ON THE COMPLEXITY OF MATHEMATICAL PROBLEMS: MEDVEDEV DEGREES AND REVERSE MATHEMATICS

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We investigate the complexity of mathematical problems from two perspectives: Medvedev degrees and reverse mathematics. In the Medvedev degrees, we calculate the complexity of its first-order theory, and we also calculate the complexities of the first-order theories of several related structures. We characterize the join-irreducible Medvedev degrees and deduce several consequences for the interpretation of propositional logic in the Medvedev degrees. We equate the size of chains of Medvedev degrees with the size of chains of sets of reals under $\subseteq$. In reverse mathematics, we analyze the strength of classical theorems of finite graph theory generalized to the countable. In particular, we consider Menger's theorem, Birkhoff's theorem, and unfriendly partitions.

BIOGRAPHICAL SKETCH

Paul was born on February 28, 1983 in Richland, Washington during the final episode *Goodbye, Farewell and Amen* of the popular television series *M*A*S*H*. He graduated from Dutch Fork High School in Irmo, South Carolina in 2001. Paul has attended Cornell University in Ithaca, New York since 2001, earning a B.S. in computer science in 2005 and a M.S. in computer science in 2010.

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CHAPTER 1
INTRODUCTION

Medvedev degrees describe the relative complexity of subsets of ω^ω in a computational sense. A set $\mathcal{Y} \subseteq \omega^\omega$ is at least as complicated as a set $\mathcal{X} \subseteq \omega^\omega$ if there is a computational procedure for producing a member of $\mathcal{X}$ given a member of $\mathcal{Y}$. We interpret a set $\mathcal{X} \subseteq \omega^\omega$ as an abstract mathematical problem. Namely, $\mathcal{X}$ corresponds to the problem of finding a member of $\mathcal{X}$. Under this interpretation, the Medvedev degrees serve as a model for studying the relative complexity of mathematical problems. We investigate the Medvedev degrees in Chapter 2, Chapter 3, and Chapter 4. See Section 1.3 for a full introduction to the Medvedev degrees and related structures.

Chapter 2 is mainly concerned with calculating the complexities of the first-order theories of the Medvedev degrees and related structures. The main results are as follows.

- The first-order theories of the Medvedev degrees and the Muchnik degrees are both recursively isomorphic to the third-order theory of arithmetic (Theorem 2.3.10).
- The first-order theories of the closed Medvedev degrees, the compact Medvedev degrees, the closed Muchnik degrees, and the compact Muchnik degrees are all recursively isomorphic to the second-order theory of arithmetic (Theorem 2.4.10 and Theorem 2.6.5).
- Neither the closed Medvedev degrees nor the compact Medvedev degrees is elementarily equivalent to either the closed Muchnik degrees or the compact Muchnik degrees (Theorem 2.7.2).
- The first-order theory of the Medvedev degrees of Π^0_1 classes is recursively isomorphic to the first-order theory of arithmetic (Theorem 2.9.4).
- For any of the above-mentioned degree structures and also for the Muchnik degrees of Π^0_1 classes, the structure's three-quantifier theory as a lattice is undecidable, and the structure's four-quantifier theory as a partial order is undecidable (Theorem 2.3.11, Theorem 2.4.11, Theorem 2.6.6, Theorem 2.9.5, and Theorem 2.11.7).
- The degree of the Medvedev degrees of Π^0_1 classes is $\mathbf{0}'''$ in the sense that there is a presentation of the Medvedev degrees of Π^0_1 classes recursive in $\mathbf{0}'''$ and that $\mathbf{0}'''$ is recursive in any such presentation (Theorem 2.10.6).

In Chapter 3, we characterize the join-irreducible Medvedev degrees and investigate the Medvedev degrees as semantics for propositional logic. The main results are as follows.

- A Medvedev degree is join-irreducible if and only if it is the degree of the complement of a Turing ideal (Theorem 3.1.3).
- There is a Medvedev degree greater than the second-least degree that bounds no join-irreducible degree greater than the second-least degree (Theorem 3.2.5).

- The filter in the Medvedev degrees generated by the non-minimum closed degrees is not prime (Theorem 3.4.3).

In Chapter 4, we provide an explicit construction demonstrating that the statement "there is no chain in $(2^{2^\omega}, \subseteq)$ of cardinality 2^{2^ω}" is consistent with ZFC (Corollary 4.2.6). We then compare the cardinalities of chains in the Medvedev degrees to the cardinalities of chains in $(2^{2^\omega}, \subseteq)$. The main results are as follows.

- For any cardinal κ, there is a chain of cardinality κ in $(2^{2^\omega}, \subseteq)$ if and only if there is a chain of cardinality κ in the Medvedev degrees (Theorem 4.3.1).
- The statements "there is a chain in the Medvedev degrees of cardinality 2^{2^ω}," "there is a chain in the Muchnik degrees of cardinality 2^{2^ω}," and "there is a chain in $(2^{2^\omega}, \subseteq)$ of cardinality 2^{2^ω}" are equivalent and are independent of ZFC (Corollary 4.3.2).

Reverse mathematics is an analysis of the logical strength of theorems from ordinary mathematics in the context of second-order arithmetic. Given a theorem, we wish to determine the weakest set of axioms required to prove that theorem. Theorems requiring stronger axioms are considered more complicated than theorems requiring weaker axioms. See Section 1.6 for a full introduction to reverse mathematics.

We consider reverse mathematics in Chapter 5 and Chapter 6. The main result is that Menger's theorem for countable graphs is provable in the system $\Pi^1_1\text{-}\mathsf{CA}_0$ (Theorem 5.2.4). We also present several partial results concerning the reverse mathematics of Birkhoff's theorem and of unfriendly partitions.

1.1 Basic concepts and notation

Let $n \in \omega$, $\sigma, \tau \in \omega^{<\omega}$, $f, g \in \omega^\omega$, and $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$. Then

- $f \upharpoonright n$ is the initial segment of f of length n,
- $|\sigma|$ is the length of σ,
- $\sigma \subseteq \tau$ means that σ is an initial segment of τ,
- $\sigma \subset f$ means that σ is an initial segment of f,
- $I(\sigma) = \{f \in \omega^\omega \mid \sigma \subset f\}$,
- $\sigma^\frown f$ is the concatenation of σ and f:

$$(\sigma^\frown f)(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n - |\sigma|) & \text{if } n \geq |\sigma|, \end{cases}$$

- $f \oplus g$ is the function defined by

$$(f \oplus g)(n) = \begin{cases} f(m) & \text{if } n = 2m \\ g(m) & \text{if } n = 2m+1, \end{cases}$$

- $\sigma^\frown\mathcal{X} = \{\sigma^\frown f \mid f \in \mathcal{X}\}$,
- $\mathcal{X} + \mathcal{Y} = \{f \oplus g \mid f \in X \wedge g \in \mathcal{Y}\}$, and
- $\mathcal{X} \times \mathcal{Y} = 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$.

The $+$ and $\times$ in the last two items above correspond to the lattice-theoretic operations of join and meet in the Medvedev and Muchnik degrees, which is explained in the introduction to these degrees below. These symbols also retain their more common meanings, such as addition for $+$ and product and cartesian product for $\times$. The meaning of a particular instance of either symbol will be clear from context.

The function $\langle\cdot,\cdot\rangle\colon \omega\times\omega\to\omega$ is a fixed recursive bijection. Φ_e denotes the e^{th} Turing functional. Φ always denotes a Turing functional, and if $f \in \omega^\omega$, then $\Phi(f)$ is the partial function computed when Φ uses f as its oracle. For $\sigma \in \omega^{<\omega}$, $\Phi(\sigma)$ is the partial function that, on input $n \in \omega$, is computed by running Φ on input n for at most $|\sigma|$ steps and using σ to answer oracle queries. The restriction on the running time of $\Phi(\sigma)$ ensures that oracle queries are only made of numbers $< |\sigma|$. Consequently, if $\Phi(\sigma)(n)\downarrow$, then $\Phi(f)(n) = \Phi(\sigma)(n)$ for all $f \supset \sigma$.

Let $A, B \subseteq \omega$. $A \leq_1 B$ if and only if there is a one-to-one recursive function f such that $\forall n(n \in A \leftrightarrow f(n) \in B)$. A and B are *recursively isomorphic* if and only if there is such an f that is a bijection. The Myhill isomorphism theorem states that A and B are recursively isomorphic if and only if $A \equiv_1 B$, that is, if and only if $A \leq_1 B$ and $B \leq_1 A$ (see [69] Section I.5).

A set $\mathcal{X} \subseteq \omega^\omega$ is *independent* if and only if $g \not\leq_{\mathrm{T}} f_0 \oplus f_1 \oplus \cdots \oplus f_{n-1}$ for any distinct $g, f_0, \ldots, f_{n-1} \in \mathcal{X}$. Let $\{f_n\}_{n\in\omega} \subseteq \omega^\omega$ be a sequence of functions, and let $m \in \omega$. Define $\bigoplus_{n\in\omega} f_n$ and $\bigoplus_{n\in\omega\setminus\{m\}} f_n$ by

$$(\bigoplus_{n\in\omega} f_n)(\langle i, j\rangle) = f_i(j) \text{ and}$$

$$(\bigoplus_{n\in\omega\setminus\{m\}} f_n)(\langle i, j\rangle) = \begin{cases} f_i(j) & \text{if } i \neq m \\ 0 & \text{if } i = m. \end{cases}$$

The sequence $\{f_n\}_{n\in\omega} \subseteq \omega^\omega$ is *strongly independent* if and only if $\forall m(f_m \not\leq_{\mathrm{T}} \bigoplus_{n\in\omega\setminus\{m\}} f_n)$. A sequence of sets $\{\mathcal{X}_n\}_{n\in\omega}$ with $\mathcal{X}_n \subseteq \omega^\omega$ for each n is *strongly independent* if and only if $\{f_n\}_{n\in\omega}$ is strongly independent whenever $\forall n(f_n \in \mathcal{X}_n)$.

We consider Baire space ω^ω and Cantor space 2^ω, both with their usual product topologies. Basic open sets in ω^ω have the form $I(\sigma)$ for $\sigma \in \omega^{<\omega}$. 2^ω has the subspace topology. When we are working in Cantor space, we usually write $I(\sigma)$ for $I(\sigma) \cap 2^\omega$.

A *tree* is a set $T \subseteq \omega^{<\omega}$ closed under initial segments: $(\forall \sigma, \tau \in \omega^{<\omega})(\sigma \in T \wedge \tau \subseteq \sigma \rightarrow \tau \in T)$. A function $f \in \omega^\omega$ is a *path* through T if and only if $(\forall n \in \omega)(f \restriction n \in T)$. If T is a tree, then $[T]$ denotes the set of all paths through T. If $\mathcal{X} \subseteq \omega^\omega$ is closed, then $\mathcal{X}$ is the set of paths through the tree $T = \{\sigma \mid (\exists f \in$

$\mathcal{X})(\sigma \subset f)\}$. Conversely, if $T \subseteq \omega^{<\omega}$ is a tree, then $[T]$ is a closed subset of ω^ω. A set $\mathcal{X} \subseteq \omega^\omega$ is compact if and only if it is closed and bounded (i.e., there is a $g\colon \omega \to \omega$ such that $(\forall f \in \mathcal{X})(\forall n \in \omega)(f(n) \leq g(n)))$ if and only if it is the set of paths through a finitely branching tree.

$\mathcal{X} \subseteq \omega^\omega$ is a Π^0_1 *class* if and only if it has a Π^0_1 definition. That is, if and only if $\mathcal{X} = \{f \in \omega^\omega \mid \forall n \varphi(f, n)\}$ for some recursive predicate φ. A useful characterization of the Π^0_1 classes are as the sets of paths through recursive trees: a set $\mathcal{X} \subseteq \omega^\omega$ is a Π^0_1 class if and only if $X = [T]$ for some recursive tree T (see [17] Lemma 2.2). For this reason, the Π^0_1 classes are sometimes called the effectively closed sets. The Π^0_1 classes have been persistent objects of study throughout computability theory, due in no small part to their applications to recursive mathematics and reverse mathematics. The surveys by Cenzer [14] and by Cenzer and Remmel [17] provide an extensive overview of the theory of the Π^0_1 classes, as does the forthcoming book by Cenzer and Remmel [16].

In this work, we only consider Π^0_1 classes that are non-empty subsets of 2^ω. Henceforth the term "Π^0_1 class" refers exclusively to a non-empty Π^0_1 subset of 2^ω.

1.2 Distributive lattices and Brouwer algebras

A lattice $\mathcal{L}$ is *distributive* if and only if $+$ and $\times$ distribute over each other:

- $(\forall x, y, z \in \mathcal{L})(x + (y \times z) = (x + y) \times (x + z))$ and
- $(\forall x, y, z \in \mathcal{L})(x \times (y + z) = (x \times y) + (x \times z))$.

An element x of a lattice $\mathcal{L}$ is *join-reducible* if and only if $(\exists y, z < x)(y + z = x)$. Otherwise x is *join-irreducible*. Dually, x is *meet-reducible* if and only if $(\exists y, z > x)(y \times z = x)$. Otherwise x is *meet-irreducible*. We frequently use the following well-known characterization without mention.

Lemma 1.2.1 (see [7] Section III.2)**.** *If $\mathcal{L}$ is a distributive lattice, then $x \in \mathcal{L}$ is join-irreducible if and only if $(\forall y, z \in \mathcal{L})(x \leq y + z \to x \leq y \vee x \leq z)$. Dually, $x \in \mathcal{L}$ is meet-irreducible if and only if $(\forall y, z \in \mathcal{L})(x \geq y \times z \to x \geq y \vee x \geq z)$.*

Proof. Suppose x is join-irreducible and $x \leq y + z$. Then

$$x = x \times (y + z) = (x \times y) + (x \times z).$$

Thus $x = x \times y$ or $x = x \times z$ which means $x \leq y$ or $x \leq z$. Conversely, if x is join-reducible, then by definition there are $y, z < x$ with $y + z = x$. The proof for the meet-irreducible case is obtained by dualizing the proof for the join-irreducible case. $\square$

A lattice $\mathcal{L}$ is *join-complete* if and only if every non-empty $X \subseteq \mathcal{L}$ has a least upper bound. $\mathcal{L}$ is *meet-complete* if and only if every non-empty $X \subseteq \mathcal{L}$ has a greatest lower bound. $\mathcal{L}$ is *complete* if and only if it is both join-complete and meet-complete. Similarly, a lattice $\mathcal{L}$ is *countably join-complete* if and only if every non-empty countable $X \subseteq \mathcal{L}$ has a least upper bound. $\mathcal{L}$ is *countably meet-complete* if and only if every non-empty countable $X \subseteq \mathcal{L}$ has a greatest lower bound. $\mathcal{L}$ is *countably complete* if and only if it is both countably join-complete and countably meet-complete. In a lattice $\mathcal{L}$, a set $X \subseteq \mathcal{L}$ is called *strongly join-incomplete* if and only if for every finite $\{y_i \mid i < n\} \subseteq X$ there is an $x \in X$ such that $x \nleq \sum_{i<n} y_i$. Dually, a set $X \subseteq \mathcal{L}$ is called *strongly meet-incomplete* if and only if for every finite $\{y_i \mid i < n\} \subseteq X$ there is an $x \in X$ such that $x \ngeq \prod_{i<n} y_i$.

Sometimes we want to ignore the lattice operations of a lattice $\mathcal{L}$ and consider $\mathcal{L}$ as a partial order. When we do, we write $(\mathcal{L}; \leq)$ to indicate that we are considering only the partial order structure on $\mathcal{L}$.

A *Brouwer algebra* is a distributive lattice $\mathcal{B}$ with least element 0 and greatest element 1 such that for all $x, y \in \mathcal{B}$ there is a least $z \in \mathcal{B}$ such that $x + z \geq y$. This least z is denoted $x \to y$.

Brouwer algebras give semantics for propositional logic. For a Brouwer algebra $\mathcal{B}$, a *valuation* is a function $\nu\colon$ propositional variables $\to \mathcal{B}$. A valuation ν extends to all propositional formulas φ by defining

$$\begin{aligned}
\nu(\varphi \wedge \psi) &= \nu(\varphi) + \nu(\psi),\\
\nu(\varphi \vee \psi) &= \nu(\varphi) \times \nu(\psi),\\
\nu(\varphi \to \psi) &= \nu(\varphi) \to \nu(\psi), \text{ and}\\
\nu(\neg\varphi) &= \nu(\varphi) \to 1.
\end{aligned}$$

A propositional formula φ is called *valid* in $\mathcal{B}$ if $\nu(\varphi) = 0$ for every valuation ν. Let $\mathrm{PTh}(\mathcal{B})$[1] denote the set of propositional formulas valid in $\mathcal{B}$. The axioms of intuitionistic logic are valid in every Brouwer algebra $\mathcal{B}$, so $\mathrm{IPC} \subseteq \mathrm{Th}(\mathcal{B}) \subseteq \mathrm{CPC}$ for every Brouwer algebra $\mathcal{B}$. Here IPC denotes intuitionistic logic and CPC denotes classical logic. Logics L for which $\mathrm{IPC} \subseteq L \subseteq \mathrm{CPC}$ are called *intermediate logics*.

1.3 Mass problems and reducibilities

A *mass problem* is a set of functions $\mathcal{X} \subseteq \omega^\omega$. Mass problem $\mathcal{X}$ *Medvedev reduces* (or *strongly reduces*) to mass problem $\mathcal{Y}$ (written $\mathcal{X} \leq_{\mathrm{s}} \mathcal{Y}$) if and only if there is a Turing functional Φ such that $(\forall f \in \mathcal{Y})(\Phi(f) \in \mathcal{X})$ (written $\Phi(\mathcal{Y}) \subseteq \mathcal{X}$). Mass problems $\mathcal{X}$ and $\mathcal{Y}$ are *Medvedev equivalent* (or *strongly equivalent*, written $\mathcal{X} \equiv_{\mathrm{s}} \mathcal{Y}$) if and only if $\mathcal{X} \leq_{\mathrm{s}} \mathcal{Y}$ and $\mathcal{Y} \leq_{\mathrm{s}} \mathcal{X}$. The relation $\equiv_{\mathrm{s}}$ is an equivalence relation on 2^{ω^ω}, and the equivalence class $\deg_{\mathrm{s}}(\mathcal{X})$ is called the *Medvedev degree* of $\mathcal{X}$. $\mathcal{D}_{\mathrm{s}} = \{\deg_{\mathrm{s}}(\mathcal{X}) \mid \mathcal{X} \subseteq \omega^\omega\}$ denotes the collection of all Medvedev degrees.

Mass problem $\mathcal{X}$ *Muchnik reduces* (or *weakly reduces*) to mass problem $\mathcal{Y}$ (written $\mathcal{X} \leq_{\mathrm{w}} \mathcal{Y}$) if and only if $(\forall f \in \mathcal{Y})(\exists g \in \mathcal{X})(g \leq_{\mathrm{T}} f)$. Muchnik reducibility is the non-uniform version of Medvedev reducibility. Mass problems $\mathcal{X}$ and $\mathcal{Y}$ are

[1]Usually what we call $\mathrm{PTh}(\mathcal{B})$ is called $\mathrm{Th}(\mathcal{B})$. In this work, $\mathrm{Th}(\mathcal{B})$ always denotes the set of first-order sentences true in $\mathcal{B}$.

Muchnik equivalent (or *weakly equivalent*, written $\mathcal{X} \equiv_{\mathrm{w}} \mathcal{Y}$) if and only if $\mathcal{X} \leq_{\mathrm{w}} \mathcal{Y}$ and $\mathcal{Y} \leq_{\mathrm{w}} \mathcal{X}$. The equivalence class $\deg_{\mathrm{w}}(\mathcal{X})$ is called the *Muchnik degree* of $\mathcal{X}$. $\mathcal{D}_{\mathrm{w}} = \{\deg_{\mathrm{w}}(\mathcal{X}) \mid \mathcal{X} \subseteq \omega^\omega\}$ denotes the collection of all Muchnik degrees.

Medvedev introduced $\mathcal{D}_{\mathrm{s}}$ in [41] as a formalization of Kolmogorov's ideas of a "calculus of problems" and a "logic of problem solving." Medvedev's intuition was that a mass problem $\mathcal{X}$ represents a mathematical problem, namely the problem of finding an element of $\mathcal{X}$. For example, if $A \subseteq \omega$, the problem "find an enumeration of A" may be formalized as "find an element of the mass problem $\mathcal{X} = \{f \in \omega^\omega \mid \mathrm{ran}(f) = A\}$." Under this interpretation, $\mathcal{X} \leq_{\mathrm{s}} \mathcal{Y}$ means that problem $\mathcal{X}$ is at least as hard as problem $\mathcal{Y}$ in a strongly intuitionistic sense: solutions to $\mathcal{Y}$ can be translated to solutions to $\mathcal{X}$ by a uniform effective procedure. Muchnik introduced his non-uniform variant in [45].

Medvedev reducibility and Muchnik reducibility induce a partial orders on the corresponding degrees: $\deg(\mathcal{X}) \leq \deg(\mathcal{Y})$ if and only if $\mathcal{X} \leq \mathcal{Y}$, where $\deg = \deg_{\mathrm{s}}$ and $\leq\, = \,\leq_{\mathrm{s}}$ in the Medvedev case, and $\deg = \deg_{\mathrm{w}}$ and $\leq\, = \,\leq_{\mathrm{w}}$ in the Muchnik case. $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ are distributive lattices. For mass problems $\mathcal{X}$ and $\mathcal{Y}$, it is an easy check that in both cases

$$\begin{aligned}\deg(\mathcal{X}) + \deg(\mathcal{Y}) &= \deg(\mathcal{X} + \mathcal{Y}),\\ \deg(\mathcal{X}) \times \deg(\mathcal{Y}) &= \deg(\mathcal{X} \times \mathcal{Y}),\end{aligned}$$

and that join and meet distribute over each other. In the Muchnik case, the equivalence $\mathcal{X} \times \mathcal{Y} \equiv_{\mathrm{w}} \mathcal{X} \cup \mathcal{Y}$ always holds, and $\deg_{\mathrm{w}}(\mathcal{X}) \times \deg_{\mathrm{w}}(\mathcal{Y}) = \deg_{\mathrm{w}}(X \cup Y)$. This equivalence is not always true in the Medvedev case.

$\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ have a least element $\mathbf{0} = \deg(\omega^\omega)$ and a greatest element $\mathbf{1} = \deg(\emptyset)$. In both structures, a mass problem has degree $\mathbf{0}$ if and only if it contains a recursive function, and a mass problem has degree $\mathbf{1}$ if and only if it is empty. $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ also both have a second-least element $\mathbf{0}' = \deg(\{f \in \omega^\omega | f >_{\mathrm{T}} 0\})$. The Medvedev degree $\mathbf{0}'$ and Muchnik degree $\mathbf{0}'$ have little to do with the Turing degree $\mathbf{0}'$ (the Turing jump of the Turing degree $\mathbf{0}$). In this work, $\mathbf{0}'$ usually refers to the second-least Medvedev or Muchnik degree, and it is clear from context which degree is meant. Finally, both $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ are Brouwer algebras. For mass problems $\mathcal{X}$ and $\mathcal{Y}$,

$$\begin{aligned}\deg_{\mathrm{s}}(\mathcal{X}) \to \deg_{\mathrm{s}}(\mathcal{Y}) &= \deg_{\mathrm{s}}(\{e^\frown g \mid (\forall f \in \mathcal{X})(\Phi_e(f \oplus g) \in \mathcal{Y})\}) \text{ and}\\ \deg_{\mathrm{w}}(\mathcal{X}) \to \deg_{\mathrm{w}}(\mathcal{Y}) &= \deg_{\mathrm{w}}(\{g \mid (\forall f \in \mathcal{X})(\exists h \in \mathcal{Y})(h \leq_{\mathrm{T}} f \oplus g)\}).\end{aligned}$$

See Sorbi's [74] for a good introduction to $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$.

1.4 Substructures of $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$

Substructures of $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ naturally arise by restricting the family of mass problems under consideration. We consider the degrees of closed mass problems and effectively closed mass problems (i.e., Π^0_1 classes).

1.4.1 Closed degrees

A Medvedev or Muchnik degree is *closed* (*compact*) if it is of the form $\deg(\mathcal{X})$ where $\mathcal{X}$ is closed (compact) in ω^ω. Let

$$\begin{aligned}
\mathcal{D}_{\mathrm{s,cl}} &= \{\deg_{\mathrm{s}}(\mathcal{X}) \mid \mathcal{X} \subseteq \omega^\omega \wedge \mathcal{X} \text{ is closed}\},\\
\mathcal{D}^{01}_{\mathrm{s,cl}} &= \{\deg_{\mathrm{s}}(\mathcal{X}) \mid \mathcal{X} \subseteq \omega^\omega \wedge \mathcal{X} \text{ is compact}\},\\
\mathcal{D}_{\mathrm{w,cl}} &= \{\deg_{\mathrm{w}}(\mathcal{X}) \mid \mathcal{X} \subseteq \omega^\omega \wedge \mathcal{X} \text{ is closed}\}, \text{ and}\\
\mathcal{D}^{01}_{\mathrm{w,cl}} &= \{\deg_{\mathrm{w}}(\mathcal{X}) \mid \mathcal{X} \subseteq \omega^\omega \wedge \mathcal{X} \text{ is compact}\}.
\end{aligned}$$

By inspecting the definitions, one can check that if $\mathcal{X}$ and $\mathcal{Y}$ are closed (compact), then so are $\mathcal{X}+\mathcal{Y}$ and $\mathcal{X}\times\mathcal{Y}$. Thus $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ are distributive sublattices of $\mathcal{D}_{\mathrm{s}}$, and $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ are distributive sublattices of $\mathcal{D}_{\mathrm{w}}$. Similarly, $\mathcal{D}^{01}_{\mathrm{s,cl}}$ is a distributive sublattice of $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ is a distributive sublattice of $\mathcal{D}_{\mathrm{w,cl}}$. $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ all have least element $\mathbf{0} = \deg(\omega^\omega) = \deg(2^\omega)$ and greatest element $\mathbf{1} = \deg(\emptyset)$. Notice that ω^ω is not compact, but it has the same degree as 2^ω.

The closed subsets of ω^ω form the topologically simplest class which yields non-trivial degree structures because every non-empty open set contains a recursive function. As such, closed degrees are worthy objects of study. For example, Bianchini and Sorbi [9] studied the filter in $\mathcal{D}_{\mathrm{s}}$ generated by the non-minimum closed degrees. Lewis, Shore, and Sorbi [39] have made a recent study of topologically-defined collections of Medvedev degrees.

In general, every $\mathcal{X} \subseteq \omega^\omega$ is Medvedev equivalent (and hence also Muchnik equivalent) to some $\mathcal{Y} \subseteq 2^\omega$.

Lemma 1.4.1. *If $\mathcal{X} \subseteq \omega^\omega$ then there is a $\mathcal{Y} \subseteq 2^\omega$ with $\mathcal{X} \equiv_{\mathrm{s}} \mathcal{Y}$.*

Proof. For $f \in \omega^\omega$, let $\mathrm{graph}(f) \in 2^\omega$ denote $\{\langle n, m\rangle \mid f(n) = m\}$. Given $\mathcal{X}$, let $\mathcal{Y} = \{\mathrm{graph}(f) \mid f \in \mathcal{X}\}$. Let Φ be the functional such that $\Phi(f)(\langle n,m\rangle) = 1$ if $f(n) = m$ and $\Phi(f)(\langle n,m\rangle) = 0$ otherwise. Then $\Phi(f) = \mathrm{graph}(f)$ for all f. Thus $\Phi(\mathcal{X}) = \mathcal{Y}$. Let Ψ be the functional such that $\Psi(g)(n)$ searches for an m such that $g(\langle n,m\rangle) = 1$ and outputs such an m if it is found. If g is the characteristic function of $\mathrm{graph}(f)$, then $\Psi(g)$ is total and equals f. Hence $\Psi(\mathcal{X}) = \mathcal{Y}$. ☐

If we let $\mathcal{D}^{01}_{\mathrm{s}}$ denote the Medvedev degrees of mass problems $\mathcal{X} \subseteq 2^\omega$ and let $\mathcal{D}^{01}_{\mathrm{w}}$ denote the Muchnik degrees of mass problems $\mathcal{X} \subseteq 2^\omega$, then Lemma 1.4.1 says $\mathcal{D}_{\mathrm{s}} = \mathcal{D}^{01}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}} = \mathcal{D}^{01}_{\mathrm{w}}$. However, if $\mathcal{X} \subseteq \omega^\omega$ is closed, then the $\mathcal{Y} \subseteq 2^\omega$ produced by Lemma 1.4.1 need not be. Turing functionals are continuous, but ω^ω and 2^ω are not homeomorphic. Nevertheless, if $\mathcal{X} \subseteq \omega^\omega$ is compact, then Lemma 1.4.1 produces a closed $\mathcal{Y} \subseteq 2^\omega$. So every compact $\mathcal{X} \subseteq \omega^\omega$ is Medvedev equivalent (and hence also Muchnik equivalent) to a closed (hence compact) $\mathcal{Y} \subseteq 2^\omega$. This explains the notations $\mathcal{D}^{01}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ for the collections of compact degrees.

Our topological considerations of Medvedev reducibility are consequences of the familiar *use property* (see [37] section I.3). If $\Phi(f)(m) = n$, then there is a finite $\sigma \subset f$ such that σ contains all the answers to the oracle queries made during the computation of $\Phi(f)(m) = n$. This is written $\Phi(\sigma)(m) = n$ and implies $\Phi(g)(m) = n$ for any $g \supset \sigma$. The starting point is the following simple lemma.

Lemma 1.4.2. *Let $m, n \in \omega$. For any program Φ, the set $\{f \in \omega^\omega \mid \Phi(f)(m) = n\}$ is open. If $\Phi(f)$ is total for all $f \in \mathcal{X}$, then $\{f \in \mathcal{X} \mid \Phi(f)(m) = n\}$ is clopen in $\mathcal{X}$.*

Proof. If $\Phi(f)(m) = n$, then by the use property there is some $\sigma \subset f$ such that $\Phi(\sigma)(m) = n$. Hence $\{f \in \omega^\omega \mid \Phi(f)(m) = n\} = \bigcup\{I(\sigma) \mid \Phi(\sigma)(m) = n\}$.

If Φ is total on $\mathcal{X}$, then

$$\begin{aligned}\{f \in \mathcal{X} \mid \Phi(f)(m) = n\} &= \mathcal{X} \cap \{f \in \omega^\omega \mid \Phi(f)(m) = n\} \\ &= \mathcal{X} \cap \big(\bigcap_{i \neq n}\{f \in \omega^\omega \mid \Phi(f)(m) \neq i\}\big).\end{aligned}$$

The last equality holds because if $\Phi(f)$ is total and $\Phi(f)(m) \neq i$ for all $i \neq n$, then it must be that $\Phi(f)(m) = n$. ☐

1.4.2 Effectively closed degrees

Recall our convention that a Π^0_1 class is a non-empty Π^0_1 subset of 2^ω. Let

$$\begin{aligned}\mathcal{E}_\mathrm{s} &= \{\deg_\mathrm{s}(\mathcal{X}) \mid \mathcal{X} \text{ is a } \Pi^0_1 \text{ class}\} \text{ and} \\ \mathcal{E}_\mathrm{w} &= \{\deg_\mathrm{w}(\mathcal{X}) \mid \mathcal{X} \text{ is a } \Pi^0_1 \text{ class}\}.\end{aligned}$$

By inspecting the definitions, one can check that if $\mathcal{X}$ and $\mathcal{Y}$ are both Π^0_1 classes, then so are $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} \times \mathcal{Y}$. Thus $\mathcal{E}_\mathrm{s}$ is a distributive sublattice of $\mathcal{D}_\mathrm{s}$ (in fact of $\mathcal{D}^{01}_{\mathrm{s,cl}}$) and $\mathcal{E}_\mathrm{w}$ is a distributive sublattice of $\mathcal{D}_\mathrm{w}$ (in fact of $\mathcal{D}^{01}_{\mathrm{w,cl}}$). Moreover, given indices for trees T_0 and T_1, we can effectively produce produce indices for trees corresponding to $[T_0]+[T_1]$ and $[T_0]\times[T_1]$. Let $T_0 + T_1 = \{\sigma \oplus \tau \mid \sigma \in T_0 \wedge \tau \in T_1 \wedge |\tau| \leq |\sigma| \leq |\tau|+1\}$. Then $[T_0]+[T_1] = [T_0 + T_1]$ and $[T_0]\times[T_1] = [0^\frown T_0 \cup 1^\frown T_1]$.

$\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$ inherit the least element $\mathbf{0} = \deg(2^\omega)$ from $\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$, respectively. The empty set is not a Π^0_1 class, so $\deg(\emptyset)$ is not in $\mathcal{E}_\mathrm{s}$ or $\mathcal{E}_\mathrm{w}$. However, $\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$ still have a greatest element $\mathbf{1}$. Let $\mathrm{DNR}_2 = \{f \in 2^\omega \mid \forall e(f(e) \neq \Phi_e(e))\}$ (DNR stands for *diagonally non-recursive*). Then $\mathbf{1} = \deg(\mathrm{DNR}_2)$ has greatest degree in both $\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$ (see [65] Lemma 3.20). There are many more natural Π^0_1 classes which also have greatest degree. For example, the class of all (appropriately Gödel numbered) complete consistent extensions of Peano arithmetic has greatest degree in both $\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$.

$\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$ are the effective counterparts of $\mathcal{D}^{01}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$. They have enjoyed considerable attention from many authors, beginning with Simpson's suggestion to the Foundations of Mathematics discussion group that $\mathcal{E}_\mathrm{w}$ is analogous to $\mathcal{E}_\mathrm{T}$, the Turing degrees of r.e. sets, but with more natural examples [64]. This analogy with $\mathcal{E}_\mathrm{T}$ drives much of the research on $\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$. For example, every non-minimum member of $\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$ is join-reducible [10], reflecting Sacks's splitting theorem for $\mathcal{E}_{\mathrm{T}}$ [51], and $\mathcal{E}_{\mathrm{s}}$ dense [15], reflecting Sacks's density theorem for $\mathcal{E}_{\mathrm{T}}$ [53]. The question of whether $\mathcal{E}_{\mathrm{w}}$ is dense remains open. See the recent surveys by Simpson [61] and Hinman [26] for an overview of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$.

The interplay between uniformity and compactness gives us a simple description of Medvedev reducibility between two Π^0_1 classes.

Lemma 1.4.3. *$[T_0] \leq_{\mathrm{s}} [T_1]$ is Σ^0_3 relative to the trees T_0 and T_1.*

Proof. For a given Turing functional Φ, we show that

$$\Phi([T_1]) \subseteq [T_0] \text{ if and only if } (\forall n \in \omega)(\exists s \in \omega)(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi(\sigma) \restriction n \in T_0),$$

where $\Phi(\sigma) \restriction n \in T_0$ abbreviates $(\forall i < n)(\Phi(\sigma)(i)\downarrow) \wedge \Phi(\sigma) \restriction n \in T_0$. It then follows that

$$[T_0] \leq_{\mathrm{s}} [T_1] \text{ if and only if } (\exists e \in \omega)(\forall n \in \omega)(\exists s \in \omega)(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi_e(\sigma) \restriction n \in T_0),$$

which gives our Σ^0_3 definition of $\leq_{\mathrm{s}}$.

For the forward direction, let $n \in \omega$ be given. Let $\Sigma = \{\sigma \in 2^{<\omega} \mid \Phi(\sigma) \restriction n \in T_0\}$. The condition $\Phi([T_1]) \subseteq [T_0]$ implies that $[T_1] \subseteq \bigcup_{\sigma \in \Sigma} I(\sigma)$. By compactness, there is a finite $\Sigma_0 \subseteq \Sigma$ such that $[T_1] \subseteq \bigcup_{\sigma \in \Sigma_0} I(\sigma)$ and an $s \in \omega$ such that $(\forall \sigma \in 2^s)(\sigma \in T_1 \to (\exists \sigma_0 \in \Sigma_0)(\sigma_0 \subseteq \sigma))$. Then $(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi(\sigma) \restriction n \in T_0)$.

For the reverse direction, consider $f \in [T_1]$. Given any $n \in \omega$, let $s \in \omega$ be such that $(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi(\sigma) \restriction n \in T_0)$. Then $\Phi(f \restriction s) \restriction n \in T_0$, so $\Phi(f) \restriction n \in T_0$. Thus $\forall n(\Phi(f) \restriction n \in T_0)$. Hence $\Phi(f) \in [T_0]$, and therefore $\Phi([T_1]) \subseteq [T_0]$. $\square$

A sequence of trees $\{T_n\}_{n \in \omega}$ is *uniformly recursive* if and only if the set $\{\langle n, \sigma \rangle \mid \sigma \in T_n\}$ is recursive. A *recursive sequence of* Π^0_1 *classes* is a sequence of Π^0_1 classes $\{\mathcal{X}_n\}_{n \in \omega}$ for which there is a uniformly recursive sequence of trees $\{T_n\}_{n \in \omega}$ such that $T_n \subseteq 2^{<\omega}$ and $\mathcal{X}_n = [T_n]$ for each $n \in \omega$. For convenience, we also allow indexing over recursive sets A and consider recursive sequences of Π^0_1 classes of the form $\{\mathcal{X}_n\}_{n \in A}$. Though not strictly necessary for our results, a convenient fact is that there is a recursive sequence of all Π^0_1 classes (with many repetitions).

Lemma 1.4.4 (see [16] Chapter XV and [17] Section 2.7)**.** *There is a uniformly recursive sequence of infinite trees $\{T_e\}_{e \in \omega}$ such that if $\{\mathcal{Z}_e\}_{e \in \omega}$ is the corresponding recursive sequence of Π^0_1 classes, then for every Π^0_1 class $\mathcal{X}$ there is an $e \in \omega$ such that $\mathcal{X} = \mathcal{Z}_e$.*

Proof. In fact, [17] Lemma 2.2 proves that every Π^0_1 class is of the form $[T]$ for a primitive recursive tree. Let $\{P_e\}_{e \in \omega}$ be a recursive sequence of all primitive recursive functions. Then define T'_e to be the tree $T'_e = \{\sigma \in 2^{<\omega} \mid (\forall \tau \subseteq \sigma)(P_e(\tau) = 1)\}$. If P_e is the characteristic function of a tree, then T'_e is that tree. Thus if $\mathcal{X}$ is a Π^0_1 class, then $\mathcal{X} = [T'_e]$ for some $e \in \omega$. We just need to make a final adjustment to ensure that every tree in the sequence is infinite. To this end, let

$$T_e = \{\sigma \in 2^{<\omega} \mid \sigma \in T'_e \vee (\forall m \leq |\sigma|)(\sigma \restriction m \notin T'_e \to (\forall \tau \in 2^m)(\tau \notin T_e))\}.$$

$\square$

1.5 PA^- and the standard model of arithmetic

In the next chapter, we code structures that model PA^- (Peano arithmetic without induction) in distributive lattices. For reference, we present the axioms of PA^- as they appear in [33].

Definition 1.5.1 (see [33] Section 2.1)**.** PA^- is the theory axiomatized by the following sentences.

(i) $\forall x, y, z((x + y) + z = x + (y + z))$

(ii) $\forall x, y(x + y = y + x)$

(iii) $\forall x, y, z((x \times y) \times z = x \times (y \times z))$

(iv) $\forall x, y(x \times y = y \times x)$

(v) $\forall x, y, z(x \times (y + z) = (x \times y) + (x \times z))$

(vi) $\forall x(x + 0 = x \wedge x \times 0 = 0)$

(vii) $\forall x(x \times 1 = x)$

(viii) $\forall x, y, z(x < y \wedge y < z \rightarrow x < z)$

(ix) $\forall x \neg(x < x)$

(x) $\forall x, y(x < y \vee x = y \vee y < x)$

(xi) $\forall x, y, z(x < y \rightarrow x + z < y + z)$

(xii) $\forall x, y, z(0 < z \wedge x < y \rightarrow x \times z < y \times z)$

(xiii) $\forall x, y(x < y \rightarrow \exists z(x + z = y))$

(xiv) $0 < 1 \wedge \forall x(0 < x \rightarrow x = 1 \vee 1 < x)$

(xv) $\forall x(x = 0 \vee 0 < x)$

To reduce the quantifier complexity of axiom (xiii) for when we analyze the fragments of $\mathrm{Th}(\mathcal{L})$ for various lattices $\mathcal{L}$, we introduce the *monus* symbol "$\dot{-}$" and Skolemize. We call the resulting theory $\mathrm{PA}^{\dot{-}}$

Definition 1.5.2. $\mathrm{PA}^{\dot{-}}$ is the theory whose axioms are the same as PA^- but with axiom (xiii) replaced by the axiom $\forall x, y(x < y \rightarrow x + (y \dot{-} x) = y)$.

The standard relational model of arithmetic is the structure $\mathcal{N} = (\omega; <, +, \times, 0, 1)$, where $<$ is a 2-ary relation on ω, $+$ and $\times$ are 3-ary relations on ω, and 0 and 1 are constants in ω interpreted as the usual less-than, plus, times, zero, and one respectively. $\mathrm{Th}(\mathcal{N})$ denotes the first-order theory of $\mathcal{N}$. We use the relational versions of $+$ and $\times$ instead of the usual functional versions because our coding techniques most naturally code relations. Any formula in which $+$ and $\times$ are relation symbols can be trivially translated into an equivalent formula in which $+$ and $\times$ are function symbols. Translations in the other direction require *unnesting*. In general, a formula is said to be *unnested* if and only if every atomic subformula is of the form $x = y$, $c = y$, $f(x_0, \dots, x_{n-1}) = y$, or $R(x_0, \dots, x_{n-1})$, where x, y, and the x_i for $i < n$ are variables, c is a constant symbol, f is a function symbol, and R is a relation symbol. Every formula can be recursively translated into an equivalent unnested formula (see [28] section 2.6).

When unnesting is applied to a first-order formula in the functional language of arithmetic, we get an equivalent formula in which every atomic subformula is of the form $x = y$, $0 = y$, $1 = y$, $x < y$, $x + y = z$, or $x \times y = z$. That is, we get an equivalent formula in the relational language of arithmetic. Therefore the relational and functional versions of $\mathrm{Th}(\mathcal{N})$ are recursively isomorphic.

$\mathrm{Th}_2(\mathcal{N})$ denotes the second-order theory of $\mathcal{N}$, in which we allow second-order variables X, quantification $\exists X$ and $\forall X$, and second-order membership $x \in X$. $\mathrm{Th}_3(\mathcal{N})$ denotes the third-order theory of $\mathcal{N}$, in which we allow second-order variables X, third-order variables $\mathcal{X}$, quantification $\exists X$, $\forall X$, $\exists \mathcal{X}$, and $\forall \mathcal{X}$, second-order membership $x \in X$, and third-order membership $X \in \mathcal{X}$.

We also make use of the structure $\mathcal{N}^{\dot{-}} = (\omega; <, +, \times, \dot{-}, 0, 1)$, where $<$, $+$, $\times$, 0, and 1 are as for $\mathcal{N}$, and $\dot{-}$ is the 3-ary relation on ω corresponding to the function

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

Clearly, $\mathcal{N} \models \mathrm{PA}^-$, $\mathcal{N}^{\dot{-}} \models \mathrm{PA}^{\dot{-}}$, and $\mathrm{PA}^{\dot{-}} \vdash \mathrm{PA}^-$.

Let $\mathcal{M} \models \mathrm{PA}^-$. An *initial segment* of $\mathcal{M}$ is a $<$-downward-closed substructure $\mathcal{M}'$ of $\mathcal{M}$: $(\forall x \in \mathcal{M}')(\forall y \in \mathcal{M})((\mathcal{M} \models y < x) \rightarrow (y \in \mathcal{M}'))$. An *initial interval* of $\mathcal{M}$ is a subset of $\mathcal{M}$ of the form $\{y \in \mathcal{M} \mid \mathcal{M} \models y < x \vee y = x\}$ for some $x \in \mathcal{M}$. The reason that we have defined initial intervals to be non-empty is simply for later convenience.

For us, the crucial facts about PA^- and its models are the following.

Lemma 1.5.3 (see [33] Theorem 2.2)**.** *If $\mathcal{M} \models \mathrm{PA}^-$, then there is an initial segment of $\mathcal{M}$ that is isomorphic to $\mathcal{N}$. In particular, $\mathcal{N}$ is the unique model of PA^-, up to isomorphism, in which every initial interval is finite.*

Lemma 1.5.4 (see [33] Corollary 2.9)**.** *If φ is a Σ^0_1 sentence and $\mathcal{N} \models \varphi$, then $\mathrm{PA}^- \vdash \varphi$.*

1.6 Reverse mathematics

Reverse mathematics, introduced by Friedman [24], is an analysis of the logical strength of the theorems of ordinary mathematics in the context of second-order arithmetic. A result in reverse mathematics typically has the form "T is equivalent to `strong stystem` over `weak system`," where `strong stystem` and `weak system` are subsystems of second-order arithmetic and T is some theorem from ordinary mathematics. This means that T is provable in `strong stystem` and that all the axioms of `strong stystem` are provable in `weak system`$\cup\{T\}$. The proof of `strong stystem` from `weak system`$\cup\{T\}$ is called a *reversal*.

We now describe the axiomatic systems relevant for our work. We follow [67], the standard reference for reverse mathematics. Also see [4] Section 2 for a thorough introduction to most of the systems we consider and for computability-theoretic interpretations of these systems.

Before we describe the systems, we need to know that the *basic axioms* are the sentences

$$\begin{aligned}
&\forall m(m+1 \neq 0)\\
&\forall m \forall n(m+1 = n+1 \rightarrow m = n)\\
&\forall m(m+0 = m)\\
&\forall m \forall n(m+(n+1) = (m+n)+1)\\
&\forall m(m \times 0 = 0)\\
&\forall m \forall n(m \times (n+1) = (m \times n)+m)\\
&\forall m \neg(m < 0)\\
&\forall m \forall n(m < n+1 \leftrightarrow (m < n \vee m = n)),
\end{aligned}$$

that the *induction axiom* is the sentence

$$\forall X((0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)),$$

and that the *comprehension scheme* consists of all universal closures of formulas of the form

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ can be any formula in the language of second-order arithmetic in which X does not occur freely. Full second-order arithmetic consists of the basic axioms, the induction axiom, and the comprehension scheme.

$\mathtt{RCA}_0$ (for *recursive comprehension axiom*) consists of the basic axioms, the Σ^0_1 *induction scheme*, and the Δ^0_1 *comprehension scheme*. The Σ^0_1 induction scheme consists of all universal closures of formulas of the form

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n \varphi(n)$$

where φ is Σ^0_1. The Δ^0_1 comprehension scheme consists of all universal closures of formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where φ is Σ^0_1, ψ is Π^0_1, and X does not occur freely in φ. $\mathtt{RCA}_0$ is the standard `weak system` for the purpose of reversals. $\mathtt{RCA}_0$ proves that the function $\langle i, j \rangle \mapsto (i+j)^2 + i$ is injective (see [67] Section II.2). For $X \subseteq \mathcal{N}$ and $n \in \mathcal{N}$, we define

$$\begin{aligned}
&(X)_n = \{i \mid \langle i, n \rangle \in X\} \text{ and}\\
&(X)^n = \{\langle i, m \rangle \mid \langle i, m \rangle \in X \wedge m < n\}.
\end{aligned}$$

$\mathtt{RCA}_0$ proves that if X exists, then so do $(X)_n$ and $(X)^n$. We interpret $(X)_n$ as the n^{th} column of X and $(X)^n$ as set of the first n columns of X.

$\mathtt{WKL}_0$ (for *weak König's lemma*) consists of $\mathtt{RCA}_0$ plus the axiom "every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path." The following equivalent formulation is useful. A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is called *bounded* if and only if there is a function $g\colon \mathbb{N} \to \mathbb{N}$ such that $(\forall \tau \in T)(\forall m < |\tau|)(\tau(m) < g(m))$. Over $\mathtt{RCA}_0$, $\mathtt{WKL}_0$ is equivalent to the statement "every bounded infinite subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path." See [67] Lemma IV.1.4.

$\mathtt{ACA}_0$ (for *arithmetical comprehension axiom*) consists of the basic axioms, the induction axiom, and the *arithmetical comprehension scheme*. The arithmetical comprehension scheme is the restriction of the comprehension scheme to formulas φ that are arithmetical.

$\mathtt{ATR}_0$ (for *arithmetical transfinite recursion*) consists of $\mathtt{ACA}_0$ plus an axiom scheme that says if a set can be constructed by iterating arithmetical comprehension along an existing well-order, then that set exists. Let $\mathrm{LO}(X, <_X)$ be a formula that says "$<_X$ is a linear order on the set X," and let $\mathrm{WO}(X, <_X)$ be a formula that says "$<_X$ is a well-order on the set X." Given a formula $\theta(n, Y)$, let $\mathrm{H}_\theta(X, <_X, Y)$ be a formula that says "$\mathrm{LO}(X, <_X)$ and $Y = \{\langle n, j \rangle \mid j \in X \wedge \theta(n, \{\langle m, i \rangle \in Y \mid i <_X j\})\}$." The axioms of $\mathtt{ATR}_0$ consist of those of $\mathtt{ACA}_0$ plus all universal closures of formulas of the form

$$\forall X \forall <_X \; (\mathrm{WO}(X, <_X) \to \exists Y \, \mathrm{H}_\theta(X, <_X, Y))$$

where θ is arithmetical. An easier-to-understand equivalent of $\mathtt{ATR}_0$ is the system Σ^1_1 *separation*, which consists of the axioms of $\mathtt{RCA}_0$ plus the all universal closures of formulas of the form

$$\neg \exists n(\varphi_0(n) \wedge \varphi_1(n)) \to \exists Z \forall n((\varphi_0(n) \to n \in Z) \wedge (\varphi_1(n) \to n \notin Z)),$$

where φ_0 and φ_1 are Σ^1_1 and Z does not occur freely in either φ_0 or φ_1 (see [67] Theorem V.5.1).

$\Sigma^1_1\text{-}\mathtt{DC}_0$ (for Σ^1_1 *dependent choice*) consists of $\mathtt{ACA}_0$ and the scheme of Σ^1_1 *dependent choice*. The scheme of Σ^1_1 dependent choice consists of all universal closures of formulas of the form

$$\forall n \forall X \exists Y \eta(n, X, Y) \to \exists Z \forall n \eta(n, (Z)^n, (Z)_n)$$

where η is Σ^1_1 and Z does not occur freely in η.

$\Pi^1_1\text{-}\mathtt{CA}_0$ (for Π^1_1 *comprehension axiom*) consists of the basic axioms, the induction axiom, and the Π^1_1 *comprehension scheme*. The Π^1_1 comprehension scheme is the restriction of the comprehension scheme to formulas φ that are Π^1_1.

$\mathtt{RCA}_0$ is strictly weaker than $\mathtt{ACA}_0$, which is strictly weaker than both $\mathtt{ATR}_0$ and $\Sigma^1_1\text{-}\mathtt{DC}_0$. $\mathtt{ATR}_0$ and $\Sigma^1_1\text{-}\mathtt{DC}_0$ are independent over $\mathtt{RCA}_0$. However, $\mathtt{ATR}_0$ proves the consistency of $\Sigma^1_1\text{-}\mathtt{DC}_0$. Both $\mathtt{ATR}_0$ and $\Sigma^1_1\text{-}\mathtt{DC}_0$ are strictly weaker than $\Pi^1_1\text{-}\mathtt{CA}_0$.

Our proof of Menger's theorem in $\Pi^1_1\text{-}\mathtt{CA}_0$ relies on two key metamathematical facts. The first key fact concerns the existence of β-models. The second key fact concerns the existence of models of $\Sigma^1_1\text{-}\mathtt{DC}_0$.

Definition 1.6.1. A *countable coded ω-model* is a set $X \subseteq \mathbb{N}$ viewed as coding the structure $\mathcal{M} = (\mathbb{N}, \{(X)_n \mid n \in \mathbb{N}\}, +, \cdot, 0, 1, <)$.

We usually identify a countable coded ω-model X with the structure $\mathcal{M}$ that it codes.

Definition 1.6.2. A *countable coded β-model* is a countable coded ω-model $\mathcal{M}$ that is absolute for Σ^1_1 formulas with parameters from $\mathcal{M}$. That is, if φ is a Σ^1_1 formula with parameters from $\mathcal{M}$, then $\mathcal{M} \models \varphi$ if and only if φ is true.

Theorem 1.6.3 (see [67] Theorem VII.2.10)**.** *The statement "for every X there is a countable coded β-model $\mathcal{M}$ with $X \in \mathcal{M}$" is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{ACA}_0$.*

It is helpful to keep in mind that $\mathtt{ACA}_0$ proves that every countable coded β-model is a model of $\mathtt{ATR}_0$ (see [67] Theorem VII.2.7).

Theorem 1.6.4 (see [67] Theorem VIII.4.20)**.** $\mathtt{ATR}_0$ *proves that for every X there is a countable coded ω-model $\mathcal{M}$ of $\Sigma^1_1\text{-}\mathtt{DC}_0$ with $X \in \mathcal{M}$.*

The statement "for every X there is a countable coded ω-model $\mathcal{M}$ of $\Sigma^1_1\text{-}\mathtt{DC}_0$ with $X \in \mathcal{M}$" is in fact equivalent to $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$. See [67] Lemma VIII.4.15 for the reversal.

CHAPTER 2
CODING TRUE ARITHMETIC IN THE MEDVEDEV AND MUCHNIK DEGREES

The results of this chapter also appear in [54] and [57], both by the author.

A classic problem in computability theory is to determine the complexity of the first-order theory of a given degree structure, such as $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{s}}$, or $\mathcal{D}_{\mathrm{w}}$. The benchmarks are theories of arithmetic, the comparisons are made via recursive isomorphisms, and the results typically express that the first-order theories of the degree structures are as complicated as possible. The original result of this sort, due to Simpson, is that the first-order theory of $\mathcal{D}_{\mathrm{T}}$ is recursively isomorphic to the second-order theory of true arithmetic [62]. We show that the first-order theories of $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ are both recursively isomorphic to the third-order theory of true arithmetic (Theorem 2.3.10). This result was obtained independently by Lewis, Nies, and Sorbi [38].

Various substructures arise in the study of degree structures, and the complexities of their first-order theories naturally come into question. In the Turing degrees, two popular substructures are $\mathcal{D}_{\mathrm{T}}(\leq_{\mathrm{T}} \mathbf{0}')$, the Turing degrees below $\mathbf{0}'$, and $\mathcal{E}_{\mathrm{T}}$, the Turing degrees of r.e. sets. Both $\mathcal{D}_{\mathrm{T}}(\leq_{\mathrm{T}} \mathbf{0}')$ and $\mathcal{E}_{\mathrm{T}}$ have first-order theories that are recursively isomorphic to the first-order theory of true arithmetic. The $\mathcal{D}_{\mathrm{T}}(\leq_{\mathrm{T}} \mathbf{0}')$ case is due to Shore [59]. The $\mathcal{E}_{\mathrm{T}}$ case is due to unpublished work of Harrington and Slaman (see also [46]). We consider the substructures $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, and $\mathcal{E}_{\mathrm{s}}$ of $\mathcal{D}_{\mathrm{s}}$ and the substructures $\mathcal{D}_{\mathrm{w,cl}}$, $\mathcal{D}^{01}_{\mathrm{w,cl}}$, and $\mathcal{E}_{\mathrm{w}}$ of $\mathcal{D}_{\mathrm{w}}$. Our results are that the first-order theories of $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ are all recursively isomorphic to the second-order theory of true arithmetic (Theorem 2.6.5 and Theorem 2.6.5), that the first-order theory of $\mathcal{E}_{\mathrm{s}}$ is recursively isomorphic to the first-order theory of true arithmetic (Theorem 2.9.4), and that the first-order theory of $\mathcal{E}_{\mathrm{w}}$ is undecidable (Theorem 2.11.7). The question of the exact complexity of the first-order theory of $\mathcal{E}_{\mathrm{w}}$ remains wide open. Cole and Simpson conjecture that the first-order theory of $\mathcal{E}_{\mathrm{w}}$ is recursively isomorphic to $\mathcal{O}^{(\omega)}$ (the ω^{th} Turing jump of Kleene's $\mathcal{O}$), the obvious upper bound [19]. As a bonus, our coding methods also yield that neither $\mathcal{D}_{\mathrm{s,cl}}$ nor $\mathcal{D}^{01}_{\mathrm{s,cl}}$ is elementarily equivalent to either $\mathcal{D}_{\mathrm{w,cl}}$ or $\mathcal{D}^{01}_{\mathrm{w,cl}}$ (Theorem 2.7.2).

We also consider the decidabilities of fragments of the first-order theories of $\mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{w}}$, and their substructures. Our results are that if $\mathcal{L}$ is any of $\mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{w}}$, $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, $\mathcal{D}^{01}_{\mathrm{w,cl}}$, $\mathcal{E}_{\mathrm{s}}$, or $\mathcal{E}_{\mathrm{w}}$, then the Σ^0_3-theory of $\mathcal{L}$ as a lattice and the Σ^0_4-theory of $\mathcal{L}$ as a partial order are undecidable (Theorem 2.3.11, Theorem 2.4.11, Theorem 2.6.6, Theorem 2.9.5, and Theorem 2.11.7). In the positive direction, Binns has shown that the Σ^0_1-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as a lattices are identical and decidable [10]. Binns's results also imply that the Σ^0_1-theories of $\mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{w}}$, $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ as lattices are decidable. Cole and Kihara have shown that the Σ^0_2-theory of $\mathcal{E}_{\mathrm{s}}$ as a partial order is decidable [18]. No further results of this sort are known. There has been a huge amount of difficult work on the decidability of various fragments of the first-order theories of $\mathcal{D}_{\mathrm{T}}$ and $\mathcal{E}_{\mathrm{T}}$. In light of the guiding analogy that $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ are like $\mathcal{E}_{\mathrm{T}}$, we summarize the results for $\mathcal{E}_{\mathrm{T}}$ for comparison (see [60] for a survey of this area). The Σ^0_1-theory of $\mathcal{E}_{\mathrm{T}}$

as an upper-semilattice is decidable [52]. The decidability of the Σ_2^0-theory of $\mathcal{E}_{\mathrm{T}}$ as a partial order and the decidability of the Σ_2^0-theory of $\mathcal{E}_{\mathrm{T}}$ as an upper-semilattice remain unknown. However, the Σ_3^0-theory of $\mathcal{E}_{\mathrm{T}}$ as a partial order is undecidable [36]. Moreover, if one extends the partial infimum function on $\mathcal{E}_{\mathrm{T}}$ (as an upper-semilattice) to any total function, then the Σ_2^0-theory of the resulting structure is undecidable [43]. These two undecidability results for $\mathcal{E}_{\mathrm{T}}$ suggest by analogy that the Σ_2^0-theory of $\mathcal{E}_{\mathrm{s}}$ as a lattice, the Σ_3^0-theory of $\mathcal{E}_{\mathrm{s}}$ as a partial order, the Σ_2^0-theory of $\mathcal{E}_{\mathrm{w}}$ as a lattice, and the Σ_3^0-theory of $\mathcal{E}_{\mathrm{w}}$ as a partial order may all be undecidable.

We prove that $\mathcal{E}_{\mathrm{s}}$ is as complicated as possible in terms of degree of presentation (Theorem 2.10.6). Specifically, we prove that the degree of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\mathbf{0}'''$. This means that $\mathbf{0}'''$ computes a presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice and that $\mathbf{0}'''$ is computable in every presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice. A corollary is that $\mathcal{E}_{\mathrm{s}}$ has no recursive presentation as a partial order. The natural presentation of $\mathcal{E}_{\mathrm{w}}$ has Turing degree $\mathcal{O}$ [19], so it is reasonable to expect that $\mathcal{E}_{\mathrm{w}}$ has degree $\mathcal{O}$, though this question remains open. For comparison, it follows from the results of [46] (though it is not stated explicitly) that the degree of $\mathcal{E}_{\mathrm{T}}$ as an upper-semilattice is $\mathbf{0}^{(4)}$.

2.1 Interpreting the Medvedev and Muchnik degrees in arithmetic

Before we code arithmetic into $\mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{w}}$, and their substructures, we show how to interpret these structures in arithmetic.

The reductions $\mathrm{Th}(\mathcal{D}_{\mathrm{s}}) \leq_1 \mathrm{Th}_3(\mathcal{N})$ and $\mathrm{Th}(\mathcal{D}_{\mathrm{w}}) \leq_1 \mathrm{Th}_3(\mathcal{N})$ follow from the fact that every mass problem $\mathcal{X}$ is equivalent to some $\mathcal{Y} \subseteq 2^\omega$ (i.e., Lemma 1.4.1) and that the Medvedev and Muchnik reducibilities are definable in third-order arithmetic.

Lemma 2.1.1. $\mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_3(\mathcal{N})$ *and* $\mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_3(\mathcal{N})$.

Proof. The relation $R(Y, e, m, n)$ expressing $\Phi_e(Y)(m) = n$ is definable by a formula which says "there exists a number s coding a sequence of configurations witnessing the computation $\Phi_e(Y)(m) = n$." The relation $S(X, Y, e)$ expressing $\Phi_e(Y) = X$ is definable by the formula

$$\forall m((m \in X \to R(Y, e, m, 1)) \wedge (m \notin X \to R(Y, e, m, 0))).$$

Thus the relation $\mathcal{X} \leq_{\mathrm{s}} \mathcal{Y}$ is definable by the formula

$$\varphi(\mathcal{X}, \mathcal{Y}) = \exists e \forall Y (Y \in \mathcal{Y} \to \exists X (X \in \mathcal{X} \wedge S(X, Y, e))).$$

Now, given a first-order sentence ψ in the language of partial orders, produce a third-order sentence ψ' in the language of arithmetic by replacing quantifications $\forall x$ and $\exists x$ with third-order quantifications $\forall \mathcal{X}$ and $\exists \mathcal{X}$, by replacing atomic formulas $x \leq y$ with $\varphi(\mathcal{X}, \mathcal{Y})$, and by replacing atomic formulas $x = y$ with $\varphi(\mathcal{X}, \mathcal{Y}) \wedge \varphi(\mathcal{Y}, \mathcal{X})$. Then $\mathcal{N} \models \psi'$ if and only if $\mathcal{D}_{\mathrm{s}} \models \psi$.

