# $f$-Vectors of Triangulated Balls

Samuel Kolins
Department of Mathematics,
Cornell University, Ithaca NY, 14853-4201, USA,
skolins@math.cornell.edu

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**Abstract**

We describe two methods for showing that a vector cannot be the $f$-vector of a homology $d$-ball. As a consequence, we disprove a conjectured characterization of the $f$-vectors of balls of dimension five and higher due to Billera and Lee. We also provide a construction of triangulated balls with various $f$-vectors. We show that this construction obtains all possible $f$-vectors of three- and four-dimensional balls and we conjecture that this result also extends to dimension five.

## 1 Introduction

A fundamental invariant of a simplicial complex is its collection of face numbers or $f$-vector. A major area of study is understanding the possible $f$-vectors of various types of simplicial complexes. In this paper we prove some new results on the $f$-vectors of simplicial complexes that are triangulations of balls.

A complete characterization of the $f$-vectors of simplicial polytopes was given in 1981 with the proof of the $g$-theorem by Billera and Lee [3] and Stanley [18]. The $g$-conjecture asserts that this characterization also holds in the more general setting of all triangulated spheres. In [2], Billera and

Lee calculate a set of conditions on the $f$-vectors of triangulated balls that would follow from the $g$-conjecture. Billera and Lee conjecture that these conditions are not only necessary but also sufficient for a characterization of the $f$-vectors of balls (see Conjecture 6). Recently, Lee and Schmidt confirmed this conjecture for three- and four-dimensional balls [10].

In this paper we present two methods that show that certain vectors are not the $f$-vectors of triangulated balls. As a consequence, we show that the Billera and Lee conditions are not sufficient in dimensions five and higher. In both approaches, we assume that a ball with a certain $f$-vector exists and then show that there must be some way to split the ball along a codimension-one face to create two new balls. For some $f$-vectors we can show that the new balls created by this splitting cannot exist. The first technique relates one of the Betti numbers of the face ring of the ball to the existence of a codimension-one face along which we can split the ball. This has the advantage of being relatively straightforward to compute in particular examples. In the second method we look at all possible one-skeletons of a ball with a given $f$-vector and show that in each case the desired type of splitting is possible. This is used to generate an infinite class of counterexamples to the Billera and Lee conjecture in every dimension greater than four.

The second portion of the paper presents a construction of balls with prescribed $f$-vectors. In dimensions three and four this result duplicates the work of Lee and Schmidt in obtaining all possible $f$-vectors of balls. For the dimension five case, we conjecture that this construction gives all possible $f$-vectors. However, in dimensions higher than five not all $f$-vectors of balls can be obtained with this approach.

The structure of this paper is as follows. In Section 2 we review the needed background material. In Section 3 we discuss previously known and conjectured conditions on the $f$-vectors of balls. In Section 4 we present our methods for creating new necessary conditions on $f$-vectors. In Section 5 we give our construction and in Section 6 we discuss some consequences of the construction as well as our conjecture for the $f$-vectors of five-dimensional balls.

# 2 Notation and Background

We begin by discussing some needed background on simplicial complexes, the face ring, and commutative algebra. Stanley's book [19] is a good reference for most of the material in this section.

## 2.1 Basics of Simplicial Complexes

A *simplicial complex* $\Delta$ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ that is closed under inclusion and contains all of the one-element sets $\{i\}$ for $i \in [n]$. The elements of $\Delta$ are called *faces*, and the *dimension of a face* $F \in \Delta$ is $\dim F := |F| - 1$. The *dimension of* $\Delta$ is equal to the maximum of the dimensions of all of its faces. A *non-face* of $\Delta$ is a subset of $[n]$ that is not in $\Delta$ and a *non-$i$-face* of $\Delta$ is an $(i+1)$-subset of $[n]$ that is not an element of $\Delta$. A *facet* of $\Delta$ is any maximal face with respect to inclusion. A simplicial complex is *pure* if all of its facets have the same dimension.

If $F \in \Delta$ then the *link of $F$ in* $\Delta$ is $\mathrm{lk}_\Delta(F) := \{G \in \Delta : G \cup F \in \Delta, G \cap F = \emptyset\}$. For $W \subset [n]$, $\Delta_W := \{\sigma \in \Delta : \sigma \subset W\}$ is the *induced subcomplex* on the vertex set $W$.

Let $\Delta$ be a simplicial complex of dimension $d-1$. The *ith face number* of $\Delta$, denoted $f_i(\Delta)$, is the number of $i$-dimensional faces of $\Delta$. When it is clear to what complex we are referring we will often just write $f_i$ for the face numbers. So $f_{-1} = 1$ (corresponding to the empty set), $f_0 = n$, and $f_i = 0$ for $i \geq d$. The *$f$-vector* of $\Delta$ is the list of the face numbers, $f(\Delta) := (f_{-1}, f_0, f_1, \ldots, f_{d-1})$. The *$h$-vector* of $\Delta$, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ contains the same combinatorial information as the $f$-vector but is often easier to use. Its entries are defined from the face numbers by

$$\sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}.$$

Define $g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta)$ for $i \geq 1$ and $g_0(\Delta) = 1$. Then the *$g$-vector* of $\Delta$ is $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$. However, sometimes it will be useful for us to consider $g_i$ where $i > \lfloor d/2 \rfloor$.

Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. The *geometric realization* $|\Delta|$ of $\Delta$ is the union of the convex hull of $\{e_{i_1}, \dots, e_{i_k}\}$ over all faces $\{i_1, \dots, i_k\}$ of $\Delta$. We say $\Delta$ is *homeomorphic* to a topological space $X$ if $|\Delta|$ is homeomorphic to $X$. A *triangulation* of a topological space $X$ is a simplicial complex that is homeomorphic to $X$.

In some of our work, instead of considering triangulated balls it will be useful to consider the larger class of homology balls. All of our homology will be taken with coefficients in the integers. A pure simplicial complex $\Delta$ of dimension $d-1$ is a *homology* $(d-1)$*-manifold* if for every non-empty face $F \in \Delta$ the link of $F$ has the same homology as the $(d-1-|F|)$-sphere or the $(d-1-|F|)$-ball. The *boundary* of a homology $(d-1)$-manifold $\Delta$ is defined to be $\partial\Delta := \{F \in \Delta \mid H_{d-1-|F|}(\mathrm{lk}_\Delta(F)) = 0\}$. From [13] we know that the boundary of a homology $(d-1)$-manifold is either empty or a homology $(d-2)$-manifold without boundary. A *homology* $(d-1)$*-sphere* is a homology $(d-1)$-manifold with empty boundary and the same homology as the $(d-1)$-sphere. A *homology* $(d-1)$*-ball* is a homology $(d-1)$-manifold with the same homology as the $(d-1)$-ball and boundary a homology $(d-2)$-sphere. From the long exact sequence of the homology of the pair $(\Delta, \partial\Delta)$, for a homology $(d-1)$-ball $\Delta$ we have $H_{d-1}(\Delta, \partial\Delta) = \mathbb{Z}$.

Note that if $\Delta$ is a $(d-1)$-dimensional simplicial complex then $h_d = \sum_{i=0}^{d}(-1)^{d-i} f_{i-1} = (-1)^{d-1}\tilde{\chi}(|\Delta|)$. In particular, if $\Delta$ is a homology $(d-1)$-ball then $h_d = 0$.

A *shelling order* of a pure simplicial complex $\Delta$ is an ordering of the facets of $\Delta$, $\{F_1, \dots, F_{f_{d-1}}\}$, such that for $j = 1, \dots, f_{d-1}$, when $F_j$ is added to $\cup_{i=1}^{j-1} F_i$ there is a unique new face that is minimal with respect to inclusion. This face is called the *restriction face* and is denoted $r(F_j)$. A complex is *shellable* if there exists a shelling order of its facets. We can obtain the $h$-vector of a shellable complex from the restriction faces of any shelling order by $h_i = |\{F_j : |r(F_j)| = i\}|$ [19, Proposition 2.3].

