24, 1-14.

[1982] On homogeneity and definability in the first order theory of the Turing degrees, Jour. Symb. Logic 47, 8-16.

[1984] The arithmetical degrees are not elementarily equivalent to the Turing degrees, Arch. Math. Logik und Grund. 24, 137-139.

[1987] A non-inversion theorem for the jump operator, to appear.

Simpson, S. G.

[1977] First-order theory of the degrees of recursive unsolvability, Ann. of Math. 105, 121-139.

Smullyan, R. M.

[1961] Theory of Formal Systems, Ann. of Math. Studies 47, Princeton, N.J.

Soare, R. I.

[1986] Recursively Enumerable Sets and Degrees, to appear, Springer-Verlag, Berlin.

Spector, C.

[1956] On degrees of recursive unsolvability, Ann. of Math. 64, 501-506.

Yates, C.E.M.

[1976] Banach-Mazur games, comeagre sets, and degrees of unsolvability, Math. Proc. Camb. Phil. Soc. 79, 195-220.