The reduction $\mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_3(\mathcal{N})$ is obtained by switching the quantifiers $\exists e$ and $\forall Y$ in the definition of the formula φ above. $\square$

The interpretations of $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}_{\mathrm{w,cl}}$ ($\mathcal{D}^{01}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$) in second-order arithmetic rely on the fact that $\mathcal{X} \subseteq \omega^\omega$ ($\mathcal{X} \subseteq 2^\omega$) is closed if and only if it is the set of paths through some tree $T \subseteq \omega^{<\omega}$ ($T \subseteq 2^{<\omega}$). Thus we quantify over all closed mass problems by quantifying over all trees. Fix some definable coding of sequences, trees, and functions in second-order arithmetic. See [67] Section II.2 for a particularly careful method.

Lemma 2.1.2. $\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$, $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$, $\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$, *and* $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$.

Proof. The relation $P(f,T)$ expressing "function f is a path through tree T" is definable by the formula

$$\forall n \exists \sigma (\sigma \in T \wedge |\sigma| = n \wedge (\forall i < |\sigma|)(\sigma(i) = f(i))).$$

Relations $R(g,e,m,n)$ expressing $\Phi_e(g)(m) = n$ and $S(f,g,e)$ expressing $\Phi_e(g) = f$ are definable as in Lemma 2.1.1. Thus the relation $[T] \leq_{\mathrm{s}} [S]$ is definable by the formula

$$\varphi(T,S) = \exists e \forall g (P(g,S) \rightarrow \exists f (P(f,T) \wedge S(f,g,e))).$$

Now, given a first-order sentence ψ in the language of partial orders, produce a second-order sentence ψ' in the language of arithmetic by replacing quantifications $\forall x$ and $\exists x$ with second-order quantifications $\forall T_x$ and $\exists T_x$ quantifying over trees $T_x \subseteq \omega^{<\omega}$, by replacing atomic formulas $x \leq y$ with $\varphi(T_x, T_y)$, and by replacing atomic formulas $x = y$ with $\varphi(T_x,T_y) \wedge \varphi(T_y,T_x)$. Then $\mathcal{N} \models \psi'$ if and only if $\mathcal{D}_{\mathrm{s,cl}} \models \psi$. This proves $\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$. The reduction $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ is exactly the same, except we quantify over trees $T \subseteq 2^{<\omega}$.

The reductions $\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ and $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ are obtained by switching the quantifiers $\exists e$ and $\forall g$ in the definition of the formula φ above. ☐

To interpret $\mathcal{E}_{\mathrm{s}}$ in first-order arithmetic, we use the recursive sequence containing all Π^0_1 classes $\{\mathcal{Z}_e\}_{e \in \omega}$ from Lemma 1.4.4 and the fact that $\mathcal{Z}_i \leq_{\mathrm{s}} \mathcal{Z}_j$ is a Σ^0_3 property of i and j.

Lemma 2.1.3. $\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}(\mathcal{N})$.

Proof. Let $\{\mathcal{Z}_e\}_{e \in \omega}$ be a recursive sequence containing all Π^0_1 classes as in Lemma 1.4.4, and let $\{T_e\}_{e \in \omega}$ be the corresponding uniformly recursive sequence of trees. Given a first-order sentence ψ in the language of partial orders, produce an equivalent sentence in the language of partial orders by replacing every atomic formula $x = y$ by the formula $x \leq y \wedge y \leq x$. Then produce a first-order sentence ψ' in the language of arithmetic by replacing every atomic formula $x \leq y$ by the Σ^0_3 formula from Lemma 1.4.3 expressing $[T_x] \leq_{\mathrm{s}} [T_y]$. Then $\mathcal{N} \models \psi'$ if and only if $\mathcal{E}_{\mathrm{s}} \models \psi$. ☐

2.2 Coding arithmetic in distributive lattices

We present our scheme for coding arithmetic in distributive lattices. Although the definitions below make sense in any lattice, they were designed with the particular goal of coding $\mathcal{N}$ into $\mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{w}}$, and their sublattices in mind. For example, meet-irreducible elements play a major role in the coding. One may dualize the coding to replace meet-irreducible by join-irreducible, but this would not suffice for our purposes because all non-zero elements of $\mathcal{E}_{\mathrm{s}}$ are join-reducible [10]. The coding presented here has been slightly modified from the original version developed in [57] in order to reduce the quantifier complexity of coded relations.

2.2.1 Coding models of relational theories

Definition 2.2.1. For elements s and w of a lattice, s *meets to* w if and only if $\exists y(y > w \wedge s \times y = w)$.

Definition 2.2.2. For a lattice $\mathcal{L}$ and a $w \in \mathcal{L}$,

$$E(w) = \{s \in \mathcal{L} \mid s \text{ is meet-irreducible} \wedge s \text{ meets to } w\}.$$

The next two lemmas prove important properties of E in distributive lattices.

Lemma 2.2.3. *If $\mathcal{L}$ is a distributive lattice and $w \in \mathcal{L}$, then $E(w)$ is an antichain.*

Proof. Suppose for a contradiction that there are $s, s' \in E(w)$ with $s > s'$. Let $y > w$ be such that $s \times y = w$. Then $s' \geq y$ because s' is meet-irreducible, $s' \geq s \times y$, and $s' \not\geq s$. Therefore $s > s' \geq y > w$, giving the contradiction $s \times y = y > w$. ☐

Lemma 2.2.4. *If $\mathcal{L}$ is a distributive lattice and $\{s_i\}_{i<n} \subseteq \mathcal{L}$ is a finite non-empty antichain of meet-irreducible, non-maximum elements, then $E(\prod_{i<n} s_i) = \{s_i\}_{i<n}$.*

Proof. Suppose $n = 1$. There is a $y > s_0$ because s_0 is not maximum. Thus $s_0 \times y = s_0$, so y witnesses that s_0 meets to s_0. Thus $s_0 \in E(s_0)$. On the other hand, if $x \in E(s_0)$, then there is a $y > s_0$ such that $x \times y = s_0$. Since s_0 is meet-irreducible and $s_0 \not\geq y$, it must be that $s_0 \geq x$. $E(s_0)$ is an antichain by Lemma 2.2.3 and $s_0 \in E(s_0)$, so it must be that $x = s_0$. Hence $E(s_0) = \{s_0\}$.

Suppose $n > 1$, and let $w = \prod_{i<n} s_i$. First we show that $s_i \in E(w)$ for each $i < n$. Fix $i < n$ and let $t_i = \prod\{s_j \mid j < n \wedge j \neq i\}$ so that $s_i \times t_i = w$. Then $s_i \not\geq t_i$ because otherwise the meet-irreducibility of s_i implies that $s_i \geq s_j$ for some $j \neq i$, contradicting that $\{s_i\}_{i<n}$ is an antichain. Thus s_i is meet-irreducible and t_i witnesses that s_i meets to w. Hence $s_i \in E(w)$. Conversely, if $x \in E(w)$, then x is meet-irreducible and $x \geq w$. Thus $x \geq s_i$ for some $i < n$, so $x = s_i$ because $E(w)$ is an antichain by Lemma 2.2.3. Thus $E(w) = \{s_i\}_{i<n}$. ☐

In the above lemma, the non-maximum hypothesis is only for the $n = 1$ case. If $n > 1$, then no member of $\{s_i\}_{i<n}$ is maximum because it is an antichain, so the non-maximum hypothesis is automatically satisfied. If $\mathcal{L}$ has a maximum

element 1, then $E(1) = \emptyset$ even though 1 is meet-irreducible. This is by the definition of "meets to," because there is no $y \in \mathcal{L}$ with $y > 1$.

Given an element w of a lattice, we think of w as code for the set $E(w)$. The symbol "E" stands for "elements," as in the elements of the set coded by w.[1]

Now we code 2-ary and 3-ary relations on $E(w_0)$ for an element w_0 of a lattice $\mathcal{L}$. The same scheme can code n-ary relations for any $n \in \omega$, but we only need to code 2-ary and 3-ary relations to code $\mathcal{N}$. The intuition behind the following definition is that if $s_0, u_0 \in E(w_0)$, then $s_0 + u_0$ should code the pair (s_0, u_0). However, this coding makes the pairs (s_0, u_0) and (u_0, s_0) indistinguishable because $s_0 + u_0 = u_0 + s_0$. To solve this problem, we fix additional parameters $w_1, w_2, m \in \mathcal{L}$. Once $w_0, w_1, w_2, m \in \mathcal{L}$ are fixed, any $c \in \mathcal{L}$ can be interpreted as coding a 2-ary relation R^2_c on $E(w_0)$ and a 3-ary relation R^3_c on $E(w_0)$.

Definition 2.2.5. Let $\mathcal{L}$ be a lattice and fix elements $w_0, w_1, w_2, m \in \mathcal{L}$. Then any $c \in \mathcal{L}$ defines a 2-ary relation R^2_c on $E(w_0)$ and a 3-ary relation R^3_c on $E(w_0)$ by

$$\begin{aligned}
R^2_c(s_0, u_0) &\text{ if and only if } s_0 \in E(w_0) \wedge u_0 \in E(w_0) \\
&\qquad \wedge \exists u_1 (u_1 \text{ meets to } w_1 \wedge u_0 + u_1 \geq m \wedge s_0 + u_1 \geq c) \\
R^3_c(s_0, u_0, v_0) &\text{ if and only if } s_0 \in E(w_0) \wedge u_0 \in E(w_0) \wedge v_0 \in E(w_0) \\
&\qquad \wedge \exists u_1 \exists v_2 (u_1 \text{ meets to } w_1 \wedge v_2 \text{ meets to } w_2 \\
&\qquad\qquad \wedge u_0 + u_1 \geq m \wedge v_0 + v_2 \geq m \wedge s_0 + u_1 + v_2 \geq c).
\end{aligned}$$

With Definition 2.2.5 in hand, we can define codes for models of various theories. For PA^- we have the following definitions.

Definition 2.2.6. In a lattice $\mathcal{L}$, a *code (for a structure in the language of arithmetic)* is a sequence of elements

$$\vec{w} = (w_0, w_1, w_2, m, \ell, p, t, z, o)$$

from $\mathcal{L}$ interpreted as coding the structure

$$\mathcal{M}_{\vec{w}} = (E(w_0); R^2_\ell, R^3_p, R^3_t, z, o)$$

where R^2_ℓ, R^3_p, and R^3_t are the relations on $E(w_0)$ defined from ℓ, p, and t, respectively, as in Definition 2.2.5.

In Definition 2.2.6, w is for "ω," m is for "match," ℓ is for "less," p is for "plus," t is for "times," z is for "zero," and o is for "one."

If $\vec{w}$ is a code in a lattice $\mathcal{L}$, then sentences in the language of arithmetic are interpreted in $\mathcal{M}_{\vec{w}}$ in the obvious way.

Definition 2.2.7. Let φ be a first-order sentence in the language of arithmetic. The *translation* of φ is the first-order formula $\varphi'(w_0, w_1, w_2, m, \ell, p, t, z, o)$ (with the displayed variables free) in the language of lattices obtained from φ by making the following replacements.

[1]In [57], $E(w)$ was called $\widetilde{E}(w)$ (see [57] Definition 4.4) and its definition required that the $s \in \widetilde{E}(w)$ also be minimal with respect being meet-irreducible and meeting to w. The minimality requirement is unnecessary by Lemma 2.2.3.

- Replace $<$ by the formula defining R_ℓ^2,
- replace $+$ by the formula defining R_p^3,
- replace $\times$ by the formula defining R_t^3,
- replace 0 by z,
- replace 1 by o,
- replace $\exists x$ by the formula expressing $\exists x \in E(w_0)$, and
- replace $\forall x$ by the formula expressing $\forall x \in E(w_0)$.

If $\mathcal{L}$ is a lattice and $\vec{w}$ is a code, then $\mathcal{M}_{\vec{w}} \models \varphi$ means that $\mathcal{L} \models \varphi'(\vec{w})$.

Definition 2.2.8. In a lattice $\mathcal{L}$, a *code for a model of* PA^- is a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \models \mathrm{PA}^-$.

If φ is a first-order sentence in the language of arithmetic, then its translation φ' is a first-order formula in the language of lattices. Thus for such a sentence φ, the property "$\vec{w}$ is a code such that $\mathcal{M}_{\vec{w}} \models \varphi$" is first-order. The property "$\vec{w}$ is a code for a model of PA^-" is therefore expressible by a first-order formula in the language of lattices.

To code true first-, second-, or third-order arithmetic in a lattice, we impose an extra first-, second-, or third-order correctness condition on a code $\vec{w}$ for a model of PA^-, and we impose additional assumptions on the properties of the lattice in which the coding is done. In this way we ensure that coded structures $\mathcal{M}_{\vec{w}}$ are isomorphic to $\mathcal{N}$ and that first-, second-, and third-order quantification over $\mathcal{M}_{\vec{w}}$ can be simulated by first-order quantification over $\mathcal{L}$.

2.2.2 The finite matching property and the first-order correctness condition

In this section we present sufficient conditions for interpreting true first-order arithmetic in a distributive lattice. Our strategy is to recognize the codes $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ as the codes $\vec{w}$ of models of PA^- such that every initial interval of $\mathcal{M}_{\vec{w}}$ is finite. The following definitions allows us to compare the cardinalities of initial intervals of coded models of PA^-.

Definition 2.2.9. Let $\mathcal{L}$ be a lattice and let $\vec{w}$ be a code for a model of PA^-. An $a \in \mathcal{L}$ *codes an initial interval of* $\mathcal{M}_{\vec{w}}$ if and only if $(\exists s \in E(w_0))(\forall b \in \mathcal{L})(b \in E(a) \leftrightarrow R_\ell^2(b, s) \vee b = s)$.

Definition 2.2.10. For a lattice $\mathcal{L}$ and elements $r, q \in \mathcal{L}$, $E(r)$ *matches* $E(q)$ if and only if there is a $z \in \mathcal{L}$ such that

(i) $(\forall x \in E(q))(\exists! y \in E(r))(x + y \in E(z))$, and

(ii) $(\forall x \in E(r))(\exists! y \in E(q))(x + y \in E(z))$.

Clearly if $E(r)$ matches $E(q)$, then $|E(r)| = |E(q)|$. The next definition enforces a weak converse of this fact.

Definition 2.2.11. A lattice $\mathcal{L}$ has the *finite matching property* if and only if whenever $q, q' \in \mathcal{L}$ are such that $|E(q)| = |E(q')| = n$ for some $n \in \omega$ then there is an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(q)$ and $E(q')$.

We can now define the first-order correctness condition and prove that a code for a model of PA^- that satisfies the first-order correctness condition always codes a structure isomorphic to $\mathcal{N}$ provided that $\mathcal{L}$ is distributive, that $\mathcal{L}$ has the finite matching property, and that some code in $\mathcal{L}$ codes a structure isomorphic to $\mathcal{N}$. It follows that $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L})$.

Definition 2.2.12. In a lattice $\mathcal{L}$, a code $\vec{w}$ satisfies the *first-order correctness condition* if and only if

(i) $(\forall s \in E(w_0))(\exists a \in \mathcal{L})(\forall b \in \mathcal{L})(b \in E(a) \leftrightarrow R^2_\ell(b, s) \vee b = s)$ (that is, every initial interval of $\mathcal{M}_{\vec{w}}$ is coded by some $a \in \mathcal{L}$), and

(ii) for every $a \in \mathcal{L}$ that codes an initial interval of $\mathcal{M}_{\vec{w}}$ and every code $\vec{w}'$ for a model of PA^- that satisfies item (i), there is an $a' \in \mathcal{L}$ that codes an initial interval of $\mathcal{M}_{\vec{w}'}$ and an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(a)$ and $E(a')$.

Observe that the property "$\vec{w}$ is a code a model of PA^- that satisfies the first-order correctness condition" can be expressed by a first-order formula in the language of lattices.

Lemma 2.2.13. *Let $\mathcal{L}$ be a distributive lattice with the finite matching property.*

(i) If $\vec{w}$ is a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$, then $\vec{w}$ is a code for a model of PA^- satisfying the first-order correctness condition.

(ii) If there is a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$, then $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$ for every $\vec{w}'$ that is a that is a code for a model of PA^- satisfying the first-order correctness condition.

Proof. For item (i), let $\vec{w}$ be a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. The code $\vec{w}$ is a code for a model of PA^- because $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. For Definition 2.2.12 item (i), let $s \in E(w_0)$ and notice that $\{b \mid R^2_\ell(b, s) \vee b = s\}$ is finite because it is an initial interval of a structure isomorphic to $\mathcal{N}$ and that it is an antichain because it is a subset of $E(w_0)$ which is an antichain by Lemma 2.2.3. Thus $a = \prod\{b \mid R^2_\ell(b, s) \vee b = s\}$ witnesses Definition 2.2.12 item (i) for s because $E(a) = \{b \mid R^2_\ell(b, s) \vee b = s\}$ by Lemma 2.2.4. For Definition 2.2.12 item (ii), let $a \in \mathcal{L}$ code an initial interval of $\mathcal{M}_{\vec{w}}$ and let $\vec{w}'$ be a code for a model of PA^- satisfying Definition 2.2.12 item (i). $|E(a)| = n$ for some $n \in \omega$ because $E(a)$ is an initial interval of a structure isomorphic to $\mathcal{N}$. $\mathcal{M}_{\vec{w}'} \models \mathrm{PA}^-$, so by Lemma 1.5.3 there is an initial interval of $\mathcal{M}_{\vec{w}'}$ of cardinality n and, by Definition 2.2.12 item (i), there is an $a' \in \mathcal{L}$ coding this initial interval. Thus $|E(a)| = |E(a')| = n$, so by the finite matching property there is an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(a)$ and $E(a')$. Thus $\vec{w}$ is indeed a code for a model of PA^- satisfying the first-order correctness condition.

For item (ii), let $\vec{w}$ be a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$, and let $\vec{w}'$ be a code for a model of PA^- satisfying the first-order correctness condition. We show that $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$. By Lemma 1.5.3, it suffices to show that every initial interval $\mathcal{M}_{\vec{w}'}$

is finite. Thus let $s' \in E(w_0')$, let $\{b' \mid R^2_{\ell'}(b', s') \vee b' = s'\}$ be the corresponding initial interval, and, by Definition 2.2.12 item (i), let $a' \in \mathcal{L}$ code this initial interval. By item (i), $\vec{w}$ is a code for a model of PA^- satisfying Definition 2.2.12 item (i), so by Definition 2.2.12 item (ii) (applied to $\vec{w}'$) there is an $a \in \mathcal{L}$ coding an initial interval of $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ such that $|E(a)| = |E(a')|$. $E(a)$ is finite, hence the initial interval $\{b' \mid R^2_{\ell'}(b', s') \vee b' = s'\}$ is finite. $\square$

Lemma 2.2.14. *Let $\mathcal{L}$ be a distributive lattice with the finite matching property such that there exists a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. Then* $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$.

Proof. It suffices to prove $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L})$ because $\mathrm{Th}(\mathcal{L}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$ as the lattice operations $+$ and $\times$ are first-order definable from the partial order $\leq$. Let φ be a first-order sentence in the language of arithmetic. Let θ be the first-order sentence

$$\begin{aligned}\theta = \exists \vec{w}(&\vec{w} \text{ is a code for a model of } \mathrm{PA}^- \\ &\wedge \vec{w} \text{ satisfies the first-order correctness condition} \\ &\wedge \mathcal{M}_{\vec{w}} \models \varphi)\end{aligned}$$

in the language of lattices. By Lemma 2.2.13, there are codes in $\mathcal{L}$ for models of PA^- satisfying the first-order correctness condition, and every code in $\mathcal{L}$ for a model of PA^- satisfying the first-order correctness condition codes a structure isomorphic to $\mathcal{N}$. Thus $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$. $\square$

2.2.3 The coding countable subsets property and the second-order correctness condition

In this section we present sufficient conditions for interpreting true second-order arithmetic in a distributive lattice. Our strategy is to use first-order quantification over the lattice to quantify over all countable subsets of a coded structure. We then recognize the codes $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ as the codes $\vec{w}$ of models of PA^- such that $\mathcal{M}_{\vec{w}}$ is well-founded.

Consider a code $\vec{w}$ for a structure $\mathcal{M}_{\vec{w}}$ in a lattice $\mathcal{L}$. Given $a \in \mathcal{L}$, let $F(a) = \{s \in \mathcal{L} \mid s \geq a\}$. Every element $a \in \mathcal{L}$ codes the subset $F(a) \cap E(w_0) \subseteq E(w_0)$. Using this coding, we extend the translation described in Definition 2.2.7 to second-order sentences φ in the language of arithmetic.

Definition 2.2.15. Let φ be a second-order sentence in the language of arithmetic. The *translation* of φ is the first-order formula $\varphi'(w_0, w_1, w_2, m, \ell, p, t, z, o)$ (with the displayed variables free) in the language of lattices obtained from φ by making the replacements described in Definition 2.2.7 and by also making the following replacements.

- Replace the second-order variable X by the first-order variable v_X,
- replace $x \in X$ by the formula expressing $x \in F(v_X) \cap E(w_0)$, and
- replace quantifiers $\exists X$ and $\forall X$ by $\exists v_X$ and $\forall v_X$ respectively.

If $\mathcal{L}$ is a lattice and $\vec{w}$ is a code, then $\mathcal{M}_{\vec{w}} \models \varphi$ means that $\mathcal{L} \models \varphi'(\vec{w})$.

Notice that even though φ is allowed to be a second-order sentence in the language of arithmetic, its translation φ' is still a first-order formula in the language of lattices.

The next definition ensures that if $\vec{w}$ is a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$, then for every $S \subseteq E(w_0)$ there is an $a \in \mathcal{L}$ such that $F(a) \cap E(w_0) = S$. That is, every subset of $E(w_0)$ has a code, so quantifying over all subsets of $E(w_0)$ is the same as quantifying over all coded subsets of $E(w_0)$.

Definition 2.2.16. A lattice $\mathcal{L}$ has the *coding countable subsets property* if and only if for every $w \in \mathcal{L}$ and every countable $S \subseteq E(w)$ there is an $a \in \mathcal{L}$ such that $F(a) \cap E(w) = S$.

The following second-order correctness condition recognizes the structures isomorphic to $\mathcal{N}$ among the coded models of PA^- in a distributive lattice with the coding countable subsets property.

Definition 2.2.17. In a lattice $\mathcal{L}$, a code $\vec{w}$ satisfies the *second-order correctness condition* if and only if for every $a \in \mathcal{L}$, if there is an $s \in F(a) \cap E(w_0)$, then there is an R^2_ℓ-least such s.

Observe that the property "$\vec{w}$ is a code a model of PA^- that satisfies the second-order correctness condition" can be expressed by a first-order formula in the language of lattices.

Lemma 2.2.18. *Let $\mathcal{L}$ be a distributive lattice with the coding countable subsets property.*

(i) If $\vec{w}$ is a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$, then $\vec{w}$ is a code for a model of PA^- *satisfying the second-order correctness condition.*

(ii) If $\vec{w}$ is a code for a model of PA^- *satisfying the second-order correctness condition, then $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$.*

Proof. Item (i) is true because $\mathcal{N}$ is a well-founded model of PA^-. For item (ii), let $\vec{w}$ be a code for a model of PA^- satisfying the second-order correctness condition. If $\mathcal{M}_{\vec{w}}$ were not well-founded, then there would be a countable $S \subseteq E(w_0)$ with no R^2_ℓ-least element. By the coding countable subsets property, there would be an $a \in \mathcal{L}$ such that $F(a) \cap E(w_0) = S$. This contradicts the second-order correctness condition. Thus $\mathcal{M}_{\vec{w}}$ is a well-founded model of PA^- and hence is isomorphic to $\mathcal{N}$. □

Lemma 2.2.19. *Let $\mathcal{L}$ be a distributive lattice with the coding countable subsets property such that there exists a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. Then $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$.*

Proof. It suffices to prove $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L})$ because $\mathrm{Th}(\mathcal{L}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$ as the lattice operations $+$ and $\times$ are first-order definable from the partial order $\leq$. Let φ

be a second-order sentence in the language of arithmetic. Let θ be the first-order sentence

$$\begin{aligned}\theta = \exists\vec{w}(&\vec{w} \text{ is a code for a model of } \mathrm{PA}^- \\ &\wedge \vec{w} \text{ satisfies the second-order correctness condition} \\ &\wedge \mathcal{M}_{\vec{w}} \models \varphi)\end{aligned}$$

in the language of lattices. By Lemma 2.2.18 item (i), there is a code $\vec{w}$ for a model of PA^- satisfying the second-order correctness condition. Thus to show that $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$, it suffices to show that, for every such $\vec{w}$, $\mathcal{N} \models \varphi$ if and only if $\mathcal{M}_{\vec{w}} \models \varphi$. Let $\vec{w}$ be such a code. Then $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ by Lemma 2.2.18 item (ii). In this case $E(w_0)$ is countable, so if $S \subseteq E(w_0)$, then there is an $a \in \mathcal{L}$ such that $F(a) \cap E(w_0) = S$ by the coding countable subsets property. That is, quantifying over all coded subsets of $E(w_0)$ quantifies over all subsets of $E(w_0)$. Therefore $\mathcal{N} \models \varphi$ if and only if $\mathcal{M}_{\vec{w}} \models \varphi$. $\square$

2.2.4 The coding all subsets property and the third-order correctness condition

In this section we present sufficient conditions for interpreting true third-order arithmetic in a distributive lattice. Our strategy for recognizing $\mathcal{N}$ among coded models of PA^- is the same as in the previous section. Given $w \in \mathcal{L}$, we show how to simulate quantification over subsets of $2^{E(w)}$ by quantification over the subsets of $E(r)$ for some other $r \in \mathcal{L}$.

Consider a code $\vec{w}$ for a structure $\mathcal{M}_{\vec{w}}$ in a lattice $\mathcal{L}$. As in the previous section, every $a \in \mathcal{L}$ codes the subset $F(a) \cap E(w_0) \subseteq E(w_0)$. Fix an $r \in \mathcal{L}$. Then every $b \in \mathcal{L}$ also codes a set $\mathcal{S}_r(b) \subseteq 2^{E(w_0)}$, where

$$\mathcal{S}_r(b) = \{X \subseteq E(w_0) \mid (\exists s \in F(b) \cap E(r))(\forall u \in E(w_0))(u \in X \leftrightarrow u \leq s)\}.$$

Here "r" stands for "reals," and we think of $E(r)$ as the set $2^{E(w)}$. $F(b) \cap E(r)$ ranges over subsets of $E(r)$ as b ranges over $\mathcal{L}$, and this is how we simulate third-order quantification over $\mathcal{M}_{\vec{w}}$ by first-order quantification over $\mathcal{L}$.

We can express $F(a) \cap E(w_0) \in \mathcal{S}_r(b)$ (that is, "the subset of $E(w_0)$ coded by a is an element of the subset of $2^{E(w_0)}$ coded by b and r") by the following first-order formula in the language of lattices:

$$(\exists s \in F(b) \cap E(r))(\forall u \in E(w_0))(a \leq u \leftrightarrow u \leq s).$$

Using this coding, we extend the translations described in Definition 2.2.7 and Definition 2.2.15 to third-order sentences φ in the language of arithmetic.

Definition 2.2.20. Let φ be a third-order sentence in the language of arithmetic. The *translation* of φ is the first-order formula $\varphi'(w_0, w_1, w_2, m, \ell, p, t, z, o, r)$ (with the displayed variables free) in the language of lattices obtained from φ by making the replacements described in Definition 2.2.7 and Definition 2.2.15 and by also making the following replacements.

- Replace the third-order variable $\mathcal{X}$ by the first-order variable $v_{\mathcal{X}}$,
- replace $X \in \mathcal{X}$ by the formula expressing $F(v_X) \cap E(w_0) \in \mathcal{S}_r(v_{\mathcal{X}})$, and
- replace quantifiers $\exists\mathcal{X}$ and $\forall\mathcal{X}$ by $\exists v_{\mathcal{X}}$ and $\forall v_{\mathcal{X}}$ respectively.

If $\mathcal{L}$ is a lattice, $\vec{w}$ is a code, and $r \in \mathcal{L}$, then $\mathcal{M}_{\vec{w},r} \models \varphi$ means that $\mathcal{L} \models \varphi'(\vec{w}, r)$.

Again, even though φ is allowed to be a third-order sentence in the language of arithmetic, its translation φ' is still a first-order formula in the language of lattices. Note the inclusion of the new parameter r in the translation φ'.

The next definitions ensure that if $\vec{w}$ is a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$, then there is an $r \in \mathcal{L}$ such that for every $\mathcal{S} \subseteq 2^{E(w_0)}$ there is a $b \in \mathcal{L}$ such that $\mathcal{S}_r(b) = \mathcal{S}$. That is, every subset of $2^{E(w_0)}$ has a code, so quantifying over all subsets of $2^{E(w_0)}$ is the same as quantifying over all coded subsets of $2^{E(w_0)}$.

Definition 2.2.21. A lattice $\mathcal{L}$ has the *coding all subsets property* if and only if for every $w \in \mathcal{L}$ and every $S \subseteq E(w)$ there is an $a \in \mathcal{L}$ such that $F(a) \cap E(w) = S$.

Lemma 2.2.22.

(i) If $\mathcal{L}$ is a meet-complete distributive lattice with 1, then $\mathcal{L}$ has the coding all subsets property.

(ii) If $\mathcal{L}$ is a countably meet-complete distributive lattice with 1, then $\mathcal{L}$ has the coding countable subsets property.

Proof. For (i), Let $w \in \mathcal{L}$, and let $S \subseteq E(w)$. If $S = \emptyset$, then take $a = 1$. Then $F(a) \cap E(w) = \emptyset$ because by definition 1 does not meet to any element of $\mathcal{L}$. If $S \neq \emptyset$, then let $a = \prod S$. Clearly $S \subseteq F(a) \cap E(w)$. For the converse, suppose for a contradiction that there is an $x \in (F(a) \cap E(w)) \setminus S$. This x meets to a because x meets to w and $x \geq a \geq w$. Let $y > a$ be such that $x \times y = a$. Then, for all $s \in S$, we have that $s \geq x \times y$, that $s \not\geq x$ because $E(w)$ is an antichain by Lemma 2.2.3 and $x \notin S$, and that s is meet-irreducible. Thus $s \geq y$ for all $s \in S$ which gives the contradiction $a \geq y$.

The same proof works for (ii) when S is countable. □

Definition 2.2.23. In a lattice $\mathcal{L}$, a code $\vec{w}$ satisfies the *third-order correctness condition* if and only if $\vec{w}$ satisfies the second-order correctness condition and there is an $r \in \mathcal{L}$ such that

$$(\forall a \in \mathcal{L})(\exists s \in E(r))(\forall u \in E(w_0))(a \leq u \leftrightarrow u \leq s).$$

Observe that the property "$\vec{w}$ is a code a model of PA^- that satisfies the third-order correctness condition" can be expressed by a first-order formula in the language of lattices.

Lemma 2.2.24. *Let $\mathcal{L}$ be a distributive lattice with the coding all subsets property such that there exists a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ and such that $\vec{w}$ satisfies the third-order correctness condition. Then $\mathrm{Th}_3(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$.*

Proof. It suffices to prove $\mathrm{Th}_3(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L})$ because $\mathrm{Th}(\mathcal{L}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$ as the lattice operations $+$ and $\times$ are first-order definable from the partial order $\leq$. Let φ be a third-order sentence in the language of arithmetic. Let θ be the first-order sentence

$$\begin{aligned} \theta = \exists \vec{w} \exists r (& \vec{w} \text{ is a code for a model of } \mathrm{PA}^- \\ & \wedge r \text{ witnesses that } \vec{w} \text{ satisfies the third-order correctness condition} \\ & \wedge \mathcal{M}_{\vec{w},r} \models \varphi) \end{aligned}$$

in the language of lattices. By hypothesis there exist $\vec{w}$ and r such that $\vec{w}$ is a code for a model of PA^- and r witnesses that $\vec{w}$ satisfies the third-order correctness condition. Thus to show that $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$, it suffices to show that, for every such $\vec{w}$ and r, $\mathcal{N} \models \varphi$ if and only if $\mathcal{M}_{\vec{w},r} \models \varphi$. Fix such a $\vec{w}$ and r. $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ by Lemma 2.2.18 item (ii). Every subset of $E(w_0)$ is of the form $F(a) \cap E(w_0)$ for some $a \in \mathcal{L}$ by the coding all subsets property. We need to show that every subset of $2^{E(w_0)}$ is of the form $\mathcal{S}_r(b)$ for some $b \in \mathcal{L}$. To this end, let $\mathcal{X} \subseteq 2^{E(w_0)}$. For each $X \in \mathcal{X}$, let $a_X \in \mathcal{L}$ be such that $F(a_X) \cap E(w_0) = X$ by the coding all subsets property, and let $s_X \in E(r)$ be such that $(\forall u \in E(w_0))(a_X \leq u \leftrightarrow u \leq s_X)$ by the third-order correctness condition. Again by the coding all subsets property, let $b \in \mathcal{L}$ be such that $F(b) \cap E(r) = \{s_X \mid X \in \mathcal{X}\}$. Then $\mathcal{S}_r(b) = \mathcal{X}$. We have shown that quantifying over all coded subsets of $E(w_0)$ quantifies over all subsets of $E(w_0)$ and that quantifying over all coded subsets of $2^{E(w_0)}$ quantifies over all subsets of $2^{E(w_0)}$. Therefore $\mathcal{N} \models \varphi$ if and only if $\mathcal{M}_{\vec{w},r} \models \varphi$. ☐

2.2.5 Counting quantifiers

An analysis of the quantifier complexity of our coding scheme shows that to determine the truth of existential sentences in $\mathcal{N}$ we only need to determine the truth of Π^0_3 sentences in $\mathcal{L}$.

We switch to coding models of $\mathrm{PA}^{\dot{-}}$ because the axioms of $\mathrm{PA}^{\dot{-}}$ are all of the form $\forall \vec{x} \psi(\vec{x})$ for quantifier-free ψ. Here *code* now means a code for a structure in the language of $\mathcal{N}^{\dot{-}}$. A code is now a sequence

$$\vec{w} = (w_0, w_1, w_2, m, \ell, p, t, d, z, o)$$

(with "d" for "difference") interpreted as coding the structure

$$\mathcal{M}^{\dot{-}}_{\vec{w}} = (E(w_0); R^2_\ell, R^3_p, R^3_t, R^3_d, z, o).$$

As in Definition 2.2.7, sentences in the language of $\mathcal{N}^{\dot{-}}$ translate to formulas in the language of lattices. The new $\dot{-}$ relation is replaced in the translation by the formula defining R^3_d. A *code for a model of* $\mathrm{PA}^{\dot{-}}$ is a code $\vec{w}$ such that $\mathcal{M}^{\dot{-}}_{\vec{w}} \models \mathrm{PA}^{\dot{-}}$.

In the language of lattices, "s is meet-irreducible" is a Π^0_1 property and "s meets to w" is a Σ^0_1 property, so "$s \in E(w)$" is a Δ^0_2 property. Hence $R^2_c(s_0, u_1)$

and $R^3_c(s_0, u_1, v_2)$ are both Δ^0_2 properties of s_0, u_1, v_2, and the coding parameters w_0, w_1, w_2, m, and c. Therefore, our coding translates atomic formulas in the language of $\mathcal{N}^{\dot{-}}$ to Δ^0_2 properties of lattices. Every Boolean combination of Δ^0_2 properties is again a Δ^0_2 property, so our coding also translates quantifier-free formulas in the language of $\mathcal{N}^{\dot{-}}$ to Δ^0_2 properties of lattices. Thus if $\varphi = \exists\vec{x}\psi(\vec{x})$ is a sentence in the language of $\mathcal{N}^{\dot{-}}$ where ψ is quantifier-free, then the translation $\varphi'(\vec{w})$ may be taken to be a Σ^0_2 formula in the language of lattices. Similarly, if $\varphi = \forall\vec{x}\psi(\vec{x})$, then the translation $\varphi'(\vec{w})$ is Π^0_2. Thus "$\mathcal{M}^{\dot{-}}_{\vec{w}} \models \mathrm{PA}^{\dot{-}}$" can be expressed by a Π^0_2 formula in the language of lattices. The axioms of $\mathrm{PA}^{\dot{-}}$ need to be unnested before they are translated, but this can be done in such a way that they all remain of the form $\forall\vec{x}\psi(\vec{x})$ for quantifier-free ψ.

In a lattice, the relations $x+y=z$ and $x \times y = z$ are definable by Π^0_1 formulas in the language of partial orders. This translation increases the quantifier-complexities calculated in the previous paragraph by one alternation. Existential sentences in the language of $\mathcal{N}^{\dot{-}}$ translate to Σ^0_3 formulas in the language of partial orders, and universal sentences in the language of $\mathcal{N}^{\dot{-}}$ translate to Π^0_3 formulas in the language of partial orders. The property "$\mathcal{M}^{\dot{-}}_{\vec{w}} \models \mathrm{PA}^{\dot{-}}$" is a Π^0_3 property of $\vec{w}$ in the language of partial orders.

Lemma 2.2.25. *Let $\mathcal{L}$ be a lattice, and let $\vec{w}$ be a code such that $\mathcal{M}^{\dot{-}}_{\vec{w}} \cong \mathcal{N}^{\dot{-}}$. Then Σ^0_3-$\mathrm{Th}(\mathcal{L})$ and Σ^0_4-$\mathrm{Th}(\mathcal{L};\leq)$ are undecidable.*

Proof. We prove

$$\{\exists\vec{x}\psi(\vec{x}) \mid \psi \text{ is quantifier-free} \wedge \mathcal{N} \models \exists\vec{x}\psi(\vec{x})\} \leq_1 \Pi^0_3\text{-}\mathrm{Th}(\mathcal{E}_{\mathrm{s}}).$$

It is well-known that the problem of determining whether $\mathcal{N} \models \exists\vec{x}\psi(\vec{x})$ for quantifier-free ψ is undecidable.[2] Clearly Σ^0_3-$\mathrm{Th}(\mathcal{L}) \equiv_1 \Pi^0_3$-$\mathrm{Th}(\mathcal{L})$.

Let $\varphi = \exists\vec{x}\psi(\vec{x})$ be a sentence in the language of arithmetic where ψ is quantifier-free. Let θ be the sentence

$$\theta = \forall\vec{w}(\mathcal{M}^{\dot{-}}_{\vec{w}} \models \mathrm{PA}^{\dot{-}} \to \mathcal{M}^{\dot{-}}_{\vec{w}} \models \varphi)$$

in the language of lattices. As calculated above, $\mathcal{M}^{\dot{-}}_{\vec{w}} \models \mathrm{PA}^{\dot{-}}$ is a Π^0_2 property of $\vec{w}$, and $\mathcal{M}^{\dot{-}}_{\vec{w}} \models \varphi$ is a Σ^0_2 property of $\vec{w}$. Thus θ is a Π^0_3 sentence in the language of lattices. We need to show $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$. Suppose $\mathcal{N} \models \varphi$. Then $\mathrm{PA}^{\dot{-}} \vdash \varphi$ by Lemma 1.5.4, which implies that $\mathcal{L} \models \theta$. Suppose $\mathcal{N} \not\models \varphi$. Then by assumption there is a code $\vec{w}$ such that $\mathcal{M}^{\dot{-}}_{\vec{w}} \cong \mathcal{N}^{\dot{-}}$. For this $\vec{w}$, $\mathcal{M}^{\dot{-}}_{\vec{w}} \models \mathrm{PA}^{\dot{-}}$ but $\mathcal{M}^{\dot{-}}_{\vec{w}} \not\models \varphi$, which implies $\mathcal{L} \not\models \theta$.

The proof that Σ^4_0-$\mathrm{Th}(\mathcal{L};\leq)$ is undecidable is the same. The above sentence θ is Π^0_4 in the language of partial orders. $\square$

[2] For example, undecidability is implied by Matiyasevich's solution to Hilbert's tenth problem [40]. It is a standard fact in computability theory that determining whether $\mathcal{N} \models \exists\vec{x}\psi(\vec{x})$ is undecidable if ψ is allowed bounded quantifiers, but allowing bounded quantifiers in ψ increases the quantifier complexity of the translated formula.

2.3 The complexities of $\mathrm{Th}(\mathcal{D}_\mathrm{s}; \leq_\mathrm{s})$ and $\mathrm{Th}(\mathcal{D}_\mathrm{w}; \leq_\mathrm{w})$

In this section we prove $\mathrm{Th}_3(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_\mathrm{w}; \leq_\mathrm{w}) \leq_1 \mathrm{Th}(\mathcal{D}_\mathrm{s}; \leq_\mathrm{s})$, thereby completing the proof that all three theories are pairwise recursively isomorphic.

2.3.1 Defining $\mathcal{D}_\mathrm{w}$ in $\mathcal{D}_\mathrm{s}$

The Muchnik degrees are definable in the Medvedev degrees [22], thereby giving $\mathrm{Th}(\mathcal{D}_\mathrm{w}; \leq_\mathrm{w}) \leq_1 \mathrm{Th}(\mathcal{D}_\mathrm{s}; \leq_\mathrm{s})$.

Definition 2.3.1. For a mass problem $\mathcal{A}$, let $C(\mathcal{A})$ denote the Turing upward-closure of $\mathcal{A}$: $C(\mathcal{A}) = \{f \mid (\exists g \in \mathcal{A})(g \leq_\mathrm{T} f)\}$.

Definition 2.3.2. A Medvedev degree $\mathbf{s}$ is called a *degree of solvability* if $\mathbf{s} = \deg_\mathrm{s}(\{f\})$ for some $f \in \omega^\omega$.

Definition 2.3.3. A Medvedev degree $\mathbf{m}$ is called a *Muchnik degree* if $\mathbf{m} = \deg_\mathrm{s}(C(\mathcal{A}))$ for some mass problem $\mathcal{A}$.

Notice that $C(\mathcal{A}) \leq_\mathrm{s} \mathcal{B}$ if and only if $\mathcal{B} \subseteq C(\mathcal{A})$. Medvedev degrees of the form $\deg_\mathrm{s}(C(\mathcal{A}))$ are called Muchnik degrees because $\mathcal{A} \leq_\mathrm{w} \mathcal{B}$ if and only if $C(\mathcal{B}) \subseteq C(\mathcal{A})$ if and only if $C(\mathcal{A}) \leq_\mathrm{s} C(\mathcal{B})$. The mapping $\deg_\mathrm{w}(\mathcal{A}) \mapsto \deg_\mathrm{s}(C(\mathcal{A}))$ embeds $\mathcal{D}_\mathrm{w}$ into $\mathcal{D}_\mathrm{s}$ as an upper-semilattice but not as a lattice [71].

Lemma 2.3.4 (Medvedev [41], Dyment [22])**.** *The degrees of solvability and the Muchnik degrees are definable in $\mathcal{D}_\mathrm{s}$.*

The formula defining the degrees of solvability is

$$\theta(x) = \exists y(x < y \wedge \forall z(x < z \rightarrow y \leq z)).$$

For a degree of solvability $\mathbf{x} = \deg_\mathrm{s}(\{f\})$, the witnessing $\mathbf{y}$ is the degree $\deg_\mathrm{s}(\{e^\frown g \mid \Phi_e(g) = f \wedge g \not\leq_\mathrm{T} f\})$. Complete proofs that θ defines the degrees of solvability are found in [22] and [74]. We reproduce the definability of the Muchnik degrees here. The result essentially appears in [22], but is not phrased in terms of definability.

Proof that the Muchnik degrees are definable in $\mathcal{D}_\mathrm{s}$.
The defining formula is $\chi(x) = \forall y(\forall z((\theta(z) \wedge y \leq z) \rightarrow x \leq z) \rightarrow x \leq y)$, where θ is the formula defining the degrees of solvability as above. Let $\deg_\mathrm{s}(C(\mathcal{A}))$ be a Muchnik degree. If $\mathcal{B}$ satisfies $(\forall f \in \omega^\omega)(\mathcal{B} \leq_\mathrm{s} \{f\} \rightarrow C(\mathcal{A}) \leq_\mathrm{s} \{f\})$, then in particular we must have $C(\mathcal{A}) \leq_\mathrm{s} \{f\}$ for all $f \in \mathcal{B}$. Hence $\mathcal{B} \subseteq C(\mathcal{A})$ and so $\chi(\deg_\mathrm{s}(C(\mathcal{A})))$ holds. Conversely, suppose $\chi(\deg_\mathrm{s}(\mathcal{A}))$. As $(\forall f \in \omega^\omega)(C(\mathcal{A}) \leq_\mathrm{s} \{f\} \rightarrow \mathcal{A} \leq_\mathrm{s} \{f\})$, we have $\mathcal{A} \leq_\mathrm{s} C(\mathcal{A})$. Thus $\mathcal{A} \equiv_\mathrm{s} C(\mathcal{A})$, so $\deg_\mathrm{s}(\mathcal{A})$ is a Muchnik degree. □

Corollary 2.3.5. $\mathrm{Th}(\mathcal{D}_\mathrm{w}; \leq_\mathrm{w}) \leq_1 \mathrm{Th}(\mathcal{D}_\mathrm{s}; \leq_\mathrm{s})$.

Proof. Interpret $\mathrm{Th}(\mathcal{D}_\mathrm{w})$ inside $\mathrm{Th}(\mathcal{D}_\mathrm{s})$ by restricting quantification in $\mathcal{D}_\mathrm{s}$ to quantify only over degrees of the form $\deg_\mathrm{s}(C(\mathcal{A}))$. That is, given a sentence ψ in the language of partial orders, generate a sentence ψ' by inductively replacing subformulas $\exists x\varphi$ and $\forall x\varphi$ by formulas $\exists x(\chi(x)\wedge\varphi)$ and $\forall x(\chi(x)\rightarrow\varphi)$. Then $\mathcal{D}_\mathrm{w}\models\psi$ if and only if $\mathcal{D}_\mathrm{s}\models\psi'$. □

In $\mathcal{D}_\mathrm{w}$, a degree $\mathbf{s}$ is also called a degree of solvability if $\mathbf{s}=\deg_\mathrm{w}(\{f\})$ for some $f\in\omega^\omega$. The formula $\theta(x)$ as above defines the degrees of solvability in $\mathcal{D}_\mathrm{w}$, and the proof is similar to that for $\mathcal{D}_\mathrm{s}$.

2.3.2 Coding third-order arithmetic in $\mathcal{D}_\mathrm{w}$

We find a code $\vec{\mathbf{w}}$ in $\mathcal{D}_\mathrm{w}$ such that $\mathcal{M}_{\vec{\mathbf{w}}}\cong\mathcal{N}$. First, it is well-known and an easy observation that $\mathcal{D}_\mathrm{w}$ is a complete lattice. Hence $\mathcal{D}_\mathrm{w}$ has the coding all subsets property by Lemma 2.2.22.

Lemma 2.3.6. *$\mathcal{D}_\mathrm{w}$ is a complete lattice.*

Proof. Let $\mathfrak{X}\subseteq\mathcal{D}_\mathrm{w}$ be non-empty, and let $\langle\,\mathcal{X}_i\mid i\in I\,\rangle$ be a selection of one representative for each degree in $\mathfrak{X}$. Then the least upper bound of $\mathfrak{X}$ is $\deg_\mathrm{w}\big(\bigcap_{i\in I}C(\mathcal{X}_i)\big)$ and the greatest lower bound of $\mathfrak{X}$ is $\deg_\mathrm{w}\big(\bigcup_{i\in I}C(\mathcal{X}_i)\big)$ (which equals $\deg_\mathrm{w}\big(\bigcup_{i\in I}\mathcal{X}_i\big)$). □

The crucial point is now the existence of the degree $\mathbf{r}$ witnessing that our $\vec{\mathbf{w}}$ satisfies the third-order correctness condition.

Lemma 2.3.7. *Let $\mathcal{W}$ be a $\leq_\mathrm{T}$-antichain, and let $\mathbf{w}=\deg_\mathrm{w}(\mathcal{W})$.*

(i) If $\mathbf{x}\in\mathcal{D}_\mathrm{w}$ meets to $\mathbf{w}$, then $\mathbf{x}\leq_\mathrm{w}\deg_\mathrm{w}(\{f\})$ for some $f\in\mathcal{W}$.

(ii) $E(\mathbf{w})=\{\deg_\mathrm{w}(\{f\})\mid f\in\mathcal{W}\}$.

Proof. (i) Let $\mathbf{x}\in\mathcal{D}_\mathrm{w}$ be such that $\mathbf{x}$ meets to $\mathbf{w}$. Suppose for a contradiction that $(\forall f\in\mathcal{W})(\mathbf{x}\not\leq_\mathrm{w}\deg_\mathrm{w}(\{f\}))$. Let $\mathbf{y}\in\mathcal{D}_\mathrm{w}$ witness that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $\mathbf{y}>_\mathrm{w}\mathbf{w}$ and $\mathbf{x}\times\mathbf{y}=\mathbf{w}$. Then, for all $f\in\mathcal{W}$, $\deg_\mathrm{w}(\{f\})$ is meet-irreducible (as it is the degree of a singleton), $\mathbf{x}\not\leq_\mathrm{w}\deg_\mathrm{w}(\{f\})$, and $\mathbf{x}\times\mathbf{y}\leq_\mathrm{w}\deg_\mathrm{w}(\{f\})$. Therefore $(\forall f\in\mathcal{W})(\mathbf{y}\leq_\mathrm{w}\deg_\mathrm{w}(\{f\}))$ which gives the contradiction $\mathbf{y}\leq_\mathrm{w}\mathbf{w}$.

(ii) Given $f\in\mathcal{W}$, it is an easy check (using the fact that $\mathcal{W}$ is a $\leq_\mathrm{T}$-antichain) that $\deg_\mathrm{w}(\mathcal{W}\setminus\{f\})$ witnesses that $\deg_\mathrm{w}(\{f\})$ meets to $\mathbf{w}$. Hence $\{\deg_\mathrm{w}(\{f\})\mid f\in\mathcal{W}\}\subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x}\in E(\mathbf{w})$. By item (i), $\mathbf{x}\leq_\mathrm{w}\deg_\mathrm{w}(\{f\})$ for some $f\in\mathcal{W}$. We have just shown that $\deg_\mathrm{w}(\{f\})\in E(\mathbf{w})$, and $E(\mathbf{w})$ is an antichain by Lemma 2.2.3. So it must be that $\mathbf{x}=\deg_\mathrm{w}(\{f\})$. □

The following lemma is proved using standard recursion theoretic techniques.

Lemma 2.3.8. *If $\mathcal{A}=\{f_i\mid i\in\omega\}$ is a countable independent set, then there exists a Turing antichain $\mathcal{R}=\{g_X\mid X\in 2^\omega\}$ such that $\{f_i\mid i\in X\}=\{f\in\mathcal{A}\mid f\leq_\mathrm{T} g_X\}$ for each $X\in 2^\omega$.*

Proof. We construct partial functions $g_\sigma\colon \omega \to \omega$ for $\sigma \in 2^{<\omega}$ and put $g_X = \cup_{n\in\omega} g_{X\restriction n}$. The g_σ will have the following properties.

(i) If $\sigma \subset \tau$ then $\operatorname{dom} g_\sigma \subseteq \operatorname{dom} g_\tau$ and the two functions agree on their common domain.

(ii) If $s < |\sigma|$ and $\sigma(s) = 0$ then $g_\sigma(\langle s, j \rangle)$ is defined for all j and equals 0 for all but finitely many j.

(iii) If $s < |\sigma|$ and $\sigma(s) = 1$ then $g_\sigma(\langle s, j \rangle)$ is defined for all j and equals $f_s(j)$ for all but finitely many j.

(iv) $g_\sigma(\langle s, j \rangle)$ is defined for only finitely many $\langle s, j \rangle$ with $s \geq |\sigma|$.

Items (i)–(iii) ensure that each g_X is a total function, and item (iii) ensures $f_s \leq_{\mathrm{T}} g_X$ for all $s \in X$. In addition we satisfy the following requirements for all $e, i \in \omega$ and all $X, Y \subseteq \omega$.

- $R^X_{e,i}$: $i \notin X \to \Phi_e(g_X) \neq f_i$
- $Q^{X,Y}_e$: $X \neq Y \to \Phi_e(g_X) \neq g_Y$

Let $g_\emptyset = \emptyset$. At stage s we have g_σ for all σ of length s.

At stage $s = 2\langle e, i \rangle$ we handle requirement $R^X_{e,i}$. For each σ of length s do the following. If $\sigma(i) = 0$, if there is a finite partial function h_σ with domain disjoint from g_σ, and if there is a number n such that $\Phi_e(g_\sigma \cup h_\sigma)(n){\downarrow} \neq f_i(n)$, then redefine g_σ to be $g_\sigma \cup h_\sigma$. Then for each σ of length s put $g_{\sigma^\frown 0} = g_\sigma \cup \{\langle \langle s, j \rangle, 0 \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$ and put $g_{\sigma^\frown 1} = g_\sigma \cup \{\langle \langle s, j \rangle, f_s(j) \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$.

At stage $s = 2e + 1$ we handle requirement $Q^{X,Y}_e$. List the pairs (σ, τ) with $|\sigma| = |\tau| = s$ and $\sigma \neq \tau$. For each such (σ, τ) do the following. Let n be least such that $n \notin \operatorname{dom} g_\tau$. If there is a finite partial function h_σ with domain disjoint from g_σ and if there is a number m such that $\Phi_e(g_\sigma \cup h_\sigma)(n){\downarrow} = m$, then redefine g_σ to be $g_\sigma \cup h_\sigma$ and redefine g_τ to be $g_\tau \cup \{\langle n, m+1 \rangle\}$. After these extensions are made for each pair (σ, τ), then for each σ of length s put $g_{\sigma^\frown 0} = g_\sigma \cup \{\langle \langle s, j \rangle, 0 \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$ and put $g_{\sigma^\frown 1} = g_\sigma \cup \{\langle \langle s, j \rangle, f_s(j) \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$.

We verify $i \notin X \to f_i \nleq_{\mathrm{T}} g_X$. Suppose that $i \notin X$ and $\Phi_e(g_X) = f_i$. Consider stage $s = 2\langle e, i \rangle$ of the construction. Let $\sigma = X \restriction s$ and let $f = \bigoplus\{f_t \mid t < s \wedge \sigma(t) = 1\}$. The function f computes the graph of the partial function g_σ. Thus we can use f to simulate the computation $\Phi_e(g_\sigma \cup h)(n)$ for any finite partial function h with domain disjoint from g_σ. We now have the contradiction $f_i \leq_{\mathrm{T}} f$ as follows. Given input n, use f to search for a finite partial function h with domain disjoint from g_σ such that $\Phi_e(g_\sigma \cup h)(n){\downarrow} = m$ for some m. There must be such an h because g_X extends g_σ and $\Phi_e(g_X)(n){\downarrow}$. Moreover, we must have $m = f_i(n)$. Otherwise at stage s we would have been able to find an h_σ such that $\Phi_e(g_\sigma \cup h_\sigma)(n){\downarrow} \neq f_i(n)$, and this would imply $\Phi_e(g_X) \neq f_i$.

We verify $X \neq Y \to g_Y \nleq_{\mathrm{T}} g_X$. Suppose for a contradiction that $\Phi(g_X) = g_Y$. Choose an index e for Φ greater than the least e such that $X(e) \neq Y(e)$, put $s = 2e + 1$, and let $\sigma = X \restriction s$, $\tau = Y \restriction s$. Consider the g_σ and g_τ we have right before we process the pair (σ, τ) in stage s. Let n be least such that $n \notin \operatorname{dom} g_\tau$. Since g_X extends g_σ and $\Phi_e(g_X)$ is total, we must have found a finite h_σ and number m such that $\Phi_e(g_\sigma \cup h_\sigma)(n)\downarrow = m$. But then we extended g_τ so that $g_\tau(n) = m+1$. Thus $\Phi_e(g_X)(n) = m \neq g_Y(n)$, a contradiction. $\square$

Lemma 2.3.9. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{w}}$ such that $\mathcal{M}^{\dot-}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot-}$ and such that $\vec{\mathbf{w}}$ satisfies the third-order correctness condition.*

Proof. Let $\mathcal{W}_0 = \{f_{0,n}\}_{n\in\omega}$, $\mathcal{W}_1 = \{f_{1,n}\}_{n\in\omega}$, and $\mathcal{W}_2 = \{f_{2,n}\}_{n\in\omega}$ be such that $\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$ is independent. Then let

$$
\begin{aligned}
\mathbf{w}_0 &= \deg_{\mathrm{w}}(\mathcal{W}_0), && \\
\mathbf{w}_1 &= \deg_{\mathrm{w}}(\mathcal{W}_1), && \\
\mathbf{w}_2 &= \deg_{\mathrm{w}}(\mathcal{W}_2), && \\
\mathbf{m} &= \deg_{\mathrm{w}}(\mathcal{M}) &\quad \text{for} \quad& \mathcal{M} = \{f_{0,n} \oplus f_{1,n}\}_{n\in\omega} \cup \{f_{0,n} \oplus f_{2,n}\}_{n\in\omega}, \\
\boldsymbol{\ell} &= \deg_{\mathrm{w}}(\mathcal{L}) &\quad \text{for} \quad& \mathcal{L} = \{f_{0,i} \oplus f_{1,j} \mid i < j\}, \\
\mathbf{p} &= \deg_{\mathrm{w}}(\mathcal{P}) &\quad \text{for} \quad& \mathcal{P} = \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i + j = k\}, \\
\mathbf{t} &= \deg_{\mathrm{w}}(\mathcal{T}) &\quad \text{for} \quad& \mathcal{T} = \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \times j = k\}, \\
\mathbf{d} &= \deg_{\mathrm{w}}(\mathcal{D}) &\quad \text{for} \quad& \mathcal{D} = \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \dot- j = k\}, \\
\mathbf{z} &= \deg_{\mathrm{w}}(\{f_{0,0}\}), &\quad \text{and} \quad& \\
\mathbf{o} &= \deg_{\mathrm{w}}(\{f_{0,1}\}). &&
\end{aligned}
$$

These degrees give the code $\vec{\mathbf{w}}$. By Lemma 2.3.8, let $\mathcal{R} = \{g_X \mid X \in 2^\omega\}$ be a Turing antichain such that $\{f_{0,i} \in \mathcal{W}_0 \mid i \in X\} = \{f_{0,i} \in \mathcal{W}_0 \mid f_{0,i} \leq_{\mathrm{T}} g_X\}$ for each $X \in 2^\omega$. Let $\mathbf{r} = \deg_{\mathrm{w}}(\mathcal{R})$.

By Lemma 2.3.7 item (ii), $E(\mathbf{w}_0) = \{\deg_{\mathrm{w}}(\{f_{0,n}\})\}_{n\in\omega}$. The map $\deg_{\mathrm{w}}(\{f_{0,n}\}) \mapsto n$ is the isomorphism witnessing $\mathcal{M}^{\dot-}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot-}$. Clearly $\mathbf{z} \mapsto 0$ and $\mathbf{o} \mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, $\times$, and $\dot-$ are similar. Let $i, j \in \omega$. If $i < j$, then $\deg_{\mathrm{w}}(\{f_{1,j}\})$ meets to $\mathbf{w}_1$ by Lemma 2.3.7 item (ii), and it is easy to see that $\deg_{\mathrm{w}}(\{f_{0,j}\}) + \deg_{\mathrm{w}}(\{f_{1,j}\}) \geq_{\mathrm{w}} \mathbf{m}$ and that $\deg_{\mathrm{w}}(\{f_{0,i}\}) + \deg_{\mathrm{w}}(\{f_{1,j}\}) \geq_{\mathrm{w}} \boldsymbol{\ell}$. Thus $R^2_{\boldsymbol{\ell}}(\deg_{\mathrm{w}}(\{f_{0,i}\}), \deg_{\mathrm{w}}(\{f_{0,j}\}))$. Conversely, suppose that $R^2_{\boldsymbol{\ell}}(\deg_{\mathrm{w}}(\{f_{0,i}\}), \deg_{\mathrm{w}}(\{f_{0,j}\}))$. Let $\mathbf{u}_1 \in \mathcal{D}_{\mathrm{w}}$ be such that $\mathbf{u}_1$ meets to $\mathbf{w}_1$, $\deg_{\mathrm{w}}(\{f_{0,j}\}) + \mathbf{u}_1 \geq_{\mathrm{w}} \mathbf{m}$, and $\deg_{\mathrm{w}}(\{f_{0,i}\}) + \mathbf{u}_1 \geq_{\mathrm{w}} \boldsymbol{\ell}$. Since $\mathbf{u}_1$ meets to $\mathbf{w}_1$, it must be that $\mathbf{u}_1 \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{f_{1,k}\})$ for some $k \in \omega$ by Lemma 2.3.7 item (i). Thus $\deg_{\mathrm{w}}(\{f_{0,j}\}) + \deg_{\mathrm{w}}(\{f_{1,k}\}) \geq_{\mathrm{w}} \mathbf{m}$. However, if $k \neq j$, then $f_{0,j} \oplus f_{1,k}$ does not compute any member of $\mathcal{M}$ by independence. Thus $\mathbf{u}_1 \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{f_{1,j}\})$, which implies that $\deg_{\mathrm{w}}(\{f_{0,i}\}) + \deg_{\mathrm{w}}(\{f_{1,j}\}) \geq_{\mathrm{w}} \boldsymbol{\ell}$. Again by independence, if $i \not< j$, then $f_{0,i} \oplus f_{1,j}$ does not compute any member of $\mathcal{L}$. Hence $i < j$.