## 2.2 The Face Ring

Let $k$ be an infinite field of arbitrary characteristic and $R := k[x_1, \dots, x_n]$. For a simplicial complex $\Delta$ on vertex set $[n]$ the *face ring* (or *Stanley-Reisner* *ring*) $k[\Delta]$ is obtained by taking the quotient of $R$ by the ideal generated by the monomials corresponding to *non-faces* of $\Delta$. More specifically, $k[\Delta] := R/I_\Delta$ where

$$I_\Delta := (x_{i_1}x_{i_2}\cdots x_{i_k} : i_1 < i_2 < \cdots < i_k, \{i_1, i_2, \ldots, i_k\} \notin \Delta).$$

A simplicial complex $\Delta$ is called *Cohen-Macaulay* if its face ring $k[\Delta]$ is Cohen-Macaulay. In [17], Reisner gives a characterization of Cohen-Macaulay complexes in terms of the homology of the links of the faces in the complex. As a consequence of this result, all homology balls and spheres are Cohen-Macaulay.

Still assuming $\dim \Delta = d - 1$, a *linear system of parameters* (l.s.o.p.) for $k[\Delta]$ is a collection of degree one elements $\theta_1, \ldots, \theta_d \in k[\Delta]$ such that $k[\Delta]/(\theta_1, \ldots, \theta_d)$ is finite-dimensional over $k$. For an infinite field $k$ a generic choice of $d$ degree one elements of $k[\Delta]$ is a l.s.o.p. Given a l.s.o.p. $\theta_1, \ldots, \theta_d$ of $\Delta$, define $k(\Delta) := k[\Delta]/(\theta_1, \ldots, \theta_d)$. Although $k(\Delta)$ does depend on the choice of a l.s.o.p., all of our results involving $k(\Delta)$ hold for any l.s.o.p. so we use the notation $k(\Delta)$ without specifying a l.s.o.p.

Let $T$ be a graded ring such that $T = R/I$ for some ideal $I$. Denote by $T_i$ the $i$th homogeneous component of $T$. The *Hilbert function* of T is given by $F(T, i) := \dim_k T_i$. For a Cohen-Macaulay complex $F(k(\Delta), i) = h_i$.

## 2.3 Algebraic Betti Numbers and Hochster's Formula

Let $S$ be one of the rings $k[\Delta]$ or $k(\Delta)$. Thinking of $S$ as an $R$-module the minimal free resolution of $S$ has the form

$$\begin{aligned} 0 \to \bigoplus_j S[-j]^{\beta_{l,j}} \to \bigoplus_j S[-j]^{\beta_{l-1,j}} \to \cdots \\ \to \bigoplus_j S[-j]^{\beta_{1,j}} \to \bigoplus_j S[-j]^{\beta_{0,j}} \to S \to 0. \end{aligned}$$

Here $S[-j]$ is the module $S$ shifted by degree $j$ and $l$ is the length of the resolution, also called the *homological dimension* or *projective dimension* of $S$. The $\beta_{i,j}$ are called the *Betti numbers* of $S$. Using the Auslander-Buchsbaum formula [19, Theorem I.11.2], for a Cohen-Macaulay complex $\Delta$ the minimal resolution of $k[\Delta]$ has length $n-d$ and the minimal resolution of $k(\Delta)$ has length $n$ (where $n$ is the number of vertices in $\Delta$ and hence also the number of variables in $R$). The Betti numbers of $k[\Delta]$ are related to the topology of the complex $\Delta$. One powerful expression of this relationship is Hochster's Formula [7],

$$\beta_{i,j}(k[\Delta]) = \sum_{W\subset V, |W|=j} \dim_k(\tilde{H}_{j-i-1}(\Delta_W; k)). \tag{1}$$

## 2.4 Orders on Monomials

We now turn our attention to terminology related to monomial ideals. A non-empty set $M$ of monomials is an *order ideal* if for all $m \in M$ and $m'|m$ we have $m' \in M$. Denote by $M_i$ the monomials in $M$ of degree $i$. The *degree sequence* of $M$ is the vector $(|M_0|, |M_1|, |M_2|, \ldots)$. A vector that is the degree sequence of some order ideal of monomials is called an *M-vector*.

Given a monomial $m = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$ in the indeterminates $X_1, \ldots, X_n$ define the *degree* of $m$ to be $\sum_{i=1}^{n} a_i$. For monomials of the same degree define the *lexicographic order* $<_l$ by $X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m} <_l X_1^{b_1} X_2^{b_2} \cdots X_m^{b_m}$ if there exists some $k \in \{1, 2, \ldots, m\}$ such that $a_i = b_i$ for $i < k$ and $a_k > b_k$. Let $L \subseteq k[x_1, \ldots, x_n]$ be an ideal generated by monomials. Then $L$ is a *lex ideal* if given any two monomials $m$ and $m'$ of the same degree with $m <_l m'$ and $m' \in L$ then $m \in L$.

Define the *reverse-lexicographic order* (or rev-lex order) $<_{rl}$ on monomials by $X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m} <_{rl} X_1^{b_1} X_2^{b_2} \cdots X_m^{b_m}$ if there exists some $k \in \{1, 2, \ldots, m\}$ such that $a_i = b_i$ for $i > k$ and $a_k < b_k$. An order ideal $M$ is *compressed* if for each $j$ the elements of $M_j$ are the first $|M_j|$ monomials of degree $j$ in the rev-lex order. Given any $M$-vector $h$ there exists a compressed order ideal of monomials with degree sequence equal to $h$ [3, Proposition 1].

Let $m$ be a monomial and $c \in \mathbb{N}$ such that the degree of $m$ is less than or equal to $c$. There is a unique way to write $m$ as $m = X_{e_1} X_{e_2} \cdots X_{e_c}$ where $0 \leq e_1 \leq e_2 \leq \cdots \leq e_c$ and we take $X_0 = 1$. This is called the *extended representation* of $m$. Define the *partial order* $<_p$ on monomials of degree less than or equal to $c$ by $X_{e_1} X_{e_2} \cdots X_{e_c} \leq_p X_{e'_1} X_{e'_2} \cdots X_{e'_c}$ if and only if $e_k \leq e'_k$ for $k = 1, \ldots, c$. Note that any initial segment of monomials in the rev-lex order is also an initial segment in this partial order. For $C \in \mathbb{N}$ we also define a partial order $<_p$ on $C$-subsets of the natural numbers. If $S = \{i_1, \ldots, i_C\}$ and $T = \{j_1, \ldots, j_C\}$ are $C$-subsets of $\mathbb{N}$ with elements listed in increasing order then $S \leq_p T$ if and only if $i_k \leq j_k$ for $k = 1, \ldots, C$.

### 2.5 $M$-vectors

In addition to the definition of an $M$-vector in terms of order ideals there is also a numerical characterization of $M$-vectors due to Macaulay [11]. Given any $l, i \in \mathbb{N}$ there is a unique expansion of $l$ of the form

$$l = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \qquad n_i > n_{i-1} > \cdots > n_j \geq j \geq 1.$$

This is called the *$i$-canonical representation of $l$*. Define $l^{<i>}$, the *$i$th pseudo-power of $l$*, by

$$l^{<i>} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j+1}.$$

An integer vector $(h_0, h_1, \ldots)$ is an $M$-vector if and only if $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{<i>}$ for $i \geq 1$.

Another characterization of $M$-vectors is in terms of Cohen-Macaulay complexes. An integer vector $h = (h_0, h_1, \ldots, h_d)$ is an $M$-vector if and only if there exists a $(d-1)$-dimensional Cohen-Macaulay complex $\Delta$ such that $h(\Delta) = h$. In fact, this result is also true for the more restrictive class of shellable complexes [19, Theorem II.3.3].

## 3 Known and Conjectured Necessary Conditions on the $h$-vectors of Homology Balls

In this section we discuss some of the previously known necessary conditions on the $h$-vectors of homology balls as well as some conjectured conditions on these $h$-vectors. Many of the results are obtained by examining the relationship between the $h$-vector of a homology ball and the $h$-vector of its boundary homology sphere. We use the conditions that the $g$-conjecture places on the $h$-vector of the boundary sphere to obtain possible restrictions on the $h$-vector of the original ball.