We now check the third-order correctness condition. To do this, we first need to check the second-order correctness condition. Let $\mathbf{a} \in \mathcal{D}_{\mathrm{s}}$ be such that $F(\mathbf{a}) \cap E(\mathbf{w}_0)$ is non-empty. Let $n \in \omega$ be least such that $\deg_{\mathrm{w}}(\{f_{0,n}\}) \in F(\mathbf{a}) \cap E(\mathbf{w}_0)$. Then this $\deg_{\mathrm{w}}(\{f_{0,n}\})$ is the $R^2_{\boldsymbol{\ell}}$-least element of $F(\mathbf{a}) \cap E(\mathbf{w}_0)$. Thus the second-order correctness condition holds. Now, given an $\mathbf{a} \in \mathcal{D}_{\mathrm{w}}$, let $X = \{i \mid \deg_{\mathrm{w}}(\{f_{0,i}\}) \in F(\mathbf{a}) \cap E(\mathbf{w}_0)\}$, and let $\mathbf{s} = \deg_{\mathrm{w}}(\{g_X\})$. Then $\mathbf{s} \in E(\mathbf{r})$ by Lemma 2.3.7 item (ii) because $\mathcal{R}$ is a $\leq_{\mathrm{T}}$-antichain. Also, for all $\deg_{\mathrm{w}}(\{f_{0,i}\}) \in E(\mathbf{w}_0)$, $\mathbf{a} \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{f_{0,i}\})$ if and only if $i \in X$ if and only if $\deg_{\mathrm{w}}(\{f_{0,i}\}) \leq_{\mathrm{w}} \mathbf{s}$. Thus the third-order correctness condition holds. $\square$

The following theorem was proved independently by Lewis, Nies, and Sorbi [38].

Theorem 2.3.10. $\mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}}) \equiv_1 \mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}}) \equiv_1 \mathrm{Th}_3(\mathcal{N})$.

Proof. $\mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_3(\mathcal{N})$ by Lemma 2.1.1. $\mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}})$ by Corollary 2.3.5. For $\mathrm{Th}_3(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}})$, by Lemma 2.3.9 let $\vec{\mathbf{w}}$ be a code in $\mathcal{D}_{\mathrm{w}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ and such that $\vec{\mathbf{w}}$ satisfies the third-order correctness condition. Removing the degree $\mathbf{d}$ from the code $\vec{\mathbf{w}}$ gives a code $\vec{\mathbf{v}}$ such that $\mathcal{M}_{\vec{\mathbf{v}}} \cong \mathcal{N}$ and such that $\vec{\mathbf{v}}$ satisfies the third-order correctness condition. $\mathcal{D}_{\mathrm{w}}$ has the coding all subsets property by Lemma 2.2.22 because $\mathcal{D}_{\mathrm{w}}$ has a greatest element and is meet-complete by Lemma 2.3.6. Hence $\mathrm{Th}_3(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}})$ by Lemma 2.2.24. $\square$

Theorem 2.3.11. *Σ^0_3-$\mathrm{Th}(\mathcal{D}_{\mathrm{w}})$ and Σ^0_4-$\mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}})$ are both undecidable.*

Proof. By Lemma 2.3.9, there is a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{w}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. The results then follow from Lemma 2.2.25. $\square$

The undecidability of Σ^0_3-$\mathrm{Th}(\mathcal{D}_{\mathrm{s}})$ and the undecidability of Σ^0_4-$\mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}})$ are proved in Theorem 2.6.6.

2.4 The complexities of $\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$ and $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$

In this section, we prove $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}), \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$, and in the next section we prove $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}), \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$. Recall from Lemma 2.3.4 that the degrees of solvability are definable in $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$. The definability of the degrees of solvability in any of $\mathcal{L} = \mathcal{D}_{\mathrm{s,cl}}, \mathcal{D}^{01}_{\mathrm{s,cl}}, \mathcal{D}_{\mathrm{w,cl}}, \mathcal{D}^{01}_{\mathrm{w,cl}}$ would give an immediate proof of $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq)$ for that $\mathcal{L}$. This is because the Turing degrees are isomorphic to the degrees of solvability and because the first-order theory of the Turing degrees is recursively isomorphic to $\mathrm{Th}_2(\mathcal{N})$ [62]. Singleton mass problems $\{f\}$ are compact, so the degrees of solvability are in all these lattices. However, we do not know if the degrees of solvability are definable in any of these lattices.

Question 2.4.1. Are the degrees of solvability definable in $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, or $\mathcal{D}^{01}_{\mathrm{w,cl}}$?

We show that $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ are countably meet-complete. Hence both lattices have the coding countable subsets property by Lemma 2.2.22.

Lemma 2.4.2. *Both $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ are countably meet-complete.*

Proof. Let $\mathfrak{X} = \{\mathbf{x}_i \mid i \in \omega\}$ be a countable set of degrees in $\mathcal{D}_{\mathrm{w,cl}}$. Let $\mathcal{X}_i \subseteq \omega^\omega$ be a closed representative of $\mathbf{x}_i$ for each i. The degree $\mathbf{x} = \deg_{\mathrm{w}}\bigl(\bigcup_{i\in\omega} i^\frown \mathcal{X}_i\bigr)$ is the greatest lower bound of $\mathfrak{X}$ in $\mathcal{D}_{\mathrm{w}}$ and $\bigcup_{i\in\omega} i^\frown \mathcal{X}_i$ is closed. Hence $\mathbf{x} \in \mathcal{D}_{\mathrm{w,cl}}$, so $\mathbf{x}$ is the greatest lower bound of $\mathfrak{X}$ in $\mathcal{D}_{\mathrm{w,cl}}$.

The above proof does not work for $\mathcal{D}^{01}_{\mathrm{w,cl}}$ because $\bigcup_{i\in\omega} i^\frown \mathcal{X}_i$ is not compact. We provide a modified proof for $\mathcal{D}^{01}_{\mathrm{w,cl}}$. Let $\mathfrak{X} = \{\mathbf{x}_i \mid i \in \omega\}$ be a countable set of degrees in $\mathcal{D}^{01}_{\mathrm{w,cl}}$. Let $\mathcal{X}_i \subseteq 2^\omega$ be a closed representative of $\mathbf{x}_i$ for each i. Choose any g in any non-empty $\mathcal{X}_i$ (if all the $\mathcal{X}_i$ are empty, then $\mathbf{1}$ is the greatest lower bound). Let $\sigma_i = (g \restriction i)^\frown(1 - g(i))$ for each $i \in \omega$. The set $\mathcal{X} = \{g\} \cup \bigl(\bigcup_{i\in\omega} \sigma_i{}^\frown \mathcal{X}_i\bigr)$ is closed in 2^ω, so let $\mathbf{x} = \deg_{\mathrm{w}}(\mathcal{X})$. Then $\mathbf{x} \in \mathcal{D}^{01}_{\mathrm{w,cl}}$, and it is easy to see that $\mathbf{x} = \deg_{\mathrm{w}}\bigl(\bigcup_{i\in\omega} i^\frown \mathcal{X}_i\bigr)$. Hence $\mathbf{x}$ is the greatest lower bound of $\mathfrak{X}$ in $\mathcal{D}^{01}_{\mathrm{w,cl}}$ (and in $\mathcal{D}_{\mathrm{w}}$). ☐

Question 2.4.3. Are $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ countably join-complete?

Lemma 2.4.4. *Let $\mathcal{W} \subseteq \omega^\omega$ be a closed $\leq_{\mathrm{T}}$-antichain, and let $\mathbf{w} = \deg_{\mathrm{w}}(\mathcal{W})$.*

(i) If $\mathbf{x} \in \mathcal{D}_{\mathrm{w,cl}}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{f\})$ for some $f \in \mathcal{W}$.

(ii) $E(\mathbf{w}) = \{\deg_{\mathrm{w}}(\{f\}) \mid f \in \mathcal{W}\}$.

The same is true with 2^ω in place of ω^ω and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ in place of $\mathcal{D}_{\mathrm{w,cl}}$.

Proof. (i) The proof of Lemma 2.3.7 item (i) works in both the $\mathcal{D}_{\mathrm{w,cl}}$ case and the $\mathcal{D}^{01}_{\mathrm{w,cl}}$ case.

(ii) First consider the $\mathcal{D}_{\mathrm{w,cl}}$ case. Let $f \in \mathcal{W}$. Let T be a tree such that $\mathcal{W} = [T]$. Let $\langle \tau_i \mid i \in \omega \rangle$ list the sequences in T that are not initial segments of f (so that $(\forall g \in \mathcal{W})(g \neq f \leftrightarrow \exists i(\tau_i \subset g))$). Let T_i denote the full subtree of T rooted at τ_i: $T_i = \{\sigma \in \omega^{<\omega} \mid \tau_i{}^\frown\sigma \in T\}$. Let R be the tree $\bigcup_{i\in\omega} i^\frown T_i$ where $i^\frown T_i = \{i^\frown\sigma \mid \sigma \in T_i\}$ for each i. Let $\mathcal{Y} = [R]$, and let $\mathbf{y} = \deg_{\mathrm{w}}(\mathcal{Y})$. If, for a mass problem $\mathcal{A}$, we let $\deg_{\mathrm{T}}(\mathcal{A}) = \{\deg_{\mathrm{T}}(f) \mid f \in \mathcal{A}\}$ denote the set of Turing degrees of the members of $\mathcal{A}$, we see that $\deg_{\mathrm{T}}(\mathcal{Y}) = \deg_{\mathrm{T}}(\mathcal{W}) \setminus \{\deg_{\mathrm{T}}(f)\}$. From this and the fact that $\mathcal{W}$ is a $\leq_{\mathrm{T}}$-antichain, it follows that $\mathbf{y} >_{\mathrm{w}} \mathbf{w}$ and $\deg_{\mathrm{w}}(\{f\}) \times \mathbf{y} = \mathbf{w}$. Hence $\deg_{\mathrm{w}}(\{f\})$ is meet-irreducible (because it is the degree of a singleton) and meets to $\mathbf{w}$. Thus $\{\deg_{\mathrm{w}}(\{f\}) \mid f \in \mathcal{W}\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{f\})$ for some $f \in \mathcal{W}$. We have just shown that $\deg_{\mathrm{w}}(\{f\}) \in E(\mathbf{w})$, and $E(\mathbf{w})$ is an antichain by Lemma 2.2.3. So it must be that $\mathbf{x} = \deg_{\mathrm{w}}(\{f\})$.

For the $\mathcal{D}^{01}_{\mathrm{w,cl}}$ case, as before let $f \in \mathcal{W}$, and let T be a tree such that $\mathcal{W} = [T]$. Let $\langle \tau_i \mid i \in \omega \rangle$ list the sequences in T that are not initial segments of f. Let T_i denote the full subtree of T rooted at τ_i. Choose any $g \in \mathcal{W} \setminus \{f\}$ (if $\mathcal{W} = \{f\}$, then the lemma is trivial). Let $\sigma_i = (g \restriction i)^\frown(1 - g(i))$ for each $i \in \omega$. Let R be the tree $\bigcup_{i\in\omega} \sigma_i{}^\frown T_i$. Let $\mathcal{Y} = [R]$. Then $\deg_{\mathrm{T}}(\mathcal{Y}) = \deg_{\mathrm{T}}(\mathcal{W}) \setminus \{\deg_{\mathrm{T}}(f)\}$. The proof now proceeds exactly as in the $\mathcal{D}_{\mathrm{w,cl}}$ case. ☐

Definition 2.4.5 (Dyment [22]). $\mathcal{W} \subseteq \omega^\omega$ is called *effectively discrete* if $(\forall f \in \mathcal{W})(\forall g \in \mathcal{W})(f \neq g \rightarrow f(0) \neq g(0))$.

An effectively discrete mass problem is closed and at most countable.

Lemma 2.4.6. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{w,cl}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.*

Proof. Let $\mathcal{W}_0 = \{n^\frown f_{0,n}\}_{n\in\omega}$, $\mathcal{W}_1 = \{n^\frown f_{1,n}\}_{n\in\omega}$, and $\mathcal{W}_2 = \{n^\frown f_{2,n}\}_{n\in\omega}$ be such that $\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$ is independent. Then let

$$
\begin{aligned}
\mathbf{w}_0 &= \deg_{\mathrm{w}}(\mathcal{W}_0), \\
\mathbf{w}_1 &= \deg_{\mathrm{w}}(\mathcal{W}_1), \\
\mathbf{w}_2 &= \deg_{\mathrm{w}}(\mathcal{W}_2), \\
\mathbf{m} &= \deg_{\mathrm{w}}(\mathcal{M}) && \text{for} \quad \mathcal{M} = \{(2n)^\frown(f_{0,n} \oplus f_{1,n})\}_{n\in\omega} \\
&&& \qquad\quad \cup \{(2n+1)^\frown(f_{0,n} \oplus f_{2,n})\}_{n\in\omega}, \\
\boldsymbol{\ell} &= \deg_{\mathrm{w}}(\mathcal{L}) && \text{for} \quad \mathcal{L} = \{\langle i, j \rangle^\frown(f_{0,i} \oplus f_{1,j}) \mid i < j\}, \\
\mathbf{p} &= \deg_{\mathrm{w}}(\mathcal{P}) && \text{for} \quad \mathcal{P} = \{\langle \langle i, j \rangle, k \rangle^\frown(f_{0,i} \oplus f_{1,j} \oplus f_{2,k}) \mid i + j = k\}, \\
\mathbf{t} &= \deg_{\mathrm{w}}(\mathcal{T}) && \text{for} \quad \mathcal{T} = \{\langle \langle i, j \rangle, k \rangle^\frown(f_{0,i} \oplus f_{1,j} \oplus f_{2,k}) \mid i \times j = k\}, \\
\mathbf{d} &= \deg_{\mathrm{w}}(\mathcal{D}) && \text{for} \quad \mathcal{D} = \{\langle \langle i, j \rangle, k \rangle^\frown(f_{0,i} \oplus f_{1,j} \oplus f_{2,k}) \mid i \dot{-} j = k\}, \\
\mathbf{z} &= \deg_{\mathrm{w}}(\{0^\frown f_{0,0}\}), && \text{and} \\
\mathbf{o} &= \deg_{\mathrm{w}}(\{1^\frown f_{0,1}\}).
\end{aligned}
$$

The above mass problems are all effectively discrete, so their degrees are all in $\mathcal{D}_{\mathrm{w,cl}}$. These degrees give the code $\vec{\mathbf{w}}$. The proof that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ is the same as in the proof of Lemma 2.3.9. Use Lemma 2.4.4 in place of Lemma 2.3.7. $\square$

An infinite effectively discrete Turing antichain is not compact, so we can no longer rely on them to provide codes. Instead we use the following definitions.

Definition 2.4.7. Let $g \in 2^\omega$. A set $\mathcal{X} \subseteq 2^\omega$ is called a *g-spine* (or just a *spine*) if it is of the form $\{g\} \cup \{\sigma_i{}^\frown f_i \mid i \in A\}$ where $A \subseteq \omega$ is infinite, $\sigma_i = (g \restriction i)^\frown(1 - g(i))$ for each $i \in A$, and $f_i \in 2^\omega$ for each $i \in A$.

Definition 2.4.8. Let $g \in 2^\omega$ and let $\mathcal{X} \subseteq 2^\omega$ be countable. Fix an enumeration $\langle f_i \mid i \in \omega \rangle$ of $\mathcal{X}$. We denote by $\mathrm{spine}(g, \mathcal{X})$ the g-spine $\{g\} \cup \{\sigma_i{}^\frown f_i \mid i \in \omega\}$ where $\sigma_i = (g \restriction i)^\frown(1 - g(i))$ for each $i \in \omega$. We denote by $\mathrm{spine}(\mathcal{X})$ the f_0-spine $\mathrm{spine}(f_0, \mathcal{X} \setminus \{f_0\})$.

Notice that a spine is a closed subset of 2^ω.

Lemma 2.4.9. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{D}^{01}_{\mathrm{w,cl}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.*

Proof. Let $\mathcal{W}'_0 = \{f_{0,n}\}_{n\in\omega}$, $\mathcal{W}'_1 = \{f_{1,n}\}_{n\in\omega}$, and $\mathcal{W}'_2 = \{f_{2,n}\}_{n\in\omega}$ be such that

$\mathcal{W}_0' \cup \mathcal{W}_1' \cup \mathcal{W}_2' \subseteq 2^\omega$ is independent. Then let

$$\begin{array}{rll} \mathbf{w}_0 = \deg_{\mathrm{w}}(\mathcal{W}_0) & \text{for} & \mathcal{W}_0 = \mathrm{spine}(\mathcal{W}_0'), \\ \mathbf{w}_1 = \deg_{\mathrm{w}}(\mathcal{W}_1) & \text{for} & \mathcal{W}_1 = \mathrm{spine}(\mathcal{W}_1'), \\ \mathbf{w}_2 = \deg_{\mathrm{w}}(\mathcal{W}_2) & \text{for} & \mathcal{W}_2 = \mathrm{spine}(\mathcal{W}_2'), \\ \mathbf{m} = \deg_{\mathrm{w}}(\mathcal{M}) & \text{for} & \mathcal{M} = \mathrm{spine}(\{f_{0,n} \oplus f_{1,n}\}_{n\in\omega} \cup \{f_{0,n} \oplus f_{2,n}\}_{n\in\omega}), \\ \boldsymbol{\ell} = \deg_{\mathrm{w}}(\mathcal{L}) & \text{for} & \mathcal{L} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \mid i < j\}), \\ \mathbf{p} = \deg_{\mathrm{w}}(\mathcal{P}) & \text{for} & \mathcal{P} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i + j = k\}), \\ \mathbf{t} = \deg_{\mathrm{w}}(\mathcal{T}) & \text{for} & \mathcal{T} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \times j = k\}), \\ \mathbf{d} = \deg_{\mathrm{w}}(\mathcal{D}) & \text{for} & \mathcal{D} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \dot{-} j = k\}), \\ \mathbf{z} = \deg_{\mathrm{w}}(\{f_{0,0}\}), & \text{and} & \\ \mathbf{o} = \deg_{\mathrm{w}}(\{f_{0,1}\}). & & \end{array}$$

These degrees give the code $\vec{\mathbf{w}}$. The proof that $\mathcal{M}_{\vec{\mathbf{w}}}^{\dot{-}} \cong \mathcal{N}^{\dot{-}}$ is the same as in the proof of Lemma 2.3.9. Use Lemma 2.4.4 in place of Lemma 2.3.7. □

Theorem 2.4.10. $\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \equiv_1 \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \equiv_1 \mathrm{Th}_2(\mathcal{N})$.

Proof. $\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ and $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ by Lemma 2.1.2. For $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$, by Lemma 2.4.6 let $\vec{\mathbf{w}}$ be a code in $\mathcal{D}_{\mathrm{w,cl}}$ such that $\mathcal{M}_{\vec{\mathbf{w}}}^{\dot{-}} \cong \mathcal{N}^{\dot{-}}$. Removing the degree $\mathbf{d}$ from the code $\vec{\mathbf{w}}$ gives a code $\vec{\mathbf{v}}$ such that $\mathcal{M}_{\vec{\mathbf{v}}} \cong \mathcal{N}$. $\mathcal{D}_{\mathrm{w,cl}}$ has the coding countable subsets property by Lemma 2.2.22 because $\mathcal{D}_{\mathrm{w,cl}}$ has a greatest element and is countably meet-complete by Lemma 2.4.2. Hence $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{w}}; \leq_{\mathrm{w}})$ by Lemma 2.2.19. The proof that $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$ is the same. Use Lemma 2.4.9 in place of Lemma 2.4.6. □

Theorem 2.4.11. *The fragments* Σ^0_3-$\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}})$, Σ^0_4-$\mathrm{Th}(\mathcal{D}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$, Σ^0_3-$\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}})$, *and* Σ^0_4-$\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{w,cl}}; \leq_{\mathrm{w}})$ *are all undecidable.*

Proof. By Lemma 2.4.6, there is a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{w,cl}}$ such that $\mathcal{M}_{\vec{\mathbf{w}}}^{\dot{-}} \cong \mathcal{N}^{\dot{-}}$. By Lemma 2.4.9, there is a code $\vec{\mathbf{w}}$ in $\mathcal{D}^{01}_{\mathrm{w,cl}}$ such that $\mathcal{M}_{\vec{\mathbf{w}}}^{\dot{-}} \cong \mathcal{N}^{\dot{-}}$. The results then follow from Lemma 2.2.25. □

2.5 $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ have the coding countable subsets property

$\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ are not countably meet-complete by Lemma 2.7.1, so Lemma 2.2.22 does not apply to them. We need to prove that both $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ have the coding countable subsets property. The next lemma is a clarifying example. It implies that a closed (compact) $\mathcal{W}$ has meet-reducible degree in $\mathcal{D}_{\mathrm{s,cl}}$ ($\mathcal{D}^{01}_{\mathrm{s,cl}}$) if and only if it has meet-reducible degree in $\mathcal{D}_{\mathrm{s}}$.

Lemma 2.5.1 (Dyment [22]). *If* $\mathcal{W} \equiv_{\mathrm{s}} \mathcal{X} \times \mathcal{Y}$, *then* $\mathcal{W} = \widehat{\mathcal{X}} \cup \widehat{\mathcal{Y}}$ *where* $\widehat{\mathcal{X}}$ *and* $\widehat{\mathcal{Y}}$ *are disjoint and clopen in* $\mathcal{W}$, $\widehat{\mathcal{X}} \geq_{\mathrm{s}} \mathcal{X}$, $\widehat{\mathcal{Y}} \geq_{\mathrm{s}} \mathcal{Y}$, *and* $\mathcal{W} \equiv_{\mathrm{s}} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$.

Proof. Let Φ be such that $\Phi(\mathcal{W}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. Put $\widehat{\mathcal{X}} = \{f \in \mathcal{W} \mid \Phi(f)(0) = 0\}$ and put $\widehat{\mathcal{Y}} = \{f \in \mathcal{W} \mid \Phi(f)(0) = 1\}$. By Lemma 1.4.2, $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are clopen in $\mathcal{W}$, and it is easily checked that $\widehat{\mathcal{X}} \geq_{\mathrm{s}} \mathcal{X}$ and $\widehat{\mathcal{Y}} \geq_{\mathrm{s}} \mathcal{Y}$ (hence $\mathcal{W} \leq_{\mathrm{s}} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$). We have $\mathcal{W} \geq_{\mathrm{s}} 0^\frown\widehat{\mathcal{X}} \cup 1^\frown\widehat{\mathcal{Y}}$ by the reduction which sends f to $0^\frown f$ if $\Phi(f)(0) = 0$ and sends f to $1^\frown f$ if $\Phi(f)(0) = 1$. □

For comparison with the Muchnik case, if a closed (compact) $\mathcal{W}$ has meet-reducible degree in $\mathcal{D}_{\mathrm{w,cl}}$ ($\mathcal{D}^{01}_{\mathrm{w,cl}}$), then it has meet-reducible degree in $\mathcal{D}_{\mathrm{w}}$. However, we do not know the converse.

Question 2.5.2. If $\mathcal{W}$ is closed (compact) and $\mathcal{W} \equiv_{\mathrm{w}} \mathcal{X} \times \mathcal{Y}$ for $\mathcal{X}, \mathcal{Y} >_{\mathrm{w}} \mathcal{W}$, then are there closed (compact) such $\mathcal{X}$ and $\mathcal{Y}$?

If the $\mathcal{X}$ in Lemma 2.5.1 has meet-irreducible degree, then we have the following refinement.

Lemma 2.5.3. *If $\mathcal{W} \equiv_{\mathrm{s}} \mathcal{X} \times \mathcal{Y}$ where $\mathcal{X}$ has meet-irreducible degree and $\mathcal{Y} >_{\mathrm{s}} \mathcal{W}$, then $\mathcal{W} = \widehat{\mathcal{X}} \cup \widehat{\mathcal{Y}}$ where $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are disjoint and clopen in $\mathcal{W}$, $\widehat{\mathcal{X}} \equiv_{\mathrm{s}} \mathcal{X}$, and $\widehat{\mathcal{X}} \not\geq_{\mathrm{s}} \widehat{\mathcal{Y}}$.*

Proof. As in Lemma 2.5.1, let Φ be such that $\Phi(\mathcal{W}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$, put $\widehat{\mathcal{X}} = \{f \in \mathcal{W} \mid \Phi(f)(0) = 0\}$, and put $\widehat{\mathcal{Y}} = \{f \in \mathcal{W} \mid \Phi(f)(0) = 1\}$. Then $\mathcal{W} = \widehat{\mathcal{X}} \cup \widehat{\mathcal{Y}}$, $\widehat{\mathcal{X}} \cap \widehat{\mathcal{Y}} = \emptyset$, $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are clopen in $\mathcal{W}$, $\widehat{\mathcal{X}} \geq_{\mathrm{s}} \mathcal{X}$, $\widehat{\mathcal{Y}} \geq_{\mathrm{s}} \mathcal{Y}$, and $\mathcal{W} \equiv_{\mathrm{s}} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$. To see that $\mathcal{X} \geq_{\mathrm{s}} \widehat{\mathcal{X}}$, observe that $\mathcal{X} \geq_{\mathrm{s}} \mathcal{W} \equiv_{\mathrm{s}} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$. $\mathcal{X}$ has meet-irreducible degree, so $\mathcal{X} \geq_{\mathrm{s}} \widehat{\mathcal{X}}$ or $\mathcal{X} \geq_{\mathrm{s}} \widehat{\mathcal{Y}}$. We cannot have $\mathcal{X} \geq_{\mathrm{s}} \widehat{\mathcal{Y}}$ because $\widehat{\mathcal{Y}} \geq_{\mathrm{s}} \mathcal{Y}$ and this would imply $\mathcal{W} \equiv_{\mathrm{s}} \mathcal{X} \times \mathcal{Y} \equiv_{\mathrm{s}} \mathcal{Y} >_{\mathrm{s}} \mathcal{W}$. Thus $\mathcal{X} \geq_{\mathrm{s}} \widehat{\mathcal{X}}$. Similarly $\widehat{\mathcal{X}} \not\geq_{\mathrm{s}} \widehat{\mathcal{Y}}$ for otherwise $\mathcal{W} \equiv_{\mathrm{s}} \widehat{\mathcal{Y}} \geq_{\mathrm{s}} \mathcal{Y} >_{\mathrm{s}} \mathcal{W}$. □

Corollary 2.5.4. *For all $\mathbf{w} \in \mathcal{D}_{\mathrm{s}}$, $E(\mathbf{w})$ is at most countable.*

Proof. Fix a representative $\mathcal{W}$ for $\mathbf{w}$. Lemma 2.5.3 shows that if $\mathbf{x} \in E(\mathbf{w})$, then $\mathbf{x}$ has a representative of the form $\{f \in \mathcal{W} \mid \Phi(f)(0) = 0\}$ for some program Φ. There are only countably many programs, so there can be at most countably many $\mathbf{x} \in E(\mathbf{w})$. □

Notice that Corollary 2.5.4 is in contrast to the Muchnik case, in which a degree may have uncountably many meet-irreducibles that meet to it. For example, if $\mathcal{W}$ is a $\leq_{\mathrm{T}}$-antichain, then, in $\mathcal{D}_{\mathrm{w}}$, $|E(\deg_{\mathrm{w}}(\mathcal{W}))| = |\mathcal{W}|$ by Lemma 2.3.7. There exist uncountable $\leq_{\mathrm{T}}$-antichains, and there even exist uncountable closed $\leq_{\mathrm{T}}$-antichains (see [69] Section VI.1). Also notice that if $\mathbf{w}$ is closed (compact) and $\mathbf{x}$ is meet-irreducible and meets to $\mathbf{w}$, then Lemma 2.5.3 produces a closed (compact) representative for $\mathbf{x}$. Thus for a closed (compact) degree $\mathbf{w}$, the meet-irreducible degrees that meet to $\mathbf{w}$ are the same whether they are computed in $\mathcal{D}_{\mathrm{s}}$ or in $\mathcal{D}_{\mathrm{s,cl}}$ ($\mathcal{D}^{01}_{\mathrm{s,cl}}$).

Lemma 2.5.5. *Let $\mathcal{W}$ be a mass problem such that $E(\deg_{\mathrm{s}}(\mathcal{W}))$ is countable, and let $\langle\, \mathcal{X}_i \mid i \in \omega \,\rangle$ be a list of representatives for the degrees in $E(\deg_{\mathrm{s}}(\mathcal{W}))$. Then there are mass problems $\langle\, \widehat{\mathcal{X}}_i \mid i \in \omega \,\rangle$ such that:*

(i) $\widehat{\mathcal{X}}_i \subseteq \mathcal{W}$ *is clopen in* $\mathcal{W}$ *for each* i,

(ii) $\widehat{\mathcal{X}}_i \cap \widehat{\mathcal{X}}_j = \emptyset$ *for* $i \neq j$,

(iii) $\widehat{\mathcal{X}}_i \equiv_{\mathrm{s}} \mathcal{X}_i$ *for each* i,

(iv) $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{s}} \mathcal{W} \setminus \widehat{\mathcal{X}}_i$ *for each* i.

Proof. Inductively construct the sequence $\langle \widehat{\mathcal{X}}_i \mid i \in \omega \rangle$. At the start of step $n+1$ we have $\langle \widehat{\mathcal{X}}_i \mid i \leq n \rangle$ satisfying (i)–(iv) for $i, j \leq n$, and we have indices $e_0, \ldots, e_n$ such that, for $i \leq n$, $\widehat{\mathcal{X}}_i = \{f \in \mathcal{W} \setminus \bigcup_{j<i} \widehat{\mathcal{X}}_j \mid \Phi_{e_i}(f)(0) = 0\}$ and $\mathcal{W} \setminus \bigcup_{j \leq i} \widehat{\mathcal{X}}_j = \{f \in \mathcal{W} \setminus \bigcup_{j<i} \widehat{\mathcal{X}}_j \mid \Phi_{e_i}(f)(0) = 1\}$.

We first show $\mathcal{W} \equiv_{\mathrm{s}} \widehat{\mathcal{X}}_0 \times \cdots \times \widehat{\mathcal{X}}_n \times (\mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i)$. The meet is $\geq_{\mathrm{s}} \mathcal{W}$ because each term is a subset of $\mathcal{W}$. To see the reverse inequality, write the meet as $\bigcup_{i \leq n} i^\frown \widehat{\mathcal{X}}_i \cup (n+1)^\frown(\mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i)$. Then apply the following reduction. For each $i \leq n$ in order, check if $\Phi_{e_i}(f)(0)$ is 0 or 1. If it is 0, send f to $i^\frown f$. If it is 1, go to the next i. If $\Phi_{e_i}(f)(0) = 1$ for each $i \leq n$, then send f to $(n+1)^\frown f$.

We now have $\mathcal{X}_{n+1} \geq_{\mathrm{s}} \mathcal{W} \equiv_{\mathrm{s}} \widehat{\mathcal{X}}_0 \times \cdots \times \widehat{\mathcal{X}}_n \times (\mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i)$. We cannot have $\mathcal{X}_{n+1} \geq_{\mathrm{s}} \widehat{\mathcal{X}}_i$ for any $i \leq n$ because $\widehat{\mathcal{X}}_i \equiv_{\mathrm{s}} \mathcal{X}_i$ and the $\mathcal{X}_i$'s are incomparable because $E(\deg_{\mathrm{s}}(\mathcal{W}))$ is an antichain by Lemma 2.2.3. However, $\mathcal{X}_{n+1}$ has meet-irreducible degree. Therefore $\mathcal{X}_{n+1} \geq_{\mathrm{s}} \mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$. Moreover, by distributivity $\deg_{\mathrm{s}}(\mathcal{X}_{n+1})$ meets to $\deg_{\mathrm{s}}(\mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i)$ because $\deg_{\mathrm{s}}(\mathcal{X}_{n+1})$ meets to $\deg_{\mathrm{s}}(\mathcal{W})$ and $\deg_{\mathrm{s}}(\mathcal{X}_{n+1}) \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i) \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{W})$. Thus, as in Lemma 2.5.3, there is an $\widehat{\mathcal{X}}_{n+1} \subseteq \mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$ clopen in $W \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$ and an e_{n+1} such that

$$\begin{aligned} \widehat{\mathcal{X}}_{n+1} &= \{f \in \mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i \mid \Phi_{e_{n+1}}(f)(0) = 0\}, \\ \mathcal{W} \setminus \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i &= \{f \in \mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i \mid \Phi_{e_{n+1}}(f)(0) = 1\}, \\ \widehat{\mathcal{X}}_{n+1} &\equiv_{\mathrm{s}} \mathcal{X}_{n+1}, \text{ and} \\ \widehat{\mathcal{X}}_{n+1} &\not\geq_{\mathrm{s}} \mathcal{W} \setminus \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i. \end{aligned}$$

Clearly $\widehat{\mathcal{X}}_{n+1}$ is disjoint from $\widehat{\mathcal{X}}_i$ for $i \leq n$. $\widehat{\mathcal{X}}_{n+1}$ is clopen in $\mathcal{W}$ because it is clopen in $\mathcal{W} \setminus \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$ which is clopen in $\mathcal{W}$. Finally, $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{s}} \mathcal{W} \setminus \widehat{\mathcal{X}}_{n+1}$ because $\widehat{\mathcal{X}}_{n+1}$ has meet-irreducible degree, $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{s}} \widehat{\mathcal{X}}_i$ for $i \leq n$, $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{s}} \mathcal{W} \setminus \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i$, and $\mathcal{W} \setminus \widehat{\mathcal{X}}_{n+1} \equiv_{\mathrm{s}} \widehat{\mathcal{X}}_0 \times \cdots \times \widehat{\mathcal{X}}_n \times (\mathcal{W} \setminus \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i)$. ☐

Lemma 2.5.6. *Let* $\mathcal{W}$ *be a mass problem. Then for any* $S \subseteq E(\deg_{\mathrm{s}}(\mathcal{W}))$ *there is an* $\mathcal{A} \subseteq \mathcal{W}$ *closed in* $\mathcal{W}$ *such that* $F(\deg_{\mathrm{s}}(\mathcal{A})) \cap E(\deg_{\mathrm{s}}(\mathcal{W})) = S$. *In particular, if* $\mathcal{W}$ *is closed, then so is* $\mathcal{A}$. *Thus* $\mathcal{D}_{\mathrm{s,cl}}$ *and* $\mathcal{D}^{01}_{\mathrm{s,cl}}$ *have the coding countable subsets property.*

Proof. We only consider the case in which $E(\mathbf{w})$ is infinite. By Corollary 2.5.4, $E(\mathbf{w})$ is countable. Let $\langle \mathcal{X}_i \mid i \in \omega \rangle$ be a list of representatives for the degrees in $E(\mathbf{w})$. Apply Lemma 2.5.5 to $\mathcal{W}$ and $\langle \mathcal{X}_i \mid i \in \omega \rangle$ to get a new set of representatives $\langle \widehat{\mathcal{X}}_i \mid i \in \omega \rangle$ disjoint and clopen in $\mathcal{W}$ with $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{s}} \mathcal{W} \setminus \widehat{\mathcal{X}}_i$ for each i. Put $\mathcal{A} = \mathcal{W} \setminus \bigcup\{\widehat{\mathcal{X}}_i \mid \deg_{\mathrm{s}}(\widehat{\mathcal{X}}_i) \notin S\}$, and note that $\mathcal{A}$ is closed in $\mathcal{W}$. We show $\widehat{\mathcal{X}}_i \geq_{\mathrm{s}} \mathcal{A}$ if and only if $\deg_{\mathrm{s}}(\widehat{\mathcal{X}}_i) \in S$. If $\deg_{\mathrm{s}}(\widehat{\mathcal{X}}_i) \in S$ then $\widehat{\mathcal{X}}_i \subseteq \mathcal{A}$ and so $\widehat{\mathcal{X}}_i \geq_{\mathrm{s}} \mathcal{A}$. If $\deg_{\mathrm{s}}(\widehat{\mathcal{X}}_i) \notin S$ then $\mathcal{A} \subseteq \mathcal{W} \setminus \widehat{\mathcal{X}}_i$ and so $\mathcal{A} \geq_{\mathrm{s}} \mathcal{W} \setminus \widehat{\mathcal{X}}_i$. Thus $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{s}} \mathcal{A}$ because $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{s}} \mathcal{W} \setminus \widehat{\mathcal{X}}_i$. ☐

2.6 The complexities of $\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$ and $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$

We can now prove that $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$ and $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$.

Lemma 2.6.1. *Let $\mathcal{W} \subseteq \omega^\omega$ be an effectively discrete $\leq_{\mathrm{T}}$-antichain, and let $\mathbf{w} = \deg_{\mathrm{s}}(\mathcal{W})$.*

(i) If $\mathbf{x} \in \mathcal{D}_{\mathrm{s,cl}}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f\})$ for some $f \in \mathcal{W}$.

(ii) $E(\mathbf{w}) = \{\deg_{\mathrm{s}}(\{f\}) \mid f \in \mathcal{W}\}$.

Proof. (i) Let $\mathbf{x} \in \mathcal{D}_{\mathrm{s,cl}}$ be such that $\mathbf{x}$ meets to $\mathbf{w}$. Suppose for a contradiction that $(\forall f \in \mathcal{W})(\mathbf{x} \not\leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f\}))$. Let $\mathbf{y} \in \mathcal{D}_{\mathrm{s,cl}}$ witness that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $\mathbf{y} \geq_{\mathrm{s}} \mathbf{w}$ and $\mathbf{x} \times \mathbf{y} = \mathbf{w}$. Let $\mathcal{X}$ be a representative for $\mathbf{x}$, and let $\mathcal{Y}$ be a representative for $\mathbf{y}$. Then $\mathcal{X} \times \mathcal{Y} \equiv_{\mathrm{s}} \mathcal{W}$, so let Φ be a Turing functional such that $\Phi(\mathcal{W}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. If $\Phi(f) \in 0^\frown\mathcal{X}$ for some $f \in \mathcal{W}$, then $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f\})$ for this f, contrary to assumption. Thus it must be that $\Phi(f) \in 1^\frown\mathcal{Y}$ for all $f \in \mathcal{W}$. That is, Φ witnesses that $\mathbf{w} \geq_{\mathrm{s}} \mathbf{y}$, a contradiction.

(ii) Given $f \in \mathcal{W}$, it is an easy check (using the fact that $\mathcal{W}$ is an effectively discrete $\leq_{\mathrm{T}}$-antichain) that $\deg_{\mathrm{s}}(\mathcal{W} \setminus \{f\})$ witnesses that $\deg_{\mathrm{s}}(\{f\})$ meets to $\mathbf{w}$. Hence $\{\deg_{\mathrm{s}}(\{f\}) \mid f \in \mathcal{W}\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f\})$ for some $f \in \mathcal{W}$. We have just shown that $\deg_{\mathrm{s}}(\{f\}) \in E(\mathbf{w})$, and $E(\mathbf{w})$ is an antichain by Lemma 2.2.3. So it must be that $\mathbf{x} = \deg_{\mathrm{s}}(\{f\})$. ☐

Lemma 2.6.2. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{s,cl}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.*

Proof. If $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{M}$, $\mathcal{L}$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{D}$, $\{0^\frown f_{0,0}\}$, and $\{1^\frown f_{0,1}\}$ are the mass problems defined in the proof of Lemma 2.4.6, then their Medvedev degrees give a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{s,cl}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. The proof that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ is the same as in the proof of Lemma 2.3.9. Use Lemma 2.6.1 in place of Lemma 2.3.7. ☐

To code $\mathcal{N}$ in $\mathcal{D}^{01}_{\mathrm{s,cl}}$, we need to reprove Lemma 2.6.1 for spines.

Lemma 2.6.3. *Let $\mathcal{W} = \{g\} \cup \{\sigma_i{}^\frown f_i \mid i \in X\} \subseteq 2^\omega$ be a g-spine that is a $\leq_{\mathrm{T}}$-antichain, and let $\mathbf{w} = \deg_{\mathrm{s}}(\mathcal{W})$.*

(i) If $\mathbf{x} \in \mathcal{D}^{01}_{\mathrm{s,cl}}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f_i\})$ for some $i \in X$.

(ii) $E(\mathbf{w}) = \{\deg_{\mathrm{s}}(\{f_i\}) \mid i \in X\}$.

Proof. (i) Let $\mathbf{x} \in \mathcal{D}^{01}_{\mathrm{s,cl}}$ be such that $\mathbf{x}$ meets to $\mathbf{w}$. Suppose for a contradiction that $(\forall f \in \mathcal{W})(\mathbf{x} \not\leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f\}))$. Let $\mathbf{y} \in \mathcal{D}^{01}_{\mathrm{s,cl}}$ witness that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $\mathbf{y} >_{\mathrm{s}} \mathbf{w}$ and $\mathbf{x} \times \mathbf{y} = \mathbf{w}$. Let $\mathcal{X}$ be a representative for $\mathbf{x}$, and let $\mathcal{Y}$ be a representative for $\mathbf{y}$. Then $\mathcal{X} \times \mathcal{Y} \equiv_{\mathrm{s}} \mathcal{W}$, so let Φ be a Turing functional such that $\Phi(\mathcal{W}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. If $\Phi(\sigma_i{}^\frown f_i) \in 0^\frown\mathcal{X}$ for some $i \in X$, then $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f_i\})$ for this i, contrary to assumption. Thus it must be that $\Phi(\sigma_i{}^\frown f_i) \in 1^\frown\mathcal{Y}$ for all $i \in X$. It must also be that $\Phi(g) \in 1^\frown\mathcal{Y}$. If not, then $\Phi(g)(0) = 0$ and there is some $\tau \subset g$ such that $\Phi(\tau)(0) = 0$. Choose $i \in X$ with $i > |\tau|$. Then $\tau \subset \sigma_i$, giving the contradiction $\Phi(\sigma_i{}^\frown f_i)(0) = 0$. Therefore $\Phi(\mathcal{W}) \subseteq 1^\frown Y$. Thus $\mathbf{w} \geq_{\mathrm{s}} \mathbf{y}$, a contradiction.

(ii) Given $i \in X$, it is an easy check (using the fact that $\mathcal{W}$ is a g-spine that is a $\leq_{\mathrm{T}}$-antichain) that $\deg_{\mathrm{s}}(\mathcal{W} \setminus \{\sigma_i{}^\frown f_i\})$ witnesses that $\deg_{\mathrm{s}}(\{f_i\})$ meets to $\mathbf{w}$. Hence $\{\deg_{\mathrm{s}}(\{f_i\}) \mid i \in X\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f_i\})$ for some $i \in X$. We have just shown that $\deg_{\mathrm{s}}(\{f_i\}) \in E(\mathbf{w})$, and $E(\mathbf{w})$ is an antichain by Lemma 2.2.3. So it must be that $\mathbf{x} = \deg_{\mathrm{s}}(\{f_i\})$. ☐

Notice the difference between Lemma 2.4.4 and Lemma 2.6.3. If $\mathcal{W}$ is a g-spine that is a $\leq_{\mathrm{T}}$-antichain, then in $\mathcal{D}^{01}_{\mathrm{w,cl}}$ we have $\deg_{\mathrm{w}}(\{g\}) \in E(\deg_{\mathrm{w}}(\mathcal{W}))$, but in $\mathcal{D}^{01}_{\mathrm{s,cl}}$ we have $\deg_{\mathrm{s}}(\{g\}) \notin E(\deg_{\mathrm{s}}(\mathcal{W}))$.

Lemma 2.6.4. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{D}^{01}_{\mathrm{s,cl}}$ such that $\mathcal{M}^{\dot-}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot-}$.*

Proof. Let g, $\mathcal{W}'_0 = \{f_{0,n}\}_{n\in\omega}$, $\mathcal{W}'_1 = \{f_{1,n}\}_{n\in\omega}$, and $\mathcal{W}'_2 = \{f_{2,n}\}_{n\in\omega}$ be such that $\{g\} \cup \mathcal{W}'_0 \cup \mathcal{W}'_1 \cup \mathcal{W}'_2 \subseteq 2^\omega$ is independent. Then let

$$\begin{array}{rll}
\mathbf{w}_0 = \deg_{\mathrm{s}}(\mathcal{W}_0) & \text{for} & \mathcal{W}_0 = \mathrm{spine}(g, \mathcal{W}'_0),\\
\mathbf{w}_1 = \deg_{\mathrm{s}}(\mathcal{W}_1) & \text{for} & \mathcal{W}_1 = \mathrm{spine}(g, \mathcal{W}'_1),\\
\mathbf{w}_2 = \deg_{\mathrm{s}}(\mathcal{W}_2) & \text{for} & \mathcal{W}_2 = \mathrm{spine}(g, \mathcal{W}'_2),\\
\mathbf{m} = \deg_{\mathrm{s}}(\mathcal{M}) & \text{for} & \mathcal{M} = \mathrm{spine}(g, \{f_{0,n} \oplus f_{1,n}\}_{n\in\omega} \cup \{f_{0,n} \oplus f_{2,n}\}_{n\in\omega}),\\
\boldsymbol{\ell} = \deg_{\mathrm{s}}(\mathcal{L}) & \text{for} & \mathcal{L} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \mid i < j\}),\\
\mathbf{p} = \deg_{\mathrm{s}}(\mathcal{P}) & \text{for} & \mathcal{P} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i + j = k\}),\\
\mathbf{t} = \deg_{\mathrm{s}}(\mathcal{T}) & \text{for} & \mathcal{T} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \times j = k\}),\\
\mathbf{d} = \deg_{\mathrm{s}}(\mathcal{D}) & \text{for} & \mathcal{D} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \dot- j = k\}),\\
\mathbf{z} = \deg_{\mathrm{s}}(\{f_{0,0}\}), & \text{and} & \\
\mathbf{o} = \deg_{\mathrm{s}}(\{f_{0,1}\}). & &
\end{array}$$

These degrees give the code $\vec{\mathbf{w}}$. The proof that $\mathcal{M}^{\dot-}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot-}$ is the same as in the proof of Lemma 2.3.9. Use Lemma 2.6.3 in place of Lemma 2.3.7. ☐

Theorem 2.6.5. $\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \equiv_1 \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \equiv_1 \mathrm{Th}_2(\mathcal{N})$.

Proof. $\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ and $\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}_2(\mathcal{N})$ by Lemma 2.1.2. For $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$, by Lemma 2.6.2 let $\vec{\mathbf{w}}$ be a code in $\mathcal{D}_{\mathrm{s,cl}}$ such that $\mathcal{M}^{\dot-}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot-}$. Removing the degree $\mathbf{d}$ from the code $\vec{\mathbf{w}}$ gives a code $\vec{\mathbf{v}}$ such that $\mathcal{M}_{\vec{\mathbf{v}}} \cong$

$\mathcal{N}$. $\mathcal{D}_{\mathrm{s,cl}}$ has the coding countable subsets property by Lemma 2.5.6. Hence $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}})$ by Lemma 2.2.19. The proof that $\mathrm{Th}_2(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$ is the same. Use Lemma 2.6.4 in place of Lemma 2.6.2. $\square$

Theorem 2.6.6. Σ^0_3-$\mathrm{Th}(\mathcal{D}_{\mathrm{s}})$, Σ^0_4-$\mathrm{Th}(\mathcal{D}_{\mathrm{s}}; \leq_{\mathrm{s}})$, Σ^0_3-$\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}})$, Σ^0_4-$\mathrm{Th}(\mathcal{D}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$, Σ^0_3-$\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}})$, *and* Σ^0_4-$\mathrm{Th}(\mathcal{D}^{01}_{\mathrm{s,cl}}; \leq_{\mathrm{s}})$ *are all undecidable.*

Proof. By Lemma 2.6.2, there is a code $\vec{\mathbf{w}}$ in $\mathcal{D}_{\mathrm{s,cl}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. One readily checks that this $\vec{\mathbf{w}}$ is also satisfies $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ when $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}}$ is interpreted in $\mathcal{D}_{\mathrm{s}}$ instead of $\mathcal{D}_{\mathrm{s,cl}}$ because Lemma 2.6.1 is valid in $\mathcal{D}_{\mathrm{s}}$. By Lemma 2.6.4, there is a code $\vec{\mathbf{w}}$ in $\mathcal{D}^{01}_{\mathrm{s,cl}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. The results then follow from Lemma 2.2.25. $\square$

2.7 A first-order sentence distinguishing $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ from $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$

$\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ are countably meet-complete by Lemma 2.4.2. In contrast, if $\mathfrak{X} \subseteq \mathcal{D}_{\mathrm{s,cl}}$ or $\mathfrak{X} \subseteq \mathcal{D}^{01}_{\mathrm{s,cl}}$ is countable and strongly meet-incomplete, then $\mathfrak{X}$ does not have a greatest lower bound by the following lemma. Recall that a subset X of a lattice $\mathcal{L}$ is strongly meet-incomplete if and only if for every finite $\{y_i \mid i < n\} \subseteq X$ there is an $x \in X$ such that $x \ngeq \prod_{i<n} y_i$.

Lemma 2.7.1 (Dyment [23]; See also [74])**.** *No countable strongly meet-incomplete* $\mathfrak{X} \subseteq \mathcal{D}_{\mathrm{s}}$ *has a greatest lower bound.*

The proof of Lemma 2.7.1 works in $\mathcal{D}_{\mathrm{s,cl}}$, and it only requires a slight modification for $\mathcal{D}^{01}_{\mathrm{s,cl}}$.

If $\vec{\mathbf{w}}$ is a code for a model of PA^- that satisfies the second-order correctness condition in any of $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, $\mathcal{D}^{01}_{\mathrm{w,cl}}$, then $E(\mathbf{w}_0)$ is countable. This observation gives us the following theorem.

Theorem 2.7.2. *Neither* $\mathcal{D}_{\mathrm{s,cl}}$ *nor* $\mathcal{D}^{01}_{\mathrm{s,cl}}$ *is elementarily equivalent to either* $\mathcal{D}_{\mathrm{w,cl}}$ *or* $\mathcal{D}^{01}_{\mathrm{w,cl}}$.

Proof. Let φ be the first-order sentence "for all $\vec{w}$, if $\vec{w}$ is a code for a model of PA^- that satisfies the second-order correctness condition, then $E(w_0)$ has a greatest lower bound." The sentence φ is true in both $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ because such an $E(\mathbf{w}_0)$ is countable and these lattices are countably meet-complete by Lemma 2.4.2. On the other hand, φ fails in both $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$. If $\vec{\mathbf{w}}$ is the code produced in either Lemma 2.6.2 or Lemma 2.6.4, then $E(\mathbf{w}_0) = \{\deg_{\mathrm{s}}(\{f_i\}) \mid i \in \omega\}$ where $\{f_i \mid i \in \omega\}$ is a $\leq_{\mathrm{T}}$-antichain. It is then easy to check that $E(\mathbf{w}_0)$ is strongly meet-incomplete and hence has no greatest lower bound by Lemma 2.7.1. $\square$

The relationship between $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ and the relationship between $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ require further study.

Question 2.7.3.

- Is every closed $\mathcal{X} \subseteq \omega^\omega$ Medvedev equivalent to some closed $\mathcal{Y} \subseteq 2^\omega$? If not, are $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ isomorphic? If not, are $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s,cl}}$ elementarily equivalent?
- Is every closed $\mathcal{X} \subseteq \omega^\omega$ Muchnik equivalent to some closed $\mathcal{Y} \subseteq 2^\omega$? If not, are $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ isomorphic? If not, are $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ elementarily equivalent?

2.8 Meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$ and r.e. separating degrees

In this section we present facts about the meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$ that allow us to implement our coding in $\mathcal{E}_{\mathrm{s}}$. We begin with a characterization of the meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$.

Lemma 2.8.1 ([6] Corollary 3.5)**.** *Let $\mathcal{Q}$ be a Π^0_1 class. Then $\deg_{\mathrm{s}}(\mathcal{Q})$ is meet-irreducible if and only if for every clopen $\mathcal{C} \subseteq 2^\omega$ either $\mathcal{Q} \cap \mathcal{C} \equiv_{\mathrm{s}} \mathcal{Q}$ or $\mathcal{Q} \cap \mathcal{C}^c \equiv_{\mathrm{s}} \mathcal{Q}$.*

Proof. We prove the contrapositive in both directions. First, suppose $\mathcal{C} \subseteq 2^\omega$ is clopen, $\mathcal{Q} \cap \mathcal{C} \not\equiv_{\mathrm{s}} \mathcal{Q}$, and $\mathcal{Q} \cap \mathcal{C}^c \not\equiv_{\mathrm{s}} \mathcal{Q}$. $\mathcal{Q} \cap \mathcal{C} \geq_{\mathrm{s}} \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{C}^c \geq_{\mathrm{s}} \mathcal{Q}$ by the identity functional, so it must be that $\mathcal{Q} \cap \mathcal{C} >_{\mathrm{s}} \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{C}^c >_{\mathrm{s}} \mathcal{Q}$. $\mathcal{C}$ is clopen, so there is a finite set of strings $\{\sigma_i\}_{i<n} \subseteq 2^{<\omega}$ such that $\mathcal{C} = \bigcup_{i<n} I(\sigma)$. Then $0^\frown(\mathcal{Q} \cap \mathcal{C}) \cup 1^\frown(\mathcal{Q} \cap \mathcal{C}^c) \leq_{\mathrm{s}} \mathcal{Q}$ by the functional

$$f \mapsto \begin{cases} 0^\frown f & \text{if } (\exists i < n)(\sigma_i \subset f) \\ 1^\frown f & \text{otherwse.} \end{cases}$$

What we have shown is $\deg_{\mathrm{s}}(\mathcal{Q} \cap \mathcal{C}) >_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{Q})$, $\deg_{\mathrm{s}}(\mathcal{Q} \cap \mathcal{C}^c) >_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{Q})$, and $\deg_{\mathrm{s}}(\mathcal{Q} \cap \mathcal{C}) \times \deg_{\mathrm{s}}(\mathcal{Q} \cap \mathcal{C}^c) \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{Q})$. Thus $\deg_{\mathrm{s}}(\mathcal{Q})$ is meet-reducible.

Conversely, suppose $\deg_{\mathrm{s}}(\mathcal{Q})$ is meet-reducible, and let $\mathcal{X}$ and $\mathcal{Y}$ be Π^0_1 classes such that $\mathcal{X} >_{\mathrm{s}} \mathcal{Q}$, $\mathcal{Y} >_{\mathrm{s}} \mathcal{Q}$, and $\mathcal{Q} \equiv_{\mathrm{s}} 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. Let Φ be such that $\Phi(\mathcal{Q}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. Consider the set $\widehat{\mathcal{X}} = \{f \in \mathcal{Q} \mid \Phi(f)(0) = 0\}$. $\Phi(f)$ is total for all $f \in \mathcal{Q}$, so we can write $\widehat{\mathcal{X}} = \mathcal{Q} \cap \{f \in 2^\omega \mid \Phi(f)(0) \neq 1\}$ (where $\Phi(f)(0) \neq 1$ includes the possibility that $\Phi(f)(0)$ diverges), which is the intersection of two closed subsets of 2^ω. Hence $\widehat{\mathcal{X}}$ is compact. Let $\Sigma = \{\sigma \in 2^{<\omega} \mid \Phi(\sigma)(0) = 0\}$. Then $\widehat{\mathcal{X}} \subseteq \bigcup_{\sigma \in \Sigma} I(\sigma)$, so by compactness there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\widehat{\mathcal{X}} \subseteq \bigcup_{\sigma \in \Sigma_0} I(\sigma)$. Let $\mathcal{C} = \bigcup_{\sigma \in \Sigma_0} I(\sigma)$ be this clopen set. Φ witnesses that $\mathcal{Q} \cap \mathcal{C} \geq_{\mathrm{s}} 0^\frown\mathcal{X}$ and that $\mathcal{Q} \cap \mathcal{C}^c \geq_{\mathrm{s}} 1^\frown\mathcal{Y}$. As $0^\frown\mathcal{X} \equiv_{\mathrm{s}} \mathcal{X} >_{\mathrm{s}} \mathcal{Q}$ and $1^\frown\mathcal{Y} \equiv_{\mathrm{s}} \mathcal{Y} >_{\mathrm{s}} \mathcal{Q}$, we have the desired clopen set $\mathcal{C} \subseteq 2^\omega$ such that $\mathcal{Q} \cap \mathcal{C} \not\equiv_{\mathrm{s}} \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{C}^c \not\equiv_{\mathrm{s}} \mathcal{Q}$. $\square$

Degrees of r.e. separating classes are the main examples of meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$.

Definition 2.8.2. For $A, B \subseteq \omega$, define

$$\mathcal{S}(A, B) = \{f \in 2^\omega \mid \forall n((n \in A \to f(n) = 1) \wedge (n \in B \to f(n) = 0))\}.$$

An $f \in \mathcal{S}(A, B)$ is said to *separate* A from B. $\mathcal{S} \subseteq 2^\omega$ is an *r.e. separating class* if and only if there are disjoint r.e. sets $A, B \subseteq \omega$ such that $\mathcal{S} = \mathcal{S}(A, B)$.

From the definition, an r.e. separating class is always a Π^0_1 class. An $\mathbf{s} \in \mathcal{E}_{\mathrm{s}}$ is an *r.e. separating degree* if and only if $\mathbf{s} = \deg_{\mathrm{s}}(\mathcal{S})$ for an r.e. separating class $\mathcal{S}$.

Lemma 2.8.3 ([15] Lemma 6)**.** *If $\mathcal{S}$ is an r.e. separating class and $\mathcal{C} \subseteq 2^\omega$ is a clopen set such that $\mathcal{S} \cap \mathcal{C} \neq \emptyset$, then $\mathcal{S} \cap \mathcal{C} \equiv_{\mathrm{s}} \mathcal{S}$.*

Proof. Let $\mathcal{S} = \mathcal{S}(A, B)$ be an r.e. separating class and let $\mathcal{C} \subseteq 2^\omega$ be a clopen set such that $\mathcal{S} \cap \mathcal{C} \neq \emptyset$. $\mathcal{S} \leq_{\mathrm{s}} \mathcal{S} \cap \mathcal{C}$ by the identity functional. To see $\mathcal{S} \geq_{\mathrm{s}} \mathcal{S} \cap \mathcal{C}$, let σ be such that $I(\sigma) \subseteq \mathcal{C}$ and $\mathcal{S} \cap I(\sigma) \neq \emptyset$. For any $f \in 2^\omega$, let f_σ be the function obtained from f by replacing the initial segment of f of length $|\sigma|$ by σ:

$$f_\sigma(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n) & \text{if } n \geq |\sigma|. \end{cases}$$

The condition $\mathcal{S} \cap I(\sigma) \neq \emptyset$ implies that σ separates $\{n \in A \mid n < |\sigma|\}$ from $\{n \in B \mid n < |\sigma|\}$. Thus if f separates A from B, then so does f_σ. Therefore the functional $f \mapsto f_\sigma$ witnesses $\mathcal{S} \geq_{\mathrm{s}} \mathcal{S} \cap \mathcal{C}$. □

Lemma 2.8.1 and Lemma 2.8.3 imply that every r.e. separating degree is meet-irreducible. It is important to note (as in [15]) that the r.e. separating classes are closed under $+$ and consequently that the r.e. separating degrees are closed under join: if $\mathcal{S}(A_0, B_0)$ and $\mathcal{S}(A_1, B_1)$ are r.e. separating classes then $\mathcal{S}(A_0, B_0) + \mathcal{S}(A_1, B_1) = \mathcal{S}(A_0 \oplus A_1, B_0 \oplus B_1)$. Thus the join of two r.e. separating degrees is meet-irreducible. In fact, the join of any r.e. separating degree and any meet-irreducible degree is again meet-irreducible.

Lemma 2.8.4. *Let $\mathbf{q} \in \mathcal{E}_{\mathrm{s}}$ be meet-irreducible and let $\mathbf{s} \in \mathcal{E}_{\mathrm{s}}$ be an r.e. separating degree. Then $\mathbf{q} + \mathbf{s}$ is meet-irreducible.*

Proof. Suppose $\mathbf{q} + \mathbf{s} \geq_{\mathrm{s}} \mathbf{x} \times \mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{E}_{\mathrm{s}}$. We show $\mathbf{q} + \mathbf{s} \geq_{\mathrm{s}} \mathbf{x}$ or $\mathbf{q} + \mathbf{s} \geq_{\mathrm{s}} \mathbf{y}$. Let $\mathcal{Q}$, $\mathcal{X}$, and $\mathcal{Y}$ be Π^0_1 classes such that $\deg_{\mathrm{s}}(\mathcal{Q}) = \mathbf{q}$, $\deg_{\mathrm{s}}(\mathcal{X}) = \mathbf{x}$, and $\deg_{\mathrm{s}}(\mathcal{Y}) = \mathbf{y}$ respectively, and let $\mathcal{S}$ be an r.e. separating class such that $\deg_{\mathrm{s}}(\mathcal{S}) = \mathbf{s}$. Let Φ be such that $\Phi(\mathcal{Q} + \mathcal{S}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. By compactness, choose a $\sigma \in 2^{<\omega}$ such that $\mathcal{S} \cap I(\sigma) \neq \emptyset$ and an $n \in \omega$ such that

$$(\forall \tau \in 2^n)((\exists f \in \mathcal{Q})(\tau \subset f) \rightarrow \Phi(\tau \oplus \sigma)(0){\downarrow}).$$

Let $\mathcal{C} = \bigcup\{I(\tau) \mid \tau \in 2^n \wedge \Phi(\tau \oplus \sigma)(0) = 0\}$. Then $\mathcal{C}$ is clopen, and Φ witnesses that $(\mathcal{Q} \cap \mathcal{C}) + (\mathcal{S} \cap I(\sigma)) \geq_{\mathrm{s}} 0^\frown\mathcal{X} \equiv_{\mathrm{s}} \mathcal{X}$ and that $(\mathcal{Q} \cap \mathcal{C}^c) + (\mathcal{S} \cap I(\sigma)) \geq_{\mathrm{s}} 1^\frown\mathcal{Y} \equiv_{\mathrm{s}} \mathcal{Y}$. Since $\mathcal{S} \cap I(\sigma) \equiv_{\mathrm{s}} \mathcal{S}$ by Lemma 2.8.3 and either $\mathcal{Q} \cap \mathcal{C} \equiv_{\mathrm{s}} \mathcal{Q}$ or $\mathcal{Q} \cap \mathcal{C}^c \equiv_{\mathrm{s}} \mathcal{Q}$ by Lemma 2.8.1, we have either $\mathcal{Q} + \mathcal{S} \geq_{\mathrm{s}} \mathcal{X}$ or $\mathcal{Q} + \mathcal{S} \geq_{\mathrm{s}} \mathcal{Y}$ as desired. □

Our proof that $\mathcal{E}_{\mathrm{s}}$ has the finite matching property uses the following lemma of Cole and Kihara. It is the main tool in their proof that the Σ^0_2-theory of $\mathcal{E}_{\mathrm{s}}$ as a partial order is decidable.

Lemma 2.8.5 ([18] Lemma 1)**.** *Let* $\{\mathbf{q}_i\}_{i<n} \subseteq \mathcal{E}_\mathrm{s}$ *and let* $m \in \omega$*. Then there is a set* $\{\mathbf{r}_i\}_{i<m} \subseteq \mathcal{E}_\mathrm{s}$ *such that*

$$\Big(\forall I \subseteq m\Big)\Big(\forall J, K \subseteq n\Big)$$
$$\Big(J \cap K = \emptyset \wedge \sum_{i\in J}\mathbf{q}_i \not\geq_\mathrm{s} \prod_{i\in K}\mathbf{q}_i \rightarrow \Big(\sum_{i\in J}\mathbf{q}_i + \sum_{i\in I}\mathbf{r}_i\Big) \not\geq_\mathrm{s} \Big(\prod_{i\in K}\mathbf{q}_i \times \prod_{i\notin I}\mathbf{r}_i\Big)\Big),$$

where $\sum_{i\in\emptyset}\mathbf{x}_i = \mathbf{0}$ *and* $\prod_{i\in\emptyset}\mathbf{x}_i = \mathbf{1} = \deg_\mathrm{s}(\mathrm{DNR}_2)$.

Cole and Kihara note that the $\{\mathbf{r}_i\}_{i<m}$ that they construct in Lemma 2.8.5 are all r.e. separating degrees. Their proof of Lemma 2.8.5 is an elaboration of Cenzer and Hinman's proof that $\mathcal{E}_\mathrm{s}$ is dense [15]. Cenzer and Hinman prove that if $\mathbf{p}, \mathbf{q} \in \mathcal{E}_\mathrm{s}$ are such that $\mathbf{q} \not\leq_\mathrm{s} \mathbf{p}$, then there is an r.e. separating degree $\mathbf{r} \in \mathcal{E}_\mathrm{s}$ such that $\mathbf{q} \times \mathbf{r} \not\leq_\mathrm{s} \mathbf{p}$ and $\mathbf{q} \not\leq_\mathrm{s} \mathbf{p} + \mathbf{r}$. Thus if $\mathbf{p} <_\mathrm{s} \mathbf{q}$, then $\mathbf{p} <_\mathrm{s} (\mathbf{p}+\mathbf{r}) \times \mathbf{q} <_\mathrm{s} \mathbf{q}$, yielding density. To make Lemma 2.8.5 somewhat easier to read and apply, we note that we only need the following special case.

Lemma 2.8.6. *Let* $\{\mathbf{q}_i\}_{i<n} \subseteq \mathcal{E}_\mathrm{s} \setminus \{\mathbf{1}\}$ *and let* $m \in \omega$*. Then there is a set of r.e. separating degrees* $\{\mathbf{r}_i\}_{i<m} \subseteq \mathcal{E}_\mathrm{s}$ *such that*

(i) $(\forall i, i' < n)(\forall j < m)(\mathbf{q}_i \not\geq_\mathrm{s} \mathbf{q}_{i'} \rightarrow \mathbf{q}_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{q}_{i'})$*,*

(ii) $(\forall i < n)(\forall j, j' < m)(j \neq j' \rightarrow \mathbf{q}_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{r}_{j'})$*, and*

(iii) $(\forall i < n)(\forall j < m)(\mathbf{q}_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{1})$.

Notice that item (iii) follows from item (ii) unless $m = 1$.

We can now show that $\mathcal{E}_\mathrm{s}$ has the finite matching property.

Lemma 2.8.7. $\mathcal{E}_\mathrm{s}$ *has the finite matching property. That is, if* $\mathbf{q}, \mathbf{q}' \in \mathcal{E}_\mathrm{s}$ *are such that* $|E(\mathbf{q})| = |E(\mathbf{q}')| = n$ *for some* $n \in \omega$*, then there is an* $\mathbf{r} \in \mathcal{E}_\mathrm{s}$ *such that* $E(\mathbf{r})$ *matches both* $E(\mathbf{q})$ *and* $E(\mathbf{q}')$.

Proof. If $n = 0$, then let $\mathbf{r} = \mathbf{q}$. Any degree $\mathbf{z}$ vacuously witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$ and that $E(\mathbf{r})$ matches $E(\mathbf{q}')$. So suppose $n > 0$, let $E(\mathbf{q}) = \{\mathbf{q}_i\}_{i<n}$, and let $E(\mathbf{q}') = \{\mathbf{q}'_i\}_{i<n}$. Apply Lemma 2.8.6 to $\{\mathbf{q}_i\}_{i<n} \cup \{\mathbf{q}'_i\}_{i<n}$ with $m = n$, noting that $\{\mathbf{q}_i\}_{i<n}$ and $\{\mathbf{q}'_i\}_{i<n}$ are both antichains by Lemma 2.2.3, to get r.e. separating degrees $\{\mathbf{r}_i\}_{i<n}$ such that

(i) $\mathbf{q}_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{q}_k$ and $\mathbf{q}'_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{q}'_k$ whenever $i, j, k < n$ are such that $i \neq k$, and

(ii) $\mathbf{q}_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{r}_k$ and $\mathbf{q}'_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{r}_k$ whenever $i, j, k < n$ are such that $j \neq k$.

(iii) $\mathbf{q}_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{1}$ and $\mathbf{q}'_i + \mathbf{r}_j \not\geq_\mathrm{s} \mathbf{1}$ whenever $i, j < n$.

(Lemma 2.8.6 applies because, by definition, $\mathbf{1}$ does not meet to any degree and so cannot be in $E(\mathbf{q})$ or $E(\mathbf{q}')$.)

Put $\mathbf{r} = \prod_{i<n}\mathbf{r}_i$, $\mathbf{z} = \prod_{i<n}(\mathbf{q}_i + \mathbf{r}_i)$, and $\mathbf{z}' = \prod_{i<n}(\mathbf{q}'_i + \mathbf{r}_i)$. We show that $\mathbf{z}$ witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$. The proof that $\mathbf{z}'$ witnesses that $E(\mathbf{r})$ matches

$E(\mathbf{q}')$ is similar. Item (ii) implies that $\{\mathbf{r}_i\}_{i<n}$ and $\{\mathbf{q}_i + \mathbf{r}_i\}_{i<n}$ are both antichains. Lemma 2.8.4 implies that $\mathbf{q}_i + \mathbf{r}_i$ is meet-irreducible for each $i < n$. Therefore $E(\mathbf{r}) = \{\mathbf{r}_i\}_{i<n}$ and $E(\mathbf{z}) = \{\mathbf{q}_i + \mathbf{r}_i\}_{i<n}$ by Lemma 2.2.4. Suppose $\mathbf{q}_i + \mathbf{r}_j \geq_{\mathrm{s}} \mathbf{z}$ for some $i, j < n$. Then $\mathbf{q}_i + \mathbf{r}_j \geq_{\mathrm{s}} \mathbf{q}_k + \mathbf{r}_k$ for some $k < n$ because $\mathbf{q}_i + \mathbf{r}_j$ is meet-irreducible by Lemma 2.8.4. Item (i) implies that $i = k$, and item (ii) implies that $j = k$. Thus for each $i < n$, $\mathbf{r}_i$ is the unique $\mathbf{y} \in E(\mathbf{r})$ such that $\mathbf{q}_i + \mathbf{y} \in E(\mathbf{z})$, and $\mathbf{q}_i$ is the unique $\mathbf{y} \in E(\mathbf{q})$ such that $\mathbf{r}_i + \mathbf{y} \in E(\mathbf{z})$. Thus $\mathbf{z}$ witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$. $\square$

We need one last fact about the r.e. separating classes to implement our coding in $\mathcal{E}_{\mathrm{s}}$.

Lemma 2.8.8 ([32] Theorem 4.1)**.** *There is a recursive sequence $\{\mathcal{S}_n\}_{n\in\omega}$ r.e. separating classes that is strongly independent.*

2.9 The complexity of $\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$

In this section we prove that $\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}}) \equiv_1 \mathrm{Th}(\mathcal{N})$ and that Π^0_3-$\mathrm{Th}(\mathcal{E}_{\mathrm{s}})$ and Π^0_4-$\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$ are undecidable. By Lemma 2.2.14 and Lemma 2.2.25, it suffices to find a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{s}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.

Definition 2.9.1. Let $\mathcal{Q}$ be a Π^0_1 class with no recursive member. Let A be an infinite recursive set, and let $\{\sigma_n\}_{n\in A}$ be a recursive sequence of pairwise incomparable strings such that $\bigcup_{n\in A} I(\sigma_n) = 2^\omega \setminus \mathcal{Q}$ (for example, let T be a recursive tree such that $\mathcal{Q} = [T]$ and let $\{\sigma_n\}_{n\in A}$ be the strings $\sigma \notin T$ of minimal length). Let $\{\mathcal{S}_n\}_{n\in A}$ be an infinite recursive sequence of Π^0_1 classes. Define $\mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_n\}_{n\in A})$ to be the Π^0_1 class

$$\mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_n\}_{n\in A}) = \mathcal{Q} \cup \bigcup_{n\in A} \sigma_n{}^\frown\mathcal{S}_n.$$

Lemma 2.9.2. *Let $\mathcal{Q}$ be a Π^0_1 class with no recursive member. Let $\{\mathcal{S}_n\}_{n\in A}$ be an infinite recursive sequence of r.e. separating classes (indexed by a recursive set A) that is an antichain and is such that $\mathcal{Q} \nleq_{\mathrm{s}} \mathcal{S}_n$ for all $n \in A$. Let $\mathbf{w} = \deg_{\mathrm{s}}(\mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_n\}_{n\in A}))$.*

(i) If $\mathbf{x} \in \mathcal{E}_{\mathrm{s}}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{S}_n)$ for some $n \in A$.

(ii) $E(\mathbf{w}) = \{\deg_{\mathrm{s}}(\mathcal{S}_n) \mid n \in A\}$.

Proof. Let $\mathcal{W} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_n\}_{n\in A})$.

(i) Let $\mathbf{x} \in \mathcal{E}_{\mathrm{s}}$ be such that $\mathbf{x}$ meets to $\mathbf{w}$. Suppose for a contradiction that $\mathbf{x} \nleq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{S}_n)$ for all $n \in A$. Let $\mathcal{X}$ be a Π^0_1 class such that $\mathbf{x} = \deg_{\mathrm{s}}(\mathcal{X})$, and let $\mathcal{Y}$ be a Π^0_1 class such that $\deg_{\mathrm{s}}(\mathcal{Y})$ witnesses that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $\mathcal{Y} >_{\mathrm{s}} \mathcal{W}$ and $\mathcal{W} \equiv_{\mathrm{s}} 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. Let Φ be such that $\Phi(\mathcal{Q} \cup \bigcup_{n\in A} \sigma_n{}^\frown\mathcal{S}_n) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$.

Claim.

(a) $\Phi(\sigma_n{}^\frown\mathcal{S}_n) \subseteq 1^\frown\mathcal{Y}$ for all $n \in A$ and

(b) $\Phi(\mathcal{Q}) \subseteq 1^\frown\mathcal{Y}$.

Proof of claim. If item (a) fails, then for some $n \in A$ there is a clopen $\mathcal{C} \subseteq 2^\omega$ such that $(\sigma_n{}^\frown\mathcal{S}_n) \cap \mathcal{C} \neq \emptyset$ and $\Phi((\sigma_n{}^\frown\mathcal{S}_n) \cap \mathcal{C}) \subseteq 0^\frown\mathcal{X}$. So $(\sigma_n{}^\frown\mathcal{S}_n) \cap \mathcal{C} \geq_{\mathrm{s}} 0^\frown\mathcal{X} \equiv_{\mathrm{s}} \mathcal{X}$. The class $\sigma_n{}^\frown\mathcal{S}_n$ is an r.e. separating class because $\mathcal{S}_n$ is, so $(\sigma_n{}^\frown\mathcal{S}_n) \cap \mathcal{C} \equiv_{\mathrm{s}} \sigma_n{}^\frown\mathcal{S}_n \equiv_{\mathrm{s}} \mathcal{S}_n$, where the first equivalence is by Lemma 2.8.3. Thus the contradiction $\mathcal{X} \leq_{\mathrm{s}} \mathcal{S}_n$.

If item (b) fails, then there is an $f \in \mathcal{Q}$ and a $\sigma \subset f$ such that $\Phi(\sigma)(0)\downarrow = 0$. Since $I(\sigma) \nsubseteq \mathcal{Q}$, there is an $n \in A$ such that $\sigma_n \supseteq \sigma$. Hence $\Phi(\sigma_n{}^\frown\mathcal{S}_n) \nsubseteq 1^\frown\mathcal{Y}$, contradicting item (a). ☐

The claim shows that $\Phi(\mathcal{Q} \cup \bigcup_{n\in\omega} \sigma_n{}^\frown\mathcal{S}_n) \subseteq 1^\frown\mathcal{Y}$. Thus $\mathcal{Y} \leq_{\mathrm{s}} \mathcal{W}$, which contradicts $\mathcal{Y} >_{\mathrm{s}} \mathcal{W}$.

(ii) Let $n \in A$. To see that $\deg_{\mathrm{s}}(\mathcal{S}_n) \in E(\mathbf{w})$, let $\mathcal{Y} = \mathcal{Q} \cup \bigcup_{i \in A\setminus\{n\}} \sigma_i{}^\frown\mathcal{S}_i$.

Claim. $\mathcal{S}_n \ngeq_{\mathrm{s}} \mathcal{Y}$

Proof of claim. Suppose for a contradiction that Φ is such that $\Phi(\mathcal{S}_n) \subseteq \mathcal{Y}$. If there is an $i \in A \setminus \{n\}$ such that $\Phi(\mathcal{S}_n) \cap (\sigma_i{}^\frown\mathcal{S}_i) \neq \emptyset$, then there is a clopen $\mathcal{C} \subseteq 2^\omega$ such that $\mathcal{S}_n \cap \mathcal{C} \neq \emptyset$ and $\Phi(\mathcal{S}_n \cap \mathcal{C}) \subseteq \sigma_i{}^\frown\mathcal{S}_i$. $\mathcal{S}_n \equiv_{\mathrm{s}} \mathcal{S}_n \cap \mathcal{C}$ by Lemma 2.8.3, and $\mathcal{S}_n \cap \mathcal{C} \geq_{\mathrm{s}} \sigma_i{}^\frown\mathcal{S}_i \equiv_{\mathrm{s}} \mathcal{S}_i$. This contradicts that $\{\mathcal{S}_n\}_{n\in A}$ is an antichain. Thus $\Phi(\mathcal{S}_n) \cap (\sigma_i{}^\frown\mathcal{S}_i) = \emptyset$ for all $n \in A$. Therefore $\Phi(\mathcal{S}_n) \subseteq \mathcal{Q}$. This contradicts $\mathcal{Q} \nleq_{\mathrm{s}} \mathcal{S}_n$. ☐

It is easy to check that $\mathcal{W} \equiv_{\mathrm{s}} 0^\frown\mathcal{S}_n \cup 1^\frown\mathcal{Y}$, so, by the claim, $\deg_{\mathrm{s}}(\mathcal{Y})$ witnesses that $\deg_{\mathrm{s}}(\mathcal{S}_n)$ meets to $\mathbf{w}$. The degree $\deg_{\mathrm{s}}(\mathcal{S}_n)$ is meet-irreducible because it is an r.e. separating degree. Thus $\deg_{\mathrm{s}}(\mathcal{S}_n) \in E(\mathbf{w})$.

We have shown that $\{\deg_{\mathrm{s}}(\mathcal{S}_n) \mid n \in A\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{S}_n)$ for some $n \in A$. $E(\mathbf{w})$ is an antichain by Lemma 2.2.3 and $\deg_{\mathrm{s}}(\mathcal{S}_n) \in E(\mathbf{w})$, so it must be that $\mathbf{x} = \deg_{\mathrm{s}}(\mathcal{S}_n)$. ☐

We now have all the ingredients to find a code for $\mathcal{N}$ in $\mathcal{E}_{\mathrm{s}}$.

Lemma 2.9.3. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{s}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.*

Proof. By Lemma 2.8.8, let $\mathcal{Q}$ be an r.e. separating class and let $\{\mathcal{S}_{0,n}\}_{n\in\omega}$, $\{\mathcal{S}_{1,n}\}_{n\in\omega}$, and $\{\mathcal{S}_{2,n}\}_{n\in\omega}$ be recursive sequences of r.e. separating classes such that $\{\mathcal{Q}\} \cup \{\mathcal{S}_{0,n}\}_{n\in\omega} \cup \{\mathcal{S}_{1,n}\}_{n\in\omega} \cup \{\mathcal{S}_{2,n}\}_{n\in\omega}$ is strongly independent. Then let

$$\begin{aligned}
\mathbf{w}_0 &= \deg_{\mathrm{s}}(\mathcal{W}_0) && \text{for} && \mathcal{W}_0 = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n}\}_{n\in\omega}),\\
\mathbf{w}_1 &= \deg_{\mathrm{s}}(\mathcal{W}_1) && \text{for} && \mathcal{W}_1 = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{1,n}\}_{n\in\omega}),\\
\mathbf{w}_2 &= \deg_{\mathrm{s}}(\mathcal{W}_2) && \text{for} && \mathcal{W}_2 = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{2,n}\}_{n\in\omega}),\\
\mathbf{m} &= \deg_{\mathrm{s}}(\mathcal{M}) && \text{for} && \mathcal{M} = \mathrm{spine}(Q, \{\mathcal{S}_{0,n} + \mathcal{S}_{1,n}\}_{n\in\omega} \cup \{\mathcal{S}_{0,n} + \mathcal{S}_{2,n}\}_{n\in\omega}),\\
\boldsymbol{\ell} &= \deg_{\mathrm{s}}(\mathcal{L}) && \text{for} && \mathcal{L} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,i} + \mathcal{S}_{1,j} \mid i < j\}),\\
\mathbf{p} &= \deg_{\mathrm{s}}(\mathcal{P}) && \text{for} && \mathcal{P} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,i} + \mathcal{S}_{1,j} + \mathcal{S}_{2,k} \mid i + j = k\}),\\
\mathbf{t} &= \deg_{\mathrm{s}}(\mathcal{T}) && \text{for} && \mathcal{T} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,i} + \mathcal{S}_{1,j} + \mathcal{S}_{2,k} \mid i \times j = k\}),
\end{aligned}$$

$$
\begin{aligned}
\mathbf{d} &= \deg_{\mathrm{s}}(\mathcal{D}) && \text{for} && \mathcal{D} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,i} + \mathcal{S}_{1,j} + \mathcal{S}_{2,k} \mid i \dot{-} j = k\}),\\
\mathbf{z} &= \deg_{\mathrm{s}}(\mathcal{S}_{0,0}), && \text{and} &&\\
\mathbf{o} &= \deg_{\mathrm{s}}(\mathcal{S}_{0,1}). &&&&
\end{aligned}
$$

By Lemma 2.9.2 item (ii), $E(\mathbf{w}_0) = \{\deg_{\mathrm{s}}(\mathcal{S}_{0,n})\}_{n\in\omega}$. The map $\deg_{\mathrm{s}}(\mathcal{S}_{0,n}) \mapsto n$ is the isomorphism witnessing $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. Clearly $\mathbf{z} \mapsto 0$ and $\mathbf{o} \mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, $\times$, and $\dot{-}$ are similar. Let $i, j \in \omega$. If $i < j$, then $\deg_{\mathrm{s}}(\mathcal{S}_{1,j})$ meets to $\mathbf{w}_1$ by Lemma 2.9.2 item (ii), and it is easy to see that $\deg_{\mathrm{s}}(\mathcal{S}_{0,j}) + \deg_{\mathrm{s}}(\mathcal{S}_{1,j}) \geq_{\mathrm{s}} \mathbf{m}$ and that $\deg_{\mathrm{s}}(\mathcal{S}_{0,i}) + \deg_{\mathrm{s}}(\mathcal{S}_{1,j}) \geq_{\mathrm{s}} \boldsymbol{\ell}$. Thus $R^2_{\boldsymbol{\ell}}(\deg_{\mathrm{s}}(\mathcal{S}_{0,i}), \deg_{\mathrm{s}}(\mathcal{S}_{0,j}))$. Conversely, suppose that $R^2_{\boldsymbol{\ell}}(\deg_{\mathrm{s}}(\mathcal{S}_{0,i}), \deg_{\mathrm{s}}(\mathcal{S}_{0,j}))$. Let $\mathbf{u}_1 \in \mathcal{E}_{\mathrm{s}}$ be such that $\mathbf{u}_1$ meets to $\mathbf{w}_1$, $\deg_{\mathrm{s}}(\mathcal{S}_{0,j}) + \mathbf{u}_1 \geq_{\mathrm{s}} \mathbf{m}$, and $\deg_{\mathrm{s}}(\mathcal{S}_{0,i}) + \mathbf{u}_1 \geq_{\mathrm{s}} \boldsymbol{\ell}$. Since $\mathbf{u}_1$ meets to $\mathbf{w}_1$, it must be that $\mathbf{u}_1 \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{S}_{1,k})$ for some $k \in \omega$ by Lemma 2.9.2 item (i). Thus $\deg_{\mathrm{s}}(\mathcal{S}_{0,j}) + \deg_{\mathrm{s}}(\mathcal{S}_{1,k}) \geq_{\mathrm{s}} \mathbf{m}$. However, if $k \neq j$, then no member of $\mathcal{S}_{0,j} + \mathcal{S}_{1,k}$ computes any member of $\mathcal{M}$ by strong independence. Thus $\mathbf{u}_1 \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{S}_{1,j})$, which implies that $\deg_{\mathrm{s}}(S_{0,i}) + \deg_{\mathrm{s}}(S_{1,j}) \geq_{\mathrm{s}} \boldsymbol{\ell}$. Again by strong independence, if $i \not< j$, then no member of $\mathcal{S}_{0,i} + \mathcal{S}_{1,j}$ computes any member of $\mathcal{L}$. Hence $i < j$. $\square$

Higuchi also used spines of recursive sequences of independent r.e. separating classes to prove that $\mathcal{E}_{\mathrm{s}}$ is not a Brower algebra [25].

Theorem 2.9.4. $\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}}) \equiv_1 \mathrm{Th}(\mathcal{N})$.

Proof. $\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}}) \leq_1 \mathrm{Th}(\mathcal{N})$ by Lemma 2.1.3. For $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$, by Lemma 2.9.3 let $\vec{\mathbf{w}}$ be a code in $\mathcal{E}_{\mathrm{s}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. Removing the degree $\mathbf{d}$ from the code $\vec{\mathbf{w}}$ gives a code $\vec{\mathbf{v}}$ such that $\mathcal{M}_{\vec{\mathbf{v}}} \cong \mathcal{N}$. $\mathcal{E}_{\mathrm{s}}$ has the finite matching property by Lemma 2.8.7, thus $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$ by Lemma 2.2.14. $\square$

Theorem 2.9.5. Σ^0_3-$\mathrm{Th}(\mathcal{E}_{\mathrm{s}})$ *and* Σ^0_4-$\mathrm{Th}(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$ *are undecidable.*

Proof. There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{s}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ by Lemma 2.9.3. The results then follow from Lemma 2.2.25. $\square$

2.10 The degree of $\mathcal{E}_{\mathrm{s}}$ is $\mathbf{0}'''$

In this section, we consider the complexities of presentations of $\mathcal{E}_{\mathrm{s}}$.

Definition 2.10.1. A *presentation of $\mathcal{E}_{\mathrm{s}}$ as a partial order* consists of a relation $\leq_{\mathcal{P}} \subseteq \omega \times \omega$ such that the structure $\mathcal{P} = (\omega; \leq_{\mathcal{P}})$ is isomorphic to $(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$. A *presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice* consists of a relation $\leq_{\mathcal{L}} \subseteq \omega \times \omega$ and functions $+_{\mathcal{L}} \colon \omega \times \omega \to \omega$ and $\times_{\mathcal{L}} \colon \omega \times \omega \to \omega$ such that the structure $\mathcal{L} = (\omega; \leq_{\mathcal{L}}, +_{\mathcal{L}}, \times_{\mathcal{L}})$ is isomorphic to $\mathcal{E}_{\mathrm{s}}$.

We measure the complexities of presentations by their Turing degrees.

Definition 2.10.2. The *degree* of a presentation $\mathcal{P}$ of $\mathcal{E}_{\mathrm{s}}$ as a partial order is $\deg_{\mathrm{T}}(\mathcal{P}) = \deg_{\mathrm{T}}(\leq_{\mathcal{P}})$. The *degree* of a presentation $\mathcal{L}$ of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\deg_{\mathrm{T}}(\mathcal{L}) = \deg_{\mathrm{T}}(\leq_{\mathcal{L}} \oplus +_{\mathcal{L}} \oplus \times_{\mathcal{L}})$.

Equivalently, the degree of a presentation is the Turing degree of its atomic diagram, suitably Gödel numbered.

Lemma 2.10.3. *There is a presentation $\mathcal{L}$ of $\mathcal{E}_{\mathrm{s}}$ as a lattice with $\deg_{\mathrm{T}}(\mathcal{L}) \leq_{\mathrm{T}} \mathbf{0}'''$.*

Proof. Let $\{\mathcal{Z}_e\}_{e\in\omega}$ be a recursive sequence containing all Π^0_1 classes as in Lemma 1.4.4, and let $\{T_e\}_{e\in\omega}$ be the corresponding uniformly recursive sequence of trees. Since $[T_i] \leq_{\mathrm{s}} [T_j]$ is a Σ^0_3 property of $\langle i, j \rangle$ by Lemma 1.4.3, we can use $\mathbf{0}'''$ to make a new sequence of trees $\{T'_e\}_{e\in\omega}$ such that $\{[T'_e]\}_{e\in\omega}$ contains exactly one representative for each degree in $\mathcal{E}_{\mathrm{s}}$. Inductively, let T'_e be T_i for the least $i \in \omega$ such that $(\forall j < e)([T_i] \not\equiv_{\mathrm{s}} [T'_j])$. Again using $\mathbf{0}'''$, for $i, j \in \omega$ define $i \leq_{\mathcal{L}} j$ if and only if $[T'_i] \leq_{\mathrm{s}} [T'_j]$, define $+_{\mathcal{L}}(i, j)$ to be the $k \in \omega$ such that $[T'_k] \equiv_{\mathrm{s}} [T'_i + T'_j]$, and define $\times_{\mathcal{L}}(i, j)$ to be the $k \in \omega$ such that $[T'_k] \equiv_{\mathrm{s}} [0^\frown T'_i \cup 1^\frown T'_j]$. Then $\mathcal{L} \cong \mathcal{E}_{\mathrm{s}}$ and $\deg_{\mathrm{T}}(\mathcal{L}) \leq_{\mathrm{T}} \mathbf{0}'''$. $\square$

We prepare to show that every presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice computes $\mathbf{0}'''$. Let $\{\mathcal{X}_n\}_{n\in\omega}$ be a recursive sequence of Π^0_1 classes, and let $m \in \omega$. Define $\sum_{n\in\omega} \mathcal{X}_n$ and $\sum_{n\in\omega\setminus\{m\}} \mathcal{X}_n$ by

$$\sum_{n\in\omega} \mathcal{X}_n = \Big\{ \bigoplus_{n\in\omega} f_n \;\Big|\; \forall n (f_n \in X_n) \Big\} \text{ and}$$
$$\sum_{n\in\omega\setminus\{m\}} \mathcal{X}_n = \Big\{ \bigoplus_{n\in\omega\setminus\{m\}} f_n \;\Big|\; \forall n (n \neq m \to f_n \in X_n) \Big\}.$$

The predicates $\forall n(f_n \in \mathcal{X}_n)$ and $\forall n(n \neq m \to f_n \in \mathcal{X}_n)$ are Π^0_1 because the sequence $\{\mathcal{X}_n\}_{n\in\omega}$ is recursive. Hence $\sum_{n\in\omega} \mathcal{X}_n$ and $\sum_{n\in\omega\setminus\{m\}} \mathcal{X}_n$ are Π^0_1 classes. If $\{\mathcal{S}(A_n, B_n)\}_{n\in\omega}$ is a recursive sequence of r.e. separating classes, then one checks that

$$\sum_{n\in\omega} \mathcal{S}(A_n, B_n) = \mathcal{S}\Big(\bigoplus_{n\in\omega} A_n, \bigoplus_{n\in\omega} B_n \Big) \text{ and}$$
$$\sum_{n\in\omega\setminus\{m\}} \mathcal{S}(A_n, B_n) = \mathcal{S}\Big(\bigoplus_{n\in\omega\setminus\{m\}} A_n, \Big(\bigoplus_{n\in\omega\setminus\{m\}} B_n \Big) \cup \{\langle m, k \rangle \mid k \in \omega\} \Big).$$

These two Π^0_1 classes are in fact r.e. separating classes because any Π^0_1 class that is a separating class must be an r.e. separating class. If T is a recursive tree such that $[T] = \mathcal{S}(A, B)$ for $A, B \subseteq \omega$, then $A = \{n \mid (\exists s > n)(\forall \sigma \in 2^s)(\sigma \in T \to \sigma(n) = 1)\}$ and $B = \{n \mid (\exists s > n)(\forall \sigma \in 2^s)(\sigma \in T \to \sigma(n) = 0)\}$, both of which are r.e.

Lemma 2.10.4. *Let $\mathcal{Q}$ be an r.e. separating class, and let $\varphi(e, m, k, \ell)$ be a recursive predicate. Then there is a recursive sequence of Π^0_1 classes $\{\mathcal{X}_{\langle e,m \rangle}\}_{\langle e,m\rangle\in\omega}$ such that for all $e, m \in \omega$*

$$\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m \rangle}) = \begin{cases} \mathbf{0} & \text{if } \forall k \exists \ell \varphi(e, m, k, \ell) \\ \deg_{\mathrm{s}}(\mathcal{Q}) & \text{if } \exists k \forall \ell \neg\varphi(e, m, k, \ell). \end{cases}$$

Proof. Let A and B be disjoint r.e. sets such that $\mathcal{Q} = \mathcal{S}(A, B)$. Let $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ be recursive stage enumerations of A and B respectively. For $e, m \in \omega$, let $\mathcal{X}_{\langle e,m \rangle}$ be the r.e. separating class $\mathcal{X}_{\langle e,m \rangle} = \mathcal{S}(C_{\langle e,m \rangle}, D_{\langle e,m \rangle})$ where

$$C_{\langle e,m \rangle} = \{\langle k, x \rangle \mid \exists s(x \in A_s \wedge (\forall \ell \leq s)(\neg\varphi(e, m, k, \ell)))\} \text{ and}$$
$$D_{\langle e,m \rangle} = \{\langle k, x \rangle \mid \exists s(x \in B_s \wedge (\forall \ell \leq s)(\neg\varphi(e, m, k, \ell)))\}.$$

For all $k \in \omega$, the k^{th} column of $C_{\langle e,m \rangle}$ is a subset of A, and the k^{th} column of $D_{\langle e,m \rangle}$ is a subset of B. Thus $C_{\langle e,m \rangle}$ and $D_{\langle e,m \rangle}$ are disjoint. The sequences $\{C_{\langle e,m \rangle}\}_{\langle e,m \rangle\in\omega}$ and $\{D_{\langle e,m \rangle}\}_{\langle e,m \rangle\in\omega}$ are uniformly r.e., which implies that the sequence $\{\mathcal{X}_{\langle e,m \rangle}\}_{\langle e,m \rangle\in\omega}$ is a recursive sequence of Π^0_1 classes.

To see that $\mathcal{X}_{\langle e,m \rangle}$ has the desired degree, first suppose that $\forall k\exists\ell\varphi(e, m, k, \ell)$. In this case, the set $C_{\langle e,m \rangle}$ is recursive. To determine if $\langle k, x \rangle \in C_{\langle e,m \rangle}$, search for the least ℓ such that $\varphi(e, m, k, \ell)$, which must exist by assumption. Once ℓ is found, enumerate A up to stage ℓ. Then $\langle k, x \rangle \in C_{\langle e,m \rangle}$ if and only if $x \in A_\ell$. $\mathcal{X}_{\langle e,m \rangle}$ contains the characteristic function of $C_{\langle e,m \rangle}$, which we have just shown is recursive, so $\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m \rangle}) = \mathbf{0}$. On the other hand, if $\exists k\forall\ell\neg\varphi(e, m, k, \ell)$, then fix a witnessing k. In this case, the k^{th} column of $C_{\langle e,m \rangle}$ is A, and the k^{th} column of $D_{\langle e,m \rangle}$ is B. Given $f \in 2^\omega$, let f_k be the function $f_k(x) = f(\langle k, x \rangle)$. If f separates $C_{\langle e,m \rangle}$ from $D_{\langle e,m \rangle}$, then f_k separates A from B. Thus the functional $f \mapsto f_k$ witnesses $\mathcal{X}_{\langle e,m \rangle} \geq_{\mathrm{s}} \mathcal{Q}$. The functional $f \mapsto g$ where $g(\langle i, x \rangle) = f(x)$ always witnesses $\mathcal{Q} \geq_{\mathrm{s}} \mathcal{X}_{\langle e,m \rangle}$. Hence $\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m \rangle}) = \deg_{\mathrm{s}}(\mathcal{Q})$. □

Lemma 2.10.5. *If $\mathcal{L}$ is a presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice, then $\mathbf{0}''' \leq_{\mathrm{T}} \deg_{\mathrm{T}}(\mathcal{L})$.*

Proof. Let $\mathcal{L} = (\omega; \leq_{\mathcal{L}}, +_{\mathcal{L}}, \times_{\mathcal{L}})$ be a presentation of $\mathcal{E}_{\mathrm{s}}$. Let $f\colon \mathcal{E}_{\mathrm{s}} \to \mathcal{L}$ be an isomorphism. Fix a Σ^0_3-complete set $C \subseteq \omega$. We show how to compute C from $\deg_{\mathrm{T}}(\mathcal{L})$.

By Lemma 2.8.8, let $\mathcal{Q}$ be an r.e. separating class and let $\{\mathcal{S}_{0,n}\}_{n\in\omega}$ and $\{\mathcal{S}_{1,n}\}_{n\in\omega}$ be recursive sequences of r.e. separating classes such that $\{\mathcal{Q}\} \cup \{\mathcal{S}_{0,n}\}_{n\in\omega} \cup \{\mathcal{S}_{1,n}\}_{n\in\omega}$ is strongly independent. Then let

$$\begin{aligned}
\mathbf{w}_0 &= \deg_{\mathrm{s}}(\mathcal{W}_0) && \text{for} && \mathcal{W}_0 = \operatorname{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n}\}_{n\in\omega}),\\
\mathbf{w}_1 &= \deg_{\mathrm{s}}(\mathcal{W}_1) && \text{for} && \mathcal{W}_1 = \operatorname{spine}(\mathcal{Q}, \{\mathcal{S}_{1,n}\}_{n\in\omega}),\\
\mathbf{v} &= \deg_{\mathrm{s}}(\mathcal{V}) && \text{for} && \mathcal{V} = \sum_{n\in\omega} \mathcal{S}_{0,n},\\
\mathbf{r} &= \deg_{\mathrm{s}}(\mathcal{R}) && \text{for} && \mathcal{R} = \operatorname{spine}(\mathcal{Q}, \{\mathcal{R}_n\}_{n\in\omega}), \text{ where } \mathcal{R}_n = \sum_{m\in\omega\setminus\{n\}} \mathcal{S}_{0,m},\\
\mathbf{m} &= \deg_{\mathrm{s}}(\mathcal{M}) && \text{for} && \mathcal{M} = \operatorname{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n} + \mathcal{S}_{1,n}\}_{n\in\omega}), \text{ and}\\
\mathbf{p} &= \deg_{\mathrm{s}}(\mathcal{P}) && \text{for} && \mathcal{P} = \operatorname{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n} + \mathcal{S}_{1,n+1}\}_{n\in\omega}).
\end{aligned}$$

Let $\{\mathcal{Z}_e\}_{e\in\omega}$ be a recursive sequence containing all Π^0_1 classes as in Lemma 1.4.4. Let $D \subseteq \omega$ be the set

$$D = \{e \mid \exists n(n \in C \wedge \mathcal{Z}_e \leq_{\mathrm{s}} \mathcal{S}_{0,n} \wedge \mathcal{V} \leq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{R}_n)\}.$$

D is Σ^0_3 because C is Σ^0_3, the sequences $\{\mathcal{Z}_e\}_{e\in\omega}$, $\{\mathcal{S}_{0,n}\}_{n\in\omega}$, and $\{\mathcal{R}_n\}_{n\in\omega}$ are recursive, and $\leq_{\mathrm{s}}$ is Σ^0_3 by Lemma 1.4.3. Let $\varphi(e,m,k,\ell)$ be a recursive predicate such that $D = \{e \mid \exists m\forall k\exists\ell\varphi(e,m,k,\ell)\}$. By Lemma 2.10.4, let $\{\mathcal{X}_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ be a recursive sequence of Π^0_1 classes such that for all $e,m\in\omega$

$$\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m\rangle}) = \begin{cases}\mathbf{0} & \text{if } \forall k\exists\ell\varphi(e,m,k,\ell)\\ \deg_{\mathrm{s}}(\mathcal{Q}) & \text{if } \exists k\forall\ell\neg\varphi(e,m,k,\ell).\end{cases}$$

Let $\mathbf{x} = \deg_{\mathrm{s}}(\mathcal{X})$ for $\mathcal{X} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{Z}_e + \mathcal{X}_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega})$.

The procedure for determining whether $n\in C$ from $\deg_{\mathrm{T}}(\mathcal{L})$ uses the fixed parameters $f(\mathbf{w}_0)$, $f(\mathbf{w}_1)$, $f(\mathbf{v})$, $f(\mathbf{r})$, $f(\mathbf{m})$, $f(\mathbf{p})$, $f(\deg_{\mathrm{s}}(\mathcal{S}_{0,0}))$, and $f(\mathbf{x})$. Given $n\in\omega$ search $\mathcal{L}$ for elements $a_{i,j}$ for $i<2$ and $1\leq j\leq n$ and for an element b satisfying the conditions

(i) $a_{i,j}$ meets to $f(\mathbf{w}_i)$ for all $i<2$ and all $1\leq j\leq n$,

(ii) $a_{0,j} +_{\mathcal{L}} a_{1,j} \geq_{\mathcal{L}} f(\mathbf{m})$ for all $1\leq j\leq n$,

(iii) $a_{0,j} +_{\mathcal{L}} a_{1,j+1} \geq_{\mathcal{L}} f(\mathbf{p})$ for all $0\leq j\leq n-1$ (where $a_{0,0} = f(\deg_{\mathrm{s}}(\mathcal{S}_{0,0}))$),

(iv) b meets to $f(\mathbf{r})$, and

(v) $a_{0,n} +_{\mathcal{L}} b \geq_{\mathcal{L}} f(\mathbf{v})$.

When the search is completed, output "yes" if $f(\mathbf{x}) \leq_{\mathcal{L}} a_{0,n}$ and output "no" otherwise.

First, observe that the above search is recursive in $\deg_{\mathrm{T}}(\mathcal{L})$ because the "meets to" relation is r.e. in $\deg_{\mathrm{T}}(\mathcal{L})$. Furthermore, the search will always terminate because the elements $a_{i,j} = f(\deg_{\mathrm{s}}(\mathcal{S}_{i,j}))$ for all $i<2$ and all $1\leq j\leq n$ and the element $b = f(\deg_{\mathrm{s}}(\mathcal{R}_n))$ satisfy conditions (i)–(v), and the search will eventually find them. Conditions (i) and (iv) follow from Lemma 2.9.2 item (ii). Notice that $\mathcal{Q}$ and $\{\mathcal{R}_j\}_{j\in\omega}$ satisfy the hypothesis of Lemma 2.9.2 because $\{\mathcal{Q}\}\cup\{\mathcal{S}_{0,j}\}_{j\in\omega}$ is strongly independent. Conditions (ii) and (iii) are easy to see. For condition (v), it is also easy to see that $\mathcal{S}_{0,n} + \mathcal{R}_n \equiv_{\mathrm{s}} \mathcal{V}$.

We need to show that the procedure outputs "yes" on input n if and only if $n\in C$. Let $a_{i,j}$ for $i<2$ and $1\leq j\leq n$ and b be the elements found in the search performed on input n.

Claim. *For all $i<2$ and all $1\leq j\leq n$, $a_{i,j} \leq_{\mathcal{L}} f(\deg_{\mathrm{s}}(\mathcal{S}_{i,j}))$.*

Proof of claim. For each $i<2$ and each $1\leq j\leq n$, let $\mathcal{A}_{i,j}$ be a Π^0_1 class such that $\deg_{\mathrm{s}}(\mathcal{A}_{i,j}) = f^{-1}(a_{i,j})$. By condition (i) of the search and Lemma 2.9.2 item (i), $\mathcal{A}_{0,1} \leq_{\mathrm{s}} \mathcal{S}_{0,m}$ and $\mathcal{A}_{1,1} \leq_{\mathrm{s}} \mathcal{S}_{1,k}$ for some $m,k\in\omega$. Condition (iii) implies that $\mathcal{S}_{0,0} + \mathcal{S}_{1,k} \geq_{\mathrm{s}} \mathcal{P}$, which is false by strong independence unless $k=1$. So $\mathcal{A}_{1,1} \leq_{\mathrm{s}} \mathcal{S}_{1,1}$. Knowing this, condition (ii) implies that $\mathcal{S}_{0,m} + \mathcal{S}_{1,1} \geq_{\mathrm{s}} \mathcal{M}$, which is false by strong independence unless $m=1$. So $\mathcal{A}_{0,1} \leq_{\mathrm{s}} \mathcal{S}_{0,1}$. Now proceed by induction. Let $1\leq j<n$ and assume that $\mathcal{A}_{0,j} \leq_{\mathrm{s}} \mathcal{S}_{0,j}$ and that $\mathcal{A}_{1,j} \leq_{\mathrm{s}} S_{1,j}$. Just as in the argument for the base case, $\mathcal{A}_{0,j+1} \leq_{\mathrm{s}} \mathcal{S}_{0,m}$ and $\mathcal{A}_{1,j+1} \leq_{\mathrm{s}} \mathcal{S}_{1,k}$ for some $m,k\in\omega$. $\mathcal{S}_{0,j} + \mathcal{S}_{1,k} \geq_{\mathrm{s}} \mathcal{P}$ by condition (iii), which implies that $k = j+1$. $\mathcal{S}_{0,m} + \mathcal{S}_{1,j+1} \geq_{\mathrm{s}} \mathcal{M}$ by condition (ii), which implies that $m = j+1$. □

At the end of the search, $a_{0,n} \leq_{\mathcal{L}} f(\deg_{\mathrm{s}}(\mathcal{S}_{0,n}))$ by the claim, b meets to $f(\mathbf{r})$ by condition (iv), and $a_{0,n} +_{\mathcal{L}} b \geq_{\mathcal{L}} f(\mathbf{v})$ by condition (v). By Lemma 2.9.2 item (ii), $b \leq_{\mathcal{L}} f(\deg_{\mathrm{s}}(\mathcal{R}_m))$ for some $m \in \omega$. However, if $m \neq n$, then $\mathcal{S}_{0,n} \leq_{\mathrm{s}} \mathcal{R}_m$, in which case $\mathcal{S}_{0,n} + \mathcal{R}_m \equiv_{\mathrm{s}} \mathcal{R}_m \not\geq_{\mathrm{s}} \mathcal{V}$. Thus it must be that $b \leq_{\mathcal{L}} f(\deg_{\mathrm{s}}(\mathcal{R}_n))$.

Suppose $n \in C$. Since $\{\mathcal{Z}_e\}_{e \in \omega}$ lists all the Π^0_1 classes, there is an $e \in \omega$ such that $\deg_{\mathrm{s}}(\mathcal{Z}_e) = f^{-1}(a_{0,n})$. This e satisfies $\exists n(n \in C \wedge \mathcal{Z}_e \leq_{\mathrm{s}} \mathcal{S}_{0,n} \wedge \mathcal{V} \leq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{R}_n)$. Thus $e \in D$, which means $\exists m \forall k \exists \ell \varphi(e, m, k, \ell)$. If m witnesses this property for e, then $\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m \rangle}) = \mathbf{0}$ and $\mathcal{Z}_e + X_{\langle e,m \rangle} \equiv_{\mathrm{s}} \mathcal{Z}_e$. Thus $\mathcal{X} \leq_{\mathrm{s}} \mathcal{Z}_e$, which means $f(\mathbf{x}) \leq_{\mathcal{L}} a_{0,n}$. Thus "yes" was the output.

Suppose $n \notin C$. We show $\mathcal{X} \not\leq_{\mathrm{s}} \mathcal{S}_{0,n}$.

Claim. *For all $e, m \in \omega$, $\mathcal{S}_{0,n} \not\geq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle}$.*

Proof of claim. If $\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m \rangle}) = \deg_{\mathrm{s}}(\mathcal{Q})$, then $\mathcal{S}_{0,n} \not\geq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle}$ because $\mathcal{S}_{0,n} \not\geq_{\mathrm{s}} \mathcal{Q}$ by strong independence. If $\deg_{\mathrm{s}}(\mathcal{X}_{\langle e,m \rangle}) = \mathbf{0}$, then $\forall k \exists \ell \varphi(e, m, k, \ell)$. Therefore $e \in D$, so there is an n' such that $n' \in C$, $\mathcal{Z}_e \leq_{\mathrm{s}} \mathcal{S}_{0,n'}$, and $\mathcal{V} \leq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{R}_{n'}$. Notice that $n \neq n'$ because $n \notin C$ and $n' \in C$. Therefore $\mathcal{S}_{0,n} \leq_{\mathrm{s}} \mathcal{R}_{n'}$. So if $\mathcal{S}_{0,n} \geq_{\mathrm{s}} \mathcal{Z}_e$, then $\mathcal{R}_{n'} \geq_{\mathrm{s}} \mathcal{Z}_e$. So $\mathcal{V} \not\leq_{\mathrm{s}} \mathcal{R}_{n'} \equiv_{\mathrm{s}} \mathcal{Z}_e + \mathcal{R}_{n'}$, a contradiction. Hence $\mathcal{S}_{0,n} \not\geq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle}$. ☐

Suppose for a contradiction that Φ is such that $\Phi(\mathcal{S}_{0,n}) \subseteq \mathcal{X}$. If there are $n, m \in \omega$ such that $\Phi(\mathcal{S}_{0,n}) \cap (\sigma_{\langle e,m \rangle}{}^\frown(\mathcal{Z}_e + X_{\langle e,m \rangle})) \neq \emptyset$, then there is a clopen $\mathcal{C} \subseteq 2^\omega$ such that $\mathcal{S}_{0,n} \cap \mathcal{C} \neq \emptyset$ and $\Phi(\mathcal{S}_{0,n} \cap \mathcal{C}) \subseteq \sigma_{\langle e,m \rangle}{}^\frown(\mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle})$. $\mathcal{S}_{0,n} \equiv_{\mathrm{s}} \mathcal{S}_{0,n} \cap \mathcal{C}$ by Lemma 2.8.3, and $\mathcal{S}_{0,n} \cap \mathcal{C} \geq_{\mathrm{s}} (\sigma_{\langle e,m \rangle}{}^\frown(\mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle})) \equiv_{\mathrm{s}} \mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle}$. This contradicts the claim. Thus $\Phi(\mathcal{S}_{0,n}) \cap (\sigma_{\langle e,m \rangle}{}^\frown(\mathcal{Z}_e + \mathcal{X}_{\langle e,m \rangle})) = \emptyset$ for all $e, m \in \omega$. Therefore $\Phi(\mathcal{S}_{0,n}) \subseteq \mathcal{Q}$. This contradicts $\mathcal{Q} \not\leq_{\mathrm{s}} \mathcal{S}_{0,n}$. Hence $\mathcal{X} \not\leq_{\mathrm{s}} \mathcal{S}_{0,n}$. It follows that $f(\mathbf{x}) \not\leq_{\mathcal{L}} a_{0,n}$ because $a_{0,n} \leq_{\mathcal{L}} f(\deg_{\mathrm{s}}(\mathcal{S}_{0,n}))$. Thus "no" was the output. ☐

Theorem 2.10.6. *The degree of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\mathbf{0}'''$. That is, there is a presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice recursive in $\mathbf{0}'''$ and $\mathbf{0}'''$ is recursive in every presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice.*

Proof. Lemma 2.10.3 proves that there is a presentation recursive in $\mathbf{0}'''$, and Lemma 2.10.5 proves that $\mathbf{0}'''$ is recursive in every presentation. ☐

Corollary 2.10.7. *$\mathcal{E}_{\mathrm{s}}$ has no recursive presentation as a partial order.*

Proof. In any lattice, the relations $x + y = z$ and $x \times y = z$ are definable from the partial order by Π^0_1 formulas. Thus if $\mathcal{E}_{\mathrm{s}}$ had a recursive presentation as a partial order, it would have a presentation as a lattice recursive in $\mathbf{0}'$. This contradicts the theorem. ☐

2.11 The undecidability of $\mathrm{Th}(\mathcal{E}_{\mathrm{w}})$ and $\mathrm{Th}(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$

In this section, we code $\mathcal{N}^{\dot-}$ in $\mathcal{E}_{\mathrm{w}}$, thereby showing that Σ^0_3-$\mathrm{Th}(\mathcal{E}_{\mathrm{w}})$ and Σ^0_4-$\mathrm{Th}(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$ are undecidable. In place of separating classes, our coding of $\mathcal{N}^{\dot-}$ in

$\mathcal{E}_\mathrm{w}$ uses Simpson's Σ^0_3 embedding lemma and his embedding of $\mathcal{E}_\mathrm{T}$ into $\mathcal{E}_\mathrm{w}$. A set $\mathcal{X} \subseteq \omega^\omega$ is Σ^0_3 if and only if it is of the form $\mathcal{X} = \{f \in \omega^\omega \mid \exists m \forall k \exists \ell \varphi(f, m, k, \ell)\}$ for some recursive predicate φ.

Lemma 2.11.1 (Σ^0_3 *embedding lemma* [66] Lemma 3.3)**.** *Let $\mathcal{S} \subseteq \omega^\omega$ be Σ^0_3 and let $\mathcal{P} \subseteq 2^\omega$ be a Π^0_1 class. Then there is a Π^0_1 class $\mathcal{Q} \subseteq 2^\omega$ such that $\mathcal{Q} \equiv_\mathrm{w} S \cup P$.*

In the Muchnik case, $\deg_\mathrm{w}(\mathcal{S}) \times \deg_\mathrm{w}(\mathcal{P}) = \deg_\mathrm{w}(\mathcal{S} \cup \mathcal{P})$ for any $\mathcal{S}, \mathcal{P} \subseteq \omega^\omega$. For this reason, the Σ^0_3 embedding lemma may be phrased as "if $\mathcal{S}$ is Σ^0_3 and $\mathcal{P}$ is a Π^0_1 class then $\deg_\mathrm{w}(\mathcal{S}) \times \deg_\mathrm{w}(\mathcal{P}) \in \mathcal{E}_\mathrm{w}$." For our purposes, $\mathcal{P}$ is always DNR_2, so $\deg_\mathrm{w}(\mathcal{P}) = \deg_\mathrm{w}(\mathrm{DNR}_2) = \mathbf{1}$, the greatest element of $\mathcal{E}_\mathrm{w}$.

If A is an r.e. set, then $\{A\}$ is a Σ^0_3 (in fact a Π^0_2) subset of 2^ω. One of Simpson's original applications of his Σ^0_3 embedding lemma is to show that the map $\deg_\mathrm{T}(A) \mapsto \deg_\mathrm{w}(\{A\}) \times \mathbf{1}$ is an upper-semilattice embedding of $\mathcal{E}_\mathrm{T}$ into $\mathcal{E}_\mathrm{w}$ preserving the least and greatest elements [66]. To show that this map is indeed an embedding, Simpson uses the following variant of the Arslanov completeness criterion, which we also employ.

Lemma 2.11.2 (see [31] Lemma 4.1 and [69] Theorem V.5.1)**.** *If A is an r.e. set, then $\mathrm{DNR}_2 \leq_\mathrm{w} \{A\}$ if and only if $A \equiv_\mathrm{T} 0'$.*

Proof. It is easy to compute a function in DNR_2 from $0'$. Conversely, if A computes a function in DNR_2, then A computes a function f such that $\forall e(W_{f(e)} \neq W_e)$, where here $\{W_e\}_{e \in \omega}$ is the standard enumeration of the r.e. sets (such an f is called *fixed-point free*; see [31] Lemma 4.1). Thus $A \equiv_\mathrm{T} 0'$ by the Arslanov completeness criterion (see [69] Theorem V.5.1). ☐

For comparison, it is not known whether $\mathcal{E}_\mathrm{T}$ embeds into $\mathcal{E}_\mathrm{s}$. See [11] for further results concerning embedding distributive lattices in $\mathcal{E}_\mathrm{s}$ and $\mathcal{E}_\mathrm{w}$.

For us, the key property of the degrees $\deg_\mathrm{w}(\{A\}) \times \mathbf{1}$ for r.e. sets A is that they are all meet-irreducible in $\mathcal{E}_\mathrm{w}$ (of course these degrees are generally meet-reducible in $\mathcal{D}_\mathrm{w}$).

Lemma 2.11.3. *If A is an r.e. set, then $\deg_\mathrm{w}(\{A\}) \times \mathbf{1}$ is meet-irreducible in $\mathcal{E}_\mathrm{w}$.*

Proof. Suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{E}_\mathrm{w}$ are such that $\deg_\mathrm{w}(\{A\}) \times \mathbf{1} \geq_\mathrm{w} \mathbf{x} \times \mathbf{y}$. Either $\deg_\mathrm{w}(\{A\}) \geq_\mathrm{w} \mathbf{x}$ or $\deg_\mathrm{w}(\{A\}) \geq_\mathrm{w} \mathbf{y}$ because $\deg_\mathrm{w}(\{A\})$ is the degree of a singleton. As $\mathbf{1} \geq_\mathrm{w} \mathbf{x}$ and $\mathbf{1} \geq_\mathrm{w} \mathbf{y}$, either $\deg_\mathrm{w}(\{A\}) \times \mathbf{1} \geq_\mathrm{w} \mathbf{x}$ or $\deg_\mathrm{w}(\{A\}) \times \mathbf{1} \geq_\mathrm{w} \mathbf{y}$. ☐

If $\{A_n\}_{n \in B}$ is a uniformly r.e. sequence of r.e. sets indexed by a recursive set B (i.e., the set $\{\langle n, m \rangle \mid n \in B \wedge m \in A_n\}$ is r.e.), then $\{A_n\}_{n \in B}$ is a Σ^0_3 subset of 2^ω and $\deg_\mathrm{w}(\{A_n\}_{n \in B}) \times \mathbf{1} \in \mathcal{E}_\mathrm{w}$. In place of Lemma 2.8.8, we use the following simpler fact.

Lemma 2.11.4 (see [69] Section VII.2)**.** *There is a uniformly r.e. sequence of r.e. sets $\{A_n\}_{n \in \omega}$ that is strongly independent.*

Notice that Lemma 2.11.4 is also a consequence of Lemma 2.8.8. If $\{\mathcal{S}(A_n, B_n)\}_{n \in \omega}$ is a recursive sequence of r.e. separating classes that is strongly independent, then $\{A_n\}_{n\in\omega}$ and $\{B_n\}_{n\in\omega}$ are both uniformly r.e. sequences of r.e. sets that are strongly independent.

Lemma 2.11.5. *Let $\{A_n\}_{n\in B}$ be an infinite uniformly r.e. sequence of r.e. sets (indexed by a recursive set B) that is a $\leq_{\mathrm{T}}$-antichain. Let $\mathbf{w} = \deg_{\mathrm{w}}(\{A_n\}_{n\in B}) \times \mathbf{1}$.*

(i) If $\mathbf{x} \in \mathcal{E}_{\mathrm{w}}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}$ for some $n \in B$.

(ii) $E(\mathbf{w}) = \{\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1} \mid n \in B\}$.

Proof. (i) Let $\mathbf{x} \in \mathcal{E}_{\mathrm{w}}$ be such that $\mathbf{x}$ meets to $\mathbf{w}$, and suppose that $\mathbf{x} \nleq_{\mathrm{w}} \deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}$ for all $n \in B$ for a contradiction. Since $\mathbf{x} \leq_{\mathrm{w}} \mathbf{1}$, it must be that $\mathbf{x} \nleq_{\mathrm{w}} \deg_{\mathrm{w}}(\{A_n\})$ for all $n \in B$. Let $\mathbf{y} \in \mathcal{E}_{\mathrm{w}}$ witness that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $\mathbf{y} >_{\mathrm{w}} \mathbf{w}$ and $\mathbf{x} \times \mathbf{y} = \mathbf{w}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be Π^0_1 classes such that $\mathbf{x} = \deg_{\mathrm{w}}(\mathcal{X})$ and $\mathbf{y} = \deg_{\mathrm{w}}(\mathcal{Y})$. Then $\mathcal{X} \cup \mathcal{Y} \leq_{\mathrm{w}} \{A_n\}$ for all $n \in B$. Thus $\mathcal{Y} \leq_{\mathrm{w}} \{A_n\}$ for all $n \in B$ because $\mathcal{X} \nleq_{\mathrm{w}} \{A_n\}$ for all $n \in B$. Therefore $\mathcal{Y} \leq_{\mathrm{w}} \{A_n\}_{n\in B}$, which implies that $\mathbf{y} \leq_{\mathrm{w}} \mathbf{w}$, a contradiction.

(ii) Let $n \in B$. To see that $\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1} \in E(\mathbf{w})$, let $\mathbf{y} = \deg_{\mathrm{w}}(\{A_i\}_{i\in B\setminus\{n\}}) \times \mathbf{1}$. It is easy to check that $(\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}) \times \mathbf{y} = \mathbf{w}$. Moreover, $\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1} \ngeq_{\mathrm{w}} \mathbf{y}$. This is because $\{A_n\} \ngeq_{\mathrm{w}} \{A_i\}_{i\in B\setminus\{n\}}$ as $\{A_i\}_{i\in B}$ is a $\leq_{\mathrm{T}}$-antichain and because $\{A_n\} \ngeq_{\mathrm{w}} \mathrm{DNR}_2$ by Lemma 2.11.2 (note that $A_n <_{\mathrm{T}} 0'$ because $\{A_i\}_{i\in B}$ is a $\leq_{\mathrm{T}}$-antichain). Thus $\mathbf{y} >_{\mathrm{w}} \mathbf{w}$, and therefore $\mathbf{y}$ witnesses that $\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}$ meets to $\mathbf{w}$. The degree $\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}$ is meet-irreducible in $\mathcal{E}_{\mathrm{w}}$ by Lemma 2.11.3. Thus $\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1} \in E(\mathbf{w})$.

We have shown that $\{\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1} \mid n \in B\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_{\mathrm{w}} \deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}$ for some $n \in B$. $E(\mathbf{w})$ is an antichain by Lemma 2.2.3 and $\deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1} \in E(\mathbf{w})$, so it must be that $\mathbf{x} = \deg_{\mathrm{w}}(\{A_n\}) \times \mathbf{1}$. ☐

We are now able to code $\mathcal{N}^{\dot-}$ in $\mathcal{E}_{\mathrm{w}}$.