**Conjecture 1.** *(The g-Conjecture) An integer vector* $(h_0, h_1, \ldots, h_d)$ *with* $h_0 = 1$ *is the h-vector of a homology d-sphere if and only if*

1. $h_i = h_{d-i}$ *for* $0 \leq i \leq \lfloor d/2 \rfloor$*, and*
2. $(1, g_1, g_2, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-vector, where* $g_i = h_i - h_{i-1}$*.*

   *(This is the g-vector of the homology d-sphere).*

The $g$-conjecture is not known to hold for all homology spheres (or all triangulated spheres) but has been proved for boundaries of simplicial polytopes [3, 18]. The relations of condition one of the $g$-conjecture are called the Dehn-Sommerville equations and are known to hold for all homology spheres [9]. Additionally, using Barnette's Lower Bound Theorem [1] and standard algebraic arguments one can show that for all triangulated spheres the initial part $(g_0, g_1, g_2)$ of the $g$-vector is an $M$-vector (in fact this result is true for the much larger class of all doubly Cohen-Macaulay complexes, which include all homology spheres [14]). Therefore the $g$-conjecture holds for all homology spheres of dimension four or lower.

In order to use these known and conjectured conditions on the $h$-vectors of homology spheres we derive a relationship between the $h$-vector of a homology ball and the $h$-vector of its boundary. In [12], MacDonald proves a generalization of the Dehn-Sommerville equations for triangulated manifolds with possibly non-empty boundary. Here we will use the equivalent result for homology manifolds expressed in terms of $h$-vectors. This form of the result is due independently to Gräbe [5, Section 2.2] and Novik and Swartz [15, Theorem 3.1].

**Theorem 2.** *Let* $\Delta$ *be a* $(d-1)$*-dimensional homology manifold with boundary. Then*

$$h_{d-i}(\Delta) - h_i(\Delta) = \binom{d}{i}(-1)^{d-1-i}\tilde{\chi}(|\Delta|) - g_i(\partial\Delta)$$

*for all* $0 \leq i \leq d$*.*

In the case where $\Delta$ is a $(d-1)$-dimensional homology ball, this reduces to $h_i(\Delta) - h_{d-i}(\Delta) = g_i(\partial\Delta)$. Let $\Delta_k$ be the cone over $\Delta$ taken $k$ times. Then $\Delta_k$ is a $(d-1+k)$-dimensional homology ball with boundary a $(d-2+k)$-dimensional homology sphere. Following the argument of Billera and Lee [2, Corollary 3.14] yields $g_i(\partial\Delta_k) = h_i(\Delta_k) - h_{d+k-i}(\Delta_k) = h_i(\Delta) - h_{d+k-i}(\Delta)$, $0 \le i \le d+k$. Combining this result with the $g$-conjecture yields the following set of conditions.

**Conjecture 3.** *If $(h_0, \ldots, h_d)$ is the $h$-vector of a homology $(d-1)$-ball and we take $h_i = 0$ for $i > d$ then $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, \ldots, h_m - h_{d+k-m})$ is an $M$-vector for $k = 0, \ldots, d+1$, $m = \lfloor (d+k-1)/2 \rfloor$.*

Even though the $g$-conjecture has not been settled for the case of homology spheres, many of the conditions in Conjecture 3 can be verified.

**Proposition 4.** *Conjecture 3 holds for $d-3 \le k \le d+1$, and for all $k$ the vector $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, h_2 - h_{d+k-2})$ is an $M$-vector.*

*Proof.* The case $k = d+1$ is the statement that the original $h$-vector of the ball, $(h_0, \ldots, h_d)$, is an $M$-vector. Since homology balls are Cohen-Macaulay complexes their $h$-vectors are $M$-vectors. For $d-2 \le k \le d$, since $h_d(\Delta) = 0$ for any homology ball $\Delta$ Conjecture 3 reduces to the fact that the $h$-vector of $\Delta$ is an $M$-vector, which was discussed above.

For the case $k = d-3$ we must show that $(h_0, h_1, \ldots, h_{d-3}, h_{d-2} - h_{d-1})$ is an $M$-vector. Since we already know that the $h$-vector of $\Delta$ is an $M$-vector, we only need to show that $h_{d-2} - h_{d-1}$ is non-negative. As a consequence of [20, Corollary 4.29] $h_{d-2} \ge h_{d-1}$ for any homology ball, giving the desired inequality.

Finally, Barnette's result implies that the initial segments $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, h_2 - h_{d+k-2})$ of the vectors in Conjecture 3 are $M$-vectors. □

As a consequence of Proposition 4 we have the following.

**Corollary 5.** *Conjecture 3 holds for homology balls of dimension less than or equal to four.*

# 4 A New Type of Necessary Condition

A *polyhedral ball* is a triangulated ball obtained from the boundary of a simplicial polytope by removing a single vertex (and all faces containing this vertex). In particular, all polyhedral balls are also homology balls. In [2, Conjecture 5.1], Billera and Lee made the following conjecture.

**Conjecture 6** (Billera and Lee)**.** *The conditions of Conjecture 3 are necessary and sufficient to describe the $h$-vectors of polyhedral balls.*

For polyhedral balls necessity follows from the $g$-theorem for polytopes [2, Corollary 3.14]. Sufficiency was verified for balls of dimension three and four by Lee and Schmidt [10]. In this section we show that in dimensions higher than four there are certain vectors that satisfy the conditions of Conjecture 3 but are not the $h$-vectors of any homology ball. This provides counterexamples to sufficiency in Conjecture 6 in dimensions five and higher.

For the results of this section we need the idea of splitting a simplicial complex along a codimension-one face. Let $\Delta$ be a $(d-1)$-homology ball and $F$ be an interior $(d-2)$-face of $\Delta$. If $\Delta_{[n]\setminus F}$ is not connected, let $W_1$ and $W_2$ be the vertex sets of the two components. Define the *splitting* of $\Delta$ along $F$ to be the creation of two new simplicial complexes $\Delta_{W_1 \cup F}$ and $\Delta_{W_2 \cup F}$. Mayer-Vietoris sequences show that these two new complexes are also homology balls.

## 4.1 The Betti Diagram of the Face Ring

Our first goal will be to understand the upper right entry of the Betti diagram of the face ring modulo a linear system of parameters, $\beta_{n-d,n-d+1}(k[\Delta])$.

**Proposition 7.** *Let $\Delta$ be a $(d-1)$-homology ball. Then $\beta_{n-d,n-d+1}(k[\Delta]) > 0$ if and only if there is a splitting of $\Delta$ along a codimension-one face that creates two homology balls.*

*Proof.* From Hochster's Formula (1)

$$\beta_{n-d,n-d+1}(k[\Delta]) = \sum_{W \subset [n], |W| = n-d+1} \dim_k(\tilde{H}_0\left(\Delta_W; k\right)). \tag{2}$$

From [6, Lemma 3.7], the group $\tilde{H}_0(\Delta_W; k)$ in equation (2) is trivial whenever $[n]\backslash W$ is not a face of $\Delta$. Therefore, $\beta_{n-d,n-d+1}(k[\Delta])$ is non-zero if and only if there exists a $(d-2)$-face $F$ of $\Delta$ such that $\Delta_{[n]\backslash F}$ is not connected. Hence $\beta_{n-d,n-d+1}(k[\Delta])$ is non-zero if and only if $\Delta$ can be split along a codimension-one face to obtain two homology balls. □

Given a homology ball $\Delta$, let $L$ be the lexicographic ideal such that the Hilbert function of $R/L$ is equal to the $h$-vector of $\Delta$. Our next proposition will allow us to detect the presence of a face along which we can split $\Delta$ by looking at the Betti Diagram of $R/L$.