Lemma 2.11.6. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{w}}$ such that $\mathcal{M}^{\dot-}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot-}$.*

Proof. By Lemma 2.11.4, let $\{A_{0,n}\}_{n\in\omega}$, $\{A_{1,n}\}_{n\in\omega}$, and $\{A_{2,n}\}_{n\in\omega}$ be uniformly r.e. sequences of r.e. sets such that $\{A_{0,n}\}_{n\in\omega} \cup \{A_{1,n}\}_{n\in\omega} \cup \{A_{2,n}\}_{n\in\omega}$ is strongly independent. Let

$$
\begin{aligned}
\mathbf{w}_0 &= \deg_{\mathrm{w}}(\mathcal{W}_0) \times \mathbf{1} && \text{for} && \mathcal{W}_0 = \{A_{0,n}\}_{n\in\omega},\\
\mathbf{w}_1 &= \deg_{\mathrm{w}}(\mathcal{W}_1) \times \mathbf{1} && \text{for} && \mathcal{W}_1 = \{A_{1,n}\}_{n\in\omega},\\
\mathbf{w}_2 &= \deg_{\mathrm{w}}(\mathcal{W}_2) \times \mathbf{1} && \text{for} && \mathcal{W}_2 = \{A_{2,n}\}_{n\in\omega},\\
\mathbf{m} &= \deg_{\mathrm{w}}(\mathcal{M}) \times \mathbf{1} && \text{for} && \mathcal{M} = \{A_{0,n} \oplus A_{1,n}\}_{n\in\omega} \cup \{A_{0,n} \oplus A_{2,n}\}_{n\in\omega},\\
\boldsymbol{\ell} &= \deg_{\mathrm{w}}(\mathcal{L}) \times \mathbf{1} && \text{for} && \mathcal{L} = \{A_{0,i} \oplus A_{1,j} \mid i < j\},\\
\mathbf{p} &= \deg_{\mathrm{w}}(\mathcal{P}) \times \mathbf{1} && \text{for} && \mathcal{P} = \{A_{0,i} \oplus A_{1,j} \oplus A_{2,k} \mid i + j = k\},\\
\mathbf{t} &= \deg_{\mathrm{w}}(\mathcal{T}) \times \mathbf{1} && \text{for} && \mathcal{T} = \{A_{0,i} \oplus A_{1,j} \oplus A_{2,k} \mid i \times j = k\},\\
\mathbf{d} &= \deg_{\mathrm{w}}(\mathcal{D}) \times \mathbf{1} && \text{for} && \mathcal{D} = \{A_{0,i} \oplus A_{1,j} \oplus A_{2,k} \mid i \dot- j = k\},\\
\mathbf{z} &= \deg_{\mathrm{w}}(\{A_{0,0}\}) \times \mathbf{1}, && \text{and} &&\\
\mathbf{o} &= \deg_{\mathrm{w}}(\{A_{0,1}\}) \times \mathbf{1}\,. &&&&
\end{aligned}
$$

To aid readability, let $\mathbf{a}_{i,j} = \deg_{\mathrm{w}}(\{A_{i,j}\})$ for all $i < 3$ and $j \in \omega$. By Lemma 2.11.5 item (ii), $E(\mathbf{w_0}) = \{\mathbf{a}_{0,n} \times \mathbf{1}\}_{n\in\omega}$. The map $\mathbf{a}_{0,n} \times \mathbf{1} \mapsto n$ is the isomorphism witnessing $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. Clearly $\mathbf{z} \mapsto 0$ and $\mathbf{o} \mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, $\times$, and $\dot{-}$ are similar. Let $i, j \in \omega$. If $i < j$, then $\mathbf{a}_{1,j} \times \mathbf{1}$ meets to $\mathbf{w}_1$ by Lemma 2.11.5 item (ii), and by distributivity

$$\begin{aligned}(\mathbf{a}_{0,j} \times \mathbf{1}) + (\mathbf{a}_{1,j} \times \mathbf{1}) &= (\mathbf{a}_{0,j} + \mathbf{a}_{1,j}) \times \mathbf{1}\\ &= \deg_{\mathrm{w}}(\{A_{0,j} \oplus A_{1,j}\}) \times \mathbf{1}\\ &\geq_{\mathrm{w}} \mathbf{m}, \text{ and}\\ (\mathbf{a}_{0,i} \times \mathbf{1}) + (\mathbf{a}_{1,j} \times \mathbf{1}) &= (\mathbf{a}_{0,i} + \mathbf{a}_{1,j}) \times \mathbf{1}\\ &= \deg_{\mathrm{w}}(\{A_{0,i} \oplus A_{1,j}\}) \times \mathbf{1}\\ &\geq_{\mathrm{w}} \boldsymbol{\ell}.\end{aligned}$$

Thus $R^2_{\boldsymbol{\ell}}(\mathbf{a}_{0,i} \times \mathbf{1}, \mathbf{a}_{0,j} \times \mathbf{1})$. Conversely, suppose that $R^2_{\boldsymbol{\ell}}(\mathbf{a}_{0,i} \times \mathbf{1}, \mathbf{a}_{0,j} \times \mathbf{1})$. Let $\mathbf{u}_1 \in \mathcal{E}_{\mathrm{w}}$ be such that $\mathbf{u}_1$ meets to $\mathbf{w}_1$, $(\mathbf{a}_{0,j} \times \mathbf{1}) + \mathbf{u}_1 \geq_{\mathrm{w}} \mathbf{m}$, and $(\mathbf{a}_{0,i} \times \mathbf{1}) + \mathbf{u}_1 \geq_{\mathrm{w}} \boldsymbol{\ell}$. Since $\mathbf{u}_1$ meets to $\mathbf{w}_1$, it must be that $\mathbf{u}_1 \leq_{\mathrm{w}} \mathbf{a}_{1,k} \times \mathbf{1}$ for some $k \in \omega$ by Lemma 2.11.5 item (i). Thus $(\mathbf{a}_{0,j} \times \mathbf{1}) + (\mathbf{a}_{1,k} \times \mathbf{1}) \geq_{\mathrm{w}} \mathbf{m}$, so by distributivity

$$\deg_{\mathrm{w}}(\{A_{0,j} \oplus A_{1,k}\}) \times \mathbf{1} = (\mathbf{a}_{0,j} + \mathbf{a}_{1,k}) \times \mathbf{1} = (\mathbf{a}_{0,j} \times \mathbf{1}) + (\mathbf{a}_{1,k} \times \mathbf{1}) \geq_{\mathrm{w}} \mathbf{m}.$$

However, if $k \neq j$, then $\{A_{0,j} \oplus A_{1,k}\} \not\geq_{\mathrm{w}} \mathcal{M}$ by strong independence and $\{A_{0,j} \oplus A_{1,k}\} \not\geq_{\mathrm{w}} \mathrm{DNR}_2$ by Lemma 2.11.2. This implies that $\deg_{\mathrm{w}}(\{A_{0,j} \oplus A_{1,k}\}) \times \mathbf{1} \not\geq_{\mathrm{w}} \mathbf{m}$, so it must be that $k = j$. Thus $\mathbf{u}_1 \leq_{\mathrm{w}} \mathbf{a}_{1,j} \times \mathbf{1}$, which implies that $(\mathbf{a}_{0,i} \times \mathbf{1}) + (\mathbf{a}_{1,j} \times \mathbf{1}) \geq_{\mathrm{w}} \boldsymbol{\ell}$. Then

$$\deg_{\mathrm{w}}(\{A_{0,i} \oplus A_{1,j}\}) \times \mathbf{1} = (\mathbf{a}_{0,i} + \mathbf{a}_{1,j}) \times \mathbf{1} = (\mathbf{a}_{0,i} \times \mathbf{1}) + (\mathbf{a}_{1,j} \times \mathbf{1}) \geq_{\mathrm{w}} \boldsymbol{\ell}.$$

So if $i \not< j$, then $\{A_{0,i} \oplus A_{1,j}\} \not\geq_{\mathrm{w}} \mathcal{L}$ by strong independence and $\{A_{0,i} \oplus A_{1,j}\} \not\geq_{\mathrm{w}} \mathrm{DNR}_2$ by Lemma 2.11.2, giving the contradiction $\deg_{\mathrm{w}}(\{A_{0,i} \oplus A_{1,j}\}) \times \mathbf{1} \not\geq_{\mathrm{w}} \boldsymbol{\ell}$. Hence $i < j$. ☐

Theorem 2.11.7. *Σ^0_3-$\mathrm{Th}(\mathcal{E}_{\mathrm{w}})$ and Σ^0_4-$\mathrm{Th}(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$ are undecidable.*

Proof. There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{w}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ by Lemma 2.11.6. The results then follow from Lemma 2.2.25. ☐

Clearly then $\mathrm{Th}(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$ is undecidable. Unfortunately we do not yet know how to prove anything like the finite matching property for $\mathcal{E}_{\mathrm{w}}$ to obtain $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$. The proof of the finite matching property for $\mathcal{E}_{\mathrm{s}}$ (Lemma 2.8.7) appeals to a lemma of Cole and Kihara that grew out of Cenzer and Hinman's proof that $\mathcal{E}_{\mathrm{s}}$ is dense. By analogy, perhaps progress must be made on the density of $\mathcal{E}_{\mathrm{w}}$ before further progress is made on the complexity of $\mathrm{Th}(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$.

CHAPTER 3
JOIN-IRREDUCIBLES AND PROPOSITIONAL LOGICS IN THE MEDVEDEV DEGREES

The results of this chapter also appear in [56] by the author.

We present solutions to three problems concerning the Medvedev degrees. First, Dyment characterized the meet-reducible elements of $\mathcal{D}_{\mathrm{s}}$ in the following theorem. Its corollary helps identify meet-irreducible Medvedev degrees.

Theorem 3.0.8 ([22])**.** *A Medvedev degree* $\mathbf{a}$ *is meet-reducible if and only if* $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$ *for a mass problem* $\mathcal{A}$ *for which there are r.e. sets* $V_0, V_1 \subseteq \omega^{<\omega}$ *such that*

- $(\forall f \in \mathcal{A})(\exists \sigma \in V_0 \cup V_1)(\sigma \subset f)$,
- *The following mass problems are* $\leq_{\mathrm{s}}$*-incomparable:*

$$\{f \in \mathcal{A} \mid (\exists \sigma \in V_0)(\sigma \subset f)\} \text{ and } \{f \in \mathcal{A} \mid (\exists \sigma \in V_1)(\sigma \subset f)\}$$

Corollary 3.0.9 ([22])**.** *If* $\mathcal{A}$ *is a mass problem such that* $\sigma^\frown\mathcal{A} \subseteq \mathcal{A}$ *for all* $\sigma \in \omega^{<\omega}$, *then* $\deg_{\mathrm{s}}(\mathcal{A})$ *is meet-irreducible.*

In particular, $\mathbf{0}'$ is meet-irreducible because $\sigma^\frown f >_{\mathrm{T}} 0$ whenever $\sigma \in \omega^{<\omega}$ and $f >_{\mathrm{T}} 0$.

The problem of characterizing the join-irreducible elements of $\mathcal{D}_{\mathrm{s}}$ was posed in [74]. We prove that $\mathbf{a} \in \mathcal{D}_{\mathrm{s}}$ is join-irreducible if and only if $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ for some Turing ideal $\mathcal{I}$ (Theorem 3.1.3).

Second, providing semantics for propositional logic was one of Medvedev's main motivations behind introducing $\mathcal{D}_{\mathrm{s}}$, and he proved $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}) = \mathrm{JAN}$ in Medvedev [42]. JAN denotes the logic $\mathrm{IPC} + \neg p \vee \neg\neg p$ named after Jankov who studied it in Jankov [30]. In any Brouwer algebra $\mathcal{B}$, the quotient of $\mathcal{B}$ by the principal filter generated by $\mathbf{a} \in \mathcal{B}$ is denoted $\mathcal{B}/\mathbf{a}$. The quotient $\mathcal{B}/\mathbf{a}$ is isomorphic to the interval $[\mathbf{0}, \mathbf{a}]$ which is a Brouwer algebra under the operations inherited from $\mathcal{B}$. Logics of the form $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a})$ have been studied in Skvortsova [68][1], Sorbi [73], and Sorbi and Terwijn [75]. Our work is motivated by the following question which remains open.

Question 3.0.10 ([75])**.** Is $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) \subseteq \mathrm{JAN}$ for all $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$?

Sorbi and Terwijn's study of Question 3.0.10 in [75] lead them to ask whether every degree $>_{\mathrm{s}} \mathbf{0}'$ bounds a join-irreducible degree $>_{\mathrm{s}} \mathbf{0}'$ because a "yes" answer to this question implies a "yes" answer to Question 3.0.10. However, Sorbi and Terwijn conjectured that there is a degree $>_{\mathrm{s}} \mathbf{0}'$ that bounds no join-irreducible degree $>_{\mathrm{s}} \mathbf{0}'$, and we prove that this is correct (Theorem 3.2.5). We also provide slight extensions to some of the results in [73], thereby widening the class of degrees $\mathbf{a}$ for which $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) \subseteq \mathrm{JAN}$ is known.

Third, we use techniques similar to those used to characterize the join-irreducible degrees to prove that the filter generated by the non-minimum degrees of closed mass problems is not prime (Theorem 3.4.3). This problem was posed in Bianchini and Sorbi [9] and in Sorbi [74].

[1]Skvortsova and Dyment are the same person. Dyment got married and became Skvortsova.

3.1 Characterizing the join-irreducible Medvedev degrees

A *Turing ideal* is a set $\mathcal{I} \subseteq \omega^\omega$ that is closed downward under $\leq_{\mathrm{T}}$ (i.e., $f \in \mathcal{I} \wedge g \leq_{\mathrm{T}} f \rightarrow g \in \mathcal{I}$) and closed under $\oplus$ (i.e., $f, g \in \mathcal{I} \rightarrow f \oplus g \in \mathcal{I}$). We prove that $\mathbf{a} \in \mathcal{D}_{\mathrm{s}}$ is join-irreducible if and only if $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ for some Turing ideal $\mathcal{I}$.

For a mass problem $\mathcal{A}$, let $C(\mathcal{A})$ denote the *Turing upward-closure* of $\mathcal{A}$: $C(\mathcal{A}) = \{f \mid (\exists g \in \mathcal{A})(f \geq_{\mathrm{T}} g)\}$. A mass problem $\mathcal{A}$ is called *Turing upward-closed* if $\mathcal{A} = C(\mathcal{A})$. The identity functional witnesses $C(\mathcal{A}) \leq_{\mathrm{s}} \mathcal{A}$ for any mass problem $\mathcal{A}$, and if $\mathcal{A}$ and $\mathcal{B}$ are mass problems such that $\mathcal{A}$ is Turing upward-closed, then $\mathcal{A} \leq_{\mathrm{s}} \mathcal{B}$ if and only if $\mathcal{B} \subseteq \mathcal{A}$. Our starting point is the following observation.

Lemma 3.1.1 ([74])**.** *If $\mathcal{A}$ is a mass problem such that $\deg_{\mathrm{s}}(\mathcal{A})$ is join-irreducible, then $\omega^\omega \setminus C(\mathcal{A})$ is a Turing ideal.*

Proof. We prove the contrapositive. If $\omega^\omega \setminus C(\mathcal{A})$ is not a Turing ideal, then there are $f, g \notin C(\mathcal{A})$ with $f \oplus g \in C(\mathcal{A})$. This means that $\{f\}, \{g\} \not\geq_{\mathrm{s}} \mathcal{A}$ but $\{f\} + \{g\} \geq_{\mathrm{s}} \mathcal{A}$. Thus $\deg_{\mathrm{s}}(\mathcal{A})$ is join-reducible. □

The next lemma is the main step in our characterization.

Lemma 3.1.2. *If $\mathcal{A}$ is a mass problem such that $\deg_{\mathrm{s}}(\mathcal{A})$ is join-irreducible, then $\mathcal{A} \equiv_{\mathrm{s}} C(\mathcal{A})$*

Proof. We prove the contrapositive. Suppose $\mathcal{A} \not\equiv_{\mathrm{s}} C(\mathcal{A})$. Then it must be that $\mathcal{A} \not\leq_{\mathrm{s}} C(\mathcal{A})$. We find mass problems $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}, \mathcal{Y} \not\geq_{\mathrm{s}} \mathcal{A}$ but $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{s}} \mathcal{A}$. Thus $\deg_{\mathrm{s}}(\mathcal{A})$ is join-reducible.

To find $\mathcal{X}$ and $\mathcal{Y}$, first find a sequence $(h_n \mid n \in \omega)$ of functions and a sequence $(e_n \mid n \in \omega)$ of indices such that

(i) $\Phi_{e_n}(h_n) \in \mathcal{A}$ for all $n \in \omega$,

(ii) $\Phi_n(h_{2n}) \notin \mathcal{A}$ and $\Phi_n(h_{2n+1}) \notin \mathcal{A}$ for all $n \in \omega$, and

(iii) $h_n(0) = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle$ for all $n \in \omega$.

We find the desired sequences by iterating the following claim.

Claim. *If $\mathcal{A} \not\leq_{\mathrm{s}} C(\mathcal{A})$, then for every $e, m \in \omega$ there is an $h \in C(\mathcal{A})$ such that $h(0) = m$ and $\Phi_e(h) \notin \mathcal{A}$.*

Proof of claim. Suppose not. Then there are $e, m \in \omega$ such that $h(0) = m$ implies $\Phi_e(h) \in \mathcal{A}$ for all $h \in C(\mathcal{A})$. Thus $h \mapsto \Phi_e(m^\frown h)$ is a reduction witnessing $\mathcal{A} \leq_{\mathrm{s}} C(\mathcal{A})$, a contradiction. □

Suppose we have h_i and e_i for all $i < n$. To find h_n and e_n, let $e = \lfloor n/2 \rfloor$ and let $m = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle$. By the claim, there is an $h_n \in C(\mathcal{A})$ such that $h_n(0) = m$ and $\Phi_e(h_n) \notin \mathcal{A}$. The fact that $h_n \in C(\mathcal{A})$ means that there is an e_n such that $\Phi_{e_n}(h_n) \in \mathcal{A}$.

Now set $\mathcal{X} = \{h_{2n} \mid n \in \omega\}$ and $\mathcal{Y} = \{h_{2n+1} \mid n \in \omega\}$. Then $\Phi_e(\mathcal{X}) \nsubseteq \mathcal{A}$ and $\Phi_e(\mathcal{Y}) \nsubseteq \mathcal{A}$ for each e by item (ii). Hence $\mathcal{X}, \mathcal{Y} \ngeq_{\mathrm{s}} \mathcal{A}$. The following reduction witnesses $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{s}} \mathcal{A}$.

Given h, decompose h as $h = f \oplus g$ and decode $f(0)$ and $g(0)$ as $f(0) = \langle 2n, x_0, x_1, \ldots, x_{2n-1} \rangle$ and $g(0) = \langle 2m+1, y_0, y_1, \ldots, y_{2m} \rangle$. If either $f(0)$ or $g(0)$ is not of the required form, then output the 0 function (as such an h cannot be in $\mathcal{X} + \mathcal{Y}$). Otherwise output $\Phi_{x_{2m+1}}(g)$ if $2n > 2m+1$ and output $\Phi_{y_{2n}}(f)$ if $2m+1 > 2n$.

Suppose this reduction is applied to some $h = h_{2n} \oplus h_{2m+1} \in \mathcal{X} + \mathcal{Y}$. In this case $f = h_{2n}$, $g = h_{2m+1}$, and $f(0)$ and $g(0)$ are of the required form by item (iii). So if $2n > 2m+1$ we output $\Phi_{e_{2m+1}}(h_{2m+1})$ and if $2m+1 > 2n$ we output $\Phi_{e_{2n}}(h_{2n})$. Both alternatives are in $\mathcal{A}$ by item (i). Thus $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{s}} \mathcal{A}$. $\square$

Theorem 3.1.3. *A Medvedev degree* $\mathbf{a}$ *is join-irreducible if and only if* $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ *for some Turing ideal* $\mathcal{I}$*.*

Proof. Suppose $\mathbf{a}$ is join-irreducible, and let $\mathcal{A}$ be a mass problem such that $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$. Then $\mathcal{I} = \omega^\omega \setminus C(\mathcal{A})$ is a Turing ideal by Lemma 3.1.1, $\mathcal{A} \equiv_{\mathrm{s}} C(\mathcal{A})$ by Lemma 3.1.2, and therefore $\mathcal{A} \equiv_{\mathrm{s}} C(\mathcal{A}) = \omega^\omega \setminus \mathcal{I}$. Hence $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ for the Turing ideal $\mathcal{I}$.

Conversely, suppose $\mathcal{I}$ is a Turing ideal and let $\mathcal{X}$ and $\mathcal{Y}$ be mass problems such that $\mathcal{X}, \mathcal{Y} \ngeq_{\mathrm{s}} \omega^\omega \setminus \mathcal{I}$. We show that $\mathcal{X} + \mathcal{Y} \ngeq_{\mathrm{s}} \omega^\omega \setminus \mathcal{I}$. Observe $\mathcal{X}, \mathcal{Y} \nsubseteq \omega^\omega \setminus \mathcal{I}$ for otherwise the identity functional would witness $\mathcal{X}, \mathcal{Y} \geq_{\mathrm{s}} \omega^\omega \setminus \mathcal{I}$. Let $f \in \mathcal{X} \cap \mathcal{I}$ and let $g \in \mathcal{Y} \cap \mathcal{I}$, thereby making $f \oplus g \in (\mathcal{X} + \mathcal{Y}) \cap \mathcal{I}$. The function $f \oplus g$ is in $\mathcal{X} + \mathcal{Y}$, but it does not compute any member of $\omega^\omega \setminus \mathcal{I}$. Therefore $\mathcal{X} + \mathcal{Y} \ngeq_{\mathrm{s}} \omega^\omega \setminus \mathcal{I}$. Hence $\deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ is join-irreducible. $\square$

Theorem 3.1.3 is also valid with $\mathcal{D}_{\mathrm{w}}$ in place of $\mathcal{D}_{\mathrm{s}}$, a fact first noticed by Terwijn [76]. The proof of Lemma 3.1.1 also works for $\mathcal{D}_{\mathrm{w}}$, and it is easy to check that $\mathcal{A} \equiv_{\mathrm{w}} C(\mathcal{A})$ for any mass problem $\mathcal{A}$ (i.e., the $\mathcal{D}_{\mathrm{w}}$ analogue of Lemma 3.1.2 is trivial). This gives the forward direction of Theorem 3.1.3 for $\mathcal{D}_{\mathrm{w}}$. The proof of the reverse direction of Theorem 3.1.3 also works for $\mathcal{D}_{\mathrm{w}}$.

3.2 Degrees that bound no join-irreducible degrees $>_{\mathrm{s}} \mathbf{0}'$

Recall that JAN is the intermediate logic $\mathrm{IPC} + \neg p \vee \neg\neg p$. The results of this section and the next are motivated by Question 3.0.10: is $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \subseteq \mathrm{JAN}$ for every $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$?

$\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{0}') = \mathrm{CPC}$ because $\mathcal{D}_{\mathrm{s}} / \mathbf{0}' \cong [\mathbf{0}, \mathbf{0}'] = \{\mathbf{0}, \mathbf{0}'\}$. In fact, $\mathbf{0}'$ is the only degree for which $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{0}') = \mathrm{CPC}$. This is because if $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$, then $\mathbf{0}' \to \mathbf{a} = \mathbf{a}$, hence $\mathbf{0}' \times (\mathbf{0}' \to \mathbf{a}) = \mathbf{0}'$. Thus let $p = \mathbf{0}'$ to see that the formula $p \vee \neg p$ is not valid in $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a})$.

Furthermore, if $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$, then we cannot have $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \supsetneq \mathrm{JAN}$. It is an easy check that in any Brouwer algebra $\mathcal{B}$ with meet-irreducible $\mathbf{0}$ (such as the algebras $\mathcal{D}_{\mathrm{s}}/\mathbf{a}$), $\neg p \vee \neg\neg p \in \mathrm{PTh}(\mathcal{B})$ if and only if $\mathbf{1}$ is join-irreducible. However, if $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ is join-irreducible, then $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) = \mathrm{JAN}$ [73]. Thus if $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ and $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) \supseteq \mathrm{JAN}$, then $\neg p \vee \neg\neg p \in \mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a})$ which implies that $\mathbf{a}$ is join-irreducible which implies that $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) = \mathrm{JAN}$. Thus a "no" answer to Question 3.0.10 must yield a degree $\mathbf{a}$ such that $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) \nsubseteq \mathrm{JAN}$ and $\mathrm{JAN} \nsubseteq \mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a})$.

The following theorem shows that to give a "yes" answer to Question 3.0.10 it suffices to show that every $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ bounds a finite meet of join-irreducible degrees $>_{\mathrm{s}} \mathbf{0}'$.

Theorem 3.2.1 ([73])**.** *If $\mathbf{a}$ is a degree such that $\mathbf{a} \geq_{\mathrm{s}} \prod_{i=0}^{n} \mathbf{d}_i$ for join-irreducible degrees $\mathbf{d}_i >_{\mathrm{s}} \mathbf{0}'$, $i \leq n$, then $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) \subseteq \mathrm{JAN}$.*

(The above theorem is stated more generally in [73]. Each degree $\mathbf{d}_i$ for $i \leq n$ is allowed to be either join-irreducible or $\mathcal{D}e$-irreducible. See the parenthetical discussion following Theorem 3.3.1 for the definition of $\mathcal{D}e$-irreducible and an explanation of why we do not consider such degrees here. Theorem 3.3.1 is a restatement of [73] Theorem 2.11 which is the main tool used to prove Theorem 3.2.1.)

The degrees of the mass problems $\mathcal{B}_f = \{g \mid g \nleq_{\mathrm{T}} f\}$ play an important role in the study of Question 3.0.10. The following lemma from Sorbi [72] encapsulates the properties of the $\deg_{\mathrm{s}}(\mathcal{B}_f)$'s that we need in this section and the next.

Lemma 3.2.2 ([72])**.**

(i) Every $\deg_{\mathrm{s}}(\mathcal{B}_f)$ is join-irreducible.

(ii) Every $\sum_{i=0}^{n} \deg_{\mathrm{s}}(\mathcal{B}_{f_i})$ is meet-irreducible.

(iii) Let V and J be finite sets and let U_v and I_j be finite sets for each $v \in V$ and $j \in J$. Let $\mathbf{x}_u^v$ and $\mathbf{y}_i^j$ be degrees of the form $\deg_{\mathrm{s}}(\mathcal{B}_f)$ for every $v \in V$, $u \in U_v$, $j \in J$, and $i \in I_j$. Let $\mathbf{a} = \sum_{v \in V} \prod_{u \in U_v} \mathbf{x}_u^v$ and $\mathbf{b} = \sum_{j \in J} \prod_{i \in I_j} \mathbf{y}_i^j$. Then $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$ if and only if

$$\big(\forall v \in V\big)\big(\exists j \in J\big)\big(\forall i \in I_j\big)\big(\exists u \in U_v\big)\big(\mathbf{x}_u^v \leq_{\mathrm{s}} \mathbf{y}_i^j\big).$$

(iv) In the notation of item (iii),

$$\mathbf{a} \to \mathbf{b} = \sum\Big\{\prod_{i \in I_j} \mathbf{y}_i^j \;\Big|\; \Big(\forall v \in V\Big)\Big(\prod_{i \in I_j} \mathbf{y}_i^j \nleq_{\mathrm{s}} \prod_{u \in U_v} \mathbf{x}_u^v\Big)\Big\}$$

(where the empty join is $\mathbf{0}$).

Proof. Item (i) is by Theorem 3.1.3 and item (ii) is by Corollary 3.0.9. Item (iv) is proved in [72]. Item (iii) follows from item (iv) because $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$ if and only if $\mathbf{b} \to \mathbf{a} = \mathbf{0}$. $\square$

In [75] it is asked if every degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ bounds a join-irreducible degree $>_{\mathrm{s}} \mathbf{0}'$, and it is conjectured that this is not the case based on the evidence provided by the following theorem.

Theorem 3.2.3 ([75])**.** *There is a degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ such that $\mathbf{a} \not\geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{B}_f)$ for every $f >_{\mathrm{T}} 0$.*

Our characterization of the join-irreducible degrees implies that every join-irreducible degree $>_{\mathrm{s}} \mathbf{0}'$ bounds some degree $\deg_{\mathrm{s}}(\mathcal{B}_f)$ with $f >_{\mathrm{T}} 0$. Thus the conjecture is correct.

Corollary 3.2.4 (to Theorem 3.1.3)**.** *If $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ is join-irreducible, then $\mathbf{a} \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{B}_f)$ for some $f >_{\mathrm{T}} 0$.*

Proof. If $\mathbf{a}$ is join-irreducible, then, by Theorem 3.1.3, $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ for some Turing ideal $\mathcal{I}$. If $\deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I}) >_{\mathrm{s}} \mathbf{0}'$, then $\mathcal{I}$ contains some function $f >_{\mathrm{T}} 0$. Thus $\omega^\omega \setminus \mathcal{I} \subseteq \mathcal{B}_f$. Hence $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I}) \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{B}_f)$. □

Theorem 3.2.5. *There is a degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ such that every degree $\mathbf{x}$ with $\mathbf{0}' <_{\mathrm{s}} \mathbf{x} \leq_{\mathrm{s}} \mathbf{a}$ is join-reducible.*

Proof. By Theorem 3.2.3, let $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ be such that $\mathbf{a} \not\geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{B}_f)$ for every $f >_{\mathrm{T}} 0$. This $\mathbf{a}$ is the desired degree because, by Corollary 3.2.4, if $\mathbf{a} \geq_{\mathrm{s}} \mathbf{x}$ for some join-irreducible $\mathbf{x} >_{\mathrm{s}} \mathbf{0}'$, then $\mathbf{a} \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{B}_f)$ for some $f >_{\mathrm{T}} 0$. □

The degree $\mathbf{a}$ satisfying Theorem 3.2.3 was constructed by diagonalization in [75]. We can give somewhat more concrete examples of degrees satisfying Theorem 3.2.3 and Theorem 3.2.5. Recall the following definitions. Functions $f, g >_{\mathrm{T}} 0$ are a *Turing minimal pair* if, for all h, $h \leq_{\mathrm{T}} f, g$ implies $h \leq_{\mathrm{T}} 0$. A function f has *minimal Turing degree* if, for all h, $h <_{\mathrm{T}} f$ implies $h \leq_{\mathrm{T}} 0$. Minimal pairs and minimal degrees exist. In fact, there are continuum many distinct minimal Turing degrees. See Lerman [37] Section II.4 and Section V.2.

Theorem 3.2.6. *If f and g are a minimal pair, then the degree $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{B}_f) \times \deg_{\mathrm{s}}(\mathcal{B}_g)$ witnesses Theorem 3.2.5.*

Proof. Let f and g be a minimal pair. Then $\deg_{\mathrm{s}}(\mathcal{B}_f), \deg_{\mathrm{s}}(\mathcal{B}_g) >_{\mathrm{s}} \mathbf{0}'$ because $f, g >_{\mathrm{T}} 0$. Thus $\deg_{\mathrm{s}}(\mathcal{B}_f) \times \deg_{\mathrm{s}}(\mathcal{B}_g) >_{\mathrm{s}} \mathbf{0}'$ because $\mathbf{0}'$ is meet-irreducible by Corollary 3.0.9. To show that $\deg_{\mathrm{s}}(\mathcal{B}_f) \times \deg_{\mathrm{s}}(\mathcal{B}_g)$ bounds no join-irreducible degree $>_{\mathrm{s}} \mathbf{0}'$, it suffices by Corollary 3.2.4 to show that $\deg_{\mathrm{s}}(\mathcal{B}_f) \times \deg_{\mathrm{s}}(\mathcal{B}_g)$ bounds no $\deg_{\mathrm{s}}(\mathcal{B}_h)$ for $h >_{\mathrm{T}} 0$. This is true because f, g is a minimal pair, so for any $h >_{\mathrm{T}} 0$, either $h \not\leq_{\mathrm{T}} f$ or $h \not\leq_{\mathrm{T}} g$. Thus either $h \in \mathcal{B}_f$ or $h \in \mathcal{B}_g$ which means $\mathcal{B}_f \times \mathcal{B}_g$ contains a function $\equiv_{\mathrm{T}} h$. $\mathcal{B}_h$ contains no function $\leq_{\mathrm{T}} h$, therefore $\mathcal{B}_f \times \mathcal{B}_g \not\geq_{\mathrm{s}} \mathcal{B}_h$. □

We can extend the idea behind Theorem 3.2.6 to find a degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ that does not bound any finite meet of join-irreducible degrees $>_{\mathrm{s}} \mathbf{0}'$. Several of our examples in this section and the next are of the form $\deg_{\mathrm{s}}\left(\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\right)$ for mass problems $\mathcal{D}_i$, $i \in \omega$.

Lemma 3.2.7. *Let* $\mathbf{d} = \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big)$ *where* $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ *for each* $i \in \omega$*. Then* $\mathbf{d} >_{\mathrm{s}} \mathbf{0}'$*.*

Proof. Suppose for a contradiction that Φ is a reduction witnessing $\mathbf{d} \leq_{\mathrm{s}} \mathbf{0}'$ (i.e., $\Phi(f) \in \bigcup_{i\in\omega} i^\frown\mathcal{D}_i$ for all $f >_{\mathrm{T}} 0$). Let $\sigma \in \omega^{<\omega}$ be such that $\Phi(\sigma)(0)\downarrow$ and let $i = \Phi(\sigma)(0)$. Then $f \mapsto \Phi(\sigma^\frown f)$ is a reduction witnessing $\mathbf{0}' \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{D}_i)$, contradicting $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$. ☐

Theorem 3.2.8. *There is a degree* $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ *such that no degree* $\mathbf{x}$ *with* $\mathbf{0}' <_{\mathrm{s}} \mathbf{x} \leq_{\mathrm{s}} \mathbf{a}$ *is of the form* $\prod_{i=0}^{n} \mathbf{d}_i$ *for join-irreducible degrees* $\mathbf{d}_i >_{\mathrm{s}} \mathbf{0}'$*,* $i \leq n$*.*

Proof. By Corollary 3.2.4, it suffices to find a degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ which is not above any degree of the form $\prod_{i=0}^{n} \deg_{\mathrm{s}}(\mathcal{B}_{f_i})$ where $f_i >_{\mathrm{T}} 0$ for each $i \leq n$. Let $\{g_i \mid i \in \omega\}$ be a countable collection of functions all of distinct minimal Turing degree. Let $\mathcal{A} = \bigcup_{i\in\omega} i^\frown\mathcal{B}_{g_i}$ and put $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$. Lemma 3.2.7 proves that $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$.

Now consider any degree $\prod_{i=0}^{n} \deg_{\mathrm{s}}(\mathcal{B}_{f_i})$, where $f_i >_{\mathrm{T}} 0$ for each $i \leq n$. There is a $j \in \omega$ such that, for each $i \leq n$, $g_j \not\geq_{\mathrm{T}} f_i$. Thus for this j, $\deg_{\mathrm{s}}(\mathcal{B}_{g_j}) \not\geq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{B}_{f_i})$ for each $i \leq n$. Therefore $\deg_{\mathrm{s}}(\mathcal{B}_{g_j}) \not\geq_{\mathrm{s}} \prod_{i=0}^{n} \deg_{\mathrm{s}}(\mathcal{B}_{f_i})$ because $\deg_{\mathrm{s}}(\mathcal{B}_{g_j})$ is meet-irreducible. Clearly $\deg_{\mathrm{s}}(\mathcal{B}_{g_j}) \geq_{\mathrm{s}} \mathbf{a}$, so $\mathbf{a} \not\geq_{\mathrm{s}} \prod_{i=0}^{n} \deg_{\mathrm{s}}(\mathcal{B}_{f_i})$ as well. ☐

For mass problems $\mathcal{A}_i$, $i \in \omega$, the Medvedev degree $\deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{A}_i\big)$ is not in general the greatest lower bound of the degrees $\deg_{\mathrm{s}}(\mathcal{A}_i)$, $i \in \omega$. Such greatest lower bounds need not even exist. For example, the degrees $\deg_{\mathrm{s}}(\mathcal{B}_{g_i})$, $i \in \omega$ from Theorem 3.2.8 do not have a greatest lower bound. This follows from results in Dyment [23] (specifically, see Lemma 2.7.1), which studies when countable collections of degrees have least upper bounds and greatest lower bounds.

If $\mathbf{a}$ is a degree such that $\mathbf{a} \not\geq_{\mathrm{s}} \mathbf{d}$ for all join-irreducible $\mathbf{d} >_{\mathrm{s}} \mathbf{0}'$, then $\mathbf{a} \to \mathbf{d} = \mathbf{d}$ for all join-irreducible $\mathbf{d} >_{\mathrm{s}} \mathbf{0}'$. The degree $\mathbf{a}$ constructed in Theorem 3.2.8 enjoys a similar property.

Theorem 3.2.9. *There is a degree* $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ *such that* $\mathbf{a} \to \prod_{i=0}^{n} \mathbf{d}_i = \prod_{i=0}^{n} \mathbf{d}_i$ *whenever* $\mathbf{d}_i >_{\mathrm{s}} \mathbf{0}'$ *and is join-irreducible for each* $i \leq n$*.*

Proof. As in Theorem 3.2.8, let $\{g_i \mid i \in \omega\}$ be a countable collection of functions all of distinct minimal Turing degree, let $\mathcal{A} = \bigcup_{i\in\omega} i^\frown\mathcal{B}_{g_i}$, and put $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$. Suppose $\mathbf{d}_i >_{\mathrm{s}} \mathbf{0}'$ and is join-irreducible for each $i \leq n$. By Theorem 3.1.3, for each $i \leq n$ let $\mathcal{I}_i \subseteq \omega^\omega$ be a Turing ideal such that $\mathbf{d}_i = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I}_i)$. Thus $\prod_{i=0}^{n} \mathbf{d}_i = \deg_{\mathrm{s}}\big(\bigcup_{i=0}^{n} i^\frown(\omega^\omega \setminus \mathcal{I}_i)\big)$ and

$$\mathbf{a} \to \prod_{i=0}^{n} \mathbf{d}_i = \deg_{\mathrm{s}}\left(\left\{ e^\frown g \;\middle|\; \Big(\forall f \in \mathcal{A}\Big)\Big(\Phi_e(f \oplus g) \in \bigcup_{i=0}^{n} i^\frown(\omega^\omega \setminus \mathcal{I}_i)\Big)\right\}\right).$$

We now describe a reduction witnessing $\mathbf{a} \to \prod_{i=0}^{n} \mathbf{d}_i \geq_{\mathrm{s}} \prod_{i=0}^{n} \mathbf{d}_i$.

Given $e^\frown g$, for each $i \leq n+1$ search for a string $i^\frown\sigma_i$ such that $\Phi_e((i^\frown\sigma_i) \oplus g)(0)\downarrow$. If there is a $k \leq n$ such that

$$\Phi_e((i^\frown\sigma_i) \oplus g)(0) = \Phi_e((j^\frown\sigma_j) \oplus g)(0) = k$$

for two distinct $i, j \leq n+1$, choose the least such k and output $k^\frown g$. Otherwise output 0.

Suppose we apply this reduction to $e^\frown g \in \mathcal{A} \to \bigcup_{i=0}^{n} i^\frown(\omega^\omega \setminus \mathcal{I}_i)$. $\Phi_e(f \oplus g)$ must be total for each $f \in \mathcal{A}$, and for each $i \in \omega$ there is an $f \in \mathcal{A}$ with $f(0) = i$. Thus for each $i \leq n+1$ the search finds a string $i^\frown\sigma_i$ such that $\Phi_e((i^\frown\sigma_i) \oplus g)(0)\downarrow$. Moreover, each $i^\frown\sigma_i$ can be extended to a function in $\mathcal{A}$, so $\Phi_e((i^\frown\sigma_i) \oplus g)(0) \leq n$ for each $i \leq n+1$. Therefore there is a least $k \leq n$ for which there are distinct $i, j \leq n+1$ with $\Phi_e((i^\frown\sigma_i) \oplus g)(0) = \Phi_e((j^\frown\sigma_j) \oplus g)(0) = k$. The reduction outputs $k^\frown g$, so we must show that $k^\frown g \in \bigcup_{i=0}^{n} i^\frown(\omega^\omega \setminus \mathcal{I}_i)$ which means we must show that $g \notin \mathcal{I}_k$. Suppose for a contradiction that $g \in \mathcal{I}_k$. The functions g_i and g_j have distinct minimal degree, so either $g \not\leq_{\mathrm{T}} g_i$ or $g \not\leq_{\mathrm{T}} g_j$ ($g >_{\mathrm{T}} 0$ because $\mathbf{a} \not\geq_{\mathrm{s}} \prod_{i=0}^{n} \mathbf{d}_i$ by Theorem 3.2.8). For the sake of argument, suppose $g \not\leq_{\mathrm{T}} g_i$. Then $\sigma_i{}^\frown g \not\leq_{\mathrm{T}} g_i$ as well, so $\sigma_i{}^\frown g \in \mathcal{B}_{g_i}$ and $i^\frown\sigma_i{}^\frown g \in \mathcal{A}$. However, $\Phi_e((i^\frown\sigma_i{}^\frown g) \oplus g) \in k^\frown(\omega^\omega \setminus \mathcal{I}_k)$ by the choice of $i^\frown\sigma_i$. This cannot be because $(i^\frown\sigma_i{}^\frown g) \oplus g \in \mathcal{I}_k$, thus anything it computes is also in $\mathcal{I}_k$. □

By Corollary 3.3.6, the degree $\mathbf{a} = \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{B}_{g_i}\big)$ used to witness Theorem 3.2.8 and Theorem 3.2.9 satisfies $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}/\mathbf{a}) \subseteq \mathrm{JAN}$ and so does any degree that bounds it. There are, however, degrees $>_{\mathrm{s}} \mathbf{0}'$ that do not bound any degree of the form $\deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big)$ where $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$.

Theorem 3.2.10. *There is a degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ such that $\mathbf{a} \not\geq_{\mathrm{s}} \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big)$ whenever $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$.*

Proof. Let $\mathcal{D}_i$ be such that $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$. By Corollary 3.2.4, for every $i \in \omega$ there is an $f_i >_{\mathrm{T}} 0$ such that $\mathcal{D}_i \geq_{\mathrm{s}} \mathcal{B}_{f_i}$. The mass problem $\mathcal{B}_{f_i}$ is Turing upward-closed for each $i \in \omega$, so $\mathcal{D}_i \subseteq \mathcal{B}_{f_i}$ for each $i \in \omega$. Thus $\bigcup_{i\in\omega} i^\frown\mathcal{D}_i \subseteq \bigcup_{i\in\omega} i^\frown\mathcal{B}_{f_i}$. Hence it suffices to find a degree $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$ that does not bound any degree of the form $\deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{B}_{f_i}\big)$, where $f_i >_{\mathrm{T}} 0$ for each $i \in \omega$.

We use the same construction used in [75] to prove Theorem 3.2.3. Build mass problems $\mathcal{A}_s \subseteq \{g \mid g >_{\mathrm{T}} 0\}$ such that $\{g \mid g >_{\mathrm{T}} 0\} \setminus \mathcal{A}_s$ is finite for each $s \in \omega$. Set $\mathcal{A}_0 = \{g \mid g >_{\mathrm{T}} 0\}$. At stage $s+1$, choose $h_s >_{\mathrm{T}} 0$ such that h_s does not compute any of the (finitely many) functions in $\{g \mid g >_{\mathrm{T}} 0\} \setminus \mathcal{A}_s$. If $\Phi_s(h_s)$ is total and $>_{\mathrm{T}} 0$, let $g_s = \Phi_s(h_s)$ and set $\mathcal{A}_{s+1} = \mathcal{A}_s \setminus \{g_s\}$. Otherwise set $\mathcal{A}_{s+1} = \mathcal{A}_s$. Put $\mathcal{A} = \bigcap_{s\in\omega} \mathcal{A}_s$ and put $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$.

To see $\mathbf{a} >_{\mathrm{s}} \mathbf{0}'$, observe that by construction $\Phi_s(h_s) \notin \mathcal{A}$ for each $s \in \omega$. Now let $f_i >_{\mathrm{T}} 0$ for each $i \in \omega$. We need to show that $\Phi_e(\mathcal{A}) \not\subseteq \bigcup_{i\in\omega} i^\frown\mathcal{B}_{f_i}$ for every index e. To do this, we first show that the functions in $\{g \mid g >_{\mathrm{T}} 0\} \setminus \mathcal{A}$ have distinct Turing degree. Suppose that g_i leaves $\mathcal{A}$ at stage $i+1$ and g_j leaves $\mathcal{A}$ at stage $j+1$ for $i+1 < j+1$ (i.e., at stage $i+1$ we had $\Phi_i(h_i) = g_i >_{\mathrm{T}} 0$, and at stage $j+1$ we had $\Phi_j(h_j) = g_j >_{\mathrm{T}} 0$). Then $g_i \not\leq_{\mathrm{T}} g_j$ because otherwise $g_i \leq_{\mathrm{T}} g_j \leq_{\mathrm{T}} h_j$, contradicting that h_j was chosen $\not\geq_{\mathrm{T}} g_i$ at stage $j+1$. Now suppose $\Phi_e(g)$ is total for all $g \in \mathcal{A}$. Fix any $\sigma \in \omega^{<\omega}$ such that $\Phi_e(\sigma)(0)\downarrow$, and let n be such that $\Phi_e(\sigma)(0) = n$. $\mathcal{A}$ is missing at most one function $\equiv_{\mathrm{T}} f_n$, so let $g \in \mathcal{A}$ be such that $\sigma \subset g$ and $g \equiv_{\mathrm{T}} f_n$. Then $\Phi_e(g)(0) = n$, but $\Phi_e(g) \notin n^\frown \mathcal{B}_{f_n}$. Hence $\Phi_e(\mathcal{A}) \nsubseteq \bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}$. $\square$

Question 3.2.11. Let $\mathbf{a}$ be the degree constructed in Theorem 3.2.10. Does $\mathbf{a} \to \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\big) = \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\big)$ whenever $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$? Is $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \subseteq \mathrm{JAN}$?

Finally, we note that the answer to Question 3.0.10 is "no" for $\mathcal{D}_{\mathrm{w}}$ in place of $\mathcal{D}_{\mathrm{s}}$. Let $f >_{\mathrm{T}} 0$ have minimal Turing degree, and let $\mathbf{a} = \deg_{\mathrm{w}}(\mathcal{B}_f)$. Then, in $\mathcal{D}_{\mathrm{w}}$, $[\mathbf{0}, \mathbf{a}] = \{\mathbf{0}, \mathbf{0}', \mathbf{a}\}$ and $\mathrm{JAN} \subsetneq \mathrm{PTh}(\mathcal{D}_{\mathrm{w}} / \mathbf{a}) \subsetneq \mathrm{CPC}$.

3.3 New degrees whose corresponding logic is contained in JAN

We extend Theorem 3.2.1 by proving that $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \subseteq \mathrm{JAN}$ for degrees $\mathbf{a}$ such that $\mathbf{a} \geq_{\mathrm{s}} \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\big)$ for some collection of join-irreducible degrees $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$, $i \in \omega$.

A propositional formula is called *positive* if the connective '$\neg$' does not appear in it. For a logic L let L^+ denote the positive formulas in L, and for a Brouwer algebra $\mathcal{B}$ let $\mathrm{PTh}^+(\mathcal{B})$ denote the set of positive formulas valid in $\mathcal{B}$. JAN is the maximum intermediate logic L for which $L^+ = \mathrm{IPC}^+$ [30]. This means that $L^+ = \mathrm{IPC}^+$ implies $L \subseteq \mathrm{JAN}$ for any intermediate logic L. Thus $\mathrm{PTh}^+(\mathcal{B}) = \mathrm{IPC}^+$ implies $\mathrm{PTh}(\mathcal{B}) \subseteq \mathrm{JAN}$ for any Brouwer algebra $\mathcal{B}$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Brouwer algebras. An injection $f\colon \mathcal{B}_1 \to \mathcal{B}_2$ is called a *B-embedding* if it preserves 0, 1, $+$, $\times$, and $\to$ (and therefore also $\neg$). An injection $f\colon \mathcal{B}_1 \to \mathcal{B}_2$ is called a *B^+-embedding* if it preserves 0, $+$, $\times$, and $\to$ (but not necessarily 1 or $\neg$). If $\mathcal{B}_1$ B-embeds into $\mathcal{B}_2$, then $\mathrm{PTh}(\mathcal{B}_2) \subseteq \mathrm{PTh}(\mathcal{B}_1)$, and if $\mathcal{B}_1$ B^+-embeds into $\mathcal{B}_2$, then $\mathrm{PTh}^+(\mathcal{B}_2) \subseteq \mathrm{PTh}^+(\mathcal{B}_1)$. Both of these facts are easily checked in light of [49] Theorem VI.2.4. If $a \leq b$ are in a Brouwer algebra $\mathcal{B}$, then $\mathcal{B}/a$ B^+-embeds into $\mathcal{B}/b$ by the identity. This implies that $\mathrm{PTh}^+(\mathcal{B}/b) \subseteq \mathrm{PTh}^+(\mathcal{B}/a)$, and it follows that the a for which $\mathrm{PTh}(\mathcal{B}/a) \subseteq \mathrm{JAN}$ is upward-closed in any Brouwer algebra $\mathcal{B}$.

Our goal is to B^+-embed a certain class of Brouwer algebras into $\mathcal{D}_{\mathrm{s}} / \mathbf{a}$. For any set X, let $\mathrm{Fr}(X)$ denote the free distributive lattice generated by X and let $0 \oplus \mathrm{Fr}(X)$ denote $\mathrm{Fr}(X)$ with a new bottom element 0. The elements of $\mathrm{Fr}(X)$ are all of the form $\sum_{v\in V} \prod_{u\in U_v} x_u^v$ where V and the U_v are finite sets of indices and the x_u^v are all in X (see for example Balbes and Dwinger [7] Section V.3). For such representations, $\sum_{v\in V} \prod_{u\in U_v} x_u^v \leq \sum_{j\in J} \prod_{i\in I_j} y_i^j$ if and only if

$$\big(\forall v \in V\big)\big(\exists j \in J\big)\big(\forall i \in I_j\big)\big(\exists u \in U_v\big)\big(x_u^v = y_i^j\big).$$

If $a, b \in \mathrm{Fr}(X)$ are such that $a \ngeq b$, then $a \to b$ exists. To see this, let $a = \sum_{v\in V} \prod_{u\in U_v} x_u^v$ and $b = \sum_{j\in J} \prod_{i\in I_j} y_i^j$ be representations for a and b. Then check

$$a \to b = \sum\Big\{\prod_{i\in I_j} y_i^j \;\Big|\; \Big(\forall v \in V\Big)\Big(\prod_{i\in I_j} y_i^j \nleq \prod_{u\in U_v} x_u^v\Big)\Big\}.$$

If $a \geq b$ are in $\mathrm{Fr}(X)$ for an infinite X, then $a \to b$ fails to exist because in this case $\mathrm{Fr}(X)$ has no least element. We see then that $a \to b$ exists for every $a, b \in 0 \oplus \mathrm{Fr}(X)$.

If X is finite, then so are $\mathrm{Fr}(X)$ and $0 \oplus \mathrm{Fr}(X)$. Hence both are Brouwer algebras. Let $\mathrm{Fr}(n)$ denote the free distributive lattice with n generators. The logic $\mathrm{LM} = \bigcap_{n \in \omega} \mathrm{PTh}(0 \oplus \mathrm{Fr}(n))$ is called the *Medvedev logic of finite problems*. (LM is usually defined in terms of Brouwer algebras isomorphic to the $0 \oplus \mathrm{Fr}(n)$. See [75] for details.) We take advantage of the fact that $\mathrm{LM}^+ = \mathrm{IPC}^+$ [42].

If X is infinite, then $0 \oplus \mathrm{Fr}(X)$ fails to be a Brouwer algebra only because it lacks a top element. Therefore the notion of a B^+-embedding makes sense when we allow $\mathcal{B}_1$ or $\mathcal{B}_2$ to be $0 \oplus \mathrm{Fr}(X)$. If we let $0 \oplus \mathrm{Fr}(X) \oplus 1$ denote $\mathrm{Fr}(X)$ with a new bottom element 0 and a new top element 1, then $0 \oplus \mathrm{Fr}(X) \oplus 1$ is always a Brouwer algebra.

For any partial order $(\mathcal{P}, \leq_{\mathcal{P}})$, let $\mathrm{Fr}(\mathcal{P}, \leq_{\mathcal{P}})$ denote the free distributive lattice generated by $(\mathcal{P}, \leq_{\mathcal{P}})$. $\mathrm{Fr}(\mathcal{P}, \leq_{\mathcal{P}})$ is the quotient $\mathrm{Fr}(\mathcal{P})/{\equiv}$ where, for $a = \sum_{v \in V} \prod_{u \in U_v} x_u^v$ and $b = \sum_{j \in J} \prod_{i \in I_j} y_i^j$ in $\mathrm{Fr}(\mathcal{P})$, $a \equiv b$ if and only if $(a \preceq b) \wedge (b \preceq a)$, and $a \preceq b$ if and only if

$$\big(\forall v \in V\big)\big(\exists j \in J\big)\big(\forall i \in I_j\big)\big(\exists u \in U_v\big)\big(x_u^v \leq_{\mathcal{P}} y_i^j\big).$$

$\mathrm{Fr}(\mathcal{P}, \leq_{\mathcal{P}})$ is always a distributive lattice, and $0 \oplus \mathrm{Fr}(\mathcal{P}, \leq_{\mathcal{P}}) \oplus 1$ is always a Brouwer algebra; also see [72].

The following lemmas facilitate our embeddings. Lemma 3.3.3 is a slight generalization of the claim in the proof of [72] Lemma 2.3 and of [68] Lemma 6. The embedding is done in Theorem 3.3.4 which is nearly identical to [73] Theorem 2.11. Part of the reason for reproducing the proof here is to (hopefully) correct the notational inconsistencies in the proof in [73]. We restate [73] Theorem 2.11 for reference.

Theorem 3.3.1 ([73] Theorem 2.11)**.** *Let $\mathbf{d} = \prod_{i=0}^{n} \mathbf{d}_i$ where $\mathbf{d}_i >_{\mathrm{s}} \mathbf{0}'$ and $\mathbf{d}_i$ is join-irreducible for each $i \leq n$. Then $\mathbf{0} \oplus \mathrm{Fr}(\mathcal{P}, \leq_{\mathcal{P}}) \oplus \mathbf{1}$ B-embeds into $\mathcal{D}_{\mathrm{s}} / \mathbf{d}$ for every countable partial order $(\mathcal{P}, \leq_{\mathcal{P}})$.*

(The above theorem is stated more generally in [73]. Each degree $\mathbf{d}_i$ for $i \leq n$ is allowed to be either join-irreducible or $\mathcal{D}e$-irreducible. A degree $\mathbf{a}$ is *dense* if it is of the form $\deg_{\mathrm{s}}(\mathcal{A})$ where $\mathcal{A}$ is dense in ω^ω, and a degree $\mathbf{d}$ is *$\mathcal{D}e$-irreducible* if $\mathbf{a} \to \mathbf{d} = \mathbf{d}$ for all dense degrees $\mathbf{a}$. We do not consider $\mathcal{D}e$-irreducible degrees in our version of [73] Theorem 2.11, which is Theorem 3.3.4, because in Theorem 3.3.4 we require that the mass problems $\mathcal{D}_i$ (which play the role of the degrees $\mathbf{d}_i$ in [73] Theorem 2.11) are Turing upward-closed. Mass problems that are Turing upward-closed are dense and hence their degrees are not $\mathcal{D}e$-irreducible.)

Lemma 3.3.2 ([22])**.** *If $\mathcal{X} \not\geq_{\mathrm{s}} \mathcal{Y}$ are mass problems, then there is a $\mathcal{W} \subseteq \mathcal{X}$ with $|\mathcal{W}| \leq \omega$ such that $\mathcal{W} \not\geq_{\mathrm{s}} \mathcal{Y}$.*

Proof. $\mathcal{X} \not\geq_{\mathrm{s}} \mathcal{Y}$ means that there is no Turing functional Φ such that $\Phi(\mathcal{X}) \subseteq \mathcal{Y}$. Thus for each functional Φ_e there must be some function $f_e \in \mathcal{X}$ such that

$\Phi_e(f_e) \notin \mathcal{Y}$. Let $\mathcal{W}$ consist of a choice of one such $f_e \in \mathcal{X}$ for each functional Φ_e. ☐

Lemma 3.3.3. *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{F}_i$ for $i \in \omega$ be mass problems such that $\bigcup_{i\in\omega} i^\frown\mathcal{F}_i \leq_{\mathrm{s}} \mathcal{U}+\mathcal{V}$ and $\sigma^\frown\mathcal{U} \subseteq \mathcal{U}$ for all $\sigma \in \omega^{<\omega}$. Then there are mass problems $\mathcal{V}_i$ for $i \in \omega$ such that $\bigcup_{i\in\omega} i^\frown\mathcal{V}_i \equiv_{\mathrm{s}} \mathcal{V}$ and $\mathcal{F}_i \leq_{\mathrm{s}} \mathcal{U}+\mathcal{V}_i$ for each $i \in \omega$.*

Proof. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{F}_i$ for $i \in \omega$ be as in the statement of the lemma. Let Φ be such that $\Phi(\mathcal{U}+\mathcal{V}) \subseteq \bigcup_{i\in\omega} i^\frown\mathcal{F}_i$. For each $i \in \omega$, define $\mathcal{V}_i = \{g \in \mathcal{V} \mid (\exists\sigma \in \omega^{<\omega})(\Phi(\sigma \oplus g)(0) = i)\}$. $\mathcal{V} \leq_{\mathrm{s}} \bigcup_{i\in\omega} i^\frown\mathcal{V}_i$ is clear. $\bigcup_{i\in\omega} i^\frown\mathcal{V}_i \leq_{\mathrm{s}} \mathcal{V}$ by the reduction which, given g, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0)\downarrow$ and outputs $\Phi(\sigma\oplus g)(0)^\frown g$. To see $i^\frown\mathcal{F}_i \leq_{\mathrm{s}} \mathcal{U}+\mathcal{V}_i$, consider the reduction which, given $f \oplus g$, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0) = i$ and outputs $\Phi((\sigma^\frown f) \oplus g)$. If $f \oplus g \in \mathcal{U}+\mathcal{V}_i$, then such a σ is found, $\sigma^\frown f$ is in $\mathcal{U}$, and $\Phi((\sigma^\frown f) \oplus g)$ is in $i^\frown\mathcal{F}_i$. ☐

Theorem 3.3.4. *Let $\mathbf{d} = \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big)$ where $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$, $\deg_{\mathrm{s}}(\mathcal{D}_i)$ is join-irreducible, and $\mathcal{D}_i$ is Turing upward-closed for each $i \in \omega$. Then $0 \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathcal{D}_{\mathrm{s}}/\mathbf{d}$.*

Proof. Let $\mathcal{D}_i$ for $i \in \omega$ be as in the statement of the theorem, let $\mathcal{D} = \bigcup_{i\in\omega} i^\frown\mathcal{D}_i$, and let $\mathbf{d} = \deg_{\mathrm{s}}(\mathcal{D})$. Lemma 3.2.7 proves that $\mathbf{d} >_{\mathrm{s}} \mathbf{0}'$. By Lemma 3.3.2, let $\mathcal{A} \subseteq \{f \mid f >_{\mathrm{T}} 0\}$ be a countable mass problem such that $\mathcal{A} \not\geq_{\mathrm{s}} \mathcal{D}$. Let $\{f_x \mid x \in 2^\omega\}$ be a collection of functions such that $f_x \,|_{\mathrm{T}}\, f_y$ for all $x, y \in 2^\omega$ with $x \neq y$ and such that $f \not\leq_{\mathrm{T}} f_x$ for all $f \in \mathcal{A}$ and $x \in 2^\omega$. Such a sequence can be constructed via standard recursion-theoretic techniques: build a perfect tree whose paths are Turing incomparable and do not compute any functions in $\mathcal{A}$. See for example [37] Section II.4. Notice that $\mathcal{B}_{f_x} \leq_{\mathrm{s}} \mathcal{A}$ (because $\mathcal{A} \subseteq \mathcal{B}_{f_x}$) for each $x \in 2^\omega$.

Define $G\colon 0 \oplus \mathrm{Fr}(2^\omega) \to \mathcal{D}_{\mathrm{s}}$ as follows. Let $G(0) = \mathbf{0}$ and let $G(x) = \deg_{\mathrm{s}}(\mathcal{B}_{f_x})$ on the generators $x \in 2^\omega$ of $\mathrm{Fr}(2^\omega)$. Then extend G to all of $0 \oplus \mathrm{Fr}(2^\omega)$ so that $G\big(\sum_{v\in V}\prod_{u\in U_v} x_u^v\big) = \sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big)$. G preserves 0, $+$, and $\times$ by definition, and G is injective and preserves $\to$ by Lemma 3.2.2 items (iii) and (iv). Hence G is a B^+-embedding (this is essentially [72] Corollary 2.5). Now define $H\colon 0 \oplus \mathrm{Fr}(2^\omega) \to \mathcal{D}_{\mathrm{s}}/\mathbf{d}$ by $H(a) = G(a) \times \mathbf{d}$ for all $a \in 0 \oplus \mathrm{Fr}(2^\omega)$. This H is the desired B^+-embedding. By definition, H preserves 0, $+$, and $\times$. We must show that H is injective and that H preserves $\to$.

Clearly $H(a) = \mathbf{0}$ if and only if $a = 0$, so to show that H is injective we let $a, b \in \mathrm{Fr}(2^\omega)$ be such that $H(a) \leq_{\mathrm{s}} H(b)$ and show that $a \leq b$. Let $a = \sum_{v\in V}\prod_{u\in U_v} x_u^v$ be a representation for a and let $b = \sum_{j\in J}\prod_{i\in I_j} y_i^j$ be a representation for b. $H(a) \leq_{\mathrm{s}} H(b)$ means that

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \times \mathbf{d} \leq_{\mathrm{s}} \sum_{j\in J}\prod_{i\in I_j} G\big(y_i^j\big) \times \mathbf{d}.$$

Therefore

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big)\times \mathbf{d} \leq_{\mathrm{s}} \sum_{j\in J}\prod_{i\in I_j} G\big(y^j_i\big) = \prod\Big\{\sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \;\Big|\; \alpha\in\prod_{j\in J} I_j\Big\}$$

where the equality is by distributivity ($\prod_{j\in J} I_j$ denotes the Cartesian product of the I_j's). Hence

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big)\times \mathbf{d} \leq_{\mathrm{s}} \sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \text{ for each } \alpha\in\prod_{j\in J} I_j.$$

Each $\sum_{j\in J} G\big(y^j_{\alpha(j)}\big)$ is meet-irreducible by Lemma 3.2.2 item (ii). Also, $\mathbf{d}\not\leq_{\mathrm{s}} \sum_{j\in J} G\big(y^j_{\alpha(j)}\big)$ for each $\alpha\in\prod_{j\in J} I_j$ because $\sum_{j\in J} G\big(y^j_{\alpha(j)}\big)\leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{A})$ but $\mathbf{d}\not\leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{A})$. Thus

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big) \leq_{\mathrm{s}} \sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \text{ for each } \alpha\in\prod_{j\in J} I_j,$$

and this implies that

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big) \leq_{\mathrm{s}} \prod\Big\{\sum_{j\in J} G\big(y^j_{\alpha(j)}\big) \;\Big|\; \alpha\in\prod_{j\in J} I_j\Big\}.$$

The left-hand side of the above inequality is $G(a)$ and the right-hand side is $G(b)$. G is a B^+-embedding, so we conclude $a\leq b$.