**Proposition 8.** *Let $\Delta$ be a $(d-1)$-homology ball. Let $L$ be the lexicographic ideal such that the Hilbert function of $R/L$ is equal to the $h$-vector of $\Delta$. If $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L) > 0$ then there is a splitting of $\Delta$ along a codimension-one face that creates two homology balls.*

*Proof.* We will prove that when $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L) > 0$ then $\beta_{n-d,n-d+1}(k[\Delta]) > 0$. The result then follows by Proposition 7.

By [19, Theorem I.12.4], $\beta_{n-d,n-d+1}(k[\Delta])$ is equal to the dimension of the degree one portion of the socle of $k(\Delta)$. Since $k(\Delta)$ has projective dimension $n$, the dimension of the degree $i$ portion of the socle of $k(\Delta)$ is given by the Betti number $\beta_{n,n+i}(k(\Delta))$ [19, Theorem I.12.4]. Combining these facts gives

$$\beta_{n-d,n-d+1}(k[\Delta]) = \dim_k(\operatorname{soc} k(\Delta))_1 = \beta_{n,n+1}(k(\Delta)).$$

Since the Hilbert Function of $k(\Delta)$ is equal to the $h$-vector of $\Delta$, we can use Peeva's cancellation technique to relate the $h$-vector of $\Delta$ to the Betti numbers of $k(\Delta)$. By [16, Theorem 1.1], the Betti number $\beta_{n,n+1}(k(\Delta))$ is bounded above by $\beta_{n,n+1}(R/L)$ and below by $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L)$. Therefore, if $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L) > 0$ we have $\beta_{n-d,n-d+1}(k[\Delta]) = \beta_{n,n+1}(k(\Delta)) > 0$, which implies the desired result. □

## 4.2 Splitting Balls

Next we investigate the effect on the $h$-vector when a homology ball is split along a single codimension-one face.

**Proposition 9.** *Let $\Delta_1$ and $\Delta_2$ be two homology $(d-1)$-balls that can be joined along a common homology $(d-2)$-ball $B$ to form a single homology $(d-1)$-ball $\Delta$. Then*

$$h_i(\Delta_1) + h_i(\Delta_2) = h_i(\Delta) + (h_i(B) - h_{i-1}(B)) \tag{3}$$

*where we take $h_{-1}(B) = h_d(B) = 0$.*

*Proof.* On the level of $f$-vectors

$$f_i(\Delta_1) + f_i(\Delta_2) = f_i(\Delta) + f_i(B).$$

The result then follows from a straightforward calculation using the definition of the $h$-vector. □

As a special case of equation (3), when we join two homology balls along a single codimension-one face the $h$-vector of the resulting complex is the sum of the $h$-vectors of the two component homology balls but with $h_0$ reduced by one (causing $h_0$ of the resulting complex to still equal one) and $h_1$ increased by one.

**Example 10 (Algebraic approach applied to $(1,4,5,7,3,2,0)$).** With these tools, we now consider the $h$-vector $(1,4,5,7,3,2,0)$. Assume there is a homology five-ball $\Delta$ with this $h$-vector. Using the result of Eliahou and Kervaire [4, Section 3], we calculate the Betti numbers of $R/L$, where $L$ is the lexicographic ideal such that the Hilbert function of $R/L$ is equal to $(1,4,5,7,3,2,0)$. This yields $\beta_{n,n+1}(R/L) = 1$ and $\beta_{n-1,n+1}(R/L) = 0$. Therefore, by Proposition 8 there exists a codimension-one face along which we can split $\Delta$.

Using formula (3) for the $h$-vector obtained by combining homology balls, we now look for possible $h$-vectors for the two homology five-balls created when we split our original homology ball. Each of the two smaller homology balls must satisfy all of the known portions of Conjecture 3 as discussed in Proposition 4.

The only two pairs of options for the $h_1$-values of our smaller homology balls are 2,1 or 3,0. If we take 2 and 1 as the $h_1$-values the largest possible corresponding values of $h_2$ are 3 and 1, which do not sum to the $h_2$-value of our original ball. If we take 3 and 0 as our $h_2$-values then one homology ball must have $h$-vector $(1, 3, 5, 7, 3, 2, 0)$. However, the $g$-vector of the boundary homology four-sphere of this ball would be (1,1,2), violating a known portion of the $g$-conjecture.

We have shown that there must be a division of our homology five-ball into two smaller homology balls, but also that no such division exists. Therefore $(1, 4, 5, 7, 3, 2, 0)$ is not the $h$-vector of a homology five-ball, even though it satisfies all of the conditions of Conjecture 3.

Similar calculations show that there are other $h$-vectors that satisfy the conditions of Conjecture 3 but are not the $h$-vectors of homology balls. Examples include $(1, 4, 6, 9, 4, 2, 0)$ and $(1, 5, 6, 8, 4, 3, 0)$.

## 4.3 A combinatorial approach

Some of the results of the previous section can also be obtained using a more combinatorial viewpoint. Given an $h$-vector $(1, h_1, \ldots, h_d)$ and corresponding $f$-vector $(1, f_0, f_1, \ldots, f_{d-1})$, look at all of the possible graphs with $f$-vector $(1, f_0, f_1)$. For each of these graphs count the maximum number of triangles possible in a simplicial complex of dimension $d-1$ with the given graph as its one-skeleton (or equivalently, the number of triangles in the $(d-1)$-skeleton of the flag complex or clique complex induced by the graph). Compute the $h$-vector $(1, h_1, h_2, h'_3, h'_4, \ldots, h'_d)$ of the complex with all possible triangles. Since removing triangles from a complex $\Delta$ decreases $h_3(\Delta)$ and adding faces of dimension greater than two does not change $h_3(\Delta)$, any one-skeleton of a complex with $h$-vector $(1, h_1, \ldots, h_d)$ must have $h'_3 \geq h_3$.

**Example 11 (Combinatorial approach applied to $(1, 4, 5, 7, 3, 2, 0)$).** Doing an exhaustive search we find that for the $h$-vector $(1, 4, 5, 7, 3, 2, 0)$ the only graphs that obtain $h'_3 \geq h_3$ have a vertex of degree less than or equal to five. In a homology ball each vertex must be contained in at least one facet. This forces each vertex in a homology five-ball to have degree at least five. If a vertex has degree exactly five then it is contained in only one facet. In this case we can remove the facet to create a homology five-ball with $h$-vector $(1, 3, 5, 7, 3, 2, 0)$. As argued in the previous section, this contradicts the known conditions of Conjecture 3. Therefore each vertex must have degree at least six, so no homology five-ball with $h$-vector $(1,4,5,7,3,2,0)$ exists.

Using this approach, we next describe an infinite collection of $h$-vectors that satisfy all of the conditions of Conjecture 3 but such that the existence of a homology ball with one of these $h$-vectors would contradict the known fact that $g_2 \leq g_1^{<1>}$ for a sphere. This will show that the conditions of Conjecture 3 are not sufficient in dimensions five and higher.

**Theorem 12.** *If $x, y$ are integers with $x > 4$ and $1 < y < x$ then*

$$\begin{aligned}\Big(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \binom{x+1}{3} - 2, \ldots,\\ \binom{x+1}{3} - 2, \binom{x}{2} - \left(\binom{y}{2} + 1\right), x-y, 0\Big)\end{aligned} \tag{4}$$

*satisfies the conditions in Conjecture 3 but is not the $h$-vector of a homology ball.*

*Proof.* We first show that the vector (4) satisfies all of the conditions in Conjecture 3.

The case $k = 0$ (the boundary sphere condition) requires that

$$\left(1, y, \binom{y}{2} + 1\right)$$

is an $M$-vector, which follows since $\binom{y}{2} < \binom{y+1}{2}$.

For the case $k = 1$ (the condition that comes from taking a cone)

$$\left(1, x, \binom{x}{2} - x + y, \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2} - 1\right)$$

must be an $M$-vector. To see this, first note that $\binom{x}{2} - x + y = \binom{x}{2} - \binom{x-1}{1} + y - 1 = \binom{x-1}{2} + \binom{y-1}{1}$. Therefore the corresponding second pseudopower is $\left(\binom{x}{2} - x + y\right)^{<2>} = \binom{x}{3} + \binom{y}{2} = \binom{x}{3} + \binom{x}{2} - \binom{x}{2} + \binom{y}{2} = \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2}$. Combining this with the fact that $x > y$ shows that the desired vector is an $M$-vector.

For the case $k = 2$

$$\left(1, x, \binom{x}{2}, \binom{x+1}{3} - 2 - x + y, \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2} - 1\right)$$

must be an $M$-vector. Since $x > y$, the step from the second to third entry satisfies Macaulay's condition. Note that since $x > y > 1$ and $x > 4$, $\binom{x}{2} - \binom{y}{2} = \frac{1}{2}(x(x-1) - y(y-1)) > \frac{1}{2}(x(x-1) - y(x-1)) > x - y$. Therefore, the step from the third to fourth entry is non-increasing. All of the remaining checks of Macaulay's conditions and non-negativity needed to show that the desired vector is an $M$-vector are straightforward.

All of the higher $k$ values result in vectors of one of the forms

$$\Big(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \ldots, \\ \binom{x+1}{3} - 2, \binom{x+1}{3} - 2 - x + y, \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2} - 1\Big),$$

$$\Big(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \ldots, \\ \binom{x+1}{3} - 2, \binom{x+1}{3} - 2 - x + y, \binom{x}{2} - \binom{y}{2} - 1\Big),$$

$$\Big(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \ldots, \binom{x+1}{3} - 2, \binom{x}{2} - x - \Big(\binom{y}{2} - y\Big) - 1\Big),$$

or the original $h$-vector itself. Using the same arguments as in the previous cases, these are also all $M$-vectors.

Assume that there exists a homology $(d-1)$-ball $\Delta$ with $h$-vector given by (4). We will show that this results in a contradiction. Calculating the $f$-vector of $\Delta$ yields

$$f_0 = d + x, \qquad \binom{d+x}{2} - f_1 = x, \qquad \binom{d+x}{3} - f_2 = \frac{x^2}{2} + \left(\frac{2d-3}{2}\right) x + 2.$$

Note that $\binom{d+x}{i+1} - f_i$ counts the number of non-$i$-faces of $\Delta$.

We claim that every vertex of $\Delta$ has degree at least $d$. To see this, first note that any vertex of degree less than $d-1$ would not be contained in any facet of our complex. If there was a vertex of degree $d-1$ this vertex would be contained in exactly one facet. Removing this facet from our homology ball would decrease $h_1$ by one, leaving us with a homology ball whose boundary homology sphere would have $g$-vector $(1, y-1, \binom{y}{2}+1, \ldots)$, contradicting the fact that for a sphere $(g_0, g_1, g_2)$ is an $M$-vector.

Let $G$ be the graph of non-edges of $\Delta$. The vertex set of $G$ is $[d+x]$, the same as the vertex set of $\Delta$, and $\{a,b\}$ is an edge of $G$ if and only if $\{a,b\} \notin \Delta$. By the above claim the maximum degree of any vertex in $G$ is $x-1$.

For each edge $\{a,b\}$ of $G$ and each vertex $c \notin \{a,b\}$ the triangle $\{a,b,c\}$ is a non-triangle. If $G$ has no vertex of degree at least two, then for each combination of a non-edge and a vertex not in that edge there is a distinct non-triangle. This results in a total of at least $x(x+d-2)$ non-triangles, far more than the $\frac{x^2}{2} + \left(\frac{2d-3}{2}\right)x + 2$ allowed non-triangles. We can therefore assume that $G$ has a vertex $v$ of degree $k$ where $2 \leq k < x$.

Label the edges of $G$ by $e_1, e_2, \ldots, e_x$ where $v$ is contained in $e_i$ for $1 \leq i \leq k$. Let $G_i$, $1 \leq i \leq x$, be the graph on the vertex set $[x+d]$ with edges $\{e_j\}_{j=1}^{i}$. Let $A_i := \{\{a,b,c\} \subset [x+d] : a,b,c$ distinct and $G_i$ contains at least one of the edges $\{a,b\},\{b,c\},\{a,c\}\}$. Then $|A_x|$ is less than or equal to the number of non-triangles in $\Delta$.

We now compare the sets $A_{i-1}$ and $A_i$ for $i = 1,\ldots,x$. First note that $A_{i-1} \subseteq A_i$. When moving from $A_{i-1}$ to $A_i$ the new elements are those containing the two endpoints of $e_i$ and one other vertex which is not adjacent in $G_{i-1}$ to either endpoint of $e_i$. For $i \leq k$ this implies $|A_i \backslash A_{i-1}| = x+d-1-i$. For $i \geq k+1$ the graph $G_{i-1}$ has $i-(k+1)$ edges that do not contain $v$. Therefore there are at least $(x+d-3-(i-(k+1))$ vertices that are not in $e_i$ and are not adjacent in $G_{i-1}$ to either endpoint of $e_i$. Thus $A_i \backslash A_{i-1}$ contains at least $(x+d-3-(i-(k+1))$ elements. In total, $|A_x|$ is bounded below by

$$\begin{aligned}
&(x+d-2)+(x+d-3)+\cdots+(x+d-1-k)+(x+d-3)+\cdots \\
&\qquad +(x+d-3-(x-(k+1)) \\
&= \Big((x+d-2)+(x+d-3)+\cdots+(d-1)\Big)+(k-1)(x-k) \\
&= \frac{x^2}{2}+\frac{2d-3}{2}\cdot x+(k-1)(x-k).
\end{aligned}$$

Since $x > 4$ and $k > 1$, $(k-1)(x-k) \geq 3$. Therefore $|A_x| > \binom{d+x}{3} - f_2$ which means there is at least one too many non-triangles and no homology ball with this $h$-vector exists. $\square$

**Corollary 13.** *The conditions of Conjecture 3 are not sufficient to characterize homology balls in dimensions five and higher. In particular, Conjecture 6 is false in all dimensions greater than four.*

# 5 Construction Methods

In this section we present a method for constructing balls with a large variety of different $h$-vectors. The main theorem of the section is the following. (We address the case where the dimension $d-1$ is even first; the small alterations needed in the case where $d-1$ is odd are discussed in Theorem 17 at the end of the section.)

**Theorem 14.** *Let $d-1$ be even and let $(1, h_1, h_2, \ldots, h_{d-1}, 0)$ satisfy the following conditions:*

- $(1, h_1 - 1, h_2 - h_1, \ldots, h_{(d-3)/2} - h_{(d-5)/2}, \max\{h_{(d-1)/2} - h_{(d-3)/2}, 0\})$ *is an M-vector.*
- $(1, h_1 - h_{d-1}, h_2 - h_{d-2}, \ldots, h_{(d-1)/2} - h_{(d+1)/2})$ *is an M-vector.*
- $h_{(d+1)/2} \geq h_{(d+3)/2} \geq \cdots \geq h_{d-1}$.

*Then there is a triangulated $(d-1)$-ball with $h$-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.*

Theorem 14 does not obtain all possible $h$-vectors of balls. In [10, Theorem 2], Lee and Schmidt show that $h$-vectors with $h_1 \geq h_2 \geq \ldots \geq h_{d-1} \geq h_d = 0$ are the $h$-vectors of triangulated balls. However, in dimensions five and higher, taking $h_i > h_{i-1}$ for $2 \leq i \leq (d/2 - 1)$ violates the conditions of Theorem 14 (or Theorem 17 for $d-1$ odd). Additionally, the construction of Billera and Lee in [3, Section 6] can be used to create balls with $h$-vector equal to any $M$-vector whose second half is all zeros, many of which cannot be obtained using our construction.

In the proof of Theorem 14 we divide a sphere into two complementary balls intersecting only along their common boundary. The following lemma describes the relationship between the $h$-vectors of the two balls and the original sphere in this situation.