If either of $a,b\in 0\oplus\mathrm{Fr}(2^\omega)$ is 0, then clearly $H(a\to b) = H(a)\to H(b)$. So as before, let $a=\sum_{v\in V}\prod_{u\in U_v} x^v_u$ and let $b=\sum_{j\in J}\prod_{i\in I_j} y^j_i$ be in $\mathrm{Fr}(2^\omega)$. We see $H(a\to b)\geq_{\mathrm{s}} H(a)\to H(b)$ because

$$H(a\to b)+H(a) = H((a\to b)+a)\geq_{\mathrm{s}} H(b).$$

To show that $H(a\to b)\leq_{\mathrm{s}} H(a)\to H(b)$, we show that if $\mathbf{z}\in\mathcal{D}_{\mathrm{s}}$ is such that $H(b)\leq_{\mathrm{s}} H(a)+\mathbf{z}$, then $H(a\to b)\leq_{\mathrm{s}}\mathbf{z}$. Suppose $H(b)\leq_{\mathrm{s}} H(a)+\mathbf{z}$. That is,

$$\sum_{j\in J}\prod_{i\in I_j} G\big(y^j_i\big)\times\mathbf{d} \leq_{\mathrm{s}} \Big(\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big)\times\mathbf{d}\Big)+\mathbf{z}. \tag{3.1}$$

Since $a\to b=\sum\big\{\prod_{i\in I_j} y^j_i \mid (\forall v\in V)\big(\prod_{i\in I_j} y^j_i\not\leq\prod_{u\in U_v} x^v_u\big)\big\}$, we have

$$\begin{aligned} H(a\to b) &= G(a\to b)\times\mathbf{d}\\ &= \sum\Big\{\prod_{i\in I_j} G\big(y^j_i\big) \;\Big|\; \Big(\forall v\in V\Big)\Big(\prod_{i\in I_j} G\big(y^j_i\big)\not\leq_{\mathrm{s}}\prod_{u\in U_v} G\big(x^v_u\big)\Big)\Big\}\times\mathbf{d}. \end{aligned}$$

It suffices to show that, given $j\in J$, if $\prod_{i\in I_j} G\big(y^j_i\big)$ satisfies

$$\Big(\forall v\in V\Big)\Big(\prod_{i\in I_j} G\big(y^j_i\big)\not\leq_{\mathrm{s}}\prod_{u\in U_v} G\big(x^v_u\big)\Big),$$

then $\prod_{i \in I_j} G(y_i^j) \times \mathbf{d} \leq_{\mathrm{s}} \mathbf{z}$. Suppose $\prod_{i \in I_j} G(y_i^j)$ is such a meet. Then we know

$$\Big(\forall v \in V\Big)\Big(\exists u \in U_v\Big)\Big(\prod_{i \in I_j} G(y_i^j) \nleq_{\mathrm{s}} G(x_u^v)\Big).$$

By choosing such a $u \in U_v$ for every $v \in V$ and by appealing to Lemma 3.2.2 items (i) and (ii), we see that there is an $\alpha \in \prod_{v \in V} U_v$ such that

$$\prod_{i \in I_j} G(y_i^j) \nleq_{\mathrm{s}} \sum_{v \in V} G(x_{\alpha(v)}^v). \tag{3.2}$$

Distributing $\sum_{v \in V} \prod_{u \in U_v} G(x_u^v)$ in the right-hand side of (3.1) yields

$$\prod_{i \in I_j} G(y_i^j) \times \mathbf{d} \leq_{\mathrm{s}} \sum_{v \in V} G(x_{\alpha(v)}^v) + \mathbf{z}.$$

The degree $\sum_{v \in V} G(x_{\alpha(v)}^v)$ is a finite join of degrees of the form $\deg_{\mathrm{s}}(\mathcal{B}_f)$ and thus has a representative $\mathcal{U}$ such that $\sigma^\frown\mathcal{U} \subseteq \mathcal{U}$ for all $\sigma \in \omega^{<\omega}$. So by Lemma 3.3.3 there are mass problems $\mathcal{Z}_i$ for $i \in I_j$ and $\widehat{\mathcal{Z}}_i$ for $i \in \omega$ such that

$$\begin{aligned} \mathbf{z} &= \Big(\prod_{i \in I_j} \deg_{\mathrm{s}}(\mathcal{Z}_i)\Big) \times \deg_{\mathrm{s}}\Big(\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i\Big), \\ G(y_i^j) &\leq_{\mathrm{s}} \sum_{v \in V} G(x_{\alpha(v)}^v) + \deg_{\mathrm{s}}(\mathcal{Z}_i) \text{ for each } i \in I_j, \text{ and} \\ \deg_{\mathrm{s}}(\mathcal{D}_i) &\leq_{\mathrm{s}} \sum_{v \in V} G(x_{\alpha(v)}^v) + \deg_{\mathrm{s}}(\widehat{\mathcal{Z}}_i) \text{ for each } i \in \omega. \end{aligned}$$

Each $G(y_i^j)$ is join-irreducible, and $G(y_i^j) \nleq_{\mathrm{s}} \sum_{v \in V} G(x_{\alpha(v)}^v)$ by (3.2). Thus $G(y_i^j) \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{Z}_i)$ for each $i \in \omega$, so $\prod_{i \in I_j} G(y_i^j) \leq_{\mathrm{s}} \prod_{i \in I_j} \deg_{\mathrm{s}}(\mathcal{Z}_i)$. Each $\deg_{\mathrm{s}}(\mathcal{D}_i)$ is join-irreducible by assumption, and $\deg_{\mathrm{s}}(\mathcal{D}_i) \nleq_{\mathrm{s}} \sum_{v \in V} G(x_{\alpha(v)}^v)$ because the right-hand side is $\leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{A})$ but the left-hand side is not. It follows that $\deg_{\mathrm{s}}(\mathcal{D}_i) \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\widehat{\mathcal{Z}}_i)$ for each $i \in \omega$, and so $\widehat{\mathcal{Z}}_i \subseteq \mathcal{D}_i$ for each $i \in \omega$ because each $\mathcal{D}_i$ is Turing upward-closed. Thus $\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i \subseteq \mathcal{D}$, so $\mathbf{d} \leq_{\mathrm{s}} \deg_{\mathrm{s}}\big(\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i\big)$. Therefore $\prod_{i \in I_j} G(y_i^j) \times \mathbf{d} \leq_{\mathrm{s}} \big(\prod_{i \in I_j} \deg_{\mathrm{s}}(\mathcal{Z}_i)\big) \times \deg_{\mathrm{s}}\big(\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i\big) = \mathbf{z}$ as desired. □

Corollary 3.3.5. *If $\mathbf{a} \geq_{\mathrm{s}} \mathbf{d}$ are degrees such that $\mathbf{d} = \deg_{\mathrm{s}}\big(\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\big)$ where $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$, then $0 \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathcal{D}_{\mathrm{s}} / \mathbf{a}$.*

Proof. Let $\mathbf{a}$, $\mathbf{d}$, and $\mathcal{D}_i$ for $i \in \omega$ be as in the statement of the corollary. Let $\mathbf{d}_0 = \deg_{\mathrm{s}}\big(\bigcup_{i \in \omega} i^\frown C(\mathcal{D}_i)\big)$ and notice that $\mathbf{d} \geq_{\mathrm{s}} \mathbf{d}_0$. $\mathcal{D}_i \equiv_{\mathrm{s}} C(\mathcal{D}_i)$ for each $i \in \omega$ by Lemma 3.1.2, so $\mathbf{d}_0$ satisfies the hypotheses of Theorem 3.3.4. Thus $0 \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathcal{D}_{\mathrm{s}} / \mathbf{d}_0$ which B^+-embeds into $\mathcal{D}_{\mathrm{s}} / \mathbf{a}$. □

Corollary 3.3.6. *If* $\mathbf{a} \geq_{\mathrm{s}} \mathbf{d}$ *are degrees such that* $\mathbf{d} = \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big)$ *where* $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$ *and is join-irreducible for each* $i \in \omega$*, then* $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}\,/\mathbf{a}) \subseteq \mathrm{JAN}$.

Proof. The Brouwer algebra $0 \oplus \mathrm{Fr}(n)$ B^+-embeds into $0 \oplus \mathrm{Fr}(2^\omega)$ for each n, and $0 \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathcal{D}_{\mathrm{s}}\,/\mathbf{a}$ by Corollary 3.3.5. Thus $\mathrm{PTh}^+(\mathcal{D}_{\mathrm{s}}\,/\mathbf{a}) \subseteq \bigcap_{n\in\omega} \mathrm{PTh}^+(0 \oplus \mathrm{Fr}(n)) = \mathrm{LM}^+ = \mathrm{IPC}^+$. So $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}}\,/\mathbf{a}) \subseteq \mathrm{JAN}$. $\square$

Theorem 3.3.4 can be modified to B-embed $0\oplus\mathrm{Fr}(2^\omega)\oplus 1$ into $\mathcal{D}_{\mathrm{s}}\,/\mathbf{d}$ for degrees $\mathbf{d}$ as in the statement of Theorem 3.3.4. However, if $a \leq b$ in a Brouwer algebra $\mathcal{B}$, it is not in general the case that $\mathcal{B}\,/a$ B-embeds into $\mathcal{B}\,/b$. So the proof of Corollary 3.3.5 fails for B-embedding $0 \oplus \mathrm{Fr}(2^\omega) \oplus 1$. Theorem 3.3.4 can also be modified to prove a more precise analogue of [73] Theorem 2.11 (restated as Theorem 3.3.1). Let $\mathbf{d} = \deg_{\mathrm{s}}\big(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\big)$ where $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$, $\deg_{\mathrm{s}}(\mathcal{D}_i)$ is join-irreducible, and $\mathcal{D}_i$ is Turing upward-closed for each $i \in \omega$. Then $0 \oplus \mathrm{Fr}(\mathcal{P}, \leq_{\mathcal{P}}) \oplus 1$ B-embeds into $\mathcal{D}_{\mathrm{s}}\,/\mathbf{d}$ for every countable partial order $(\mathcal{P}, \leq_{\mathcal{P}})$.

3.4 $\mathfrak{F}_{\mathrm{cl}}$ is not prime

Recall that a filter $\mathfrak{F}$ in a lattice is called *prime* if $a+b \in \mathfrak{F} \to a \in \mathfrak{F} \vee b \in \mathfrak{F}$ for all a and b in the lattice. Theorem 3.1.3 can be phrased as a characterization of the prime principal filters in $\mathcal{D}_{\mathrm{s}}$: a degree $\mathbf{a}$ generates a prime filter if and only if $\mathbf{a} = \deg_{\mathrm{s}}(\omega^\omega \setminus \mathcal{I})$ for some Turing ideal $\mathcal{I}$. There is little general theory of the non-principal filters in $\mathcal{D}_{\mathrm{s}}$, but several specific cases have been studied in Dyment [22], Sorbi [70], Bianchini and Sorbi [9], and Lewis, Shore, and Sorbi [39]. See also [74] for a summary of many of the results appearing in these works. We consider the filters $\mathfrak{F}$ and $\mathfrak{F}_{\mathrm{cl}}$.

Definition 3.4.1.

- A degree $\mathbf{a}$ is called *dense* (*closed*) if $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$ for an $\mathcal{A}$ that is dense (closed) in ω^ω.
- $\mathfrak{I}$ denotes the ideal generated by $\{\mathbf{a} \mid \mathbf{a} \text{ is dense}\}$.
- $\mathfrak{F}$ denotes $\mathcal{D}_{\mathrm{s}} \setminus \mathfrak{I}$.
- $\mathfrak{F}_{\mathrm{cl}}$ denotes the filter generated by $\{\mathbf{a} \mid \mathbf{a} >_{\mathrm{s}} \mathbf{0} \text{ and is closed}\}$.

The join and meet of two dense degrees is dense [22], and the join and meet of two closed degrees is closed [9]. It follows that $\mathfrak{I} = \{\mathbf{b} \mid (\exists \mathbf{a} \geq_{\mathrm{s}} \mathbf{b})(\mathbf{a} \text{ is dense})\}$ and $\mathfrak{F}_{\mathrm{cl}} = \{\mathbf{b} \mid (\exists \mathbf{a} \leq_{\mathrm{s}} \mathbf{b})(\mathbf{a} >_{\mathrm{s}} \mathbf{0} \text{ and is closed})\}$. The basic properties of $\mathfrak{I}$, $\mathfrak{F}$, and $\mathfrak{F}_{\mathrm{cl}}$ are as follows. $\mathfrak{I}$ is a prime ideal [70], $\mathfrak{F}$ is a prime filter [9], $\mathfrak{I}$ is not principal [22], $\mathfrak{F}$ and $\mathfrak{F}_{\mathrm{cl}}$ are not principal [9], and $\mathfrak{F}_{\mathrm{cl}} \subsetneq \mathfrak{F}$ [9]. Both [9] and [74] ask for a proof that $\mathfrak{F}_{\mathrm{cl}}$ is not prime. We provide a proof of this fact now.

Lemma 3.4.2. *For any* $f \in \omega^\omega$ *there are* $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ *such that* $\mathcal{A} + \mathcal{B} \geq_{\mathrm{s}} \{f\}$ *and, for any closed* $\mathcal{C} \subseteq \omega^\omega$*, if* $\mathcal{A} \geq_{\mathrm{s}} \mathcal{C}$ *or* $\mathcal{B} \geq_{\mathrm{s}} \mathcal{C}$*, then* $\mathcal{C}$ *contains a recursive function.*

Proof. Fix a recursive bijection $\omega \leftrightarrow \omega^{<\omega}$. For $e, n \in \omega$, if

$$\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n^\frown \tau)(m)\downarrow),$$

then define $\eta(e,n,i) \in \omega^{<\omega}$ by induction on $i \in \omega$ as follows. Let $\eta(e,n,0) = n^\frown \sigma$, where σ is the least string such that $\Phi_e(n^\frown \sigma)(0)\downarrow$. Given $\eta(e,n,i)$, let $\eta(e,n,i+1) = \eta(e,n,i)^\frown 0^\frown \sigma$, where σ is the least string such that $\Phi_e(\eta(e,n,i)^\frown 0^\frown \sigma)(i+1)\downarrow$.

Let $f \in \omega^\omega$. We construct $\mathcal{A}$ and $\mathcal{B}$ such that the following conditions hold.

- If $g \in \mathcal{A}$, then $g(0)$ has the form

  $$g(0) = \langle \ell, \langle n_0, x_0, y_0 \rangle, \ldots, \langle n_{\ell-1}, x_{\ell-1}, y_{\ell-1} \rangle \rangle,$$

  where $\ell \in \omega$ and $n_i \in \omega$, $x_i \in \{0,1\}$, and $y_i \in \omega$ for each $i < \ell$.
- If $g \in \mathcal{A}$ and $\langle n_e, 0, y_e \rangle$ is in the e^{th} position of $g(0)$, then
  - $\exists m \exists \sigma (\forall \tau \supseteq \sigma)(\Phi_e({n_e}^\frown \tau)(m)\uparrow)$, and
  - any $h \in \mathcal{B}$ with $h(0) = n_e$ is of the form $h = {n_e}^\frown \sigma^\frown f$, where $|\sigma| = y_e$.
- If $g \in \mathcal{A}$ and $\langle n_e, 1, y_e \rangle$ is the e^{th} position of $g(0)$, then
  - $\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e({n_e}^\frown \tau)(m)\downarrow)$, and
  - any $h \in \mathcal{B}$ with $h(0) = n_e$ is of the form $h = \eta(e, n_e, i)^\frown 1^\frown f$ for some $i \in \omega$.
- The above properties hold with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed.

We construct $\mathcal{A}$ and $\mathcal{B}$ in stages. The construction is similar to the construction in Lemma 3.1.2 in that if g goes into $\mathcal{A}$ before h goes into $\mathcal{B}$, then $h(0)$ codes how to recover f from g, and similarly with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed. Start at stage 0 with $\mathcal{A} = \emptyset$, $\mathcal{B} = \emptyset$, $s = \langle\,\rangle$, and $t = \langle\,\rangle$.

Stage $e+1$: Set $n_e = e^\frown t$.

Case 1: $\exists m \exists \sigma (\forall \tau \supseteq \sigma)(\Phi_e({n_e}^\frown \tau)(m)\uparrow)$. Choose such a σ and put ${n_e}^\frown \sigma^\frown f$ in $\mathcal{A}$. Update $s = s^\frown \langle n_e, 0, |\sigma| \rangle$.

Case 2: $\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e({n_e}^\frown \tau)(m)\downarrow)$. Put the functions $\eta(e, n_e, i)^\frown 1^\frown f$ in $\mathcal{A}$ for each $i \in \omega$. Update $s = s^\frown \langle n_e, 1, 0 \rangle$.

Repeat the above procedure with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed and the roles of s and t reversed. This completes stage $e+1$. Then go on to stage $e+2$. This completes the construction.

Suppose $\mathcal{A} \geq_{\mathrm{s}} \mathcal{C}$ where $\mathcal{C}$ is closed. We show that $\mathcal{C}$ contains a recursive function. The proof with $\mathcal{B}$ in place of $\mathcal{A}$ is the same. Let $\Phi_e(\mathcal{A}) \subseteq C$. Consider stage $e+1$ of the above construction. Case 1 must not have occurred because otherwise $\mathcal{A}$ would contain a function ${n_e}^\frown \sigma^\frown f$ such that $\Phi_e({n_e}^\frown \sigma^\frown f)$ is not total. Thus case 2 occurred, and so $\mathcal{A}$ contains the function $\eta(e, n_e, i)^\frown 1^\frown f$ for each $i \in \omega$. Let k be the recursive function $k = {n_e}^\frown \sigma_0{}^\frown 0^\frown \sigma_1{}^\frown 0^\frown \sigma_2{}^\frown 0^\frown \cdots$, where $\eta(e, n_e, i) =$

$n_e{}^\frown\sigma_0{}^\frown 0^\frown\cdots{}^\frown 0^\frown\sigma_i$ for each $i \in \omega$ (think of k as the "limit" of the strings $\eta(e, n_e, i)$ as $i \to \infty$). Then $\Phi_e(\eta(e, n_e, i)^\frown 1^\frown f) \in \mathcal{C}$ and $\Phi_e(\eta(e, n_e, i)^\frown 1^\frown f) \restriction i = \Phi_e(k) \restriction i$ for each $i \in \omega$. Thus $\mathcal{C}$ contains the recursive function $\Phi_e(k)$ because $\mathcal{C}$ is closed.

We now describe a uniform procedure for producing f from $g \oplus h \in \mathcal{A} + \mathcal{B}$. First decode $h(0)$ as $h(0) = \langle \ell, \langle n_0, x_0, y_0 \rangle, \dots, \langle n_{\ell-1}, x_{\ell-1}, y_{\ell-1} \rangle \rangle$ and look for $g(0)$ among the n_e. If $\langle g(0), 0, y_e \rangle$ appears in $h(0)$ at position e, then output g from position $y_e + 1$ onward as in this case $g = \sigma^\frown f$ for some string σ of length $y_e + 1$. If $\langle g(0), 1, 0 \rangle$ appears in $h(0)$ at position e, then $g = \eta(e, g(0), i)^\frown 1^\frown f$ for some $i \in \omega$. Compute which i by successively computing the $\eta(e, g(0), j)$, matching them against g, and checking if the next bit of g is 0 (in which case compute $\eta(e, g(0), j+1)$) or 1 (in which case $j = i$). Output f once i is found.

The number $g(0)$ appears among the n_e coded into $h(0)$ if g went into $\mathcal{A}$ before h went into $\mathcal{B}$. Otherwise h went into $\mathcal{B}$ before g went into $\mathcal{A}$, so $h(0)$ appears among the n_e coded in $g(0)$. In this case, switch the roles of g and h and apply the above procedure to compute f. □

Theorem 3.4.3. *$\mathfrak{F}_{\mathrm{cl}}$ is not prime. In fact, if $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$, $\mathfrak{G} \neq \{\mathbf{1}\}$ is a filter, then $\mathfrak{G}$ is not prime.*

Proof. Suppose $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$ is a filter such that $\mathfrak{G} \neq \{\mathbf{1}\}$. Let $f >_{\mathrm{T}} 0$ be such that $\deg_{\mathrm{s}}(\{f\}) \in \mathfrak{G}$. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ be as in Lemma 3.4.2 for this f. Let $\mathbf{a} = \deg_{\mathrm{s}}(\mathcal{A})$ and $\mathbf{b} = \deg_{\mathrm{s}}(\mathcal{B})$. Then $\mathbf{a}, \mathbf{b} \notin \mathfrak{G}$ because $\mathbf{a}, \mathbf{b} \notin \mathfrak{F}_{\mathrm{cl}}$, but $\mathbf{a} + \mathbf{b} \in \mathfrak{G}$ because $\mathbf{a} + \mathbf{b} \geq_{\mathrm{s}} \deg_{\mathrm{s}}(\{f\})$. □

If $\mathbf{x}$ and $\mathbf{y}$ are degrees such that $\mathbf{y}$ is closed and $\mathbf{y} \not\leq_{\mathrm{s}} \mathbf{x}$, then there is no dense degree $\mathbf{z}$ such that $\mathbf{y} \leq_{\mathrm{s}} \mathbf{x} + \mathbf{z}$ [39]. Therefore, if $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$, $\mathfrak{G} \neq \{\mathbf{1}\}$ is a filter, then any degrees $\mathbf{a}$ and $\mathbf{b}$ witnessing that $\mathfrak{G}$ is not prime must both be in $\mathfrak{F} \setminus \mathfrak{G}$.

The results of the previous section suggest two new filters to study.

Definition 3.4.4.

- $\mathfrak{G}$ denotes the filter generated by

$$\{\mathbf{d} \mid \mathbf{d} >_{\mathrm{s}} \mathbf{0}' \text{ and is join-irreducible}\}.$$

- $\mathfrak{H}$ denotes the filter generated by

$$\left\{\deg_{\mathrm{s}}\Bigl(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\Bigr) \;\Big|\; (\forall i \in \omega)(\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}' \text{ and is join-irreducible})\right\}.$$

$\mathfrak{G}$ is exactly the set of all degrees $\mathbf{b}$ for which $\mathbf{b} \geq_{\mathrm{s}} \prod_{i=0}^{n} \mathbf{d}_i$ for some join-irreducible degrees $\mathbf{d}_i >_{\mathrm{s}} \mathbf{0}'$, $i \leq n$, and $\mathfrak{H}$ is exactly the set of all degrees $\mathbf{b}$ for which $\mathbf{b} \geq_{\mathrm{s}} \deg_{\mathrm{s}}\bigl(\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\bigr)$ for some join-irreducible degrees $\deg_{\mathrm{s}}(\mathcal{D}_i) >_{\mathrm{s}} \mathbf{0}'$, $i \in \omega$.

Theorem 3.4.5. *$\mathfrak{F}_{\mathrm{cl}} \subsetneq \mathfrak{G} \subsetneq \mathfrak{H} \subsetneq \{\mathbf{a} \mid \mathbf{a} >_{\mathrm{s}} \mathbf{0}'\}$. $\mathfrak{G} \not\subseteq \mathfrak{F}$ (hence also $\mathfrak{H} \not\subseteq \mathfrak{F}$). Neither $\mathfrak{G}$ nor $\mathfrak{H}$ is principal.*

Proof. Every closed degree $>_{\mathrm{s}} \mathbf{0}$ bounds a join-irreducible degree $>_{\mathrm{s}} \mathbf{0}'$ [75]. Hence $\mathfrak{F}_{\mathrm{cl}} \subseteq \mathfrak{G}$. $\mathfrak{G} \subseteq \mathfrak{H}$ is clear. To see $\mathfrak{G} \nsubseteq \mathfrak{F}$, observe that every $\mathcal{B}_f$ is dense, so if $f >_{\mathrm{T}} 0$, then $\deg_{\mathrm{s}}(\mathcal{B}_f) \in \mathfrak{G} \setminus \mathfrak{F}$. This also shows $\mathfrak{G} \nsubseteq \mathfrak{F}_{\mathrm{cl}}$. The degree constructed in Theorem 3.2.8 witnesses $\mathfrak{H} \nsubseteq \mathfrak{G}$. The degree constructed in Theorem 3.2.10 witnesses $\{\mathbf{a} \mid \mathbf{a} >_{\mathrm{s}} \mathbf{0}'\} \nsubseteq \mathfrak{H}$. We show that $\mathfrak{G}$ is not principal. The proof for $\mathfrak{H}$ is the same. First, if $\mathcal{A}$ is countable and contains no recursive functions, then there is a function $f >_{\mathrm{T}} 0$ such that $g \nleq_{\mathrm{T}} f$ for all $g \in \mathcal{A}$. Thus $\mathcal{B}_f \leq_{\mathrm{s}} \mathcal{A}$ (as $\mathcal{A} \subseteq \mathcal{B}_f$) for this f. Every $\deg_{\mathrm{s}}(\mathcal{B}_f)$ for $f >_{\mathrm{T}} 0$ is in $\mathfrak{G}$, so every $\deg_{\mathrm{s}}(\mathcal{A})$ where $\mathcal{A}$ is countable and contains no recursive function is in $\mathfrak{G}$. If $\mathfrak{G}$ were principal, it would be generated by a degree $\mathbf{x}$ such that $\mathbf{x} \leq_{\mathrm{s}} \deg_{\mathrm{s}}(\mathcal{A})$ for all countable $\mathcal{A}$ not containing a recursive function. By Lemma 3.3.2, the only such $\mathbf{x}$ are $\mathbf{0}$ and $\mathbf{0}'$. We know $\mathbf{0}$ and $\mathbf{0}'$ are not in $\mathfrak{G}$, so $\mathfrak{G}$ cannot be principal. ☐

We end with a question.

Question 3.4.6.

- Is $\mathfrak{F} \subseteq \mathfrak{G}$? Is $\mathfrak{F} \subseteq \mathfrak{H}$?
- Is $\mathfrak{G}$ prime? Is $\mathfrak{H}$ prime?
- Is $\{\mathbf{a} \mid \mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \subseteq \mathrm{JAN}\}$ a filter?

To prove that $\{\mathbf{a} \mid \mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \subseteq \mathrm{JAN}\}$ is a filter, it suffices to prove that $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / (\mathbf{a} \times \mathbf{b})) \subseteq \mathrm{JAN}$ whenever both $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a})$ and $\mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{b})$ are $\subseteq \mathrm{JAN}$ because $\{\mathbf{a} \mid \mathrm{PTh}(\mathcal{D}_{\mathrm{s}} / \mathbf{a}) \subseteq \mathrm{JAN}\}$ is upward-closed in $\mathcal{D}_{\mathrm{s}}$.

CHAPTER 4
FORCING NO BIG CHAINS IN THE POWER SET OF THE REALS

Let $\kappa \leq \lambda$ be infinite cardinals. Independence phenomena concerning the question of whether or not there is a chain (i.e. linearly ordered subset) in $(2^\kappa, \subseteq)$ of cardinality λ were studied extensively by Baumgartner [8] and Mitchell [44]. However, the particular case of $\kappa = 2^\omega$ and $\lambda = 2^{2^\omega}$ was not discussed explicitly. In the positive direction, the following results appear in Baumgartner [8]. Baumgartner attributes his techniques to Sierpinski.

Theorem 4.0.7 (see [8])**.** *If $2^{<\kappa} = \kappa$, then there is a chain in $(2^\kappa, \subseteq)$ of cardinality 2^κ.*

Corollary 4.0.8 (see [8])**.** $\mathrm{ZFC} + \mathrm{CH}$ *proves that there is a chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω}. Thus if* ZFC *is consistent, then so is* ZFC *+ "there is a chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω}."*

Proof. CH implies $2^{<\omega_1} = \omega_1$. So apply Theorem 4.0.7 with $\kappa = \omega_1$. ☐

In the negative direction, Comfort and Remus noticed in [20] that the consistency of ZFC + "there is no chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω}" indeed follows from Baumgartner's results in [8]. The main purpose of this chapter is to give a more explicit proof of the consistency of ZFC + "there is no chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω}" (Corollary 4.2.6). We thank both Todorcevic and Mitchell for independently suggesting the proof that we give.

The author was introduced to the foregoing problem via the Medvedev degrees. In [77], Terwijn uses Corollary 4.0.8 to prove that if ZFC is consistent, then so is ZFC + "there is a chain in the Medvedev degrees of cardinality 2^{2^ω}." We prove the following theorem in ZFC. For a cardinal λ, $\left(2^{2^\omega}, \subseteq\right)$ has a chain of cardinality λ if and only if the Medvedev degrees do (Theorem 4.3.1). Thus the statement "there is a chain in the Medvedev degrees of cardinality 2^{2^ω}" is independent of ZFC.

4.1 Forcing prerequisites

This section contains the basic facts about forcing and about chain conditions that we need in the next section. We follow [35] when possible. The reader familiar with this material may skip ahead.

First, a partial order may be replaced by an isomorphic partial order without changing the forcing extension.

Lemma 4.1.1 (see [35] section VII.7)**.** *Let $\mathbb{P}, \mathbb{Q} \in M$ and let $i\colon \mathbb{P} \to \mathbb{Q}$ be an isomorphism in M. Let $G \subseteq \mathbb{P}$. Then G is $\mathbb{P}$-generic over M if and only if $i''G$ is $\mathbb{Q}$-generic over M. In this case $M[G] = M[i''G]$.*

Forcing with a partial order $\mathbb{P} \times \mathbb{Q}$ is equivalent to forcing with the partial orders $\mathbb{P}$ and $\mathbb{Q}$ one-at-a-time in either order.

Lemma 4.1.2 (see [35] section VIII.1). *Let $\mathbb{P}, \mathbb{Q} \in M$ and let $G_0 \subseteq \mathbb{P}$, $G_1 \subseteq \mathbb{Q}$. Then the following are equivalent:*

(i) $G_0 \times G_1$ is $\mathbb{P} \times \mathbb{Q}$-generic over M,

(ii) G_0 is $\mathbb{P}$-generic over M and G_1 is $\mathbb{Q}$-generic over $M[G_0]$, and

(iii) G_1 is $\mathbb{Q}$-generic over M and G_0 is $\mathbb{P}$-generic over $M[G_1]$.

If (i)–(iii) hold, then $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$. Moreover, if G is $\mathbb{P} \times \mathbb{Q}$-generic over M, then $G = H_0 \times H_1$ for some $H_0 \subseteq \mathbb{P}$ and $H_1 \subseteq \mathbb{Q}$ which thus satisfy (i)–(iii).

Let κ be a cardinal. A partial order $\mathbb{P}$ is *κ-closed* if every descending chain of conditions having order-type $< \kappa$ has a lower bound. A partial order $\mathbb{P}$ has the *κ-chain condition* (abbreviated *κ-cc*) if every $X \subseteq \mathbb{P}$ of cardinality κ has two compatible elements. Notice that $\mathbb{P}$ is κ-cc if and only if for every sequence of conditions $(p_\xi \mid \xi \in \kappa)$ there are distinct $\xi, \zeta \in \kappa$ such that p_ξ and p_ζ are compatible. A partial order $\mathbb{P}$ has *property K_κ* if every $X \subseteq \mathbb{P}$ of cardinality κ has a pairwise compatible subset of cardinality κ. Notice that if κ is regular, then $\mathbb{P}$ has property K_κ if and only if for every sequence of conditions $(p_\xi \mid \xi \in \kappa)$ there is a set $X \subseteq \kappa$ of cardinality κ such that if $\xi, \zeta \in X$ then p_ξ and p_ζ are compatible. Clearly property K_κ implies κ-cc. We consider $\kappa = \omega_1$ and $\kappa = \omega_2$. Traditionally ω_1-cc is called ccc and property K_{ω_1} is called property K.

Partial orders that are κ-closed preserve cardinals $\leq \kappa$, and partial orders that are κ-cc preserve cardinals $\geq \kappa$. It follows that a partial order that is both κ-closed and κ^+-cc preserves all cardinals.

The following lemmas easily generalize to other cardinals, but we state the versions that we use which are in terms of ω_2.

Lemma 4.1.3. *If $\mathbb{P}$ and $\mathbb{Q}$ have property K_{ω_2}, then so does $\mathbb{P} \times \mathbb{Q}$.*

Proof. Let $((p_\xi, q_\xi) \mid \xi \in \omega_2)$ be a sequence of conditions from $\mathbb{P} \times \mathbb{Q}$. Applying $\mathbb{P}$'s property K_{ω_2} to the sequence $(p_\xi \mid \xi \in \omega_2)$ yields a set $Y \subseteq \omega_2$ such that $|Y| = \omega_2$ and p_ξ and p_ζ are compatible for every $\xi, \zeta \in Y$. Applying $\mathbb{Q}$'s property K_{ω_2} to the sequence $(q_\xi \mid \xi \in Y)$ yields a set $X \subseteq Y$ such that $|X| = \omega_2$ and q_ξ and q_ζ are compatible for every $\xi, \zeta \in X$. The conditions (p_ξ, q_ξ) and (p_ζ, q_ζ) are compatible for every ξ, ζ in X, so X witnesses property K_{ω_2} for the sequence $((p_\xi, q_\xi) \mid \xi \in \omega_2)$. ☐

Lemma 4.1.4. *If $\mathbb{P}$ is ω_2-cc and $X \subseteq \mathbb{P}$ has cardinality ω_2, then there is a $p \in \mathbb{P}$ such that $p \Vdash |\dot{G} \cap \check{X}| = |\check{\omega}_2|$.*

Proof. Fix a bijection $f : \omega_2 \leftrightarrow X$. If there is no such p, then there is a name $\dot{\alpha}$ such that

$$1_{\mathbb{P}} \Vdash (\dot{\alpha} \in \check{\omega}_2) \wedge (\forall \xi \in \check{\omega}_2)(\check{f}(\xi) \in \dot{G} \cap \check{X} \rightarrow \xi \in \dot{\alpha}).$$

Now let $A \subseteq \mathbb{P}$ be a maximal antichain of conditions with the property $(\forall p \in A)(\exists \beta \in \omega_2)(p \Vdash \dot{\alpha} = \check{\beta})$. We have that $|A| < \omega_2$ because $\mathbb{P}$ is ω_2-cc. Thus we are able to choose $\gamma \in \omega_2$ such that if $p \in A$ and $p \Vdash \dot{\alpha} = \check{\beta}$ then $\beta \in \gamma$. Clearly $f(\gamma) \Vdash \check{f}(\check{\gamma}) \in \dot{G} \cap \check{X}$; therefore $f(\gamma) \Vdash \check{\gamma} \in \dot{\alpha}$. Choose $p \leq f(\gamma)$ such that $p \Vdash \dot{\alpha} = \check{\beta}$ for some $\beta \in \omega_2$. This β must be $\geq \gamma$. Hence p is incompatible with everything in A, thereby contradicting A's maximality. $\square$

Lemma 4.1.4 is stated in its most natural way, but we make greater use of the following reformulation.

Lemma 4.1.5. *Let $\mathbb{P}$ be ω_2-cc, let $X \subseteq \omega_2$ have cardinality ω_2, and let $(p_\xi \mid \xi \in X)$ be a sequence of conditions from $\mathbb{P}$. Let $\dot{Z} = \{\langle \check{\xi}, p_\xi \rangle \mid \xi \in X\}$ be a name for the set $\{\xi \in X \mid p_\xi \in G\}$. Then there is a $p \in \mathbb{P}$ such that $p \Vdash |\dot{Z}| = |\check{\omega}_2|$.*

Proof. If $|\{p_\xi \mid \xi \in X\}| < \omega_2$, choose p such that $|\{\xi \in X \mid p_\xi = p\}| = \omega_2$. Otherwise choose p by applying Lemma 4.1.4 to the set $\{p_\xi \mid \xi \in X\}$. $\square$

A partial order with property K_{ω_2} retains property K_{ω_2} in an extension created by forcing with an ω_2-cc partial order.

Lemma 4.1.6. *If $\mathbb{P}$ is ω_2-cc and $\mathbb{Q}$ has property K_{ω_2}, then $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ has property K_{ω_2}.*

Proof. Suppose not. Then there is a $\mathbb{P}$-name $\dot{f}$ and a condition $p \in \mathbb{P}$ such that

$$p \Vdash (\dot{f} \colon \check{\omega}_2 \to \check{\mathbb{Q}}) \wedge (\forall X \subseteq \check{\omega}_2)(|X| = |\check{\omega}_2| \to (\exists \xi, \zeta \in X)(\dot{f}(\xi) \perp \dot{f}(\zeta))).$$

That is, p forces that the sequence $(f(\xi) \mid \xi \in \omega_2)$ of conditions from $\mathbb{Q}$ is a counterexample to $\mathbb{Q}$ having property K_{ω_2} (as $\mathbb{P}$ preserves ω_2). For each $\xi \in \omega_2$, choose a condition $p_\xi \in \mathbb{P}$ and a condition $q_\xi \in \mathbb{Q}$ such that $p_\xi \leq p$ and $p_\xi \Vdash \dot{f}(\check{\xi}) = \check{q}_\xi$. This defines a sequence of conditions $(p_\xi \mid \xi \in \omega_2)$ from $\mathbb{P}$ and a sequence of conditions $(q_\xi \mid \xi \in \omega_2)$ from $\mathbb{Q}$.

We show that the sequence $(q_\xi \mid \xi \in \omega_2)$ is a counterexample to $\mathbb{Q}$ having property K_{ω_2}. To this end, let $X \subseteq \omega_2$ have cardinality ω_2, let $\dot{Z}$ be a $\mathbb{P}$-name for the set $\{\xi \in X \mid p_\xi \in G\}$, and apply Lemma 4.1.5 to the sequence $(p_\xi \mid \xi \in X)$ to get a condition $p' \leq p$ such that $p' \Vdash |\dot{Z}| = |\check{\omega}_2|$. We now have that $p' \Vdash (\dot{Z} \subseteq \check{\omega}_2) \wedge (|\dot{Z}| = |\check{\omega}_2|)$. Thus, as $p' \leq p$, we also have that $p' \Vdash (\exists \xi, \zeta \in \dot{Z})(\dot{f}(\xi) \perp \dot{f}(\zeta))$. Let $\dot{\xi}$ and $\dot{\zeta}$ be $\mathbb{P}$-names such that $p' \Vdash (\dot{\xi}, \dot{\zeta} \in \dot{Z}) \wedge (\dot{f}(\dot{\xi}) \perp \dot{f}(\dot{\zeta}))$. Finally, choose $p'' \leq p'$ and choose $\xi, \zeta \in X$ such that $p'' \Vdash (\dot{\xi} = \check{\xi}) \wedge (\dot{\zeta} = \check{\zeta})$. Then $p'' \Vdash \check{\xi}, \check{\zeta} \in \dot{Z}$, and so, by the definition of $\dot{Z}$, $p'' \Vdash \check{p}_\xi, \check{p}_\zeta \in \dot{G}$. Hence $p'' \Vdash (\dot{f}(\check{\xi}) \perp \dot{f}(\check{\zeta})) \wedge (\dot{f}(\check{\xi}) = \check{q}_\xi) \wedge (\dot{f}(\check{\zeta}) = \check{q}_\zeta)$. So it must be that $q_\xi \perp q_\zeta$. $\square$

We work with partial orders of the form $\mathrm{Fn}(I, 2, \kappa)$, which is the set of partial functions $p \colon I \to 2$ of cardinality $< \kappa$ ordered by $p \leq q$ if and only if p extends q. If $|I| \geq \kappa$, forcing with $\mathrm{Fn}(I, 2, \kappa)$ adds $|I|$ subsets of κ, thereby making $2^\kappa \geq |I|$ in the extension (see [35] section VII.6).

In order to show that our partial orders have the desired chain conditions, we recall the notion of a Δ-*system* and the corresponding Δ-*system lemma*. A set A is called a Δ-*system* if there is a set r such that $a \cap b = r$ whenever a and b are distinct members of A.

Lemma 4.1.7 (Δ-system lemma; see [35] section II.1)**.** *Let κ be an infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $(\forall \alpha < \theta)(|\alpha^{<\kappa}| < \theta)$. Assume $|A| \geq \theta$ and $(\forall x \in A)(|x| < \kappa)$. Then there is a Δ-system $B \subseteq A$ with $|B| = \theta$.*

Finally, in order to show that the cardinal arithmetic works out as desired in our extensions, we recall that sets in an extension $M[G]$ have *nice names* in M. If $\mathbb{P} \in M$ is a partial order and σ is a name, then a *nice name* for a subset of σ is a name τ of the form $\bigcup\{\{\pi\} \times A_\pi \mid \pi \in \operatorname{dom}(\sigma)\}$ where each A_π is an antichain in $\mathbb{P}$.

Lemma 4.1.8 (see [35] section VII.5)**.** *If $\mathbb{P} \in M$ is a partial order and σ, μ are names, then there is a nice name τ for a subset of σ such that $1_{\mathbb{P}} \Vdash (\mu \subseteq \sigma) \rightarrow (\mu = \tau)$.*

A nice name for a subset of an existing set $x \in M$ is particularly nice. A nice name τ for a subset of the canonical name $\check{x} = \{(\check{y}, 1_{\mathbb{P}}) \mid y \in x\}$ can be taken to be of the form $\bigcup\{\{\check{y}\} \times A_y \mid y \in x\}$.

4.2 Forcing no chains in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω}

Recall that a *(set-theoretic) tree* is a partial order $(T, \leq_T)$ such that, for each $t \in T$, the set of predecessors $\{s \in T \mid s <_T t\}$ is well-ordered by $<_T$. We usually write T for $(T, \leq_T)$. The *level* $\ell(t)$ of a $t \in T$ is the order-type of $\{s \in T \mid s <_T t\}$, the ξ^{th} *level* of T is $\mathrm{Lev}_\xi = \{t \in T \mid \ell(t) = \xi\}$, and the *height* of T is the least ξ such that $\mathrm{Lev}_\xi = \emptyset$. A *branch* through T is a maximal linearly ordered subset of T. The *length* of a branch is its order-type. We rephrase our question about chains as a question about trees via the following theorem.

Theorem 4.2.1 (Baumgartner, Mitchell; see [8])**.** *Let $\kappa \leq \lambda$ be infinite cardinals. The following are equivalent:*

- *There is a tree of height $\leq \kappa$ and cardinality $\leq \kappa$ with $\geq \lambda$ branches.*
- *There is a chain in $(2^\kappa, \subseteq)$ of cardinality λ.*

In light of Theorem 4.2.1 (with $\kappa = 2^\omega$ and $\lambda = 2^{2^\omega}$), we force there to be no chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω} by forcing there to be no tree of height $\leq 2^\omega$ and cardinality $\leq 2^\omega$ with $\geq 2^{2^\omega}$ branches. Our proof is similar to Silver's proof that one can force there to be no Kurepa trees given that there is an inaccessible cardinal (see [35] section VIII.3).

Our strategy is as follows.

(i) Let $M \models \mathrm{GCH}$, and, in M, let $\mathbb{P} = \mathrm{Fn}(\omega_2, 2, \omega)$ and $\mathbb{Q} = \mathrm{Fn}(\omega_{\omega_2}, 2, \omega_1)$. Force with $\mathbb{P} \times \mathbb{Q}$.

(ii) $\mathbb{P} \times \mathbb{Q}$ preserves cardinals. If G is $\mathbb{P} \times \mathbb{Q}$-generic over M, then, in $M[G]$, $2^\omega = \omega_2$, $2^{\omega_1} = \omega_{\omega_2}$, and $2^{\omega_2} = \omega_{\omega_2}^+$.

(iii) Suppose $T \subseteq \omega_2 \times \omega_2$ is a tree in $M[G]$ of height $\leq \omega_2$ and of cardinality $\leq \omega_2$. T has $\leq |\omega_2^{<\omega_2}| = \omega_{\omega_2}$ branches of cardinality $< \omega_2$. Thus we need only show that T has $< \omega_{\omega_2}^+$ branches of cardinality ω_2. To do this, factor $\mathbb{P} \times \mathbb{Q}$ into an isomorphic partial order $\mathbb{R}_0 \times \mathbb{R}_1$ with the following properties.

- $M[G] = M[G_0][G_1]$ where G_0 is $\mathbb{R}_0$-generic over M and G_1 is $\mathbb{R}_1$-generic over $M[G_0]$,
- $T \in M[G_0]$, and, in $M[G_0]$, T has $\leq \omega_3$ branches, and
- any branch of T in $M[G_0][G_1]$ of cardinality ω_2 is already in $M[G_0]$.

Thus T has $\leq \omega_3 < \omega_{\omega_2}^+$ branches of cardinality ω_2 in $M[G]$.

Let M, $\mathbb{P}$, and $\mathbb{Q}$ be as in step (i) of the strategy.

Lemma 4.2.2. *In M, the partial order $\mathbb{P}$ has property K, and the partial orders $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{P} \times \mathbb{Q}$ all have property K_{ω_2}.*

Proof. To see that $\mathbb{P}$ has property K, let $X \subseteq \mathbb{P}$ have cardinality ω_1 and let A be the set of domains of the functions in X. Then $|A| = \omega_1$ and $(\forall x \in A)(|x| < \omega)$. As $(\forall \alpha < \omega_1)(|\alpha^{<\omega}| < \omega_1)$, Lemma 4.1.7 gives us a Δ-system $B \subseteq A$ of cardinality ω_1 with finite root r. There are only a finite number of functions $f\colon r \to 2$, so there must be one such f for which the set $\{g \in X \mid (\operatorname{dom} g \in B) \wedge (g \supseteq f)\}$ has cardinality ω_1. This is the desired pairwise compatible subset of X. The proof that $\mathbb{P}$ has property K_{ω_2} is the same as the above, but with ω_2 in place of ω_1.

For $\mathbb{Q}$, let $X \subseteq \mathbb{Q}$ have cardinality ω_2, and let A be the set of domains of the functions in X. $|A| = \omega_2$ because for a fixed countable $x \subseteq \omega_{\omega_2}$ there are only $2^\omega = \omega_1$ possible functions $x \to 2$. We have $(\forall x \in A)(|x| < \omega_1)$ and, as $|\omega_1^{<\omega_1}| = \omega_1$, we have $(\forall \alpha < \omega_2)(|\alpha^{<\omega_1}| < \omega_2)$. Thus Lemma 4.1.7 gives us a Δ-system $B \subseteq A$ of cardinality ω_2 with countable root r. There are only ω_1 functions $f\colon r \to 2$, so there must be one such f for which the set $\{g \in X \mid (\operatorname{dom} g \in B) \wedge (g \supseteq f)\}$ has cardinality ω_2. This is the desired pairwise compatible subset of X.

Finally, $\mathbb{P} \times \mathbb{Q}$ has property K_{ω_2} by Lemma 4.1.3. ☐

The next lemma completes step (ii).

Lemma 4.2.3. *$\mathbb{P} \times \mathbb{Q}$ preserves cardinals. If G is $\mathbb{P} \times \mathbb{Q}$-generic over M, then, in $M[G]$, $2^\omega = \omega_2$, $2^{\omega_1} = \omega_{\omega_2}$, and $2^{\omega_2} = \omega_{\omega_2}^+$.*

Proof. We use Lemma 4.1.2 to equate forcing with $\mathbb{P} \times \mathbb{Q}$ to forcing with $\mathbb{Q}$ then with $\mathbb{P}$. Let G be $\mathbb{P} \times \mathbb{Q}$-generic over M. Factor G as $G = G_0 \times G_1$ where $G_0 \subseteq \mathbb{P}$ and $G_1 \subseteq \mathbb{Q}$ so that $M[G] = M[G_1][G_0]$. Forcing with $\mathbb{Q}$ adds ω_{ω_2} subsets of ω_1, and $\mathbb{Q}$ is both ω_1-closed and ω_2-cc. Thus, in $M[G_1]$, $2^{\omega_1} \geq \omega_{\omega_2}$ and $M[G_1]$ has the same cardinals as M. Forcing cannot add any new finite functions, so $M[G_1]$'s version of $\mathrm{Fn}(\omega_2, 2, \omega)$ is exactly $\mathbb{P}$ because forcing with $\mathbb{Q}$ preserves ω_2. Below we need that $(\omega_2)^\omega = \omega_2$ in $M[G_1]$. This equality is true in M by GCH and remains true in $M[G_1]$ because of the fact that $\mathbb{Q}$ is ω_1-closed implies that forcing with $\mathbb{Q}$ adds no new functions $\omega \to \omega_2$.

$\mathbb{P}$ is ccc, and forcing with $\mathbb{P}$ adds ω_2 subsets of ω. Thus $M[G_1][G_0]$ has the same cardinals as $M[G_1]$, and, in $M[G_1][G_0]$, $2^\omega \geq \omega_2$. So we have that $M[G] = M[G_1][G_0]$ has the same cardinals as M (i.e. $\mathbb{P} \times \mathbb{Q}$ preserves cardinals) and that, in $M[G]$, $2^\omega \geq \omega_2$ and $2^{\omega_1} \geq \omega_{\omega_2}$. Furthermore, in $M[G]$, $2^{\omega_2} \geq \omega_{\omega_2}^+$ because $2^{\omega_2} \geq 2^{\omega_1} \geq \omega_{\omega_2}$ and 2^{ω_2} cannot have cofinality ω_2. We show the desired reverse inequalities by counting nice names.

To see $2^\omega \leq \omega_2$, count the nice $\mathbb{P}$-names for subsets of ω in $M[G_1]$. Such a name has the form $\bigcup\{\{\check{n}\} \times A_n \mid n \in \omega\}$ where each A_n is an antichain in $\mathbb{P}$. $\mathbb{P}$ is ccc and has cardinality ω_2, hence $\mathbb{P}$ has $(\omega_2)^\omega = \omega_2$ antichains. Therefore there are $\leq (\omega_2)^\omega = \omega_2$ nice names for subsets of ω in $M[G_1]$. Forcing with $\mathbb{P}$ preserves cardinals, so there are $\leq \omega_2$ subsets of ω in $M[G]$.

To see $2^{\omega_1} \leq \omega_{\omega_2}$, count the nice $\mathbb{P} \times \mathbb{Q}$-names for subsets of ω_1 in M. Such a name has the form $\bigcup\{\{\check{\alpha}\} \times A_\alpha \mid \alpha \in \omega_1\}$ where each A_α is an antichain in $\mathbb{P} \times \mathbb{Q}$. $\mathbb{P} \times \mathbb{Q}$ has property K_{ω_2} by Lemma 4.2.2, so its antichains have cardinality $\leq \omega_1$. The fact that GCH holds in M implies that $\mathbb{P} \times \mathbb{Q}$ has cardinality ω_{ω_2} and consequently that $\mathbb{P} \times \mathbb{Q}$ has $\leq (\omega_{\omega_2})^{\omega_1} = \omega_{\omega_2}$ antichains. Therefore there are $\leq (\omega_{\omega_2})^{\omega_1} = \omega_{\omega_2}$ nice names for subsets of ω_1 in M. Forcing with $\mathbb{P} \times \mathbb{Q}$ preserves cardinals, so there are $\leq \omega_{\omega_2}$ subsets of ω_1 in $M[G]$.

Finally, to see $2^{\omega_2} \leq \omega_{\omega_2}^+$, count the nice $\mathbb{P} \times \mathbb{Q}$-names for subsets of ω_2 in M. Such a name has the form $\bigcup\{\{\check{\alpha}\} \times A_\alpha \mid \alpha \in \omega_2\}$ where each A_α is an antichain in $\mathbb{P} \times \mathbb{Q}$. There are $\leq (\omega_{\omega_2})^{\omega_2} = \omega_{\omega_2}^+$ nice names for subsets of ω_1 in M. Forcing with $\mathbb{P} \times \mathbb{Q}$ preserves cardinals, so there are $\leq \omega_{\omega_2}^+$ subsets of ω_2 in $M[G]$. $\square$

The next lemma is the crux of step (iii).

Lemma 4.2.4. *Suppose, in some model N, that $T \subseteq \omega_2 \times \omega_2$ is a tree of height ω_2 and that $\mathbb{R}$ is a partial order with property K_{ω_2}. If G is $\mathbb{R}$-generic over N, then there are no new branches of T in $N[G]$ of cardinality ω_2.*

Proof. In N, let $B \subseteq 2^{\omega_2}$ be the set of branches through T and let $\dot{b}$ be a name for a branch through T. Suppose for a contradiction that there is a condition $p \in \mathbb{R}$ forcing that $\dot{b}$ is a new branch of cardinality ω_2. That is, $p \Vdash (\dot{b}$ is a branch$) \wedge (|\dot{b}| = |\check{\omega}_2|) \wedge (\dot{b} \notin \check{B})$ (note $\mathbb{R}$ preserves ω_2). For each $\xi < \omega_2$, we can find an $x_\xi \in \mathrm{Lev}_\xi$ and a $p_\xi \leq p$ such that $p_\xi \Vdash \check{x}_\xi \in \dot{b}$. Apply $\mathbb{R}$'s property K_{ω_2} to the sequence $(p_\xi \mid \xi \in \omega_2)$ to get a set $X \subseteq \omega_2$ such that $|X| = \omega_2$ and such that p_ξ and p_ζ are compatible for every $\xi, \zeta \in X$. Notice that if $\xi < \zeta$ are in X, then the compatibility of p_ξ and p_ζ implies that $x_\xi <_T x_\zeta$. Therefore the set $c = \{x \in \omega_2 \mid (\exists \xi \in X)(x \leq_T x_\xi)\}$ is a branch in B. Let $\dot{Z}$ be a name for the set $\{\xi \in X \mid p_\xi \in G\}$, and, by Lemma 4.1.5, choose $p' \leq p$ such that $p' \Vdash |\dot{Z}| = |\check{\omega}_2|$. We now show that $p' \Vdash \dot{b} = \check{c}$, which contradicts that $p \Vdash \dot{b} \notin \check{B}$. To this end, suppose that H is $\mathbb{R}$-generic with $p' \in H$, and now work in $N[H]$. Suppose that $x \in b \cap \mathrm{Lev}_\xi$. There is some $\zeta > \xi$ with $p_\zeta \in H$ because $|Z| = |\omega_2|$. Therefore $x_\zeta \in b$ because $p_\zeta \Vdash \check{x}_\zeta \in \dot{b}$. Hence $x <_T x_\zeta$ and so $x \in c$. This proves $b \subseteq c$, and equality follows from the fact that both b and c are branches of cardinality ω_2 in a tree of height ω_2. $\square$

We can now carry out step (iii).

Theorem 4.2.5. *If* ZFC *is consistent, then so is* ZFC + *"there is no tree of height* $\leq 2^\omega$ *and cardinality* $\leq 2^\omega$ *with* $\geq 2^{2^\omega}$ *branches."*

Proof. Let $M \models \mathrm{GCH}$. In M, let $\mathbb{P} = \mathrm{Fn}(\omega_2, 2, \omega)$ and let $\mathbb{Q} = \mathrm{Fn}(\omega_{\omega_2}, 2, \omega_1)$. Let G be $\mathbb{P} \times \mathbb{Q}$-generic over M. By Lemma 4.2.3,

$$M[G] \models (2^\omega = \omega_2) \wedge (2^{\omega_1} = \omega_{\omega_2}) \wedge (2^{\omega_2} = \omega_{\omega_2}^+).$$

We show that in $M[G]$ there is no tree of height $\leq \omega_2$ and cardinality $\leq \omega_2$ with $\geq \omega_{\omega_2}^+$ branches. A tree of cardinality ω_2 is isomorphic to a tree $\subseteq \omega_2 \times \omega_2$, so it suffices to show that in $M[G]$ there is no tree $\subseteq \omega_2 \times \omega_2$ of height $\leq \omega_2$ with $\geq \omega_{\omega_2}^+$ branches.

Suppose $T \subseteq \omega_2 \times \omega_2$ is a tree of height $\leq \omega_2$ in the extension $M[G]$. Then, in M, T has a nice name of the form $\dot{T} = \bigcup \{\{\check{s}\} \times A_s \mid s \in \omega_2 \times \omega_2\}$ where each A_s is an antichain in $\mathbb{P} \times \mathbb{Q}$, a partial order with property K_{ω_2} by Lemma 4.2.2. Thus each A_s has cardinality $\leq \omega_1$, and hence the set $X = \{\alpha \in \omega_{\omega_2} \mid (\exists (p,q) \in \bigcup_{s \in \omega_2 \times \omega_2} A_s)(\alpha \in \operatorname{dom} q)\}$ has cardinality $\leq \omega_2$. Add elements of ω_{ω_2} to X so that it has cardinality exactly ω_2. Let $\mathbb{Q}_X = \mathrm{Fn}(X, 2, \omega_1)$ and let $\mathbb{Q}_{\omega_{\omega_2} \setminus X} = \mathrm{Fn}(\omega_{\omega_2} \setminus X, 2, \omega_1)$. It is easy to check that $\mathbb{Q}_X \times \mathbb{Q}_{\omega_{\omega_2} \setminus X} \cong \mathbb{Q}$, so factor $\mathbb{P} \times \mathbb{Q}$ as $(\mathbb{P} \times \mathbb{Q}_X) \times \mathbb{Q}_{\omega_{\omega_2} \setminus X}$. In terms of step (iii) as described above, $\mathbb{P} \times \mathbb{Q}_X$ is $\mathbb{R}_0$ and $\mathbb{Q}_{\omega_{\omega_2} \setminus X}$ is $\mathbb{R}_1$. By Lemma 4.1.1, there is a $(\mathbb{P} \times \mathbb{Q}_X) \times \mathbb{Q}_{\omega_{\omega_2} \setminus X}$-generic set H such that $M[H] = M[G]$, and, by Lemma 4.1.2, $H = H_0 \times H_1$ where H_0 is $\mathbb{P} \times \mathbb{Q}_X$-generic over M and H_1 is $\mathbb{Q}_{\omega_{\omega_2} \setminus X}$-generic over $M[H_0]$. By our choice of X, the $\mathbb{P} \times \mathbb{Q}$-name $\dot{T}$ is also a $\mathbb{P} \times \mathbb{Q}_X$-name. So our tree T is in the intermediate extension $M[H_0]$. The models M, $M[H_0]$, and $M[H_0][H_1] = M[H] = M[G]$ all have the same cardinals because M and $M[G]$ do by Lemma 4.2.3. Also, $M[H_0] \models (2^\omega = \omega_2) \wedge (2^{\omega_1} = \omega_2) \wedge (2^{\omega_2} = \omega_3)$ by a proof similar to that of Lemma 4.2.3. Furthermore, $\mathbb{Q}_{\omega_{\omega_2} \setminus X}$ has property K_{ω_2} in $M[H_0]$ by Lemma 4.1.6 because $\mathbb{Q}_{\omega_{\omega_2} \setminus X}$ and $\mathbb{P} \times \mathbb{Q}_X$ have property K_{ω_2} in M by a proof similar to that of Lemma 4.2.2. Now, T has $\leq 2^{\omega_2} = \omega_3$ branches in $M[H_0]$. Lemma 4.2.4 tells us that T does not have any more branches of cardinality ω_2 in $M[H_0][H_1]$ than it does in $M[H_0]$. So, in $M[G]$, T has $\leq \omega_3$ branches of cardinality ω_2 and $\leq |\omega_2^{<\omega_2}| = \omega_{\omega_2}$ branches of cardinality $< \omega_2$ for a grand total of $\leq \omega_{\omega_2}$ branches. ☐

In light of Theorem 4.2.1, our main goal is a corollary of Theorem 4.2.5.

Corollary 4.2.6. *If* ZFC *is consistent, then so is* ZFC + *"there is no chain in* $\left(2^{2^\omega}, \subseteq\right)$ *of cardinality* 2^{2^ω}*."*

Corollary 4.2.7. *The statement "there is a chain in* $\left(2^{2^\omega}, \subseteq\right)$ *of cardinality* 2^{2^ω}*" is independent of* ZFC*.*

Proof. Corollary 4.0.8 gives the consistency of the statement, and Corollary 4.2.6 gives the consistency of its negation. ☐

4.3 Big chains in the Medvedev degrees

We show, in ZFC, that the cardinalities of chains in $\left(2^{2^\omega}, \subseteq\right)$ are the same as the cardinalities of chains in $\mathcal{D}_{\mathrm{s}}$. The forward direction of Theorem 4.3.1 is due to Terwijn [77].

Theorem 4.3.1. *For any cardinal κ, there is a chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality κ if and only if there is a chain in $\mathcal{D}_{\mathrm{s}}$ of cardinality κ.*

Proof. Let $\mathfrak{C} \subseteq 2^{2^\omega}$ be a chain of cardinality κ. Let $(f_\xi \mid \xi \in 2^\omega)$ be a $\leq_{\mathrm{T}}$-antichain (in the pairwise incomparable sense) of functions in ω^ω. For each $\mathcal{C} \in \mathfrak{C}$, let $\mathcal{X}_\mathcal{C} = \{f_\xi \mid \xi \in \mathcal{C}\}$. Then, for $\mathcal{C}, \mathcal{D} \in \mathfrak{C}$, $\mathcal{C} \subsetneq \mathcal{D}$ implies $\mathcal{X}_\mathcal{C} >_{\mathrm{s}} \mathcal{X}_\mathcal{D}$. The inequality $\mathcal{X}_\mathcal{C} \geq_{\mathrm{s}} \mathcal{X}_\mathcal{D}$ holds because $\mathcal{X}_\mathcal{C} \subseteq \mathcal{X}_\mathcal{D}$. To see that the inequality is strict, let $\zeta \in \mathcal{D} \setminus \mathcal{C}$ so that $f_\zeta \in \mathcal{X}_\mathcal{D} \setminus \mathcal{X}_\mathcal{C}$. The function f_ζ does not compute any function in $\mathcal{X}_\mathcal{C}$ because the f_ξ's were chosen to be a $\leq_{\mathrm{T}}$-antichain. Thus there is no Turing functional Φ for which $\Phi(\mathcal{X}_\mathcal{D}) \subseteq \mathcal{X}_\mathcal{C}$. The set $\{\deg_{\mathrm{s}}(\mathcal{X}_\mathcal{C}) \mid \mathcal{C} \in \mathfrak{C}\}$ is therefore a chain in $\mathcal{D}_{\mathrm{s}}$ of cardinality κ.