**Lemma 15.** *Let $\Delta$ be a $(d-1)$-dimensional triangulated sphere and $B \subset \Delta$ be a $(d-1)$-dimensional triangulated ball. Let $C := (\Delta \backslash B) \cup (\partial B)$ be the complementary $(d-1)$-ball to $B$ in $\Delta$. Then $h_i(C) = h_i(\Delta) - h_{d-i}(B)$.*

*Proof.* Since $B$ and $C$ intersect only along $\partial B$ and $B \cup C = \Delta$

$$f_i(B) + f_i(C) = f_i(\Delta) + f_i(\partial B).$$

A straightforward calculation shows

$$h_i(B) + h_i(C) = h_i(\Delta) + (h_i(\partial B) - h_{i-1}(\partial B)) = h_i(\Delta) + g_i(\partial B), \qquad (5)$$

where we take $g_0(\partial B) = h_0(\partial B) = 1$. From Theorem 2, $g_i(\partial B) = h_i(B) - h_{d-i}(B)$. Substituting this into equation (5) and simplifying yields $h_i(C) = h_i(\Delta) - h_{d-i}(B)$, as desired. $\square$

Since the proof of Theorem 14 relies heavily on the ideas in the Billera-Lee construction [3, Section 6] and Kalai's paper [8] we review some of those concepts and notation here.

Let $d > 0$ be an odd integer. Define $F_d(n)$, a collection of $(d+1)$-subsets of $[n]$, by $F \in F_d(n)$ if and only if $F = \cup_{j=1}^{(d+1)/2}\{i_j, i_j + 1\}$ where $i_1, \ldots, i_{(d+1)/2}$ are elements of $[n-1]$ such that $i_{j+1} > i_j + 1$ for every $j$. Let $J$ be any initial segment in $F_d(n)$ with respect to the partial order $<_p$. Kalai showed that $B(J)$, the simplicial complex with facets the elements of $J$, is a shellable ball. A shelling order is given by any linear order of the facets consistent with $<_p$ and the size of the restriction face of a facet $F$ is given by the number of pairs of vertices of $F$ not in their leftmost possible position, $|r(F)| = |\{j : i_j \neq 2j - 1\}|$.

Let $\mathcal{F}_{b,c}$ be the set of all monomials in the variables $Y_1, Y_2, \ldots, Y_b$ of degree at most $c$. Define a bijection $\alpha : F_d(n) \to \mathcal{F}_{n-d-1,(d+1)/2}$ by $\alpha(F) = \prod_{j=1}^{(d+1)/2} Y_{e_j}$, where $e_j = i_j - 2j + 1$ is the amount that the $j$th pair of $F$ is displaced from its leftmost possible position and we take $Y_0 = 1$. This bijection is order preserving between the partial orders $<_p$ on monomials and subsets of $[n]$. Therefore, given an initial segment $I$ of monomials in $\mathcal{F}_{b,c}$ using the partial order $<_p$, the corresponding facets (under the map $\alpha^{-1}$) form a shellable ball $B(\alpha^{-1}(I))$ with $|r(F)|$ equal to the degree of $\alpha(F)$ for all facets $F$. For ease of notation, when $I$ is an initial segment of monomials in the $<_p$ order we write $B(\alpha^{-1}(I))$ as $B(I)$.

In this way, any rev-lex initial segment of monomials $I$ gives rise to a shellable ball $B(I)$. So given an $M$-vector, the image under $\alpha^{-1}$ of the corresponding compressed order ideal of monomials is the set of facets of a shellable ball with $h$-vector equal to the original $M$-vector (this is what was done in the Billera and Lee paper).

In the case where $d$ is even, the construction is altered by adding an additional vertex $\{0\}$ to each facet.

*Proof of Theorem 14.* Define

$$(1, g_1, g_2, \ldots, g_{(d-3)/2}, g_{(d-1)/2}) := \\ (1, h_1 - 1, h_2 - h_1, \ldots, h_{(d-3)/2} - h_{(d-5)/2}, \max\{h_{(d-1)/2} - h_{(d-3)/2}, 0\}).$$

We first consider the case where $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$. Since we know $(1, g_1, g_2, \ldots, g_{(d-1)/2}, 0, 0, \ldots)$ is an $M$-vector, there is a compressed order ideal $I$ with this vector as its degree sequence. We know $I \subseteq \mathcal{F}_{g_1,(d-1)/2} \subseteq \mathcal{F}_{g_1,(d+1)/2}$, so using the Billera-Lee method we construct the $d$-ball $B(I)$ with $h$-vector $(1, g_1, g_2, \ldots, g_{(d-1)/2}, 0, 0, \ldots)$. Note that each facet of $B(I)$ contains the elements 1 and 2. The boundary of $B(I)$ is a $(d-1)$-sphere. Theorem 2, the definition of the $g_i$, and the Dehn-Sommerville equations combine to give

$$h(\partial B(I)) = (1, h_1, h_2, \ldots, h_{(d-1)/2}, h_{(d-1)/2}, \ldots, h_2, h_1, 1).$$

Define

$$(1, G_1, \ldots, G_{(d-1)/2}) := (1, h_1 - h_{d-1}, \ldots, h_{(d-1)/2} - h_{(d+1)/2}).$$

We now construct a $(d-1)$-ball $\mathcal{B}$ in the sphere $\partial B(I)$ such that $h(\mathcal{B}) = (1, G_1, G_2, \ldots, G_{(d-1)/2}), 0, 0, \ldots)$. Using Lemma 15 we have that the complementary ball $\partial B(I) \backslash (\mathcal{B} \backslash \partial \mathcal{B})$ is the desired $(d-1)$-ball.

The $(d-1)$-ball $\mathcal{B}$ uses the same correspondence $\alpha$ between facets and monomials except that the vertex names are shifted by one. Given a monomial $m = \prod_{j=1}^{(d-1)/2} Y_{e_j}$ we define the potential corresponding facet of $\mathcal{B}$ by $(\alpha')^{-1}(m) := \{1\} \cup \left( \cup_{j=1}^{(d-1)/2} \{i_j, i_j + 1\} \right)$ where $i_j = e_j + 2j$ (instead of $i_j = e_j + 2j - 1$, as is the case for the correspondence $\alpha^{-1}$).

Next we characterize the facets of $\partial B(I)$ that will be used in the construction of $\mathcal{B}$. A set of $d$ vertices is a facet of $\partial B(I)$ if and only if it is in exactly one facet of $B(I)$. Note that the only possible facet of $B(I)$

that can contain the face $(\alpha')^{-1}(\prod_{j=1}^{(d-1)/2} Y_{e_j})$ is $\alpha^{-1}(Y_0 \cdot \prod_{j=1}^{(d-1)/2} Y_{e'_j})$ where $e'_j = \max\{e_j - 1, 0\}$. It follows that $(\alpha')^{-1}(\prod_{j=1}^{(d-1)/2} Y_{e_j})$ is a facet of $\partial B(I)$ if and only if $\prod_{j=1}^{(d-1)/2} Y_{e'_j} \in I$. Additionally, since all of the facets of $B(I)$ contain the element 1, the face $F \backslash \{1\}$ is in $\partial B(I)$ for all facets $F$ of $B(I)$.

We now build the ball $\mathcal{B}$. For each $k$ we will select a set $\mathcal{M}_k$ of degree $k$ monomials such that $|\mathcal{M}_k| = G_k$. We will show that $\cup_{i=0}^{(d-1)/2} \mathcal{M}_i$ is an initial segment in the partial order $<_p$. Then the facets of $\mathcal{B}$ will be the faces $(\alpha')^{-1}(m)$ for $m \in \cup_{i=0}^{(d-1)/2} \mathcal{M}_i$.

By the above discussion, since $1 \in I$ we know that $(\alpha')^{-1}(1)$ is a facet of $\partial B(I)$. We therefore set $\mathcal{M}_0 = \{1\}$.

Assume that for some $k > 0$ we have already chosen $\mathcal{M}_i$ for $i \leq k$ with $|\mathcal{M}_i| = G_i$. Define the set $S_{k+1}$ to be $\{Y_1 \cdot m : m \in \mathcal{M}_k\}$ (call these type one elements) as well as all of the monomials $\prod_{i=1}^{k+1} Y_{e_i}$ such that all of the $e_i > 1$ and $\prod_{i=1}^{k+1} Y_{e_i - 1} \in I$ (these are called type two elements). There are $G_k$ elements of the first type and $g_{k+1}$ elements of the second type giving a total of

$$G_k + g_{k+1} = (h_k - h_{d-k}) + (h_{k+1} - h_k) = h_{k+1} - h_{d-k} \geq h_{k+1} - h_{d-(k+1)} = G_{k+1}$$

elements in $S_{k+1}$.

Select the first $G_{k+1}$ elements of $S_{k+1}$ in the rev-lex order to be the monomials in $\mathcal{M}_{k+1}$. We complete the proof of the case $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$ with the following proposition.

**Proposition 16.** *For $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$, $\cup_{i=0}^{(d-1)/2} \mathcal{M}_i$ is an initial segment in the order $<_p$.*

We prove Proposition 16 inductively; given that the monomials in $\cup_{i=0}^{k} \mathcal{M}_i$ are an initial segment in $<_p$ we show that the monomials in $\cup_{i=0}^{k+1} \mathcal{M}_i$ still form an initial segment. The base case $k = 0$ follows since $\mathcal{M}_0 = \{1\}$ is an initial segment in $<_p$. We divide the proof of the inductive step into five claims.

**Claim 1.** *Let $\tau$ be a type two element in $\mathcal{M}_k$. Then $\{m \in \mathcal{M}_k : m \leq_{rl} \tau\}$ is a rev-lex initial segment in degree $k$.*

*Proof of Claim 1.* Let $m$ be a degree $k$ monomial such that $m <_{rl} \tau$. Let $m'$ be the monomial $m$ with all of the $Y_1$'s changed to $Y_2$'s (if $m$ does not contain

$Y_1$ then $m' = m$). Both $m$ and $\tau$ are degree $k$ monomials and $\tau$ contains no $Y_1$'s which implies $m' \leq_{rl} \tau$. Since $I$ is a compressed order ideal the type two elements of $S_k$ form a compressed order ideal in the variables $Y_2, Y_3, \ldots$. Therefore $m'$ must be a type two element of $S_k$ and $m' \in \mathcal{M}_k$. However, $m <_p m'$ which by the inductive assumption implies $m \in \mathcal{M}_k$, proving the desired claim. □

**Claim 2.** *Let $i \leq k$. If there exists a type two element $\tau \in S_i$ such that $\tau \notin \mathcal{M}_i$ then $\mathcal{M}_i$ is a rev-lex initial segment in degree $i$.*

*Proof of Claim 2.* Our proof is by induction on $i$. The result is trivial in the base case $i = 1$ where all initial segments in the order $<_p$ are also rev-lex initial segments.

Assume the claim holds for $i = l - 1 \geq 1$. Let $M$ be the rev-lex largest element of $\mathcal{M}_l$. If $M$ is a type two element then we are done by Claim 1, so assume that $M$ is a type one element. Let $m$ be a degree $l$ monomial such that $m <_{rl} M$. We must show that $m \in \mathcal{M}_l$.

If $m$ does not contain the variable $Y_1$, then the fact that $m <_{rl} \tau$ means that $m$ is a type two element of $S_l$. Then the fact that $m <_{rl} M$ forces $m \in \mathcal{M}_l$.

If $m$ contains $Y_1$ then $m/Y_1 <_{rl} M/Y_1 \leq_{rl} \tau/Y_j$ where $Y_j$ is the smallest (positive) index such that $Y_j$ is in $\tau$. We also know that $M/Y_1 \in \mathcal{M}_{l-1}$ and $\tau/Y_j$ is a type two element in $S_{l-1}$. If $\tau/Y_j \in \mathcal{M}_{l-1}$ then by Claim 1 we have $m/Y_1 \in \mathcal{M}_{l-1}$. This means $m$ is a type one element of $S_l$ which forces $m \in \mathcal{M}_l$. If $\tau/Y_j \notin \mathcal{M}_{l-1}$ then by the inductive hypothesis and the fact that $M/Y_1 \in \mathcal{M}_{l-1}$ we have $m/Y_1 \in \mathcal{M}_{l-1}$ which forces $m \in \mathcal{M}_l$, completing the proof of the claim. □

**Claim 3.** *Let $\rho$ be a type two element in $\mathcal{M}_{k+1}$. Then $\{m \in \mathcal{M}_{k+1} : m \leq_{rl} \rho\}$ is a rev-lex initial segment in degree $k+1$.*

*Proof of Claim 3.* Our proof of Claim 3 is in two cases. First consider the case where there exists a type two element $\tau \in S_k$ such that $\tau \notin \mathcal{M}_k$. By Claim 2 the set $\mathcal{M}_k$ is a rev-lex initial segment in degree $k$. Let $N$ be the rev-lex smallest degree $k+1$ monomial not in $\mathcal{M}_{k+1}$. It is sufficient to show $\rho <_{rl} N$.

If $N$ is not one of the first $G_{k+1}$ monomials of degree $k+1$ in the rev-lex order then all of the elements of $\mathcal{M}_{k+1}$ are rev-lex less than $N$, proving the desired claim. We therefore assume that $N$ is one of the first $G_{k+1}$ monomials of degree $k+1$ in the rev-lex order. If $N$ contains the variable $Y_1$ then since $\mathcal{M}_k$ is a rev-lex initial segment and the $G_i$ form an $M$-vector $N/Y_1 \in \mathcal{M}_k$. This means $N$ is a type one element in $S_{k+1}$ and therefore $N \in \mathcal{M}_{k+1}$, contradicting our definition of $N$. Thus $N$ does not contain the variable $Y_1$. Since the type two elements of $\mathcal{M}_{k+1}$ are a rev-lex initial segment in $Y_2, Y_3, \ldots$ all of the type two elements of $\mathcal{M}_{k+1}$ are rev-lex less than $N$, proving the desired claim for the first case.

Now consider the case where all of the type two elements of $S_k$ are in $\mathcal{M}_k$. Let $n$ be a degree $k+1$ monomial with $n <_{rl} \rho$. We need to show that $n \in \mathcal{M}_{k+1}$. If $n$ does not contain $Y_1$ then the result follows from the initial segment property of the type two elements of $S_{k+1}$. If $n$ contains $Y_1$, then consider $n/Y_1 \leq_{rl} \rho/Y_j$ where $Y_j$ is the smallest variable in $\rho$. Since the $g_k$ form an $M$-vector and all of the type two elements of degree $k$ are in $\mathcal{M}_k$ we know $\rho/Y_j$ is a type two element in $\mathcal{M}_k$. Claim 1 then shows that $n/Y_1$ is in $\mathcal{M}_k$ which forces $n \in S_{k+1}$ and $n \in \mathcal{M}_{k+1}$. This completes the proof of Claim 3. □

**Claim 4.** *$\mathcal{M}_{k+1}$ is an initial segment of degree $k+1$ monomials in the partial order $<_p$.*

*Proof of Claim 4.* To prove Claim 4 we take any monomial $m \in \mathcal{M}_{k+1}$ and any degree $k+1$ monomial $m'$ with $m' <_p m$ and show that $m' \in \mathcal{M}_{k+1}$. Claim 3 implies that $\mathcal{M}_{k+1}$ consists of a rev-lex initial segment of degree $k+1$ monomials along with a (possibly empty) collection of additional type one monomials added in rev-lex order. Since any rev-lex initial segment is also an initial segment in the partial order, we need only consider the case where $m$ is one of the type one monomials that is not part of the rev-lex initial segment in $\mathcal{M}_{k+1}$. Because $m$ is a type one element $m$ contains the variable $Y_1$. The fact that $m' <_p m$ forces $m'$ to contain $Y_1$. Then $m'/Y_1 <_p m/Y_1$ and $m/Y_1 \in \mathcal{M}_k$ which combined with the inductive hypothesis implies $m'/Y_1 \in \mathcal{M}_k$. This means $m'$ is a type one element of $S_{k+1}$. Since $m' <_{rl} m$ we know $m' \in \mathcal{M}_{k+1}$, proving Claim 4. □

**Claim 5.** *Let $m \in \mathcal{M}_{k+1}$ and let $m'$ be a monomial of degree less than or equal to $k$ with $m' <_p m$. Then $m' \in \cup_{i=0}^{k} \mathcal{M}_i$.*

*Proof of Claim 5.* Let $Y_j$ be the smallest variable in $m$, so that $m' <_p m/Y_j$. By the inductive hypothesis it is sufficient to show $m/Y_j \in \mathcal{M}_k$.