Conversely, let $\mathfrak{X} \subseteq \mathcal{D}_{\mathrm{s}}$ be a chain of cardinality κ. We produce a chain in $\left(2^{([\omega^\omega]^{\leq\omega})}, \subseteq\right)$ of cardinality κ, where $[\omega^\omega]^{\leq\omega} = \{\mathcal{X} \subseteq \omega^\omega \mid |\mathcal{X}| \leq \omega\}$. This suffices because $|[\omega^\omega]^{\leq\omega}| = 2^\omega$. Let $\hat{\mathfrak{X}} \subseteq 2^{\omega^\omega}$ consist of a choice of one representative $\mathcal{X} \subseteq \omega^\omega$ from each degree in $\mathfrak{X}$ so that $\hat{\mathfrak{X}}$ is a chain of cardinality κ under $\leq_{\mathrm{s}}$. For each $\mathcal{X} \in \hat{\mathfrak{X}}$, let $\mathcal{D}_\mathcal{X} = \{\mathcal{W} \subseteq \mathcal{X} \mid |\mathcal{W}| \leq \omega\}$ and let $\mathcal{C}_\mathcal{X} = \bigcup\{\mathcal{D}_\mathcal{Y} \mid (\mathcal{Y} \in \hat{\mathfrak{X}}) \wedge (\mathcal{Y} \geq_{\mathrm{s}} \mathcal{X})\}$. Then, for $\mathcal{X}, \mathcal{Y} \in \hat{\mathfrak{X}}$, $\mathcal{X} <_{\mathrm{s}} \mathcal{Y}$ implies $\mathcal{C}_\mathcal{X} \supsetneq \mathcal{C}_\mathcal{Y}$. The inequality $\mathcal{C}_\mathcal{X} \supseteq \mathcal{C}_\mathcal{Y}$ is clear. To see that the inequality is strict, apply Lemma 3.3.2 (as $\mathcal{X} \not\geq_{\mathrm{s}} \mathcal{Y}$) to get a $\mathcal{W} \subseteq \mathcal{X}$ such that $|\mathcal{W}| \leq \omega$ and $\mathcal{W} \not\geq_{\mathrm{s}} \mathcal{Y}$. $\mathcal{W}$ is in $\mathcal{D}_\mathcal{X}$ and hence in $\mathcal{C}_\mathcal{X}$, but $\mathcal{W}$ cannot be in $\mathcal{C}_\mathcal{Y}$ as this would imply $\mathcal{W} \geq_{\mathrm{s}} \mathcal{Y}$. Thus $\{\mathcal{C}_\mathcal{X} \mid \mathcal{X} \in \hat{\mathfrak{X}}\}$ is the desired chain. □

Theorem 4.3.1 is also valid for $\mathcal{D}_{\mathrm{w}}$ in place of $\mathcal{D}_{\mathrm{s}}$. In the forward direction of Theorem 4.3.1, we actually proved that, for $\mathcal{C}, \mathcal{D} \in \mathfrak{C}$, $\mathcal{C} \subsetneq \mathcal{D}$ implies $\mathcal{X}_\mathcal{C} >_{\mathrm{w}} \mathcal{X}_\mathcal{D}$. The proof given for the reverse direction is also valid for $\mathcal{D}_{\mathrm{w}}$, but a simpler one is possible. Notice that a mass problem $\mathcal{X} \subseteq \omega^\omega$ is Muchnik-equivalent to its Turing upward-closure $\{f \in \omega^\omega \mid (\exists g \in \mathcal{X})(f \geq_{\mathrm{T}} g)\}$. Furthermore, if $\mathcal{X}$ and $\mathcal{Y}$ are Turing upward-closed then $\mathcal{X} \leq_{\mathrm{w}} \mathcal{Y}$ if and only if $\mathcal{X} \supseteq \mathcal{Y}$. So if $\mathfrak{X} \subseteq \mathcal{D}_{\mathrm{w}}$ is a chain of cardinality κ, choose a set of representatives $\hat{\mathfrak{X}} \subseteq 2^{\omega^\omega}$ that are Turing upward-closed. Then $\hat{\mathfrak{X}}$ is a chain in $\left(2^{\omega^\omega}, \subseteq\right)$ of cardinality κ.

Corollary 4.3.2. *The following statements are both independent of* ZFC. *"There is a chain in the Medvedev degrees of cardinality 2^{2^ω}" and "there is a chain in the Muchnik degrees of cardinality 2^{2^ω}."*

Proof. By Theorem 4.3.1, both statements are equivalent under ZFC to the statement "there is a chain in $\left(2^{2^\omega}, \subseteq\right)$ of cardinality 2^{2^ω}." The result then follows from Corollary 4.2.7. □

In contrast, it is a ZFC theorem that $\left(2^{2^\omega}, \subseteq\right)$, $\mathcal{D}_{\mathrm{s}}$, and $\mathcal{D}_{\mathrm{w}}$ all have antichains (in the pairwise incomparable sense) of cardinality 2^{2^ω} [47].

CHAPTER 5
MENGER'S THEOREM IN Π^1_1-CA_0

The results of this chapter also appear in [55] by the author.

König's duality theorem for finite bipartite graphs is a classic theorem in graph theory and one of the pillars of matching theory. It expresses a duality between matchings and covers in bipartite graphs. Let (X, Y, E) be a bipartite graph. A *matching* is a set of edges $M \subseteq E$ such that no two edges in M share a vertex. A *cover* is a set of vertices $C \subseteq X \cup Y$ such that every edge in E has a vertex in C. Finite König's duality theorem says that the cardinalities of matchings and the cardinalities of covers meet in the middle.

Finite König's Duality Theorem. *In every finite bipartite graph, the maximum cardinality of a matching equals the minimum cardinality of a cover.*

Finite Menger's theorem generalizes finite König's duality theorem from bipartite graphs to arbitrary graphs. Let G be a graph with vertices $V(G)$ and edges $E(G)$. A *web* is a triple (G, A, B) where G is a graph and A and B are distinguished sets of vertices $A, B \subseteq V(G)$. The notion of a matching in a bipartite graph is generalized by the notion of a *set of disjoint A-B paths*[1] in a web. An A-B path in a web (G, A, B) is a path that starts in A and ends in B. Two paths are disjoint if they have no vertices in common. The notion of a cover in a bipartite graph is generalized by the notion of an *A-B separator* in a web. An A-B separator in a web (G, A, B) is a set of vertices $C \subseteq V(G)$ such that every A-B path in G contains a vertex of C (so that removing C from the graph separates A from B).

Finite Menger's Theorem. *In every finite web (G, A, B), the maximum cardinality of a set of disjoint A-B paths equals the minimum cardinality of an A-B separator.*

Finite Menger's theorem is itself a special case of the famous max-flow min-cut theorem for network flows. See [21] Section 2.1 for a full treatment of finite König's duality theorem, [21] Section 3.3 for finite Menger's theorem, and [21] Section 6.2 for the max-flow min-cut theorem.

The conclusions of finite König's duality theorem and finite Menger's theorem remain true for infinite bipartite graphs and infinite webs, but they are more an exercise in cardinal arithmetic than they are in combinatorics. To deepen the combinatorial content of these theorems, Erdős conjectured that there always exist a matching and a cover that simultaneously witness each other's optimality. His reformulations are what we now call König's duality theorem and Menger's theorem.

König's Duality Theorem. *In every bipartite graph (X, Y, E), there is a matching M and a cover C such that C consists of exactly one vertex from each edge in M.*

Menger's Theorem. *In every web (G, A, B), there is a set of disjoint A-B paths M and an A-B separator C such that C consists of exactly one vertex from each path in M.*

[1] For us, "path" always means "simple path," that is, no repeated vertices.

The most general case, Menger's theorem for webs of arbitrary cardinality, is now known to be true. The proof took more than forty years to discover. The first progress was by Podewski and Steffens, who proved König's duality theorem for countable bipartite graphs [48]. Aharoni next proved König's duality theorem for arbitrary bipartite graphs [1]. He then proved Menger's theorem for countable webs [2]. Finally, Aharoni and Berger proved Menger's theorem for arbitrary webs [3].

The question motivating our work is the following.

Question 5.0.3. What is the axiomatic strength of Menger's theorem for countable webs in the context of second-order arithmetic?

Aharoni, Magidor, and Shore [4] and Simpson [63] answered Question 5.0.3 for König's duality theorem for countable bipartite graphs. Aharoni, Magidor, and Shore noticed that Aharoni's proof of König's duality theorem in [1] actually proves a stronger statement, which they call *extended König's duality theorem*. They proved that extended König's duality theorem is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$, and they proved that König's duality theorem implies $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$ [4]. Simpson produced a new proof of König's duality theorem in $\mathtt{ATR}_0$ by exploiting the fact that $\mathtt{ATR}_0$ proves the existence of models of $\Sigma^1_1\text{-}\mathtt{AC}_0$ [63]. Therefore König's duality theorem for countable bipartite graphs is equivalent to $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$.

A priori, Menger's theorem for countable webs implies $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$ because it implies König's duality theorem for countable bipartite graphs over $\mathtt{RCA}_0$. Here we provide a proof of Menger's theorem for countable webs in $\Pi^1_1\text{-}\mathtt{CA}_0$ (Theorem 5.2.4). The general plan for our proof is inspired by Aharoni's proof in [2] and Diestel's presentation of it in [21] Section 8.4. As with König's duality theorem, we notice that this proof in fact proves a stronger statement, which we call *extended Menger's theorem*, that is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$. By general considerations, Menger's theorem cannot imply $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$. Menger's theorem can be written as a Π^1_2 sentence in the language of second-order arithmetic, and no true Π^1_2 sentence implies $\Pi^1_1\text{-}\mathtt{CA}_0$, even over $\mathtt{ATR}_0$ (see [4] Proposition 4.17). Question 5.0.3 now becomes more specific.

Question 5.0.4. Is Menger's theorem for countable webs provable in $\mathtt{ATR}_0$?

5.1 Warps, waves, and alternating walks

In this section we use $\mathtt{ACA}_0$ to develop the basic tools we need to prove Menger's theorem in $\Pi^1_1\text{-}\mathtt{CA}_0$. Our notation and terminology mostly follows [2] with some ideas borrowed from [21] Section 8.4.

All the graphs that we consider are countable because we are working in second-order arithmetic. All the graphs that we consider are directed. Menger's theorem for undirected graphs follows from Menger's theorem for directed graphs by the usual trick of replacing an undirected edge by two directed edges. Henceforth a "graph" is a countable directed graph. If G is a graph and

$x \in V(G)$, then $\text{in-deg}_G(x)$ denotes the in-degree of x, and $\text{out-deg}_G(x)$ denotes the out-degree of x.

As defined above, a web is a triple (G, A, B) where G is a graph and A and B are distinguished sets of vertices $A, B \subseteq V(G)$. We often abuse this notation by writing G for (G, A, B). For convenience, we always assume that there are no edges directed into A, that there are no edges directed out of B, and that $A \cap B = \emptyset$.

If H and H' are subgraphs of a graph G, then $H \cup H'$ is the subgraph of G induced by $V(H) \cup V(H')$, and $G \setminus H$ is the subgraph induced by $V(G) \setminus V(H)$.

Let G be a graph. If P is a path in G, we write $\text{in}(P)$ for the first vertex of P (if it exists) and $\text{ter}(P)$ for the last vertex of P (if it exists). If P is a path with $\text{in}(P) \in A$ and $\text{ter}(P) \in B$ for some $A, B \subseteq V(G)$, then we call P an A-B path. If P is a path and $x \in V(P)$, then Px denotes the subpath of P consisting of all the vertices up to and including x, and $\overline{Px}$ denotes the subpath of P consisting of all the vertices up to and not including x. Similarly, xP denotes the subpath of P consisting of all the vertices following x and including x, and $\underline{xP}$ denotes the subpath of P consisting of all the all the vertices following x and not including x. If P and Q are paths with $V(P) \cap V(Q) = \{x\}$, then PxQ is the path obtained by concatenating the paths Px and xQ. If $V(P) \cap V(Q) = \{\text{ter}(P)\} = \{\text{in}(Q)\}$, then PQ denotes $P\,\text{ter}(P)Q$, the concatenation of the paths P and Q.

For the purposes of this chapter, a *(graph-theoretic) tree* is a directed acyclic graph T that has a distinguished root $r \in V(T)$ such that for any $x \in V(T)$ there is a unique path in T from r to x. The path in a tree T from its root to an $x \in V(T)$ is denoted Tx. If P is a finite path, a *tree with trunk* P is a tree T of the form $P \cup T'$ where T' is a tree rooted at $\text{ter}(P)$. A tree with trunk P has root $\text{in}(P)$. If $G = (G, A, B)$ is a web, an *A-B tree in G* is a subgraph of G that is a tree with root in A and exactly one vertex in B.

Definition 5.1.1. A *warp* in a web $G = (G, A, B)$ is a subgraph W of G such that

- $A \subseteq V(W)$,
- every $x \in V(W)$ has $\text{in-deg}_W(x) \leq 1$ and $\text{out-deg}_W(x) \leq 1$, and
- every $x \in V(W)$ is reachable from some $a \in A$ by a path in W.

A warp is thus a collection of disjoint paths in G with each path starting at a distinct vertex in A and such that for every $a \in A$ there is a path in the warp starting at a. Such paths may be one-way infinite. It is often convenient to think of a warp W as the collection of its component paths $\{P_a \mid a \in A \wedge \text{in}(P_a) = a\}$ with the understanding that this collection is coded by the set $\{\langle a, \langle n, x\rangle\rangle \mid x \text{ is the } n^{\text{th}} \text{ vertex of } P_a\}$. "$P$ is a path in W" always means that P is one of these component paths.

If W is a warp, then let $\text{ter}(W) = \{x \in V(W) \mid \text{out-deg}_W(x) = 0\}$. That is, $\text{ter}(W)$ is the set of terminal vertices of the paths in W. The statement "if W is a warp then $\text{ter}(W)$ exists" is equivalent to ACA_0 over RCA_0, hence our assumption of ACA_0 throughout this section.

Definition 5.1.2. A *wave* in a web $G = (G, A, B)$ is a warp W such that $\operatorname{ter}(W)$ is an A-B separator.

The warp $\{P_a \mid a \in A\}$ in which each path P_a is the trivial path (a) is always a wave, and we call it the *trivial wave*.

Definition 5.1.3. For warps W and Y in a web $G = (G, A, B)$, Y is an *extension* of W (written $W \leq Y$) if and only if W is a subgraph of Y.

Definition 5.1.4. If $(W_i \mid i \in \mathbb{N})$ is a sequence of warps such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} W_i$ denotes the *limit warp* defined by $V\big(\bigcup_{i\in\mathbb{N}} W_i\big) = \bigcup_{i\in\mathbb{N}} V(W_i)$ and $E\big(\bigcup_{i\in\mathbb{N}} W_i\big) = \bigcup_{i\in\mathbb{N}} E(W_i)$.

It is easy to check in $\mathtt{RCA}_0$ that a limit warp, if it exists, is indeed a warp. However, the statement "if $(W_i \mid i \in \mathbb{N})$ is a sequence of warps such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} W_i$ exists" is equivalent to $\mathtt{ACA}_0$ over $\mathtt{RCA}_0$.

Definition 5.1.5. Let $W = \{P_a \mid a \in A\}$ be a wave in a web $G = (G, A, B)$. Then

- P_a is *W-essential in G* if and only if P_a is finite and there is a $\operatorname{ter}(P_a)$-B path in G disjoint from $V(W) \setminus \{\operatorname{ter}(P_a)\}$,
- $a \in A$ is *W-essential in G* if and only if P_a is W-essential in G, and
- $\operatorname{ess}_G(W) = \{a \in A \mid a \text{ is } W\text{-essential in } G\}$.

The motivation behind the definition of W-essential in G is that if P is a path in a wave W that is W-essential in G, then $\operatorname{ter}(W)$ needs $\operatorname{ter}(P)$ to separate A from B. If Q is a $\operatorname{ter}(P)$-B path disjoint from $V(W) \setminus \{\operatorname{ter}(P)\}$, then PQ is an A-B path disjoint from $\operatorname{ter}(W) \setminus \{\operatorname{ter}(P)\}$. One readily checks that $\{\operatorname{ter}(P) \mid P \text{ is a } W\text{-essential path in } G\}$ is an A-B separator.

Definition 5.1.6. If W and Y are waves in a web $G = (G, A, B)$ with $W \leq Y$, then Y is a *good* extension of W if and only if $\operatorname{ess}_G(W) = \operatorname{ess}_G(Y)$, and Y is a *bad* extension of W if and only if $\operatorname{ess}_G(W) \neq \operatorname{ess}_G(Y)$.

If W and Y are waves in a web $G = (G, A, B)$ with $W \leq Y$, then it is always the case that $\operatorname{ess}_G(Y) \subseteq \operatorname{ess}_G(W)$. Thus Y is a good extension of W if and only if $\operatorname{ess}_G(W) \subseteq \operatorname{ess}_G(Y)$.

Lemma 5.1.7 (in $\mathtt{ACA}_0$; see [2] Lemma 2.5)**.** *If $(W_i \mid i \in \mathbb{N})$ is a sequence of waves in a web $G = (G, A, B)$ such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} W_i$ is a wave in G.*

Proof. Let $W = \bigcup_{i\in\mathbb{N}} W_i$. It is easy to check that W is a warp. We need to show that $\operatorname{ter}(W)$ is an A-B separator. Let P be an A-B path, and let $X = \{\langle x, i\rangle \mid x \in V(P) \cap \operatorname{ter}(W_i)\}$, which exists by arithmetical comprehension. Each W_i is a wave, hence X is infinite. As $V(P)$ is finite, there must be an $x \in V(P)$ such that $\{i \mid x \in \operatorname{ter}(W_i)\}$ is infinite. Then $x = \operatorname{ter}(Q)$ for the path Q in W containing x. If not, then there is a vertex following x on Q, the corresponding edge must appear in W_n for some n, and so $x \notin \operatorname{ter}(W_i)$ for all $i \geq n$. $\square$

Definition 5.1.8. Let W and Y be warps in a web $G = (G, A, B)$ with $W \leq Y$. Let Q be a finite path in both Y and W (i.e., the path Q is in W and is not properly extended in Y). A $(Y \setminus W)$*-alternating walk from* $\operatorname{ter}(Q)$ is a walk $R = x_0e_0x_1e_1 \cdots e_{n-1}x_n$ such that

(i) $x_0 = \operatorname{ter}(Q)$,

(ii) for all $i \leq n$, $x_i \in (V(G) \setminus V(W)) \cup \operatorname{ter}(W)$,

(iii) for all $i < n$, if $e_i \notin E(Y)$, then $e_i = (x_i, x_{i+1})$,

(iv) for all $i < n$, if $e_i \in E(Y)$, then $e_i = (x_{i+1}, x_i)$ (i.e., R traverses e_i backwards),

(v) for all $i, j \leq n$ with $i \neq j$, if $x_i = x_j$, then $x_i \in V(Y)$, and

(vi) for all $i, j \leq n$ with $i \neq j$, $e_i \neq e_j$.

(vii) for all $0 < i \leq n$, if $x_i \in V(Y)$, then either e_{i-1} or e_i is in $E(Y)$.

Note that if x_n is the last vertex on a $(Y \setminus W)$-alternating walk from $\operatorname{ter}(Q)$ and $x_n \in V(Y)$, then item (vii) implies that $e_{n-1} \in E(Y)$. A $(Y \setminus W)$-alternating walk from $\operatorname{ter}(Q)$ is similar to a Y-walk as defined in [2] and to a walk which alternates with respect to Y as defined in [21] Section 3.3. The difference is that a $(Y \setminus W)$-alternating walk from $\operatorname{ter}(Q)$ is not allowed to use the vertices in $V(W) \setminus \operatorname{ter}(W)$, hence the notation "$Y \setminus W$."

Definition 5.1.9. Let $W = \{P_a \mid a \in A\}$ and $Y = \{Q_a \mid a \in A\}$ be warps in a web $G = (G, A, B)$ with $W \leq Y$. Let Q_{a_0} be a finite path in both Y and W. Then $\operatorname{alt}_G(Y \setminus W, \operatorname{ter}(Q_{a_0}))$ denotes the warp $\{Q'_a \mid a \in A\}$ where $Q'_a = Q_a x$ if x is the last vertex on Q which lies on a $(Y \setminus W)$-alternating walk from $\operatorname{ter}(Q_{a_0})$ and $Q'_a = P_a$ if no such x exists.

Our definition of $\operatorname{alt}_G(Y \setminus W, \operatorname{ter}(Q_{a_0}))$ is analogous to the definition of $M(a_0, W)$ in [2]. Also, note that $W \leq \operatorname{alt}_G(Y \setminus W, \operatorname{ter}(Q_{a_0})) \leq Y$. The first inequality is by Definition 5.1.8 item (ii) and the second inequality is clear.

The crucial lemma from this section is Lemma 5.1.12. Lemma 5.1.10 and Lemma 5.1.11 are used to prove Lemma 5.1.12.

Lemma 5.1.10 (in $\mathtt{ACA}_0$)**.** *Let W and Y be warps in a web $G = (G, A, B)$ with $W \leq Y$. Let Q be a finite path in both Y and W. Let R be a $(Y \setminus W)$-alternating walk from* $\operatorname{ter}(Q)$. *Then there is a warp $Z \geq W$ in G with* $\operatorname{ter}(Z) = (\operatorname{ter}(Y) \setminus \{\operatorname{ter}(Q)\}) \cup \{\operatorname{ter}(R)\}$.

Proof. Let $R = x_0e_0x_1e_1 \cdots e_{n-1}x_n$ where $x_0 = \operatorname{ter}(Q)$ and $x_n = \operatorname{ter}(R)$. Assume $n > 0$, for otherwise we may take $Z = Y$. Let Z' be the subgraph of G with $E(Z') = E(Y) \triangle E(R)$ and $V(Z') = A \cup \{x \mid (\exists e \in E(Z'))(x \text{ is a vertex of } e)\}$. One readily checks the following equalities.

- If $x \in V(Y) \setminus V(R)$, then $\text{in-deg}_{Z'}(x) = \text{in-deg}_Y(x)$ and $\text{out-deg}_{Z'}(x) = \text{out-deg}_Y(x)$,
- for $0 < i < n$, if $x_i \in V(R) \setminus V(Y)$, then $\text{in-deg}_{Z'}(x_i) = 1$ and $\text{out-deg}_{Z'}(x_i) = 1$,

- for $0 < i < n$, if $x_i \in V(R) \cap V(Y)$ is in $V(Z')$, then $\text{in-deg}_{Z'}(x_i) = \text{in-deg}_Y(x_i)$ and $\text{out-deg}_{Z'}(x_i) = \text{out-deg}_Y(x_i)$,
- $\text{in-deg}_{Z'}(x_0) = \text{in-deg}_Y(x_0)$ and $\text{out-deg}_{Z'}(x_0) = 1$, and
- $\text{in-deg}_{Z'}(x_n) = 1$ and $\text{out-deg}(x_n) = 0$.

It follows that $\text{in-deg}_{Z'}(x) \leq 1$ and $\text{out-deg}_{Z'}(x) \leq 1$ for all $x \in V(Z')$, which means that every component of Z' is either a path or a cycle. Let Z be the subgraph of Z' consisting of the component paths of Z' (i.e., Z is the subgraph of Z' induced by $\{x \in V(Z') \mid x \text{ is not on a cycle in } Z'\}$). Z contains every vertex $x \in V(Z')$ with $\text{in-deg}_{Z'}(x) = 0$ or $\text{out-deg}_{Z'}(x) = 0$. In particular, $A \subseteq V(Z)$ and $\text{ter}(Z) = (\text{ter}(Y) \setminus \{\text{ter}(Q)\}) \cup \{\text{ter}(R)\}$. To show that Z is a warp, we need only show that $\text{in}(P)$ exists and is in A for every path P in Z. The above equations imply that if $x \in V(P) \setminus A$, then $\text{in-deg}_Z(x) \neq 0$ and hence that x has an immediate predecessor on P. This fact together with the fact that R is finite implies that there is an $x \in V(P)$ such that $(V(Px) \cap V(R)) \setminus A = \emptyset$. Thus the edges of Px must all be edges of Y, which means that Px must be an initial segment of some path in Y. Hence $\text{in}(P)$ exists and is in A. Finally, $Z \geq W$ by Definition 5.1.8 item (ii). □

Lemma 5.1.11 (in $\texttt{ACA}_0$; see [2] Lemma 2.7). *Let W and Y be waves in a web $G = (G, A, B)$ with $W \leq Y$. Let Q be a finite path in both Y and W. Then $\text{alt}_G(Y \setminus W, \text{ter}(Q))$ is a wave.*

Proof. Let $U = \text{alt}_G(Y \setminus W, \text{ter}(Q))$. Suppose for a contradiction that P is an A-B path disjoint from $\text{ter}(U)$. W is a wave, so the last vertex on P that is in $V(W)$ must be in $\text{ter}(W)$. Let w be this vertex, and let S the path in W with $\text{ter}(S) = w$. The path SwP is an A-B path disjoint from $\text{ter}(U)$. Y is a wave, so wP intersects $\text{ter}(Y)$, which must happen at a vertex in $V(Y) \setminus V(U)$. Let y be the first vertex on wP in $V(Y) \setminus V(U)$. Let z be the last vertex on wPy in $V(U)$, which exists because $w \in V(U)$.

Claim. *There is a $(Y \setminus W)$-alternating walk from $\text{ter}(Q)$ ending at z.*

Proof of claim. Let Q' be the path in Y containing z. We show that there is a $(Y \setminus W)$-alternating walk R from $\text{ter}(Q)$ that meets Q' at a vertex r which is past z on Q'. If r is the first such vertex on R, then $RrQ'z$ (following the edges of Q' backwards) is the desired walk. If $z = w$, then Q' extends S, so if there is no such walk R then by Definition 5.1.9 S is a path in Z which contradicts that P is disjoint from $\text{ter}(U)$. On the other hand, if $z \neq w$, then $z \notin V(W)$ by choice of w. As $z \in V(U) \setminus V(W)$ and $z \notin \text{ter}(U)$, again by Definition 5.1.9 it must be the case that some $(Y \setminus W)$-alternating walk R from $\text{ter}(Q)$ meets Q' at a vertex past z. □

Now let R be the walk provided by the claim, let r be the last vertex of zPy on R, and let y' be the vertex immediately preceding y on the path in Y containing y. Then $RrPy(y', y)y'$ is a $(Y \setminus W)$-alternating walk from $\text{ter}(Q)$ on which y lies which contradicts $y \notin V(U)$. □

Lemma 5.1.12 (in $\mathtt{ACA}_0$; see [2] Lemma 2.8). *Let W be a wave in a web $G = (G, A, B)$ that has no bad extensions in G. Let $x \in V(G) \setminus V(W)$ be such that there is a wave $Y \geq W$ in $G \setminus \{x\} = (G \setminus \{x\}, A \setminus \{x\}, B \setminus \{x\})$ with $\mathrm{ess}_{G\setminus\{x\}}(Y) \subsetneq \mathrm{ess}_G(W)$. Then there is a wave $Z \geq W$ in G with $x \in \mathrm{ter}(Z)$.*

Proof. Let $G = (G, A, B)$, $W = \{P_a \mid a \in A\}$, x, and $Y = \{Q_a \mid a \in A\}$ be as in the statement of the lemma. Let $a_0 \in \mathrm{ess}_G(W) \setminus \mathrm{ess}_{G\setminus\{x\}}(Y)$. If we replace Q_{a_0} with P_{a_0} in Y, then we retain that this path is not Y-essential in $G \setminus \{x\}$. Thus we may assume $Q_{a_0} = P_{a_0}$. In particular, Q_{a_0} is a finite path that is not Y-essential in $G \setminus \{x\}$.

Claim. *In G, there is an alternating $(Y \setminus W)$-walk from $\mathrm{ter}(Q_{a_0})$ ending at x.*

Proof of claim. If there is a $\mathrm{ter}(Q_{a_0})$-x path disjoint from $V(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}$, then this path is the desired walk. So suppose instead there is no such path. Let $U = \mathrm{alt}_{G\setminus\{x\}}(Y \setminus W, \mathrm{ter}(Q_{a_0}))$. U is a wave in $G \setminus \{x\}$ by Lemma 5.1.11. Furthermore, $a_0 \notin \mathrm{ess}_{G\setminus\{x\}}(Y)$ implies that $a_0 \notin \mathrm{ess}_{G\setminus\{x\}}(U)$ because if P is a $\mathrm{ter}(Q_{a_0})$-B path in $G \setminus \{x\}$, then the first vertex on P in $V(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}$ is also in $V(U)$. We prove that U is not a wave in G. To do this, it suffices to show that every $\mathrm{ter}(Q_{a_0})$-B path in G intersects $V(U) \setminus \{\mathrm{ter}(Q_{a_0})\}$. Therefore if U were a wave in G, it would be a bad extension of W in G because a_0 would be in $\mathrm{ess}_G(W) \setminus \mathrm{ess}_G(U)$. This is a contradiction. Consider a $\mathrm{ter}(Q_{a_0})$-B path P. If $x \notin V(P)$, then P is a path in $G \setminus \{x\}$ and hence P intersects $V(U) \setminus \{\mathrm{ter}(Q_{a_0})\}$ because $a_0 \notin \mathrm{ess}_{G\setminus\{x\}}(U)$. If $x \in V(P)$, then by assumption Px intersects $V(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}$. Again, the first vertex on P in $V(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}$ is also in $V(U)$.

We now know that U is a wave in $G \setminus \{x\}$ but not in G. Thus there is an A-B path S in G avoiding $\mathrm{ter}(U)$, and x must lie on S. Let z be the last vertex of Sx that is in $V(U)$. It must be that $z \in ((V(U) \setminus V(W)) \cup \mathrm{ter}(W)) \setminus \mathrm{ter}(U)$. Hence there must be an alternating $(Y \setminus W)$-walk R from $\mathrm{ter}(Q_{a_0})$ to z. Let y be the last vertex of zSx which lies on R. Then $RyPx$ is the desired alternating $(Y \setminus W)$-walk from $\mathrm{ter}(Q_{a_0})$ to x. ☐

By the claim, let R be an alternating $(Y \setminus W)$-walk from $\mathrm{ter}(Q_{a_0})$ ending at x. Apply Lemma 5.1.10 to get a warp $Z \geq W$ in G with $\mathrm{ter}(Z) = (\mathrm{ter}(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}) \cup \{x\}$. Z is a wave because $\mathrm{ter}(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}$ is an $(A \setminus \{x\})$-$(B \setminus \{x\})$ separator in $G \setminus \{x\}$, thus $(\mathrm{ter}(Y) \setminus \{\mathrm{ter}(Q_{a_0})\}) \cup \{x\}$ is an A-B separator in G. ☐

5.2 Menger's theorem in Π^1_1-$\mathtt{CA}_0$

We plan to prove Menger's theorem as follows. Given a web $G = (G, A, B)$, start with W a $\leq$-maximal wave in G. Let C be the terminal vertices of the paths in W that are W-essential. Then extend these W-essential paths to be the collection of disjoint A-B paths M. Lemma 5.2.1 provides the $\leq$-maximal wave W, and Lemma 5.2.2 is the tool we use to extend the W-essential paths to a collection of disjoint A-B paths. The proof of Lemma 5.2.1 is the only argument in which we employ the full strength of Π^1_1-$\mathtt{CA}_0$.

Lemma 5.2.1 (in Π^1_1–$\mathtt{CA}_0$; see [2] Corollary 2.5a). *In every web there is a $\leq$-maximal wave.*

Proof. Let $G = (G, A, B)$ be a web, let $(g_n \mid n \in \mathbb{N})$ be an enumeration of $V(G)$, and by Theorem 1.6.3 let $\mathcal{M}$ be a countable coded β-model with $(G, A, B) \in \mathcal{M}$. Using $\mathtt{ACA}_0$ outside $\mathcal{M}$, we construct a sequence of integers $(i_n \mid n \in \mathbb{N})$ such that $(\mathcal{M})_{i_n}$ is a wave for each $n \in \mathbb{N}$ and $(\mathcal{M})_{i_n} \leq (\mathcal{M})_{i_{n+1}}$ for each $n \in \mathbb{N}$. Let i_0 be an index such that $(\mathcal{M})_{i_0}$ is the trivial wave $\{(a) \mid a \in A\}$. Suppose we have $i_0, \dots, i_n$. If there is an $i \in \mathbb{N}$ such that $(\mathcal{M})_i$ is a wave with $(\mathcal{M})_i \geq (\mathcal{M})_{i_n}$ and $g_n \in V((\mathcal{M})_i)$, then let i_{n+1} be such an i. Otherwise let $i_{n+1} = i_n$. With the desired sequence $(i_n \mid n \in \mathbb{N})$ in hand, let W be the limit $W = \bigcup_{n \in \mathbb{N}} (\mathcal{M})_{i_n}$, which is a wave by Lemma 5.1.7. This W is $\leq$-maximal in (G, A, B). If not, there is a wave $Y \geq W$ with some $g_n \in V(Y) \setminus V(W)$. As $(\mathcal{M})_{i_n} \leq W \leq Y$, at stage $n+1$ in the construction the Σ^1_1 formula $(\exists Y)(Y \text{ is a wave} \wedge Y \geq (\mathcal{M})_{i_n} \wedge g_n \in V(Y))$ is true and hence is true in $\mathcal{M}$ because $\mathcal{M}$ is a β-model. Therefore we chose i_{n+1} so that $g_n \in V((\mathcal{M})_{i_{n+1}})$, contradicting $g_n \notin V(W)$. $\square$

In [2] and [21], Lemma 5.2.1 is obtained by a simple application of Zorn's lemma. Our proof above is the most effective proof possible, in the sense that Lemma 5.2.1 is equivalent to Π^1_1–$\mathtt{CA}_0$ over $\mathtt{RCA}_0$ (see Corollary 5.3.3).

The following Lemma 5.2.2 is the key tool used to complete the proof of Menger's theorem. We first give a proof of Lemma 5.2.2 in the style of ordinary mathematics for the sake of clarity. We then explain how to formalize Lemma 5.2.2 in a way that will allow us to complete the proof of Menger's theorem in Π^1_1–$\mathtt{CA}_0$.

Lemma 5.2.2 (see [2] Theorem 3.4). *Let W be a wave in $G = (G, A, B)$ that has no bad extensions in G, let $a \in \mathrm{ess}_G(W)$, and let P_a be the component path of W starting at a. Then there is a finite a-B tree T with trunk P_a such that $V(T) \cap (V(W) \setminus V(P_a)) = \emptyset$, and there is a wave Y in $G \setminus T$ such that $Y \geq W \setminus P_a$, $\mathrm{ess}_{G \setminus T}(Y) = \mathrm{ess}_G(W) \setminus \{a\}$, and Y has no bad extensions in $G \setminus T$.*

Proof. We assume that the conclusion of the lemma is false and construct a bad extension of W in G, which is a contradiction.

Let $(Q_n \mid n \in \mathbb{N})$ list all the $\mathrm{ter}(P_a)$-B paths with each path occurring on the list infinitely often. We construct sequences $(T_n \mid n \in \mathbb{N})$ and $(Y_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$

(i) T_n is a finite tree in G with trunk P_a,

(ii) Y_n is a wave in $G \setminus T_n$,

(iii) $T_{n-1} \subseteq T_n$ (if $n > 0$),

(iv) $Y_{n-1} \leq Y_n$ (if $n > 0$),

(v) $\mathrm{ess}_{G \setminus T_n}(Y_n) = \mathrm{ess}_G(W) \setminus \{a\}$, and

(vi) Y_n has no bad extensions in $G \setminus T_n$.

Start with $T_0 = P_a$ and $Y_0 = W \setminus P_a$. Items (i), (ii), and (v) are easily checked for $n = 0$. Furthermore, if Y were a bad extension of Y_0 in $G \setminus T_0$, then $Y \cup \{P_a\}$ would be a bad extension of W in G. Hence we have item (vi) for $n = 0$ as well.

Suppose we have constructed T_n and Y_n. If $V(Q_n) \cap V(Y_n) \neq \emptyset$ or $T = T_n \cup Q_n$ is not a tree with trunk P_a, set $T_{n+1} = T_n$ and $Y_{n+1} = Y_n$. Otherwise $V(Q_n) \cap V(Y_n) = \emptyset$ and $T = T_n \cup Q_n$ is a tree with trunk P_a. The situation now is that

- T is a finite a-B tree with trunk P_a such that $V(T) \cap (V(W) \setminus V(P_a)) = \emptyset$ and
- Y_n is a wave in $G \setminus T$ such that $Y_n \geq W \setminus P_a$.

We are assuming that the lemma is false, so either $\mathrm{ess}_{G\setminus T}(Y_n) \subsetneq \mathrm{ess}_G(W) \setminus \{a\}$ or Y_n has a bad extension in $G \setminus T$. Both cases imply the existence of a wave $Y \geq Y_n$ in $G \setminus T$ with $\mathrm{ess}_{G\setminus T}(Y) \subsetneq \mathrm{ess}_G(W) \setminus \{a\} = \mathrm{ess}_{G\setminus T_n}(Y_n)$ (where the equality is by item (v)). Let x be the first vertex on Q_n such that there exists a wave $Y \geq Y_n$ in $G \setminus (T_n \cup Q_n x)$ with $\mathrm{ess}_{G\setminus(T_n\cup Q_n x)}(Y) \subsetneq \mathrm{ess}_{G\setminus T_n}(Y_n)$ (note $x \neq \mathrm{ter}(P_a)$ by item (vi)). Let $T_{n+1} = T_n \cup \overline{Q_n x}$. By the choice of x, Y_n has no bad extensions in $G \setminus T_{n+1}$, but there is an extension $Y \geq Y_n$ in $(G \setminus T_{n+1}) \setminus \{x\}$ with $\mathrm{ess}_{(G\setminus T_{n+1})\setminus\{x\}}(Y) \subsetneq \mathrm{ess}_{G\setminus T_{n+1}}(Y_n)$. Thus by Lemma 5.1.12 there is a wave $Y_{n+1} \geq Y_n$ in $G \setminus T_{n+1}$ with $x \in \mathrm{ter}(Y_{n+1})$. With this T_{n+1} and Y_{n+1}, items (i)–(iv) are clear for $n+1$. Item (v) is by the choice of x, which implies that $\mathrm{ess}_{G\setminus T_{n+1}}(Y_{n+1}) = \mathrm{ess}_{G\setminus T_n}(Y_n) = \mathrm{ess}_G(W) \setminus \{a\}$. Item (vi) is again by the choice of x because a bad extension $Y \geq Y_{n+1}$ in $G \setminus T_{n+1}$ would be a $Y \geq Y_n$ in $G \setminus T_{n+1}$ with $\mathrm{ess}_{G\setminus T_{n+1}}(Y) \subsetneq \mathrm{ess}_{G\setminus T_n}(Y_n)$.

Let $T = \bigcup_{n\in\mathbb{N}} T_n$, and let $Y = \bigcup_{n\in\mathbb{N}} Y_n$. By construction, $V(Y_n) \cap V(T_m) = \emptyset$ for all $n, m \in \mathbb{N}$, which means that each Y_n is a wave in $G \setminus T$. Therefore Y is a wave in $G \setminus T$ by Lemma 5.1.7. It remains to show that $\mathrm{ter}(Y)$ is an A-B separator in G. Our desired contradiction follows because then $Y \cup \{P_a\}$ would be a bad extension of W in G because $a \in \mathrm{ess}_G(W) \setminus \mathrm{ess}_G(Y \cup \{P_a\})$.

Let P be an A-B path in G. It suffices to show that there is a final segment S of P that lies in $G \setminus T$ and intersects $V(Y)$. This is because if x is the last vertex of S in $V(Y)$ and Q is the component path of Y containing x, then QxS is an A-B path in $G \setminus T$ which means that xS (and hence P) must intersect $\mathrm{ter}(Y)$. Thus let x be the last vertex of P on T (if there is no such x, then P is a path in $G \setminus T$ and thus intersects $\mathrm{ter}(Y)$), let n be such that $x \in T_n$, and let $m > n$ be such that $Q_m = \mathrm{ter}(P_a)T_n xP$. Consider stage $m+1$ of the construction. If $V(Q_m) \cap V(Y_m) \neq \emptyset$, then it must be that $\underline{xP}$ intersects Y_m and hence intersects Y as desired. Otherwise $V(Q_m) \cap V(Y_m) = \emptyset$ and $T_m \cup Q_m$ is a tree with trunk P_a. Thus we choose Y_{m+1} to contain a vertex of $\underline{xP}$, so Y intersects $\underline{xP}$ as desired. $\square$

Lemma 5.2.3 (in $\mathtt{ACA}_0$). *If $\mathcal{M}$ is a countable coded ω-model of Σ^1_1-$\mathtt{DC}_0$, then Lemma 5.2.2 holds in $\mathcal{M}$.*

Proof. Consider the formula $\varphi(G, Y, P, x)$ which says there exists a number z such that

(i) z codes a finite subset of $V(P)$,

(ii)

$$\forall y(\forall Y'(Y' \text{ is a wave} \geq Y \text{ in } G \setminus \overline{Py} \rightarrow \mathrm{ess}_{G\setminus\overline{Py}}(Y') = \mathrm{ess}_G(Y)) \rightarrow y \in z),$$

(iii)

$$\begin{aligned}&\forall s(s \text{ codes a finite set}\\ &\qquad \wedge \forall y(\exists Y'(Y' \text{ is a wave} \geq Y \text{ in } G \setminus \overline{Py} \wedge \mathrm{ess}_{G\setminus\overline{Py}}(Y') \subsetneq \mathrm{ess}_G(Y)) \rightarrow y \in s)\\ &\quad \rightarrow V(P) \setminus z \subseteq s), \text{ and}\end{aligned}$$

(iv) x is the first vertex on P not in z.

Claim. *In $\mathcal{M}$, suppose that $G = (G, A, B)$ is a web, Y is a wave in G, and P is a finite path in G disjoint from $V(Y)$ such that Y has no bad extensions in G but there exists a wave $Y' \geq Y$ in $G \setminus P$ with $\mathrm{ess}_{G\setminus P}(Y') \subsetneq \mathrm{ess}_G(Y)$. Then $\mathcal{M} \models \varphi(G, Y, P, x)$ if and only if, in $\mathcal{M}$, x is the first vertex on P such that there exists a wave $Y' \geq Y$ in $G \setminus Px$ with $\mathrm{ess}_{G\setminus Px}(Y') \subsetneq \mathrm{ess}_G(Y)$.*

Proof of claim. For the forward direction, using $\mathtt{ACA}_0$ outside $\mathcal{M}$, let

$$\begin{aligned}Z = \{y \in V(P) \mid\ & \mathrm{ess}_{G\setminus\overline{Py}}(Y') = \mathrm{ess}_G(Y)\\ & \text{for all waves } Y' \geq Y \text{ in } G \setminus \overline{Py} \text{ that are in } \mathcal{M}\}.\end{aligned}$$

Let z be a number coding Z and let s be a number coding $V(P) \setminus Z$. In $\mathcal{M}$, z and s code the same sets that they do outside of $\mathcal{M}$, and $\mathcal{M}$ interprets that

$$z \text{ codes } \{y \in V(P) \mid \mathrm{ess}_{G\setminus\overline{Py}}(Y') = \mathrm{ess}_G(Y) \text{ for all waves } Y' \geq Y \text{ in } G \setminus \overline{Py}\} \text{ and}$$
$$s \text{ codes } \{y \in V(P) \mid \mathrm{ess}_{G\setminus\overline{Py}}(Y') \subsetneq \mathrm{ess}_G(Y) \text{ for some wave } Y' \geq Y \text{ in } G \setminus \overline{Py}\}.$$

Hence in $\mathcal{M}$, this z is the only z which satisfies items (i)–(iii). Thus if $\varphi(G, Y, P, x)$ holds in $\mathcal{M}$, x must be the first vertex on P not in z for this z. Thus x must be the first vertex on P such that, in $\mathcal{M}$, there exists a wave $Y' \geq Y$ in $G \setminus Px$ with $\mathrm{ess}_{G\setminus Px}(Y') \subsetneq \mathrm{ess}_G(Y)$.

For the converse, by using $\mathtt{ACA}_0$ outside of $\mathcal{M}$, let x be the first vertex on P such that, in $\mathcal{M}$, there exists a wave $Y' \geq Y$ in $G \setminus Px$ with $\mathrm{ess}_{G\setminus Px}(Y') \subsetneq \mathrm{ess}_G(Y)$. Let z be a number coding $V(\overline{Px})$. In $\mathcal{M}$, z also codes $V(\overline{Px})$, and this z witnesses $\mathcal{M} \models \varphi(G, Y, P, x)$. ☐

The reason for the somewhat convoluted definition of φ is that prenexing this φ yields a Σ^1_1 formula.

Suppose for a contradiction that Lemma 5.2.2 is false in $\mathcal{M}$ and, in $\mathcal{M}$, let W, $G = (G, A, B)$, and P_a be a counterexample to Lemma 5.2.2. We use $\Sigma^1_1\text{-}\mathtt{DC}_0$ in $\mathcal{M}$ to run the construction from Lemma 5.2.2. This produces in $\mathcal{M}$ a bad extension of W in G, which is a contradiction.

We apply $\Sigma^1_1\text{-}\mathtt{DC}_0$ to the formula $\eta(n, X, Y)$ below. Our η has fixed parameters G, W, P_a, and $(Q_n \mid n \in \mathbb{N})$ (a list of all $\mathrm{ter}(P_a)$-B paths with each occurring infinitely often). We think of a set $Y \subseteq \mathbb{N}$ as coding a pair $Y = (tY, wY)$ where wY is a wave in $G \setminus tY$. Formally, $tY = (Y)_0$ and $wY = (Y)_1$. Our formula $\eta(n, X, Y)$ says that if $t(X)_{n-1}$ is the tree and $w(X)_{n-1}$ is the wave constructed at stage $n-1$ in Lemma 5.2.2, then tY is the tree and wY is the wave constructed at stage n in Lemma 5.2.2. Formally, $\eta(n, X, Y)$ says the following.

- If $n = 0$, then $tY = P_a$ and $wY = W \setminus P_a$.
- If $n > 0$, if $t(X)_{n-1}$ is a finite tree with trunk P_a such that $V(t(X)_{n-1}) \cap (V(W) \setminus V(P_a)) = \emptyset$, if $w(X)_{n-1} \geq W \setminus P_a$ is a wave in $G \setminus t(X)_{n-1}$, if $\mathrm{ess}_{G \setminus t(X)_{n-1}}(w(X)_{n-1}) = \mathrm{ess}_G(W) \setminus \{a\}$, and if $w(X)_{n-1}$ has no bad extensions in $G \setminus t(X)_{n-1}$, then
  - if $V(Q_{n-1}) \cap V(w(X)_{n-1}) \neq \emptyset$ or if $t(X)_{n-1} \cup Q_{n-1}$ is not a tree, then $tY = t(X)_{n-1}$ and $wY = w(X)_{n-1}$, and
  - if $V(Q_{n-1}) \cap V(w(X)_{n-1}) = \emptyset$ and if $t(X)_{n-1} \cup Q_{n-1}$ is a tree, then there is an x such that $tY = t(X)_{n-1} \cup \overline{Q_{n-1}x}$, wY is a wave in $G \setminus tY$, $x \in \mathrm{ter}(wY)$, $wY \geq w(X)_{n-1}$, and $\varphi(G \setminus t(X)_{n-1}, w(X)_{n-1}, Q_{n-1}, x)$.

Prenexing η yields a Σ^1_1 formula. To see this, observe that all the subformulas of η are arithmetic, with the exception of "$w(X)_{n-1}$ has no bad extensions in $G \setminus t(X)_{n-1}$," which is Π^1_1 and appears in the antecedent of η, and φ, which is Σ^1_1 and appears in the consequent of η.

We show that $\mathcal{M} \models \forall n \forall X \exists Y \eta(n, X, Y)$. The interesting case is when $n > 0$ and $X \in \mathcal{M}$ is such that, in $\mathcal{M}$,

- $t(X)_{n-1}$ is a finite tree in G with trunk P_a such that $V(t(X)_{n-1}) \cap (V(W) \setminus V(P_a)) = \emptyset$,
- $w(X)_{n-1} \geq W \setminus P_a$ is a wave in $G \setminus t(X)_{n-1}$,
- $\mathrm{ess}_{G \setminus t(X)_{n-1}}(w(X)_{n-1}) = \mathrm{ess}_G(W) \setminus \{a\}$,
- $w(X)_{n-1}$ has no bad extensions in $G \setminus t(X)_{n-1}$,
- $V(Q_{n-1}) \cap V(w(X)_{n-1}) = \emptyset$, and
- $t(X)_{n-1} \cup Q_{n-1}$ is a tree in G with trunk P_a.

By applying $\mathtt{ACA}_0$ outside $\mathcal{M}$, let x be the first vertex on Q_{n-1} such that, in $\mathcal{M}$, there is a wave $Z \geq w(X)_{n-1}$ in $G \setminus (t(X)_{n-1} \cup Q_{n-1}x)$ with $\mathrm{ess}_{G \setminus (t(X)_{n-1} \cup Q_{n-1}x)}(Z) \subsetneq \mathrm{ess}_{G \setminus t(X)_{n-1}}(w(X)_{n-1})$. Such an x exists by the assumption that G, W, and P_a are a counterexample to Lemma 5.2.2 in $\mathcal{M}$. By the claim, $\varphi(G \setminus t(X)_{n-1}, w(X)_{n-1}, Q_{n-1}, x)$ holds in $\mathcal{M}$. As $\mathcal{M} \models \mathtt{ACA}_0$, apply Lemma 5.1.12 inside $\mathcal{M}$ to get a wave $Z \geq w(X)_{n-1}$ in $G \setminus (t(X)_{n-1} \cup \overline{Q_{n-1}x})$ with $x \in \mathrm{ter}(Z)$. Set $tY = t(X)_{n-1} \cup \overline{Q_{n-1}x}$ and $wY = Z$ to get a $Y \in \mathcal{M}$ witnessing $\exists Y \eta(n, X, Y)$.

Now apply $\Sigma^1_1\text{-}\mathtt{DC}_0$ inside $\mathcal{M}$ to conclude that $\mathcal{M} \models \exists Z \forall n \eta(n, (Z)^n, (Z)_n)$, and let $Z \in \mathcal{M}$ be such a Z. By induction (on an arithmetic formula) outside $\mathcal{M}$, verify that, for all $n \in \mathbb{N}$,

- $t(Z)_n$ is a finite tree in G with trunk P_a such that $V(t(Z)_n) \cap (V(W) \setminus V(P_a)) = \emptyset$,
- $w(Z)_n \geq W \setminus P_a$ is a wave in $G \setminus t(Z)_n$,
- $t(Z)_{n-1} \subseteq t(Z)_n$ (if $n > 0$),
- $w(Z)_{n-1} \leq w(Z)_n$ (if $n > 0$),
- $\mathrm{ess}_{G \setminus t(Z)_n}(w(Z)_n) = \mathrm{ess}_G(W) \setminus \{a_0\}$,
- $w(Z)_n$ has no bad extensions in $G \setminus t(Z)_n$,

and additionally that if $V(Q_n) \cap V(w(Z)_n) = \emptyset$ and $t(Z)_n \cup Q_n$ is a finite tree in G with trunk P_a, then $\mathrm{ter}(w(Z)_{n+1})$ contains a vertex of Q_n. Inside $\mathcal{M}$, let $T = \bigcup_{n \in \mathbb{N}} t(Z)_n$ and $Y = \bigcup_{n \in \mathbb{N}} w(Z)_n$. Just as in the proof of Lemma 5.2.2, Y is a wave in $G \setminus T$ and $\mathrm{ter}(Y)$ is an A-B separator in G. Thus $Y \cup \{P_a\} \in \mathcal{M}$ is the desired bad extension of W in G, which gives the contradiction. $\square$

Theorem 5.2.4. *Menger's theorem for countable webs is provable in* $\Pi^1_1\text{-}\mathtt{CA}_0$.

Proof. Let $G = (G, A, B)$ be a countable web. By Lemma 5.2.1, let $W = \{P_a \mid a \in A\}$ be a $\leq$-maximal wave in G. Let $C = \{\mathrm{ter}(P_a) \mid a \in \mathrm{ess}_G(W)\}$. We extend the paths in $\{P_a \mid a \in \mathrm{ess}_G(W)\}$ to be a collection of disjoint A-B paths M. M and C then witness Menger's theorem for G.

By Theorem 1.6.4, let $\mathcal{M}$ be a countable coded ω-model of $\Sigma^1_1\text{-}\mathtt{DC}_0$ containing G and W. By Lemma 5.2.3, Lemma 5.2.2 holds in $\mathcal{M}$. Also, $\mathcal{M} \models$ "W is a $\leq$-maximal wave", therefore $\mathcal{M} \models$ "W has no bad extensions in G" because W has no proper extensions in G whatsoever. Let $(a_n \mid n \in \mathbb{N})$ enumerate $\mathrm{ess}_G(W)$. Outside $\mathcal{M}$, we construct sequences $(X_n \mid n \in \mathbb{N})$, $(Y_n \mid n \in \mathbb{N})$, and $(Q_n \mid n \in \mathbb{N})$ such that, for all $n \in \mathbb{N}$,

- $X_n \in \mathcal{M}$, $Y_n \in \mathcal{M}$, and $Q_n \in \mathcal{M}$,
- $X_n \subseteq V(G)$ is a finite set, $X_n \cap A = \{a_i \mid i \leq n\}$, and $X_n \subseteq X_{n+1}$,
- Y_n is a wave in $G \setminus X_n$ such that $Y_n \geq W \setminus \bigcup_{i \leq n} P_{a_i}$, Y_n has no bad extensions in $G \setminus X_n$, and $\mathrm{ess}_{G \setminus X_n}(Y_n) = \{a_i \mid i > n\}$, and
- Q_n is an A-B path extending P_{a_n}.

To get started, by Lemma 5.2.3 let $T \in \mathcal{M}$ be a finite a_0-B tree in G with trunk P_{a_0} such that $V(T) \cap (V(W) \setminus V(P_{a_0})) = \emptyset$, and let $Y_0 \in \mathcal{M}$ be a wave in $G \setminus T$ such that $Y_0 \geq W \setminus P_{a_0}$, $\mathrm{ess}_{G \setminus T}(Y_0) = \{a_i \mid i > 0\}$, and Y_0 has no bad extensions in $G \setminus T$. Let $X_0 = T$, and let Q_0 be the a_0-B path in T. Suppose we have X_n, Y_n and Q_n. Let $P'_{a_{n+1}}$ be the path in Y_n starting at a_{n+1}, and note that $P'_{a_{n+1}}$ extends $P_{a_{n+1}}$. By Lemma 5.2.3, let $T \in \mathcal{M}$ be a finite a_{n+1}-B tree in $G \setminus X_n$ with trunk $P'_{a_{n+1}}$ such that $V(T) \cap (V(Y_n) \setminus V(P'_{a_{n+1}})) = \emptyset$, and let $Y_{n+1} \in \mathcal{M}$ be a wave in $G \setminus (X_n \cup T)$ such that $Y_{n+1} \geq Y_n \setminus P'_{a_{n+1}}$, $\mathrm{ess}_{G \setminus (X_n \cup T)}(Y_{n+1}) = \{a_i \mid i > n+1\}$, and Y_{n+1} has no bad extensions in $G \setminus (X_n \cup T)$. Let $X_{n+1} = X_n \cup T$ and let Q_{n+1} be the a_{n+1}-B path in T. In the end, the collection $M = \{Q_n \mid n \in \mathbb{N}\}$ consists of disjoint A-B paths in G, and $C = \{\mathrm{ter}(P_a) \mid a \in \mathrm{ess}_G(W)\}$ is an A-B separator containing exactly one vertex from each path in M. $\square$

5.3 Extended Menger's theorem

Although Menger's theorem for countable webs cannot be equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$ as discussed in the introduction, the proof given in Theorem 5.2.4 is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ in the sense that it proves a stronger statement, called extended Menger's theorem, that is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$. This additional strength comes from our application of Lemma 5.2.1.

Extended Menger's Theorem. *Let (G, A, B) be a countable web. Then there is a set of disjoint A-B paths M and an A-B separator C such that C consists of exactly one vertex from each path in M. Furthermore, C is the set of terminal vertices of the essential paths in a $\leq$-maximal wave.*

Let G be a graph. For $x \in V(G)$, $N(x) = \{y \in V(G) \mid (x, y) \in E(G)\}$ denotes the set of *neighbors* of x. For $X \subseteq V(G)$, $D(X) = \{y \in V(G) \mid N(y) \subseteq X\}$ denotes the *demand* of X. In the proof of König's duality theorem for countable bipartite graphs in [4], the following lemma plays the role that Lemma 5.2.1 plays in the proof of Menger's theorem for countable webs given in Theorem 5.2.4.

Lemma 5.3.1 ([4] Lemma 3.2)**.** *Let (X, Y, E) be a countable bipartite graph. Then there is a $\subseteq$-maximum $Y_0 \subseteq Y$ for which there is a matching of Y_0 into $D(Y_0)$.*

The application of Lemma 5.3.1 yields a stronger form of König's duality theorem, called extended König's duality theorem.

Extended König's Duality Theorem. *Let (X, Y, E) be a countable bipartite graph. Then there is a matching M and a cover C such that C consists of exactly one vertex from each edge in E. Furthermore, for every $y \in Y$, $y \in C$ if and only if there is a $Y_0 \subseteq Y$ containing y and a matching of Y_0 into $D(Y_0)$.*

Extended König's duality theorem is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$ by [4] Theorem 4.18. In fact, Lemma 5.3.1 itself is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$ by [4] Corollary 4.20. In contrast, recall from the introduction that König's duality theorem is equivalent to $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$. We show that the existence of a $\leq$-maximal wave, that is, Lemma 5.2.1, implies Lemma 5.3.1 over $\mathtt{RCA}_0$. It follows that both Lemma 5.2.1 and extended Menger's theorem are equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$.

Lemma 5.3.2. *Lemma 5.2.1 implies Lemma 5.3.1 over $\mathtt{RCA}_0$.*

Proof. We prove the lemma in two steps. First, we prove that Lemma 5.2.1 implies $\mathtt{ACA}_0$ over $\mathtt{RCA}_0$. Second, we prove that Lemma 5.2.1 implies Lemma 5.3.1 over $\mathtt{ACA}_0$.

First work in $\mathtt{RCA}_0$. We use the fact that $\mathtt{ACA}_0$ is equivalent to the statement "for every injection $f\colon \mathbb{N} \to \mathbb{N}$ there is a $Z \subseteq \mathbb{N}$ such that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$" (see [67] Lemma III.1.3). So let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. Let (X, Y, E) be the bipartite graph with sides $X = \{x_n \mid n \in \mathbb{N}\}$ and $Y = \{y_n \mid n \in \mathbb{N}\}$ and edges $E = \{(x_m, y_n) \mid f(m) = n\}$. Let G be the web $G = ((X, Y, E), X, Y)$, and by Lemma 5.2.1 let W be a $\leq$-maximal wave in G. Let $Z = \{n \mid y_n \in V(W)\}$.

We show that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$. If $f(m) = n$, then (x_m, y_n) is the only edge incident to either x_n or y_n because f is an injection. Thus the path in W starting at x_m is either the trivial path (x_m) or the path (x_m, y_n). If the path is (x_m), then the path could be extended to (x_m, y_n), giving a proper extension of the wave W and contradicting maximality. Thus the path is (x_m, y_n), hence $y_n \in V(W)$ and $n \in Z$. Conversely, if $n \in Z$, then $y_n \in V(W)$ so (x_m, y_n) must be an edge for some $m \in \mathbb{N}$. This can only happen if $f(m) = n$.

Now work in $\mathtt{ACA}_0$. Let (X, Y, E) be a countable bipartite graph. By Lemma 5.2.1, let W be a $\leq$-maximal wave in the web $G = ((X, Y, E), X, Y)$. Let $Y_0 = Y \cap \operatorname{ter}(W)$. We show that Y_0 witnesses Lemma 5.3.1 for (X, Y, E). Let M be the matching consisting of the paths in W of length 1. If $y \in Y_0$, then by choice of Y_0 and M there is an $x \in X$ such that (x, y) in M. If $x \notin D(Y_0)$, then there is a $y' \in Y \setminus Y_0$ such that $(x, y') \in E$. Clearly $y' \notin \operatorname{ter}(W)$, and $x \notin \operatorname{ter}(W)$ as well because (x, y) is a path in W. Thus (x, y') is an X-Y path in G avoiding $\operatorname{ter}(W)$, contradicting that W is a wave. Therefore M is a matching of Y_0 into $D(Y_0)$.

To see that Y_0 is $\subseteq$-maximum, suppose for a contradiction that there is a $Y' \subseteq Y$ and a matching M' of Y' into $D(Y')$ such that $Y' \nsubseteq Y_0$. Let W' be the subgraph of (X, Y, E) with vertices $V(W) \cup Y'$ and edges $E(W) \cup \{(x, y) \in M' \mid y \notin Y_0\}$. W is a proper subgraph of W', so if we can show that W' is a wave, then we have that $W < W'$, contradicting the maximality of W. Consider an edge $(x, y) \in M'$ with $y \notin Y_0$. It must be that $x \in \operatorname{ter}(W)$ because otherwise (x, y) would be an X-Y path in G avoiding $\operatorname{ter}(W)$. It follows that W' is a warp. To see that $\operatorname{ter}(W')$ is an X-Y separator, consider an edge $(x, y) \in E$. We know $\operatorname{ter}(W)$ is an X-Y separator, so either $x \in \operatorname{ter}(W)$ or $y \in \operatorname{ter}(W)$. If $y \in \operatorname{ter}(W)$ then $y \in \operatorname{ter}(W')$, so assume $x \in \operatorname{ter}(W)$. If $x \notin \operatorname{ter}(W')$, then there must have been an edge $(x, y') \in M'$ for some $y' \in Y' \setminus Y_0$. By assumption, M' is a matching from Y' into $D(Y')$, so $x \in D(Y')$ because M' matches y' and x. Therefore $y \in Y'$ because $x \in D(Y')$ and (x, y) is an edge. Clearly $Y' \subseteq \operatorname{ter}(W')$, so $y \in \operatorname{ter}(W')$ as desired. $\square$

Corollary 5.3.3. *Lemma 5.2.1 is equivalent to* $\Pi^1_1\text{-}\mathtt{CA}_0$ *over* $\mathtt{RCA}_0$.

Proof. The given proof of Lemma 5.2.1 is in $\Pi^1_1\text{-}\mathtt{CA}_0$. By Lemma 5.3.2, Lemma 5.2.1 implies Lemma 5.3.1 over $\mathtt{RCA}_0$. By [4] Corollary 4.20, Lemma 5.3.1 is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$. $\square$

Corollary 5.3.4. *Extended Menger's theorem is equivalent to* $\Pi^1_1\text{-}\mathtt{CA}_0$ *over* $\mathtt{RCA}_0$.

Proof. Theorem 5.2.4 proves extended Menger's theorem in $\Pi^1_1\text{-}\mathtt{CA}_0$. Extended Menger's theorem asserts the existence of a $\leq$-maximal wave, which is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$ by Corollary 5.3.3. $\square$

CHAPTER 6
PARTIAL ANALYSES OF BIRKHOFF'S THEOREM AND OF UNFRIENDLY PARTITIONS

In this chapter, we present a few partial results concerning the reverse mathematics of Birkhoff's theorem and of unfriendly partitions.

6.1 Countable Birkhoff's theorem in $\mathtt{WKL}_0$

Three classic theorems in finite matching theory are Hall's theorem, König's duality theorem, and Birkhoff's theorem. The reverse mathematics of König's duality theorem was begun in [4] and was completed in [63]. The theorem is that König's duality theorem (for countable graphs) is equivalent to $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$. The reverse mathematics of Menger's theorem, a generalization of König's duality theorem, was analyzed in the previous chapter.