If $m$ is a type one element $Y_j = Y_1$ and the claim follows from the definition of a type one element. If $m$ is a type two element we consider two cases. For the case where there exists a type two element $\tau \in S_k$ such that $\tau \notin \mathcal{M}_k$ the result follows from Claim 2 and Claim 3 along with the fact that the $G_i$ form an $M$-vector. In the case where all of the type two elements of $S_k$ are in $\mathcal{M}_k$ the result was already shown in the proof of Claim 3. $\square$

Combining Claim 4 and Claim 5 we have that the monomials in $\cup_{i=0}^{k+1} \mathcal{M}_i$ form an initial segment in the order $<_p$. This completes the inductive step, finishing the proof of Proposition 16.

We now address the case where $h_{(d-1)/2} - h_{(d-3)/2} < 0$. In this case $g_{(d-1)/2} = 0$ which implies

$$h(\partial B(I)) = (1, h_1, h_2, \ldots, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-3)/2}, \ldots, h_2, h_1, 1).$$

Therefore, we alter our definition of the $G_i$,

$$\begin{aligned}&(1, G_1, \ldots, G_{(d-3)/2}, G_{(d-1)/2}, G_{(d+1)/2}) := \\ &\quad (1, h_1 - h_{d-1}, \ldots, h_{(d-3)/2} - h_{(d+3)/2}, h_{(d-3)/2} - h_{(d+1)/2}, h_{(d-3)/2} - h_{(d-1)/2}).\end{aligned}$$

We use the same argument as above to construct the $\mathcal{M}_k$ with $k < (d-1)/2$. Note that in this case the decreasing assumption on the $h_i$'s implies that $G_{(d-3)/2} \geq G_{(d-1)/2} \geq G_{(d+1)/2}$. This allows us to choose $\mathcal{M}_{(d-1)/2}$ to be the first $G_{(d-1)/2}$ type one elements of degree $(d-1)/2$ in the rev-lex order. Then $\cup_{k=0}^{(d-1)/2} \mathcal{M}_k$ is an initial segment in $<_p$, so the corresponding facets form a shellable ball with $h$-vector $(1, G_1, \ldots, G_{(d-3)/2}, G_{(d-1)/2}, 0, 0, \ldots)$.

Let $E$ be the first $G_{(d+1)/2}$ monomials in $\mathcal{M}_{(d-1)/2}$ using the rev-lex order and let $m \in E$. The degree of $m$ is $(d-1)/2$ meaning that all pairs of vertices in $(\alpha')^{-1}(m)$ are shifted to the right and $2 \notin (\alpha')^{-1}(m)$. However, 2 is an element of every facet of $B(I)$ and since $m \in \mathcal{M}_{(d-1)/2}$ we know $(\alpha')^{-1}(m)$ is contained in exactly one facet of $B(I)$. Therefore $(\alpha')^{-1}(m) \cup \{2\}$ must be a facet of $B(I)$. As argued above this implies that $\gamma(m) := (\alpha')^{-1}(m) \cup \{2\} \setminus \{1\}$ is in $\partial B(I)$ for all $m \in E$.

In order to get the desired $(d+1)/2$ entry of our $h$-vector, for each $m \in E$ we add the facet $\gamma(m)$ to our complementary ball $\mathcal{B}$. Though the facets in this last step do not correspond to monomials using the map $\alpha'$, because of the relationship between $(\alpha')^{-1}(m)$ and $\gamma(m)$ it is straightforward to check that adding the $\gamma(m)$ to the end of the shelling in the same order as the $(\alpha')^{-1}(m)$ still gives a shellable ball with the correct $h$-vector. □

For the case where $d-1$ is odd we can prove the follwing result using the same argument as the above proof with appropriate changes in notation and parity.

**Theorem 17.** *Let $d-1$ be odd and let $(1, h_1, h_2, \ldots, h_{d-1}, 0)$ satisfy the following conditions:*

- $(1, h_1 - 1, h_2 - h_1, \ldots, h_{d/2-1} - h_{d/2-2}, \max\{h_{d/2} - h_{d/2-1}, 0\})$ *is an M-vector.*
- $(1, h_1 - h_{d-1}, h_2 - h_{d-2}, \ldots, h_{d/2-1} - h_{d/2+1})$ *is an M-vector.*
- $h_{d/2} \geq h_{d/2+1} \geq \cdots \geq h_{d-1}$.

*Then there is a triangulated $(d-1)$-ball with $h$-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.*

# 6 Consequences of the Construction

As noted in the previous section, the conditions of Theorem 14 are not in general necessary for the existence of a ball with a given $h$-vector. However, in dimensions three and four it is straightforward to check that the conditions of Conjecture 3 imply the conditions of Theorem 14. Since we have already shown the necessity of the conditions of Conjecture 3 in dimensions three and four, these conditions give a complete characterization of the $h$-vectors of three- and four-dimensional balls. As mentioned previously, this result was first obtained by Lee and Schmidt in [10].

Starting in dimension five we know that the conditions of Conjecture 3 are no longer sufficient and we also know that the conditions of Theorem 14 are no longer necessary. In particular, our construction cannot create any five-balls with $h_2 < h_1$, even though many such balls exist. Given any five-ball we can attach to it a single five-simplex by gluing along a single codimension-one face of the boundary of each ball. This process adds one new vertex and one new facet to the original ball. As described in Section 4.2, this increases the $h_1$-value of the original ball by one without changing any of the other entries of the $h$-vector. Repeating this process we can create many different balls with $h_2 < h_1$.

While there exist balls with $h_2 < h_1$ that do not arise from adding vertices to other balls as described above, we have so far been unable to find any five-ball whose $h$-vector cannot be realized by adding vertices to a ball constructed using Theorem 14. In fact, using the methods of Section 4 it can be shown that many of the 'small' $h$-vectors that cannot be obtained by adding vertices to balls constructed using Theorem 14 cannot be the $h$-vectors of five-balls. We therefore make the following conjecture.

**Conjecture 18.** *A vector* $h = (1, h_1, h_2, h_3, h_4, h_5, 0)$ *is the* $h$*-vector of a five-ball if any only if there exists some integer* $m > 0$ *such that* $h = (1, h_1 - m, h_2, h_3, h_4, h_5, 0)$ *satisfies the conditions of Theorem 14.*

If any two balls satisfying the condition in Conjecture 18 are joined along a single codimension-one face, the $h$-vector of the resulting ball still satisfies the conditions of the conjecture. However, it is not true that the conditions of Theorem 14 give all of the $h$-vectors of balls that cannot be split along some codimension-one face (i.e. all the balls $\Delta$ with $\beta_{n-d,n-d+1}(k(\Delta)) = 0$). As an example, combining the ball with $h$-vector (1,3,6,10,6,3,0) formed from the construction of Theorem 14 with the shellable ball with $h$-vector $(1, 2, 0, 0, 0, 0, 0)$ by gluing along two boundary faces gives a ball with $h$-vector (1,5,7,10,5,3,0) but no codimension-one face along which to split.

Beyond dimension five we know that Conjecture 3 does not give a description of the $h$-vectors of balls but we do not have any conjecture to replace it. Determining even just a conjectural description of the $h$-vectors of these higher dimensional balls remains an interesting open problem.

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