If G is a graph and $S \subseteq V(G)$, let $N(S) = \{y \in V(G) \mid (\exists s \in S)((x, y) \in E)\}$ denote the *neighbors* of S. A bipartite graph (X, Y, E) satisfies *Hall's condition* if and only if, for all finite $S \subseteq X$, $|S| \leq |N(S)|$. A bipartite graph (X, Y, E) satisfies *symmetric Hall's condition* if and only if, for all finite $S \subseteq X$ and all finite $S \subseteq Y$, $|S| \leq |N(S)|$. The classical versions of Hall's theorem are the following.

Finite Hall's Theorem. *In every finite bipartite graph (X, Y, E), there is a complete matching of X into Y (i.e., a matching in which every vertex of X is matched) if and only if (X, Y, E) satisfies Hall's condition.*

Finite Symmetric Hall's Theorem. *In every finite bipartite graph (X, Y, E), there is a perfect matching if and only if (X, Y, E) satisfies symmetric Hall's condition.*

Call a graph *locally finite* if every vertex is has finitely many neighbors. By easy compactness arguments, a locally finite bipartite graph (X, Y, E) has a complete matching of X into Y if and only if it satisfies Hall's condition, and a locally finite bipartite graph has a perfect matching if and only if it satisfies symmetric Hall's condition.

Hirst analyzed the reverse mathematics of Hall's theorem for countable locally finite graphs in [27]. For reference, we summarize Hirst's results in the following theorems.

Theorem 6.1.1 ([27] Theorem 2.2 and [27] Theorem 3.1)**.** *The following are equivalent over $\mathtt{RCA}_0$.*

(i) $\mathtt{ACA}_0$.

(ii) Every countable locally finite bipartite graph (X, Y, E) satisfying Hall's condition has a complete matching of X into Y.

(iii) Every countable locally finite bipartite graph satisfying symmetric Hall's condition has a perfect matching.

Call a countable graph (with $V(G) = \mathbb{N}$) *bounded* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\forall x \forall y((x, y) \in E(G) \to y \leq f(x))$. That is, $f(x)$ is a bound on the neighbors of x. Of course a countable graph is locally finite if and only if it is bounded, but it takes $\mathtt{ACA}_0$ to prove this. In $\mathtt{WKL}_0$, Hall's theorem is provable for bounded graphs, but locally finite graphs are not in general bounded. For $k \in \mathbb{N}$, call a graph *k-regular* if and only if every vertex has exactly k neighbors. Under $\mathtt{RCA}_0$, a countable k-regular graph is bounded.

Theorem 6.1.2 ([27] Theorem 2.3 and [27] Theorem 3.3)**.** *The following are equivalent over* $\mathtt{RCA}_0$.

(i) $\mathtt{WKL}_0$.

(ii) Every countable bounded bipartite graph (X, Y, E) satisfying Hall's condition has a complete matching of X into Y.

(iii) Every countable bounded bipartite graph satisfying symmetric Hall's condition has a perfect matching.

(iv) For every $k \geq 1$, every countable k-regular bipartite graph has a perfect matching.

(v) Every countable 2-regular bipartite graph has a perfect matching.

We make use of the implication (v) $\Rightarrow$ (i), and we now provide a proof that is simpler than the original proof in [27].

Proof of Theorem 6.1.2 (v) $\Rightarrow$ (i). By [67] Lemma IV.4.4, it suffices to show that if $f, g\colon \mathbb{N} \to \mathbb{N}$ are injections such that $\forall m \forall n(f(m) \neq g(n))$ then $\exists Z \forall m(f(m) \in Z \wedge g(m) \notin Z)$. Thus let f and g be injections such that $\forall m \forall n(f(m) \neq g(n))$. We build a 2-regular bipartite graph $G = (X, Y, E)$ such that any matching in G determines a set Z such that $\forall m(f(m) \in Z \wedge g(m) \notin Z)$. Let

$$
\begin{aligned}
X &= \{x_n \mid n \in \mathbb{N}\} \cup \{x^j_{n,i} \mid n, i \in \mathbb{N} \wedge j < 4\}\\
Y &= \{y_n \mid n \in \mathbb{N}\} \cup \{y^0_{n,0}, y^1_{n,0} \mid n \in \mathbb{N}\} \cup \{y^j_{n,i} \mid n \in \mathbb{N} \wedge i \geq 1 \wedge j < 4\}\\
E &= \{(x_n, y^0_{n,0}) \mid n \in \mathbb{N}\} \cup \{(x_n, y^1_{n,0}) \mid n \in \mathbb{N}\}\\
&\quad \cup \{(x^2_{n,0}, y_n) \mid n \in \mathbb{N}\} \cup \{(x^3_{n,0}, y_n) \mid n \in \mathbb{N}\}\\
&\quad \cup \{(x^0_{n,i}, y^0_{n,i+1}) \mid n, i \in \mathbb{N}\} \cup \{(x^1_{n,i}, y^1_{n,i+1}) \mid n, i \in \mathbb{N}\}\\
&\quad \cup \{(x^0_{n,i}, y^0_{n,i}) \mid f(i) \neq n \wedge g(i) \neq n\} \cup \{(x^1_{n,i}, y^1_{n,i}) \mid f(i) \neq n \wedge g(i) \neq n\}\\
&\quad \cup \{(x^2_{n,i}, y^2_{n,i}) \mid n \in \mathbb{N} \wedge i \geq 1\} \cup \{(x^3_{n,i}, y^3_{n,i}) \mid n \in \mathbb{N} \wedge i \geq 1\}\\
&\quad \cup \{(x^2_{n,i}, y^2_{n,i+1}) \mid f(i) \neq n \wedge g(i) \neq n\} \cup \{(x^3_{n,i}, y^3_{n,i+1}) \mid f(i) \neq n \wedge g(i) \neq n\}\\
&\quad \cup \{(x^2_{n,i}, y^0_{n,i}) \mid f(i) = n\} \cup \{(x^3_{n,i}, y^1_{n,i}) \mid f(i) = n\}\\
&\quad \cup \{(x^2_{n,i}, y^1_{n,i}) \mid g(i) = n\} \cup \{(x^3_{n,i}, y^0_{n,i}) \mid g(i) = n\}\\
&\quad \cup \{(x^0_{n,i}, y^2_{n,i+1}) \mid f(i) = n \vee g(i) = n\} \cup \{(x^1_{n,i}, y^3_{n,i+1}) \mid f(i) = n \vee g(i) = n\}
\end{aligned}
$$

Clearly G is bipartite. To see that G is 2-regular, we check that every vertex has exactly two neighbors.

- x_n is adjacent to $y^0_{n,0}$ and $y^1_{n,0}$, and y_n is adjacent to $x^2_{n,0}$ and $x^3_{n,0}$.

- $x^0_{n,i}$ is adjacent to $y^0_{n,i+1}$, and it is adjacent to $y^0_{n,i}$ (if $f(i) \neq n \wedge g(i) \neq n$) or to $y^2_{n,i+1}$ (otherwise).
- $x^1_{n,i}$ is adjacent to $y^1_{n,i+1}$, and it is adjacent to $y^1_{n,i}$ (if $f(i) \neq n \wedge g(i) \neq n$) or to $y^3_{n,i+1}$ (otherwise).
- $x^2_{n,i}$ is adjacent to y_n (if $i = 0$) or $y^2_{n,i}$ (if $i \geq 1$), and it is adjacent to $y^2_{n,i+1}$ (if $f(i) \neq n \wedge g(i) \neq n$) or $y^0_{n,i}$ (if $f(i) = n$) or $y^1_{n,i}$ (if $g(i) = n$).
- $x^3_{n,i}$ is adjacent to y_n (if $i = 0$) or $y^3_{n,i}$ (if $i \geq 1$), and it is adjacent to $y^3_{n,i+1}$ (if $f(i) \neq n \wedge g(i) \neq n$) or $y^1_{n,i}$ (if $f(i) = n$) or $y^0_{n,i}$ (if $g(i) = n$).
- $y^0_{n,i}$ is adjacent to x_n (if $i = 0$) or $x^0_{n,i-1}$ (if $i \geq 1$), and it is adjacent to $x^0_{n,i}$ (if $f(i) \neq n \wedge g(i) \neq n$) or $x^2_{n,i}$ (if $f(i) = n$) or $x^3_{n,i}$ (if $g(i) = n$).
- $y^1_{n,i}$ is adjacent to x_n (if $i = 0$) or $x^1_{n,i-1}$ (if $i \geq 1$), and it is adjacent to $x^1_{n,i}$ (if $f(i) \neq n \wedge g(i) \neq n$) or $x^3_{n,i}$ (if $f(i) = n$) or $x^2_{n,i}$ (if $g(i) = n$).
- $y^2_{n,i}$ is adjacent to $x^2_{n,i}$, and it is adjacent to $x^2_{n,i-1}$ (if $f(i) \neq n \wedge g(i) \neq n$) or $x^0_{n,i-1}$ (otherwise).
- $y^3_{n,i}$ is adjacent to $x^3_{n,i}$, and it is adjacent to $x^3_{n,i-1}$ (if $f(i) \neq n \wedge g(i) \neq n$) or $x^1_{n,i-1}$ (otherwise).

Thus G is a 2-regular bipartite graph. Let M be a perfect matching, and by Δ^0_1 comprehension, let

$$Z = \{n \mid ((x_n, y^0_{n,0}) \in M \wedge (x^2_{n,0}, y_n) \in M) \vee ((x_n, y^1_{n,0}) \in M \wedge (x^3_{n,0}, y_n) \in M)\}.$$

Suppose $f(m) = n$. Exactly one of $(x_n, y^0_{n,0})$ and $(x_n, y^1_{n,0})$ is in M. If $(x_n, y^0_{n,0})$ is in M, then so are $(x^0_{n,i}, y^0_{n,i+1})$ for $0 \leq i < m$, $(x^2_{n,i}, y^2_{n,i})$ for $0 < i \leq m$, and $(x^2_{n,0}, y_n)$. If $(x_n, y^1_{n,0})$ is in M, then so are $(x^1_{n,i}, y^1_{n,i+1})$ for $0 \leq i < m$, $(x^3_{n,i}, y^3_{n,i})$ for $0 < i \leq m$, and $(x^3_{n,0}, y_n)$. Hence $((x_n, y^0_{n,0}) \in M \wedge (x^2_{n,0}, y_n) \in M) \vee ((x_n, y^1_{n,0}) \in M \wedge (x^3_{n,0}, y_n) \in M)$, thus $n \in Z$.

Conversely, suppose $g(m) = n$. Exactly one of $(x_n, y^0_{n,0})$ and $(x_n, y^1_{n,0})$ is in M. If $(x_n, y^0_{n,0})$ is in M, then so are $(x^0_{n,i}, y^0_{n,i+1})$ for $0 \leq i < m$, $(x^3_{n,i}, y^3_{n,i})$ for $0 < i \leq m$, and $(x^3_{n,0}, y_n)$. If $(x_n, y^1_{n,0})$ is in M, then so are $(x^1_{n,i}, y^1_{n,i+1})$ for $0 \leq i < m$, $(x^2_{n,i}, y^2_{n,i})$ for $0 < i \leq m$, and $(x^2_{n,0}, y_n)$. Hence $((x_n, y^0_{n,0}) \in M \wedge (x^3_{n,0}, y_n) \in M) \vee ((x_n, y^1_{n,0}) \in M \wedge (x^2_{n,0}, y_n) \in M)$, thus $n \notin Z$.

Thus $\forall m(f(m) \in Z \wedge g(m) \notin Z)$ as desired. □

We are now ready to discuss Birkhoff's theorem. For $n \in \mathbb{N}$, an $n \times n$ matrix of non-negative reals is called *doubly stochastic* if every row sums to one and every column sums to one. A permutation matrix is a matrix whose entries are either 0 or 1 and such that every row and every column contains exactly one 1. A linear combination $\lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_n P_n$ of permutation matrices $P_0, P_1, \ldots, P_n$ is *convex* if and only if $\lambda_i \geq 0$ for each $0 \leq i \leq n$ and $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$.

Finite Birkhoff's Theorem. *Every $n \times n$ doubly stochastic matrix is a convex linear combination of $n \times n$ permutation matrices.*

Birkhoff asked in problem 111 of [12] for an $\mathbb{N} \times \mathbb{N}$ version of this theorem. The solution we consider was provided by Isbell in [29]. Isbell makes the following definitions.

Definition 6.1.3. For an $\mathbb{N} \times \mathbb{N}$ matrix A, let

$$\|A\| = \max\left(\sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |A(i,j)|, \sup_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} |A(i,j)|\right).$$

Definition 6.1.4. A doubly stochastic $\mathbb{N} \times \mathbb{N}$ matrix A satisfies $(*)$ if and only if for every $\epsilon > 0$ there is an n such that, in any row or column, the sum of the n largest entries is $\geq 1 - \epsilon$.

Isbell's solution to Birkhoff's problem 111, what we call *countable Birkhoff's theorem*, is the following.

Countable Birkhoff's Theorem ([29] Theorem 3)**.** *If A is an $\mathbb{N} \times \mathbb{N}$ doubly stochastic matrix satisfying $(*)$, then for every $\epsilon > 0$ there is a convex linear combination of $\mathbb{N} \times \mathbb{N}$ permutation matrices B such that $\|A - B\| \leq \epsilon$.*

Countable Birkhoff's theorem may be rephrased as follows. In the Banach space of $\mathbb{N} \times \mathbb{N}$ matrices A with $\|A\|$ finite and norm $\|\cdot\|$, the convex closure of the set of permutation matrices is the set of doubly stochastic matrices satisfying $(*)$.

The main result of this section is a proof of countable Birkhoff's theorem in $\mathtt{WKL}_0$. In order to give this proof, we first need a version of Theorem 6.1.2 for bipartite multigraphs. The key to approximating a doubly stochastic matrix by a convex combination of permutation matrices is the following fact. Any doubly stochastic matrix consisting of rational entries with a fixed denominator m is a convex linear combination of permutation matrices. In the language of matchings, this says that every m-regular bipartite multigraph is the union of m edge-disjoint perfect matchings. We use multigraphs because p/m at entry (i, j) in the matrix corresponds to p edges (i, j) in the graph. Classically, the fact is proved by applying Hall's theorem for regular graphs m times. However, iterating Hall's theorem this way is not formalizable in $\mathtt{WKL}_0$. Instead, we use a compactness argument that produces the m matchings simultaneously. To get started, we need the following finite cases. The arguments are standard and are formalizable in $\mathtt{RCA}_0$.

Lemma 6.1.5 (in $\mathtt{RCA}_0$)**.** *Let $G = (A, B, E)$ be a finite bipartite multigraph graph with $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $(\forall S \subseteq A_0)(|S| \leq |N(S)|)$ and $(\forall S \subseteq B_0)(|S| \leq |N(S)|)$. Then there is a matching (not necessarily perfect) covering $A_0 \cup B_0$.*

Proof. Induct on $\langle |A_0|, |B_0| \rangle$. If there are no edges between A_0 and B_0, then we can apply finite Hall's theorem to A_0 and B_0 independently to get a matching of

A_0 into $B \setminus B_0$ and a matching of B_0 into $A \setminus A_0$. The union of these two matchings is a matching covering $A_0 \cup B_0$. This handles the cases where $|A_0| = 0$ or $|B_0| = 0$.

Suppose that $|N(S)| \geq |S| + 1$ for all non-empty $S \subseteq A_0$ and all non-empty $S \subseteq B_0$. We may assume that there are $a \in A_0$ and $b \in B_0$ such that (a, b) is an edge. Let $A' = A \setminus \{a\}$, $A'_0 = A_0 \setminus \{a\}$, $B' = B \setminus \{b\}$, and $B'_0 = B_0 \setminus \{b\}$. Then $(\forall S \subseteq A'_0)(|S| \leq |N(S)|)$ and $(\forall S \subseteq B'_0)(|S| \leq |N(S)|)$. By induction, there is a matching covering $A'_0 \cup B'_0$. Adding the edge (a, b) gives a matching covering $A_0 \cup B_0$.

Suppose instead that $|N(S)| = |S|$ for some non-empty $S \subseteq A_0$ (the case where $S \subseteq B_0$ is symmetric). Finite Hall's theorem gives a matching M_1 of S into $N(S)$, and this matching is perfect because $|N(S)| = |S|$. Put $A' = A \setminus S$, $A'_0 = A_0 \setminus S$, $B' = B \setminus N(S)$, and $B'_0 = B_0 \setminus N(S)$. Let $H = (A', B', E \cap (A' \times B'))$. We need to show that $|T| \leq |N_H(T)|$ for all $T \subseteq A'_0$ and all $T \subseteq B'_0$. Let $T \subseteq A'_0$. If $|N_H(T)| < |T|$ then $|N_G(T \cup S)| = |N_H(T)| + |N_G(S)| < |T| + |S| = |T \cup S|$, which is a contradiction because $T \cup S \subseteq A_0$. Let $T \subseteq B'_0$. $N_H(T) = N_G(T)$ because if $b \in T$ had a neighbor in S, then b would be in $N(S)$ and hence not in T. Since $B'_0 \subseteq B_0$, it follows that $|T| \leq |N_G(T)| = |N_H(T)|$. Thus by induction there is a matching M_2 in H covering $A'_0 \cup B'_0$. The matching $M_1 \cup M_2$ covers $A_0 \cup B_0$.

The formula $\varphi(\langle n, m \rangle)$ on which we induct says "for every finite bipartite multigraph (A, B, E) and for every $A_0 \subseteq A$ and $B_0 \subseteq B$ with $|A_0| = n$ and $|B_0| = m$ such that $(\forall S \subseteq A_0)(|S| \leq |N(S)|)$ and $(\forall S \subseteq B_0)(|S| \leq |N(S)|)$, there is a matching covering $A_0 \cup B_0$." Given a sensible coding of finite multigraphs $G = (A, B, E)$ over $\mathbb{N}$, bounds on the possible codes for subsets of A, B, and E can be calculated from the code for G. Thus the only unbounded quantifier in φ is "for every bipartite multigraph," hence φ can be written as a Π^0_1 formula. $\mathtt{RCA}_0$ proves the Π^0_1 induction scheme (see [67] Corollary II.3.10), so the foregoing argument is formalizable in $\mathtt{RCA}_0$. ☐

Lemma 6.1.6 (in $\mathtt{RCA}_0$)**.** *Let $G = (A, B, E)$ be a finite bipartite multigraph with $A_0 \subseteq A$ and $B_0 \subseteq B$ such that every vertex of $A \cup B$ has multidegree $\leq m$ and every vertex of $A_0 \cup B_0$ has multidegree $= m$. Then there are m edge-disjoint matchings (not necessarily perfect), each covering $A_0 \cup B_0$.*

Proof. Induct on m. Suppose $m = 1$. Let $S \subseteq A_0$. The number of edges incident to S is $|S|$, so the number of edges incident to $N(S)$ is $\geq |S|$. Hence the average multidegree of a vertex in $N(S)$ is $\geq |S|/|N(S)|$. If $|N(S)| < |S|$, then this average is > 1, which means some vertex of B has multidegree > 1, a contradiction. Hence $|S| \leq |N(S)|$ for all $S \subseteq A_0$, and similarly this holds for all $S \subseteq B_0$. By Lemma 6.1.5, there is a matching covering $A_0 \cup B_0$.

Now consider the case $m + 1$. Remove all edges without an endpoint in $A_0 \cup B_0$. If $a \in A \setminus A_0$ has $m+1$ edges incident to B_0, then put a in A_0. Similarly, if $b \in B \setminus B_0$ has $m + 1$ edges incident to A_0, then put b in B_0. In the resulting multigraph, all vertices of $A_0 \cup B_0$ have multidegree $= m + 1$, and all vertices of $(A \setminus A_0) \cup (B \setminus B_0)$ have multidegree $\leq m$. Let $S \subseteq A_0$. The number of edges incident to S is $(m + 1)|S|$, so the number of edges incident to $N(S)$ is $\geq (m + 1)|S|$. Hence the average multidegree of a vertex in $N(S)$ is $\geq (m +$

$1)|S|/|N(S)|$. If $|N(S)| < |S|$, then this average is $> m+1$, which means that some vertex of B has multidegree $> m+1$. Hence $|S| \leq |N(S)|$ for all $S \subseteq A_0$, and similarly this holds for all $S \subseteq B_0$. By Lemma 6.1.5, there is a matching M covering $A_0 \cup B_0$. Remove the edges of this matching from the multigraph. In the resulting multigraph, all vertices of $A_0 \cup B_0$ have multidegree $= m$, and all vertices of $A \cup B$ have multidegree $\leq m$. By induction there exist m edge-disjoint matchings, each covering $A_0 \cup B_0$. Add M to this list and we have $m+1$ such matchings.

As in Lemma 6.1.5, the property on which we induct can be written as a Π^0_1 formula, and hence the foregoing argument is formalizable in $\mathtt{RCA}_0$. □

We can now decompose a countable m-regular bipartite multigraph into m edge-disjoint perfect matchings in $\mathtt{WKL}_0$.

Lemma 6.1.7 (in $\mathtt{WKL}_0$)**.** *Let (X, Y, E) be a countable m-regular bipartite multigraph. Then (X, Y, E) is the union of m edge-disjoint perfect matchings.*

Proof. Assume X and Y are copies of $\mathbb{N}$. We will define a tree $T \subseteq ((\mathbb{N}^2)^m)^{<\mathbb{N}}$ ordered by extension. An element of T looks like

$$\langle \langle \langle y^0_0, x^0_0 \rangle, \ldots, \langle y^{m-1}_0, x^{m-1}_0 \rangle \rangle, \ldots, \langle \langle y^0_n, x^0_n \rangle, \ldots, \langle y^{m-1}_n, x^{m-1}_n \rangle \rangle \rangle .$$

The idea is that $\langle y^i_0, x^i_0 \rangle, \ldots, \langle y^i_n, x^i_n \rangle$ codes the i^{th} partial matching: for $k \leq n$, $k \in X$ is matched to $y^i_k \in Y$ and $k \in Y$ is matched to $x^i_k \in X$. Thus we define T to be the set of all such sequences with the following properties.

(i) If $k \leq n$ and $i < m$ then (k, y^i_k) and (x^i_k, k) are edges.

(ii) If $k, l \leq n$ and $i < m$ then $y^i_k = l$ if and only if $x^i_l = k$ (this guarantees that the i^{th} partial matching is a matching).

(iii) For each $k \leq n$, the number of times each $y \in Y$ occurs as some y^i_k for $i < m$ is $\leq$ the multiplicity of the edge (k, y), and the number of times each $x \in X$ occurs as some x^i_k for $i < m$ is $\leq$ the multiplicity of the edge (x, k) (this guarantees that the matchings are disjoint).

One readily proves that T is bounded using the fact that (X, Y, E) is m-regular (hence bounded). Thus to apply weak König's lemma, we need only to show that for each $n \in \mathbb{N}$ there is a sequence in T of length n. Let $A_0 = \{0, \ldots, n-1\} \subseteq X$ be the first n elements of X and $B_0 = \{0, \ldots, n-1\} \subseteq Y$ be the first n elements of Y. Let N_1 be the largest neighbor of a vertex in B_0 and let N_2 be the largest neighbor of a vertex in A_0. Let $A = \{0, \ldots, N_1\} \subseteq X$, and let $B = \{0, \ldots, N_2\} \subseteq Y$. By Lemma 6.1.6, there are m edge-disjoint matchings in the finite bipartite multigraph $(A, B, E \cap (A \times B))$, each covering $A_0 \cup B_0$. Such a sequence of matchings corresponds to an element of the tree of length n. Apply weak König's lemma to T. Any infinite path through T gives a decomposition of (X, Y, E) into m edge-disjoint perfect matchings. □

Lemma 6.1.7 is in fact equivalent to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$.

Theorem 6.1.8. *The following are equivalent over* RCA_0.

(i) WKL_0.
(ii) Every countable m*-regular bipartite multigraph is the union of* m *edge-disjoint perfect matchings.*

Proof. Lemma 6.1.7 proves (i) $\Rightarrow$ (ii). For (ii) $\Rightarrow$ (i), observe that (ii) implies that every 2-regular bipartite graph has a perfect matching which implies (i) by Theorem 6.1.2. $\square$

Theorem 6.1.9. *Countable Birkhoff's theorem is provable in* WKL_0.

Proof. Let A be an $\mathbb{N} \times \mathbb{N}$ doubly stochastic matrix of reals satisfying $(*)$. Let $\epsilon > 0$ be rational. By $(*)$, let n be such that, in any row or column, the sum of the n largest entries is $\geq 1 - \epsilon/4$. We first do some pre-processing of A in RCA_0. Let $m = \lceil \frac{n}{\epsilon/4} \rceil$. Define an $\mathbb{N} \times \mathbb{N}$ matrix of rationals C by letting $C(i,j)$ be the largest $p/m < A(i,j)$. Then $\|A - C\| \leq \epsilon/2$ as follows. Fix a row i, and let S be a set of n indices such that $\sum_{j \in S} A(i,j) \geq 1 - \epsilon/4$. Then

$$\sum_{j \in \mathbb{N}} (|A(i,j) - C(i,j)|) = \sum_{j \in S} (|A(i,j) - C(i,j)|) + \sum_{j \notin S} (|A(i,j) - C(i,j)|).$$

By choice of C, $C(i,j) < A(i,j)$ and $|A(i,j) - C(i,j)| \leq 1/m$ for all i and j. Therefore $\sum_{j \in S} (|A(i,j) - C(i,j)|) \leq n(1/m) \leq \epsilon/4$ and $\sum_{j \notin S} (|A(i,j) - C(i,j)|) \leq \sum_{j \notin S} A(i,j) \leq \epsilon/4$ (because $\sum_{j \in S} A(i,j) \geq 1 - \epsilon/4$). Thus $\sum_{j \in \mathbb{N}} (|A(i,j) - C(i,j)|) \leq \epsilon/2$. Similarly, each of the absolute column sums of $A - C$ is $\leq \epsilon/2$, so $\|A - C\| \leq \epsilon/2$.

Since each entry of C is of the form p/m, and since $C(i,j) < A(i,j)$ for all i and j, it follows that each row and column of C sums to a rational of the form $p/m < 1$. Therefore we can recursively define an $\mathbb{N} \times \mathbb{N}$ doubly stochastic matrix B by alternating adding entries of the form $1/m$ in each row and column of C. It must be that every row and column of C sums to at least $1 - \epsilon/2$ because $\|A - C\| \leq \epsilon/2$. Therefore at most $\epsilon/2$ is added to each row and column of C to get B, hence $\|C - B\| \leq \epsilon/2$. Thus $\|A - B\| \leq \epsilon$. We use WKL_0 to show that B is a convex linear combination of permutation matrices.

The matrix B is an $\mathbb{N} \times \mathbb{N}$ doubly stochastic matrix in which every entry is of the form p/m. View B as a bipartite multigraph (X, Y, E) where $X = \{x_i \mid i \in \mathbb{N}\}$, $Y = \{y_i \mid i \in \mathbb{N}\}$, and E contains p copies of the edge (x_i, y_j) if $B(i,j) = p/m$. (X, Y, E) is m-regular because B is doubly stochastic. By Lemma 6.1.7, let M_0, $M_1, \dots, M_{m-1}$ be m edge-disjoint perfect matchings such that $E = \bigcup_{k<m} M_k$. For each $k < m$, let P_k be the permutation matrix corresponding to M_k (i.e., $P_k(i,j) = 1$ if and only if $(x_i, y_j) \in M_k$). Then $B = (1/m)P_0 + (1/m)P_1 + \cdots + (1/m)P_{m-1}$ as desired. $\square$

Unfortunately we do not have a reversal of Theorem 6.1.9.

Isbell's solution to Birkhoff's problem 111 is not the only one. Solutions by Kendall [34] and Révész [50], for example, also deserve reverse mathematical attention.

6.2 Variations on countable Birkhoff's theorem

In the phrasing of countable Birkhoff's theorem above, we are given a doubly stochastic $\mathbb{N}\times\mathbb{N}$ matrix A satisfying $(*)$ and an $\epsilon > 0$, and we ask for a $\mathbb{N}\times\mathbb{N}$ matrix B that is a convex linear combination of permutation matrices such that $\|A-B\| \leq \epsilon$. Instead, we could ask for a sequence of convex linear combinations of permutation matrices converging to A. That is, we could ask for a sequence $\langle\, \lambda_0^n P_0^n + \cdots + \lambda_{m_n}^n P_{m_n}^n \mid n \in \mathbb{N}\,\rangle$ such that $\|A - (\lambda_0^n P_0^n + \cdots + \lambda_{m_n}^n P_{m_n}^n)\| \leq 1/n$ for each n.

Another variation on countable Birkhoff's theorem is to strengthen $(*)$.

Definition 6.2.1. If A is an $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrix, a function $f\colon \mathbb{N}\to\mathbb{N}$ *witnesses* $(*)$ *for* A if and only if, for every $n \in \mathbb{N}$, in any row or column the sum of the $h(n)$ largest entries is $\geq 1 - 1/n$.

Theorem 6.2.2. *The following statement is provable in* $\mathtt{WKL}_0$. *If A is an $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrix and h witnesses $(*)$ for A, then there is a sequence $\langle\, \lambda_0^n P_0^n + \cdots + \lambda_{m_n}^n P_{m_n}^n \mid n \in \mathbb{N}\,\rangle$ such that $\|A - (\lambda_0^n P_0^n + \cdots + \lambda_{m_n}^n P_{m_n}^n)\| \leq 1/n$ for each n.*

Proof. In the proof of Theorem 6.1.9, given $\epsilon > 0$, we found a $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrix B such that $\|A - B\| \leq \epsilon$ and every entry of B is a rational of the form p/m for fixed m. Then, using Lemma 6.1.7, we define from B an infinite bounded tree T such that any infinite path of T codes a decomposition of B into a convex linear combination of permutation matrices.

Now, the function h witnessing $(*)$ for A lets us make the argument from Theorem 6.1.9 uniformly in $1/n$. In $\mathtt{RCA}_0$ we can find a sequence $\langle\, B_n \mid n \in \mathbb{N}\,\rangle$ of $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrices such that every entry of B_n is a rational of the form p/m_n and $\|A - B_n\| \leq 1/n$ for each n. From the sequence $\langle\, B_n \mid n \in \mathbb{N}\,\rangle$, produce a sequence of infinite bounded trees $\langle\, T_n \mid n \in \mathbb{N}\,\rangle$ such that every infinite path of T_n codes a decomposition of B_n into a convex linear combination of permutation matrices. Then code the sequence of trees $\langle\, T_n \mid n \in \mathbb{N}\,\rangle$ into a single infinite bounded tree T:

$$(\forall \sigma \in \mathbb{N}^{<\mathbb{N}})(\sigma \in T \leftrightarrow (\forall \langle\, i, j\,\rangle < |\sigma|)(\sigma(\langle\, i, 0\,\rangle)\sigma(\langle\, i, 1\,\rangle)\cdots\sigma(\langle\, i, j\,\rangle) \in T_i)).$$

Any infinite path through T codes infinite paths through every T_n and hence codes a sequence $\langle\, \lambda_0^n P_0^n + \cdots + \lambda_{m_n}^n P_{m_n}^n \mid n \in \mathbb{N}\,\rangle$ such that $\|A - (\lambda_0^n P_0^n + \cdots + \lambda_{m_n}^n P_{m_n}^n)\| \leq 1/n$ for each n. □

Unfortunately we do not have a reversal of Theorem 6.2.2.

If A is an $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrix satisfying $(*)$, then finding an h witnessing $(*)$ for A requires $\mathtt{ACA}_0$.

Lemma 6.2.3. *The following are equivalent over* $\mathtt{RCA}_0$.

(i) $\mathtt{ACA}_0$.

(ii) If A is an $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrix satisfying $(*)$, then there is an h witnessing $(*)$ for A.

Proof. (i) $\Rightarrow$ (ii): Let A be an $\mathbb{N}\times\mathbb{N}$ doubly stochastic matrix satisfying $(*)$. In $\mathtt{ACA}_0$, we may define a function f by

$$\begin{aligned} f(\langle n,i\rangle) = {} & \text{the least } m \text{ such that, for all } j \le i, \\ & \text{there are } m \text{ entries in row } j \text{ that sum to } \ge 1-1/n, \text{ and} \\ & \text{there are } m \text{ entries in column } j \text{ that sum to } \ge 1-1/n. \end{aligned}$$

For each n, $\lim_i f(\langle n,i\rangle)$ exists because A satisfies $(*)$. Again in $\mathtt{ACA}_0$, define $h(n) = \lim_i f(\langle n,i\rangle)$.

(ii) $\Rightarrow$ (i): By [67] Lemma III.1.3, it suffices to show that if $f\colon \mathbb{N}\to\mathbb{N}$ is an injection, then there is a $Z \subseteq \mathbb{N}$ such that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$. Let $f\colon \mathbb{N}\to\mathbb{N}$ be an injection. Define a $\mathbb{N}\times\mathbb{N}$ block diagonal matrix

$$A = \begin{pmatrix} C_0 & & & \\ & C_1 & & \\ & & C_2 & \\ & & & \ddots \end{pmatrix}$$

as follows. Each C_m will be a finite doubly stochastic matrix, which means that A will be doubly stochastic. If $f(m) = n$, then C_m will witness $h(n) \ge m$. Define the matrix C_m as follows. Suppose $f(m) = n$. Let $N = \max(m,n)$. Then let

$$C_m = \begin{pmatrix} a_1 & \cdots & a_{n-1} & b_1 & \cdots & b_{\ell_m} & c_1 & \cdots & c_{\ell'_m} \\ \vdots & \ddots & & & & & & & \\ a_{n-1} & & 1-a_{n-1} & & & & & & \\ b_1 & & & 1-b_1 & & & & & \\ \vdots & & & & \ddots & & & & \\ b_{\ell_m} & & & & & 1-b_{\ell_m} & & & \\ c_1 & & & & & & 1-c_1 & & \\ \vdots & & & & & & & \ddots & \\ c_{\ell'_m} & & & & & & & & 1-c_{\ell'_m} \end{pmatrix}$$

where

- $a_1 = a_2 = 1/4$,
- $1-(a_1+\cdots+a_k) = 1/k$ for $2 \le k \le n-1$,
- $b_1 = \cdots = b_{\ell_m} = 1/\ell_m(n^2-n)$, where ℓ_m is least $\ge N$ such that $1/\ell_m(n^2-n) < a_{n-1}$ (this makes $1-(a_1+\cdots+a_{n-1}+b_1+\cdots+b_{\ell_m}) = 1/n$), and
- $c_1 = \cdots = c_{\ell'_m} = 1/\ell'_m n$, where ℓ'_m is least such that $1/\ell'_m n < 1/\ell_m(n^2-n)$.

We show that A satisfies $(*)$. Let $\epsilon > 0$, and let $1/n < \epsilon$. The function f is an injection, so there must be a greatest k for which $f(k) \le n$. Let N be the maximum of 2, n, and the maximum size (i.e., number of rows) of $C_0, C_1, \ldots, C_k$. Then in any row or column of A, the largest N entries sum to at least $1-1/n$. This is true for any row or column which goes through a block C_i for $i \le k$ because such rows and columns have at most N non-zero entries. Consider a row or column going through a block C_i with $i > k$. If the row or column contains two non-zero entries which sum to one, then the N largest entries sum to 1. Otherwise, the row or column contains $a_1, \dots, a_{f(i)-1}$, $f(i) > n$ by choice of k, and $a_1 + \dots + a_n = 1 - 1/n$ by construction.

By (ii), let h be a function witnessing $(*)$ for A. By Δ^0_1 comprehension, let $Z = \{n \mid (\exists m \le h(n))(f(m) = n)\}$. Clearly if $n \in Z$, then $\exists m(f(m) = n)$. Conversely, suppose that $\exists m(f(m) = n)$. In the first row through block C_m, the m largest entries are among $a_1, \dots, a_{n-1}, b_1, \dots, b_{\ell_m}$, and $a_1 + \dots + a_{n-1} + b_1 + \dots + b_{\ell_m} = 1 - 1/n$. Hence $h(n) \ge m$, so $(\exists m \le h(n))(f(m) = n)$. Thus $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$. □

The strongest version of countable Birkhoff's theorem we consider is equivalent to $\mathtt{ACA}_0$ over $\mathtt{RCA}_0$.

Theorem 6.2.4. *The following are equivalent over* $\mathtt{RCA}_0$.

(i) $\mathtt{ACA}_0$.

(ii) If A is an $\mathbb{N} \times \mathbb{N}$ doubly stochastic matrix satisfying $(*)$, then there is a sequence $\langle\, \lambda^n_0 P^n_0 + \dots + \lambda^n_{m_n} P^n_{m_n} \mid n \in \mathbb{N} \,\rangle$ such that $\|A - (\lambda^n_0 P^n_0 + \dots + \lambda^n_{m_n} P^n_{m_n})\| \le 1/n$ for each n.

Proof. (i) ⇒ (ii): By Lemma 6.2.3, let h witness $(*)$ for A. Then apply Theorem 6.2.2.

(ii) ⇒ (i): Let A be a $\mathbb{N} \times \mathbb{N}$ doubly stochastic matrix satisfying $(*)$. By (ii), let $\langle\, \lambda^n_0 P^n_0 + \dots + \lambda^n_{m_n} P^n_{m_n} \mid n \in \mathbb{N} \,\rangle$ be such that $\|A - (\lambda^n_0 P^n_0 + \dots + \lambda^n_{m_n} P^n_{m_n})\| \le 1/n$ for each n. Let $h(n) = m_n$. Then h witnesses $(*)$ for A, so (i) follows from Lemma 6.2.3. □

6.3 Reverse mathematics and unfriendly partitions

A partition $V(G) = X_0 \cup X_1$ of the vertices of a graph G is *unfriendly* if and only if $(\forall i < 2)(\forall v \in X_i)(|N(v) \cap X_i| \le |N(v) \cap X_{1-i}|)$. That is, every vertex in the graph has at least as many neighbors outside its side of the partition as inside its side of the partition. Every finite graph has an unfriendly partition: simply choose any partition that maximizes the number of edges that have an endpoint in each side of the partition. It follows by compactness that every locally finite graph has an unfriendly partition. Shelah and Milner give an example of a graph that has no unfriendly partition [58]. The cardinality of their example is the ω^{th} successor of the continuum. Amazingly, it is not known whether every countable graph has an unfriendly partition. The assertion that every countable graph has an unfriendly partition is now known as the *unfriendly partition conjecture*.

Aharoni, Milner, and Prikry show that a graph with only finitely many vertices of infinite degree has an unfriendly partition [5]. They also show that if G is a graph for which there exist a finite number of infinite cardinals $\kappa_0 < \kappa_1 < \dots < \kappa_k$ with κ_i regular for $1 \le i \le k$ such that $< \kappa_0$ vertices have finite degree and every vertex with infinite degree has degree κ_i for some $0 \le i \le k$, then G has an unfriendly partition. More recently, Bruhn, Diestel, Georgakopoulos, and Sprüssel show that every graph with no infinite path has an unfriendly partition [13]. Although an infinite graph may not have an unfriendly partition, Shelah and Milner also show that every graph, regardless of cardinality, has an unfriendly 3-partition, that is, a partition $V(G) = X_0 \cup X_1 \cup X_2$ such that $(\forall i < 3)(\forall v \in X_i)(|\{u \in N(v) \mid u \in X_i\}| \le |\{u \in N(v) \mid u \notin X_i\}|)$ [58].

The unfriendly partition conjecture motivates our study of the reverse mathematics of unfriendly partitions. We present some initial results.

Theorem 6.3.1. *The following are equivalent over* RCA_0.

(i) ACA_0.

(ii) Every locally finite graph has an unfriendly partition.

Proof. (i) $\Rightarrow$ (ii): Let $G = (V, E)$ be a locally finite graph and enumerate V as $\{v_i \mid i \in \mathbb{N}\}$. By arithmetical comprehension, let $f\colon \mathbb{N} \to \mathbb{N}$ be such that $(\forall i)((v_i, v_j) \in E \to j < f(i))$. Define a tree $T \subseteq 2^{<\mathbb{N}}$ by

$$\begin{aligned}\sigma \in T \Leftrightarrow &(\forall i < |\sigma|)(|\sigma| \le f(i) \to \\ &|\{j < |\sigma| \mid v_j \in N(v_i) \wedge \sigma(j) = \sigma(i)\}| \le |\{j < |\sigma| \mid v_j \in N(v_i) \wedge \sigma(j) \neq \sigma(i)\}|).\end{aligned}$$

The fact that every finite graph has an unfriendly partition implies that T is infinite. Apply König's lemma to T. Any infinite path through T codes an unfriendly partition of G.

(ii) $\Rightarrow$ (i): By [67] Lemma III.1.3, it suffices to show that if $f\colon \mathbb{N} \to \mathbb{N}$ is an injection, then there is a $Z \subseteq \mathbb{N}$ such that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. By Δ^0_1 comprehension, let $G = (V, E)$ be the graph where

$$\begin{aligned}V &= \{a_n, b_n, c_n, d_n, e_n \mid n \in \mathbb{N}\} \\ E &= \{(a_n, b_n), (c_n, d_n), (d_n, e_n) \mid n \in \mathbb{N}\} \\ &\quad \cup \{(a_n, c_m), (a_n, e_m), (b_n, c_m), (b_n, e_m) \mid f(m) = i\}.\end{aligned}$$

By (ii), let $X_0 \cup X_1$ be an unfriendly partition of V. By Δ^0_1 comprehension, let $Z = \{n \in \mathbb{N} \mid (a_n \in X_0 \wedge b_n \in X_0) \vee (a_n \in X_1 \wedge b_n \in X_1)\}$. If $\neg\exists m(f(m) = n)$, then a_n and b_n are only adjacent to each other and so must be in opposite sides of the partition. Hence $n \notin Z$. Conversely, suppose $f(m) = n$, and suppose for a contradiction that a_n and b_n are in opposite sides of the partition. Say $a_n \in X_0$ and $b_n \in X_1$. The set $\{a_n, b_n, c_m, d_m, e_m\}$ is a connected component because f is an injection. Vertices c_m and e_m cannot both be in X_0 because otherwise a_n has two neighbors in X_0 and only one neighbor in X_1. Similarly, c_m and e_m cannot both be in X_1. Thus one of c_m and e_m is in X_0 and the other is in X_1. Say $c_m \in X_0$ and $e_m \in X_1$. Then d_m cannot be in X_0 because this would give c_m two neighbors (a_n and d_m) in X_0 and only one neighbor (b_n) in X_1. Similarly, d_m cannot be in X_1. This contradicts the fact that $X_0 \cup X_1$ is an unfriendly partition of V. Thus a_n and b_n are either both in X_0 or both in X_1. Hence $n \in Z$. $\square$

We see from the proof of Theorem 6.3.1 (i) $\Rightarrow$ (ii) that $\mathtt{WKL}_0$ proves that every bounded graph has an unfriendly partition. In that proof, $\mathtt{ACA}_0$ is used to produce the bounding function f, and the rest of the proof goes through in $\mathtt{WKL}_0$. Unfortunately we do not have a reversal for the bounded version of Theorem 6.3.1.

Shelah and Milner's method of producing unfriendly 3-partitions in [58] goes through in $\mathtt{ACA}_0$ in the countable case. It is convenient to make the following definition.

Definition 6.3.2. Let $G = (V, E)$ be a graph. A function $f\colon V \to \mathbb{N}$ is *unfriendly at* $v \in V$ if and only if $|\{u \in N(v) \mid f(u) = f(v)\}| \leq |\{u \in N(v) \mid f(u) \neq f(v)\}|$. A function $f\colon V \to \{0, 1, 2\}$ is an *unfriendly 3-partition of* G if and only if f is unfriendly at every $v \in V$.

Lemma 6.3.3 (in $\mathtt{ACA}_0$). *Let $G = (V, E)$ be a graph. Let $A, B \subseteq V$ be such that $A \cap B = \emptyset$ and $(\forall v \in A)(N(v) \cap (A \cup B)$ is finite). Then any function $g\colon B \to \{0, 1, 2\}$ can be extended to a function $h\colon A \cup B \to \{0, 1, 2\}$ such that $h(A) \subseteq \{0, 1\}$ and h is unfriendly at every $v \in A$ in the subgraph induced by $A \cup B$.*

Proof. First assume that A is finite. Choose $g'\colon A \to \{0, 1\}$ so that $h = g \cup g'$ maximizes the number $n = |\{(v, u) \in E \mid v \in A \wedge u \in A \cup B \wedge h(v) \neq h(u)\}|$. Then h is unfriendly at each $v \in A$ in the subgraph induced by $A \cup B$. If not, then there is a $v \in A$ such that

$$|\{u \in N(v) \cap (A \cup B) \mid h(u) = h(v)\}| > |\{u \in N(v) \cap (A \cup B) \mid h(u) \neq h(v)\}|.$$

Thus we could increase n by replacing $g'(v)$ with $1 - g'(v)$, contradicting that h maximizes n.

If A is infinite, let $A = \{a_i \mid i \in \mathbb{N}\}$, and let $A_n = \{a_i \mid i < n\}$. For $\sigma \in 2^{<\mathbb{N}}$ with $|\sigma| = n$, define $g'_\sigma\colon A_n \to \{0, 1\}$ by $(\forall i < n)(g'_\sigma(a_i) = \sigma(i))$. Let $h_\sigma = g \cup g'_\sigma$. By arithmetical comprehension, define a tree $T \subseteq 2^{<\mathbb{N}}$ by

$$\begin{aligned}\sigma \in T \Leftrightarrow &(\forall i < |\sigma|)(N(a_i) \cap A \subseteq A_{|\sigma|} \rightarrow \\ &h_\sigma \text{ is unfriendly at } a_i \text{ in the subgraph induced by } A_{|\sigma|} \cup B).\end{aligned}$$

The finite case implies that T is infinite, so by König's lemma let $p\colon \mathbb{N} \to \{0, 1\}$ be a path through T. Define $f\colon A \to \{0, 1\}$ by $f(a_n) = p(n)$ for each n. We show that $h = g \cup f$ is unfriendly at every $a_n \in A$ in the subgraph induced by $A \cup B$. Let $a_n \in A$, and let m be large enough so that $N(a_n) \cap A \subseteq A_m$. Let $\sigma = \langle p(0), \ldots, p(m) \rangle$. Then h_σ is unfriendly at a_n in the subgraph induced by $A_m \cup B$ because $\sigma \in T$. However, $N(a_n) \cap (A \cup B) \subseteq N(a_n) \cap (A_m \cup B)$, and h agrees with h_σ on $A_m \cup B$. Thus h is unfriendly at a_n in the subgraph induced by $A \cup B$. $\square$

Theorem 6.3.4. *The following statement is provable in $\mathtt{ACA}_0$. Every graph has an unfriendly 3-partition.*

Proof. Let $G = (V, E)$ be a graph. By arithmetical comprehension, let $X = \{v \in V \mid N(v)$ is finite$\}$, and let $Y = \{v \in V \mid N(v) \cap X$ is infinite$\}$. Let $Z = V \setminus (X \cup Y)$. Define $g\colon Y \cup Z \to \{0, 1, 2\}$ by setting $g(Y) = 2$ and by defining $g(Z) \subseteq \{0, 1\}$

by the following recursive procedure. Let $Z = \{z_i \mid i \in \mathbb{N}\}$. Initialize an empty priority queue. For $n = 0, 1, 2, \ldots$ do the following. If z_n is not a neighbor any z in the queue, then set $g(z_n) = 0$. Otherwise, let z be the highest priority neighbor of z_n. Set $g(z_n) = 1 - g(z)$. If z is at the front of the queue (i.e., has highest priority), then move z to the back of the queue. If $N(z_n) \cap Y$ is finite, then add z_n to the back of the queue. This completes the definition of g.

By Lemma 6.3.3, let h be a function extending g such that $h(X) \subseteq \{0, 1\}$ and that h is unfriendly at every $v \in X$. We show that h is an unfriendly 3-partition of G. We already know that h is unfriendly at every $v \in X$. The function h is also unfriendly at every $v \in Y$ because $h(v) = 2$ for every $v \in Y$ and $h(u)$ is either 0 or 1 for every $u \in X$. Let $z \in Z$. If $N(z) \cap Y$ is infinite, then h is unfriendly at z because $h(z) \in \{0, 1\}$ and $h(Y) = 2$. Suppose $N(z) \cap Y$ is finite. Then $N(z) \cap Z$ must be infinite. Every time z reaches the front of the queue, we eventually define $g(z') \neq g(z)$ for some neighbor of z' of z. Thus it suffices to show that z reaches the front of the queue infinitely many times. If not, then z never reaches the front of the queue after some step n. Since the priority of z either increases or stays the same at each step after n, there is some step m after which the priority of z never changes. This contradicts the fact that the vertex with priority one at step m has infinitely many neighbors in Z. Thus h is our desired unfriendly partition. $\square$

We do not have a reversal for Theorem 6.3.4, but we can prove that $\mathtt{RCA}_0$ is not strong enough to produce unfriendly 3-partitions.

Theorem 6.3.5. *There is a recursive locally finite graph with no recursive unfriendly 3-partition. It follows that the statement "every graph has an unfriendly 3-partition" is not provable in* $\mathtt{RCA}_0$.

Proof. Our graph $G = (V, E)$ has $V = \mathbb{N}$, and we construct E in stages $\langle e, s \rangle$. For fixed e, in stages $\langle e, s \rangle$ we diagonalize against Φ_e in the e^{th} column of $\mathbb{N}$. This diagonalization occurs in three phases. Together phases I and II force Φ_e to take on the values 0, 1, and 2 four times each. Phase III consists of several subphases which finish the diagonalization. Call column e good at stage $\langle e, s \rangle$ if the diagonalization succeeds at this stage. Once column e is good at stage $\langle e, s \rangle$, the construction is finished in the e^{th} column and we skip all subsequent stages $\langle e, t \rangle$.

Initially, E has edge $(\langle e, i, 0, 0 \rangle, \langle e, i, 1, 0 \rangle)$ for each e and for each $i = 0, 1, 2, 3$. Each column begins in phase I. At stage $\langle e, s \rangle$ do the following.

- If column e is in phase I, then do the following.
  - If $\Phi_{e,s}(\langle e, i, j, 0 \rangle)\uparrow$ for some $i = 0, 1, 2, 3$ and some $j = 0, 1$, then go to the next stage.
  - Otherwise, column e is good at $\langle e, s \rangle$ if either of the following conditions are met:
    (i) $\Phi_{e,s}(\langle e, i, j, 0 \rangle) > 2$ for some $i = 0, 1, 2, 3$ and some $j = 0, 1$, or
    (ii) $\Phi_{e,s}(\langle e, i, 0, 0 \rangle) = \Phi_{e,s}(\langle e, i, 1, 0 \rangle)$ for some $i = 0, 1, 2, 3$.

If column e is not good, then add edges $(\langle e,i,j,s\rangle, \langle e,i,k,0\rangle)$ for $i = 0,1,2,3$, $j = 2,3$, and $k = 0,1$, and add edges $(\langle e,i,4,s\rangle, \langle e,i,j,s\rangle)$ for $i = 0,1,2,3$ and $j = 2,3$. Enter phase II.

- If column e is in phase II, then let $\langle e,t\rangle$ be the stage at which column e entered phase II and do the following.
  - If $\Phi_{e,s}(\langle e,i,j,t\rangle)\uparrow$ for some $i = 0,1,2,3$ and some $j = 2,3,4$, then go to the next stage.
  - Otherwise, column e is good at $\langle e,s\rangle$ if either of the following conditions are met:
    (i) $\Phi_{e,s}(\langle e,i,j,t\rangle) > 2$ for some $i = 0,1,2,3$ and some $j = 2,3,4$, or
    (ii) for some $i = 0,1,2,3$, $\Phi_{e,s}$ takes on only two values on the five vertices $\langle e,i,0,0\rangle$, $\langle e,i,1,0\rangle$, $\langle e,i,2,t\rangle$, $\langle e,i,3,t\rangle$, and $\langle e,i,4,t\rangle$.

    If column e is not good, then add edge $(\langle e,4,0,s\rangle, \langle e,4,1,s\rangle)$. Enter phase III.

- Phase III consists of sixteen subphases and begins in subphase 1. Let $\langle e,r\rangle$ be the stage at which column e entered phase III, and do the following.
  - If column e is in phase III, subphase 1, do the following.
    * If $\Phi_{e,s}(\langle e,4,j,r\rangle)\uparrow$ for some $j = 0,1$, then go to the next stage.
    * Otherwise, column e is good at $\langle e,s\rangle$ if either of the following conditions are met:
      (i) $\Phi_{e,s}(\langle e,4,j,r\rangle) > 2$ for some $j = 0,1$, or
      (ii) $\Phi_{e,s}(\langle e,4,0,r\rangle) = \Phi_{e,s}(\langle e,4,1,r\rangle)$.

      If column e is not good, then $\Phi_{e,s}(\langle e,4,0,r\rangle) = x \le 2$ and $\Phi_{e,s}(\langle e,4,1,r\rangle) = y \le 2$, where $x \ne y$. Let $z \le 2$ be the number not equal to x or y. For each $i \le 3$, let b_i be the least of $\langle e,i,0,0\rangle$, $\langle e,i,1,0\rangle$, $\langle e,i,2,t\rangle$, $\langle e,i,3,t\rangle$, and $\langle e,i,4,t\rangle$ such that $\Phi_{e,s}(b_i) = z$, where t is such that column e entered phase II at stage $\langle e,t\rangle$ (the b_i exist by the fact that column e was not good in phase II). Add edges $(\langle e,4,j,s\rangle, \langle e,4,k,r\rangle)$ for $j = 2,3$ and $k = 0,1$, and add edges $(\langle e,4,j,s\rangle, b_i)$ for $j = 2,3$ and $i = 0,1,2$. Enter subphase 2.
  - If column e is in phase III, subphase 2, do the following. Let $\langle e,t\rangle$ be the stage at which column e entered subphase 2.
    * If $\Phi_{e,s}(\langle e,4,j,t\rangle)\uparrow$ for some $j = 2,3$, then go to the next stage.
    * Otherwise, column e is good at $\langle e,s\rangle$ if any of the following conditions are met:
      (i) $\Phi_{e,s}(\langle e,4,j,t\rangle) > 2$ for some $j = 2,3$,
      (ii) $\Phi_{e,s}(\langle e,4,j,t\rangle) = z$ for some $j = 2,3$ (where z is as in subphase 1), or
      (iii) $\Phi_{e,s}(\langle e,4,2,t\rangle) = \Phi_{e,s}(\langle e,4,3,t\rangle)$.

      If column e is not good, then $\Phi_{e,s}$ is x on one of the two vertices $\langle e,4,2,t\rangle$ and $\langle e,4,3,t\rangle$, and $\Phi_{e,s}$ is y on the other vertex. Let v_x

be the vertex on which $\Phi_{e,s}$ is x, and let v_y be the vertex on which $\Phi_{e,s}$ is y. Add edges $(\langle e,4,4,s\rangle, \langle e,4,k,r\rangle)$ for $k=0,1$ and edges $(\langle e,4,4,s\rangle, b_i)$ for $i=0,1,2$. Enter subphase 3.

– If column e is in phase III, subphase $2n+1$ for $1 \le n \le 7$, then do the following. Let $\langle e,t\rangle$ be the stage at which column e entered subphase $2n+1$.
  * If $\Phi_{e,s}(\langle e,4,2n+2,t\rangle)\uparrow$ then go to the next stage.
  * Otherwise, column e is good at $\langle e,s\rangle$ if either of the following conditions are met:
    (i) $\Phi_{e,s}(\langle e,4,2n+2,t\rangle) > 2$, or
    (ii) $\Phi_{e,s}(\langle e,4,2n+2,t\rangle) = z$.
    If column e is not good, then add edges $(\langle e,4,2n+3,s\rangle, \langle e,4,k,r\rangle)$ for $k=0,1$ and edges $(\langle e,4,2n+3,s\rangle, b_i)$ for $i=0,1,2,3$. Additionally, if $\Phi_{e,s}(\langle e,4,2n+2,t\rangle) = x$, then add edge $(\langle e,4,2n+3,s\rangle, v_y)$, and if $\Phi_{e,s}(\langle e,4,2n+2,t\rangle) = y$, then add edge $(\langle e,4,2n+3,s\rangle, v_x)$. Enter subphase $2n+2$.

– If column e is in phase III, subphase $2n+2$ for $1 \le n \le 7$, then do the following. Let $\langle e,t\rangle$ be the stage at which column e entered subphase $2n+2$, and let $\langle e,p\rangle$ be the stage at which column e entered subphase $2n+1$.
  * If $\Phi_{e,s}(\langle e,4,2n+3,t\rangle)\uparrow$, then go to the next stage.
  * Otherwise, column e is good at $\langle e,s\rangle$ if any of the following conditions are met:
    (i) $\Phi_{e,s}(\langle e,4,2n+3,t\rangle) > 2$,
    (ii) $\Phi_{e,s}(\langle e,4,2n+3,t\rangle) = z$,
    (iii) $\Phi_{e,s}(\langle e,4,2n+3,t\rangle) = \Phi_{e,s}(\langle e,4,2n+2,p\rangle)$, or
    (iv) this is subphase 16.
    If column e is not good, then add edges $(\langle e,4,2n+4,s\rangle, \langle e,4,k,r\rangle)$ for $k=0,1$, and add edges $(\langle e,4,2n+4,s\rangle, b_i)$ for $i=0,1,2$. Enter subphase $2n+3$.

This completes the construction of G. Suppose Φ_e is a recursive function $\mathbb{N} \to \{0,1,2\}$. We show that Φ_e is not an unfriendly 3-partition of G.

Φ_e is total, so the construction in column e will progress through the phases until column e becomes good. Suppose column e becomes good at stage $\langle e,s\rangle$. If this stage is in phase I, then $\Phi_e(\langle e,i,0,0\rangle) = \Phi_e(\langle e,i,1,0\rangle)$ for some $i=0,1,2,3$. These two vertices only have each other as neighbors, so Φ_e is not an unfriendly partition.

If column e is good at stage $\langle e,s\rangle$ in phase II, then, for some $i=0,1,2,3$, Φ_e uses only two values on the five vertices $\langle e,i,0,0\rangle$, $\langle e,i,1,0\rangle$, $\langle e,i,2,t\rangle$, $\langle e,i,3,t\rangle$, $\langle e,i,4,t\rangle$ (where t is such that column e entered phase II at stage $\langle e,t\rangle$), and $\Phi_e(\langle e,i,0,0\rangle) \neq \Phi_e(\langle e,i,1,0\rangle)$. In this case these five vertices have no other neighbors, and one checks that Φ_e cannot be an unfriendly partition of the subgraph induced by these vertices.

If column e is good at stage $\langle e, s \rangle$ in phase III, subphase 1, then $\Phi_e(\langle e, 4, 0, r \rangle) = \Phi_e(\langle e, 4, 1, r \rangle)$ (where r is such that column e entered phase III, subphase 1 at stage $\langle e, r \rangle$). These two vertices only have each other as neighbors, so Φ_e is not an unfriendly partition.

If column e is good at stage $\langle e, s \rangle$ in phase III, subphase 2, then either $\Phi_e(\langle e, 4, j, t \rangle) = z$ for some $j = 2, 3$, or $\Phi_e(\langle e, 4, 2, t \rangle) = \Phi_e,(\langle e, 4, 3, t \rangle)$ (where t is such that column e entered phase III, subphase 2 at stage $\langle e, t \rangle$). If $\Phi_e(\langle e, 4, j, t \rangle) = z$ then $\langle e, 4, j, t \rangle$ has five neighbors, and Φ_e is z on three of these neighbors. Hence Φ_e is not an unfriendly partition. If $\Phi_e(\langle e, 4, 2, t \rangle) = \Phi_e(\langle e, 4, 3, t \rangle) \neq z$, then the value of Φ_e on these two vertices is the same as the value of Φ_e on $\langle e, 4, j, r \rangle$ for either $j = 0$ or $j = 1$ (where r is such that column e entered phase III, subphase 1 at stage $\langle e, r \rangle$). For the witnessing j, $\langle e, 4, j, r \rangle$ has three neighbors, and Φ_e agrees with $\Phi_e(\langle e, 4, j, r \rangle)$ on two of these neighbors. Hence Φ_e is not an unfriendly partition.

If column e is good at stage $\langle e, s \rangle$ in phase III, subphase $2n + 1$, then $\Phi_e(\langle e, 4, 2n + 2, t \rangle) = z$ (where t is such that column e entered phase III, subphase $2n + 1$ at stage $\langle e, t \rangle$). In this case $\langle e, 4, 2n + 2, t \rangle$ has five neighbors, and Φ_e is z on three of these neighbors. Thus Φ_e is not an unfriendly partition.

If column e is good at stage $\langle e, s \rangle$ in phase III, subphase $2n + 2$, then either $\Phi_e(\langle e, 4, 2n + 3, t \rangle) = z$, $\Phi_e(\langle e, 4, 2n + 3, t \rangle) = \Phi_e(\langle e, 4, 2n + 2, p \rangle)$, or this is subphase 16 (where t and p are such that column e entered phase III, subphase $2n + 2$ at stage $\langle e, t \rangle$ and entered phase III, subphase $2n + 1$ at stage $\langle e, p \rangle$). If $\Phi_e(\langle e, 4, 2n + 3, t \rangle) = z$, then $\langle e, 4, 2n + 3, t \rangle$ has seven neighbors, and Φ_e is z on four of these neighbors. Hence Φ_e is not an unfriendly partition. If $\Phi_e(\langle e, 4, 2n+3, t \rangle) = \Phi_e(\langle e, 4, 2n+2, p \rangle)$, then for either $j = 0$ or $j = 1$ the vertex $\langle e, 4, j, r \rangle$ has $2n + 3$ neighbors, and Φ_e agrees with $\Phi_e(\langle e, 4, j, r \rangle)$ on $n + 2$ of these neighbors (where r is such that column e entered phase III, subphase 1 at stage $\langle e, r \rangle$). Hence Φ_e is not an unfriendly partition. If this is subphase 16, and $\Phi_e(\langle e, 4, 2n + 3, t \rangle)$ is neither z nor $\Phi_e(\langle e, 4, 2n + 2, p \rangle)$, then either Φ_e is x on over half of the neighbors of v_x, or Φ_e is y on over half of the neighbors of v_y. In either case Φ_e is not an unfriendly partition. $\square$

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